

# A VISCOPLASTIC CONSTITUTIVE THEORY FOR MONOLITHIC CERAMIC MATERIALS--I

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## Introduction

With increasing use of ceramic materials in high temperature structural applications such as advanced heat engine components, the need arises to accurately predict thermomechanical behavior (Fig. 1). This paper, which is the first of two in a series, will focus on inelastic deformation behavior associated with these service conditions by providing an overview of a viscoplastic constitutive model that accounts for time-dependent hereditary material deformation (e.g., creep, stress relaxation, etc.) in monolithic structural ceramics.

Early work in the field of metal plasticity indicated that inelastic deformations are essentially unaffected by hydrostatic stress. This is not the case, however, for ceramic-based material systems, unless the ceramic is fully dense. The theory presented here allows for fully dense material behavior as a limiting case. In addition, ceramic materials exhibit different time-dependent behavior in tension and compression. Thus, inelastic deformation models for ceramics must be constructed in a fashion that admits both sensitivity to hydrostatic stress and differing behavior in tension and compression. A number of constitutive theories for materials that exhibit sensitivity to the hydrostatic component of stress have been proposed that characterize deformation using time-independent classical plasticity as a foundation. However, none of these theories allow different behavior in tension and compression. In addition, these theories are somewhat lacking in that they are unable to capture creep, relaxation, and rate-sensitive phenomena exhibited by ceramic materials at high temperature.

When subjected to elevated service temperatures, ceramic materials exhibit complex thermomechanical behavior that is inherently time-dependent, and hereditary in the sense that current behavior depends not only on current conditions, but also on thermo-mechanical history. The objective of this work (Fig. 2) is to present the formulation of a macroscopic continuum theory that captures these time-dependent phenomena. Specifically, the overview contained in this paper focuses on the multi-axial derivation of the constitutive model, and examines the scalar threshold function and its attending geometrical implications. For complete details of the model, the reader is directed to the recent publication by Janosik and Duffy (ref. 1).

## Flow Potential

Using continuum principles of engineering mechanics, the complete viscoplastic theory (Fig. 3) is derived from a scalar dissipative potential function, identified here as  $\Omega$  (Fig. 4), first proposed by Robinson (ref. 2) for a  $J_2$  model, and later extended to sintered powder metals by Duffy (ref. 3). The specific form adopted for the flow potential is an

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integral format proposed by Robinson (ref. 2) that has similar geometrical interpretations (e.g., convexity and normality) as the yield function encountered in classical plasticity. This isothermal formulation includes parameters for viscosity, hardening, recovery, and unitless stress exponents. Also included is the octahedral threshold shear stress,  $K$ , which is generally considered a scalar state variable that accounts for isotropic hardening (or softening). However, since isotropic hardening is often negligible at high homologous temperatures ( $\geq 0.5$ ), to a first approximation  $K$  is taken to be a constant for metals. This assumption is adopted in the present work regarding ceramic materials. Specific details regarding the experimental test matrix needed to characterize these parameters will appear in a second article.

### Threshold Function

The specific formulation used here for the threshold function,  $F$  (a component of the flow potential function), was originally proposed by Willam and Warnke (ref. 4) in order to formulate constitutive equations for time-independent classical plasticity behavior observed in cement and unreinforced concrete. Willam and Warnke (ref. 4) proposed a yield criterion for concrete that admits a dependence on the hydrostatic component of stress and explicitly allows different material responses in tension and compression (Fig. 5). Several formulations of their model exist, i.e., a three-parameter formulation and a five-parameter formulation. For simplicity, the work presented here builds on the three-parameter formulation shown in Fig. 6.

The Willam-Warnke criterion uses stress invariants to define the functional dependence on the Cauchy stress ( $\sigma_{ij}$ ) and internal state variable ( $\alpha_{ij}$ ). The invariants  $\bar{I}_1$  and  $\bar{J}_3$  admit a sensitivity to hydrostatic stress, and account for different behavior in tension and compression (since this invariant changes sign when the direction of a stress component is reversed), respectively. Note that a threshold flow stress is similar in nature to a yield stress in classical plasticity. The specific details involved in deriving the final form of the function  $F$  can be found in Willam and Warnke (ref. 4). A similar functional form is adopted for the scalar state function  $G$ . This formulation assumes a threshold does not exist for the scalar function  $G$ , and follows the framework of previously proposed constitutive models based on Robinson's (ref. 2) viscoplastic law.

For the Willam-Warnke three-parameter formulation, the model parameters include  $\sigma_t$ , the tensile uniaxial threshold stress,  $\sigma_c$ , the compressive uniaxial threshold stress, and  $\sigma_{bc}$ , the equal biaxial compressive threshold stress (Fig. 7). The Willam-Warnke model yields a flow surface in the shape of a pyramid with a triangular base in the Haigh-Westergaard stress space, as depicted in Fig. 5. As a reference, typical  $J_2$  plasticity models have yield surfaces that are right circular cylinders in the Haigh-Westergaard stress space.

### Flow Surfaces - Interpretation

As in Robinson's original theory, the current model is closely tied to the concepts of a potential function and normality. It is this potential-normality structure that provides a consistent framework. According to the stability postulate of Drucker (ref. 5), the concepts of normality and convexity are important requirements which must be imposed on the development of a flow or yield surface. Constitutive relationships developed on the basis of these requirements assure that the inelastic boundary-value problem is well posed, and solutions obtained are unique. Experimental work by Robinson and Ellis (ref. 6) has demonstrated the validity of the potential-normality structure relative to an isotropic  $J_2$  alloy (i.e., type 316 stainless steel). With this structure, the direction of the inelastic strain rate vector for each stress point on a given surface is directed normal to the flow surface  $F=constant$ , as illustrated in Fig. 8. Without experimental evidence to the contrary, it is postulated that this structure is similarly valid for isotropic monolithic ceramic materials. The convexity of the proposed flow surface assures stable material behavior, i.e., positive dissipation of inelastic work, which is based on thermodynamic principles. The convexity requirement (Fig. 8) also implies that level surfaces of a function are closed surfaces, since an open region of the flow surface allows the existence of a load path along which failure will never occur. Finally, the Willam-Warnke flow criterion (and the constitutive theory presented herein) degenerates to simpler models (e.g., the two-parameter Drucker-Prager and the one-parameter Von Mises) under special limiting conditions (Fig. 9).

### Constitutive Model

Constitutive equations formulated for the flow law (strain rate) and evolutionary law are given in Fig. 10. These relationships employ stress invariants to define the functional dependence on the Cauchy stress and a tensorial state

variable. The potential nature of  $\Omega$  is exhibited by the manner in which the flow and evolutionary laws are derived. The flow law,  $\dot{\epsilon}_{ij}$ , is derived from  $\Omega$  by taking the partial derivative with respect to the applied stress. The adoption of a flow potential and the concept of normality, as expressed in the flow law, were introduced by Rice (ref. 7). In his work this relationship was established using thermodynamic arguments. The authors wish to point out that this concept holds for each individual inelastic state. The evolutionary law,  $\dot{\alpha}_{ij}$ , is similarly derived from the flow potential. The rate of change of the internal stress is expressed by taking the partial derivative of  $\Omega$  with respect to the internal stress, and multiplying by a scalar hardening function dependent on the inelastic state variable (i.e., the internal stress) only. Using arguments similar to Rice's, Ponter and Leckie (ref. 8) have demonstrated the appropriateness of this type of evolutionary law.

### Example

A preliminary example was performed to illustrate some of the capabilities of the multiaxial constitutive model developed herein, and to compare the results with those obtained utilizing a  $J_2$  model. In addition, the effects of varying the Willam-Warnke parameters were demonstrated. Figs. 11-13 depict the applied loading condition, approach, and observed results.

A second article will examine specific time-dependent stress-strain behavior that can be modeled with the constitutive relationship presented in this article. No attempt is made here to assess the accuracy of the model in comparison to experiment. A quantitative assessment is reserved for a later date, after the material constants have been suitably characterized for a specific ceramic material.

### Summary/Conclusion

The overview presented in this paper is intended to provide a qualitative assessment of the capabilities of this viscoplastic model in capturing the complex thermomechanical behavior exhibited by ceramic materials at elevated service temperatures. Constitutive equations for the flow law (strain rate) and evolutionary law have been formulated based on a threshold function which exhibits a sensitivity to hydrostatic stress and allows different behavior in tension and compression. Further, inelastic deformation is treated as inherently time-dependent. A rate of inelastic strain is associated with every state of stress. As a result, creep, stress relaxation, and rate sensitivity are phenomena resulting from applied boundary conditions and are not treated separately in an ad hoc fashion. Incorporating this model into a non-linear finite element code would provide industry the means to numerically simulate the inherently time-dependent and hereditary phenomena exhibited by these materials in service.

### References

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# CERAMIC LIFE PREDICTION ANALYSIS

## POTENTIAL FAILURE MODES IN HEAT ENGINE APPLICATIONS:

- Time-Independent Failure Modes:
  - Fast fracture (tension, compression, shear)
  - Buckling
- Time-Dependent Failure Modes:
  - Static, dynamic, and cyclic fatigue
  - Creep crack growth
  - Creep deformation response
  - Stress corrosion and oxidation
  - Impact and contact loading response

## Current Capabilities:

- ✓ Fast-fracture
- ✓ Subcritical Crack Growth
  - High loading
  - Low to intermediate temperatures (~2000 °F)

## Needed Capabilities:

- Creep deformation
- Creep rupture
  - Extended duration loading
  - Elevated temperatures (~2400 °F)

Fig. 1

## OBJECTIVE

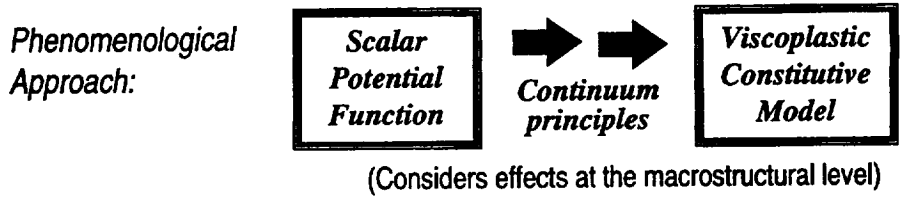
To present a multiaxial continuum theory which accounts for time-dependent hereditary material deformation behavior of isotropic structural ceramics.

## OUTLINE:

- Introduction to Viscoplasticity
- Flow Potential
- Threshold Function
- Flow Surfaces-Interpretation
- Constitutive Model
- Example
- Conclusions/Enhancements

Fig. 2

# VISCOPLASTICITY



## Plasticity:

$F < 0$  Elastic Behavior  
 $F = 0$  Yield Surface  
 $F > 0$  Not Allowed

## Viscoplasticity:

$F < 0$  Elastic Behavior  
 $F = 0$  Threshold Flow Surface  
 $F \geq 0$  Viscoplastic Flow

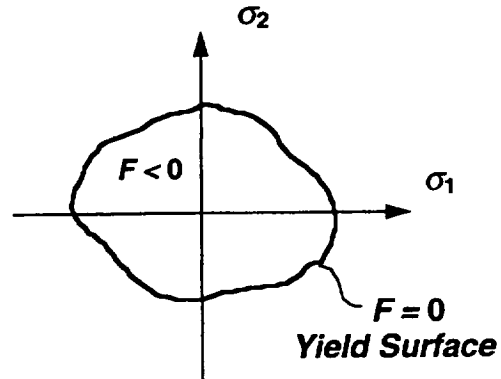


Fig. 3

# FLOW POTENTIAL FUNCTION

General form:

$$\Omega = \Omega(\sigma_{ij}, \alpha_{ij}, T) \xrightarrow{\text{Assume isothermal conditions}} \Omega = \Omega(\sigma_{ij}, \alpha_{ij})$$

applied stress
internal state variable

Specific form:

$$\Omega = K^2 \left[ \left( \frac{1}{2\mu} \right) \int F^n dF + \left( \frac{R}{H} \right) \int G^m dG \right] \quad (\text{Robinson, 1978})$$

where  $\mu, R, H, n, m,$  and  $K$  are material constants characterizing viscoplasticity

$$F = F(\Sigma_{ij}, \eta_{ij})$$

$$G = G(a_{ij}, \alpha_{ij})$$

where:

$$\Sigma_{ij} = S_{ij} - a_{ij}$$

$$\eta_{ij} = \sigma_{ij} - \alpha_{ij}$$

$$S_{ij} = \sigma_{ij} - (1/3)\sigma_{kk}\delta_{ij}$$

$$a_{ij} = \alpha_{ij} - (1/3)\alpha_{kk}\delta_{ij}$$

Fig. 4

## THRESHOLD FUNCTION

- Originally proposed to formulate constitutive equations for time-independent classical plasticity behavior in cement and unreinforced concrete
- Three-parameter model
- Assumes material is isotropic
- Allows different behavior in tension & compression through the invariant  $J_3$
- Admits dependence on the hydrostatic stress through the invariant  $I_1$

$$F = F(I_1, J_2, J_3)$$

where

$$I_1 = (1/3) \sigma_{ii}$$

$$J_2 = (1/2) S_{ij} S_{ji}$$

$$J_3 = (1/3) S_{ij} S_{jk} S_{ki}$$

### Willam-Warnke Model

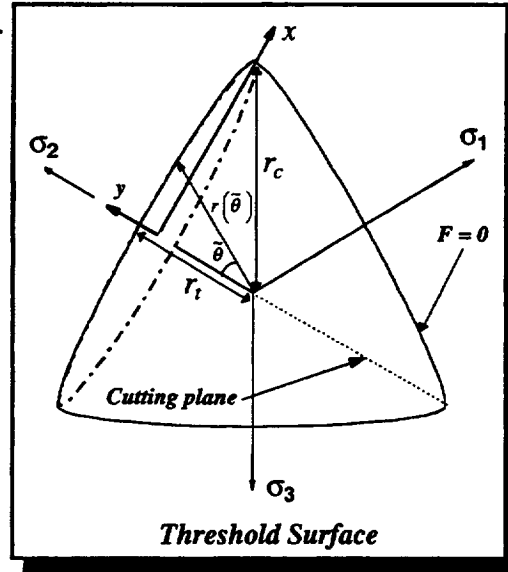


Fig. 5

## THRESHOLD FUNCTION

Inelastic deformation occurs for stress states where

$$F(\Sigma_{ij}, \eta_{ij}) > 0$$

Willam-Warnke  
3-Parameter Formulation:

$$F(\tilde{I}_1, \tilde{J}_2, \tilde{J}_3) = \frac{1}{\sigma_c} \left[ \frac{1}{r(\tilde{\theta})} \right] \left[ \frac{2\tilde{J}_2}{5} \right]^{1/2} + \frac{\tilde{I}_1}{3\rho\sigma_c} - 1$$

(Willam and Warnke, 1975)

The dependence on  $\tilde{J}_3$  is introduced through the angle of similitude  $\tilde{\theta}$

$$\cos(3\tilde{\theta}) = \frac{(3\sqrt{3})\tilde{J}_3}{2(\tilde{J}_2)^{3/2}}$$

Note that similar relationships exist for scalar function G.

Fig. 6

# THRESHOLD PARAMETERS

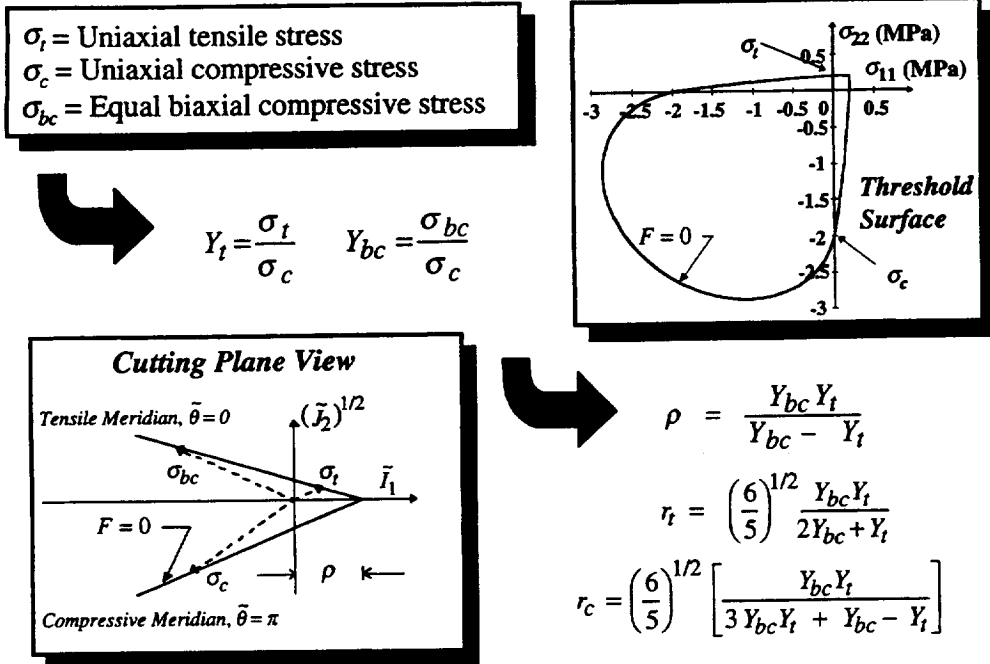
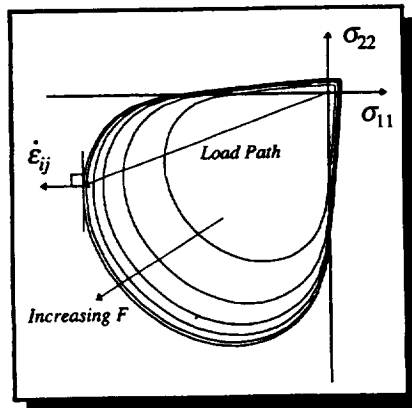


Fig. 7

## FLOW SURFACES

### Normality Condition:

The direction of the inelastic strain rate vector for each stress point on a given surface is directed normal to the flow surface  $F=constant$ .



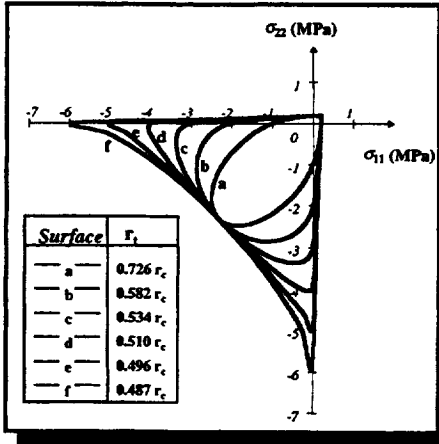
- Flow surfaces eventually cluster forming a limiting surface.
- Implies large changes in elastic strain rate for only small stress changes.

Fig. 8a

# FLOW SURFACES

Convexity Condition:

**Willam-Warnke:  $0.5r_c < r_t \leq r_c$**



- Assures stable material behavior (i.e., positive dissipation of inelastic work).
- Implies that level surfaces of a function are closed surfaces.

Fig. 8b

## FLOW SURFACES (Degeneration to “simpler” models)

Limiting Conditions:

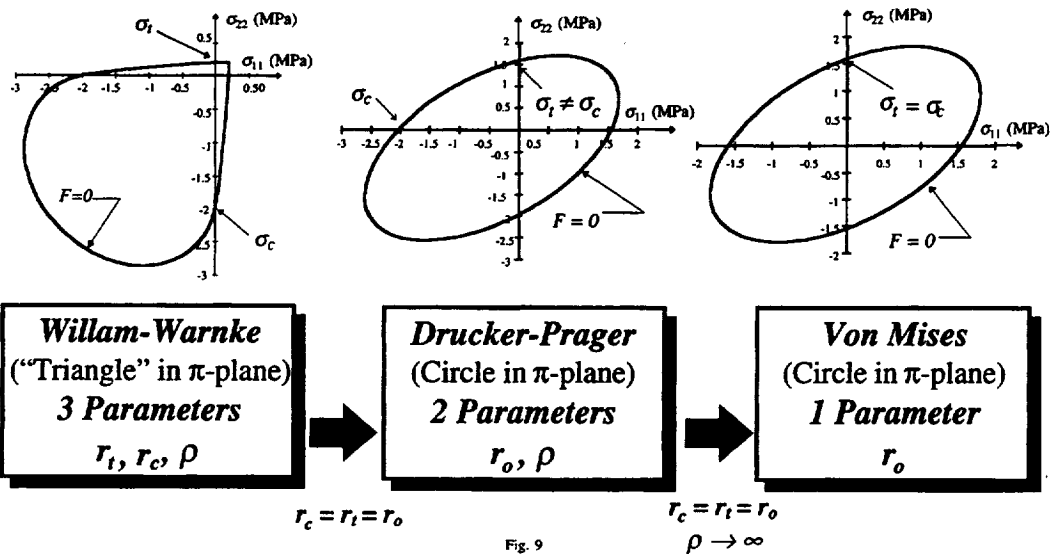


Fig. 9



# CONSTITUTIVE MODEL

The potential nature of  $\Omega$  is exhibited in the extended normality structure

## Flow Law:

$$\dot{\epsilon}_{ij} = \frac{\partial \Omega}{\partial \sigma_{ij}}$$

[Rice (1967), (1971)]



$$\dot{\epsilon}_{ij} = C_0 \left[ C_1 \delta_{ij} + C_2 \Sigma_{ij} + C_3 \left( \Sigma_{jq} \Sigma_{qi} - \frac{2 \tilde{J}_2 \delta_{ij}}{3} \right) \right]$$

Limiting Conditions:

$$r_c = r_t = r_o \text{ and } \rho \rightarrow \infty$$



J<sub>2</sub> Model

$$\dot{\epsilon}_{ij} = C_0 [C_2 \Sigma_{ij}]$$

## Evolutionary Law:

$$\dot{\alpha}_{ij} = -h \frac{\partial \Omega}{\partial \alpha_{ij}}$$

[Ponter & Leckie (1976),  
Ponter (1976)]



$$\dot{\alpha}_{ij} = h \left\{ \dot{\epsilon}_{ij} - C_4 \left[ C_1 \delta_{ij} + C_5 a_{ij} + C_6 \left( a_{jq} a_{qi} - \frac{2 J_2 \delta_{ij}}{3} \right) \right] \right\}$$

where h is a hardening function dependent on the internal stress

Limiting Conditions:

$$r_c = r_t = r_o \text{ and } \rho \rightarrow \infty$$



J<sub>2</sub> Model

$$\dot{\alpha}_{ij} = h \left\{ \dot{\epsilon}_{ij} - C_4 [C_5 a_{ij}] \right\}$$

Fig. 10

## Example: Willam-Warnke Model vs. J<sub>2</sub> Model



$$\sigma_{ij} = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ ksi}$$

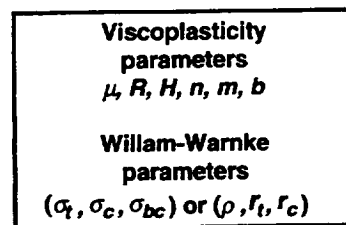
### Willam-Warnke Model:

Subcase A	Subcase B	Subcase C
$\sigma_t = 0.2$	$\sigma_t = 0.2$	$\sigma_t = 0.5$
$\sigma_c = 2.0$	$\sigma_c = 2.0$	$\sigma_c = 1.0$
$\sigma_{bc} = 2.32$	$\sigma_{bc} = 0.2$	$\sigma_{bc} = 1.5$

### J<sub>2</sub> Model:

$$\begin{aligned} \sigma_t &= 1.732 \\ \sigma_c &= 1.732 \\ \sigma_{bc} &= 1.732 \end{aligned}$$

## APPROACH



Computer Algorithm  
(Constitutive Model)

- Creep Curve (Strain vs. Time)
- State Variable vs. Time
- Flow Surface

Fig. 11

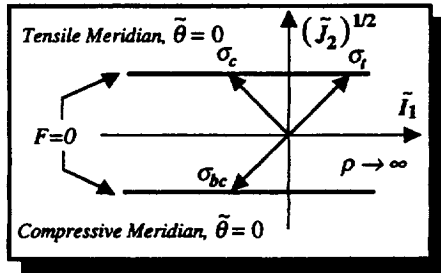
## Willam-Warnke Model vs. $J_2$ Model

### CUTTING PLANE VIEW

#### $J_2$ Model

• **Threshold Parameters**

$$\sigma_t = \sigma_c = \sigma_{bc}$$



$$\rho \rightarrow \infty$$

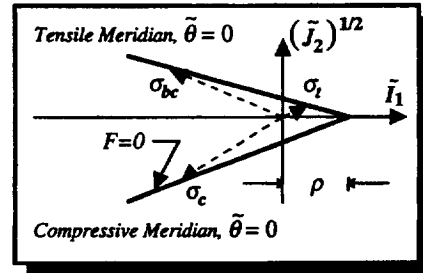
- Meridians are linear
- Meridians are parallel

#### Willam-Warnke Model

• **Threshold Parameters**

$$\sigma_t \neq \sigma_c \neq \sigma_{bc}$$

(For typical ceramics,  $\sigma_t < \sigma_c < \sigma_{bc}$ )



$$\rho = \frac{\sigma_t \sigma_{bc}}{\sigma_c (\sigma_{bc} - \sigma_t)}$$

- Meridians are linear
- Meridians intersect hydrostatic axis

Fig. 12

## Willam-Warnke Model vs. $J_2$ Model

### $J_2$ Model

- **Poisson effect ( $\nu \approx 0.5$ )**
- $\epsilon_{11} = -2\epsilon_{22} = -2\epsilon_{33}$

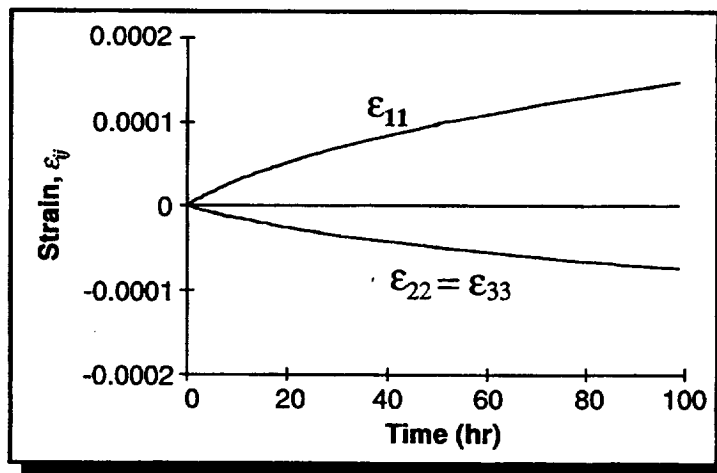


Fig. 13a

## *Willam-Warnke Model vs. $J_2$ Model*

### *Willam-Warnke Model*

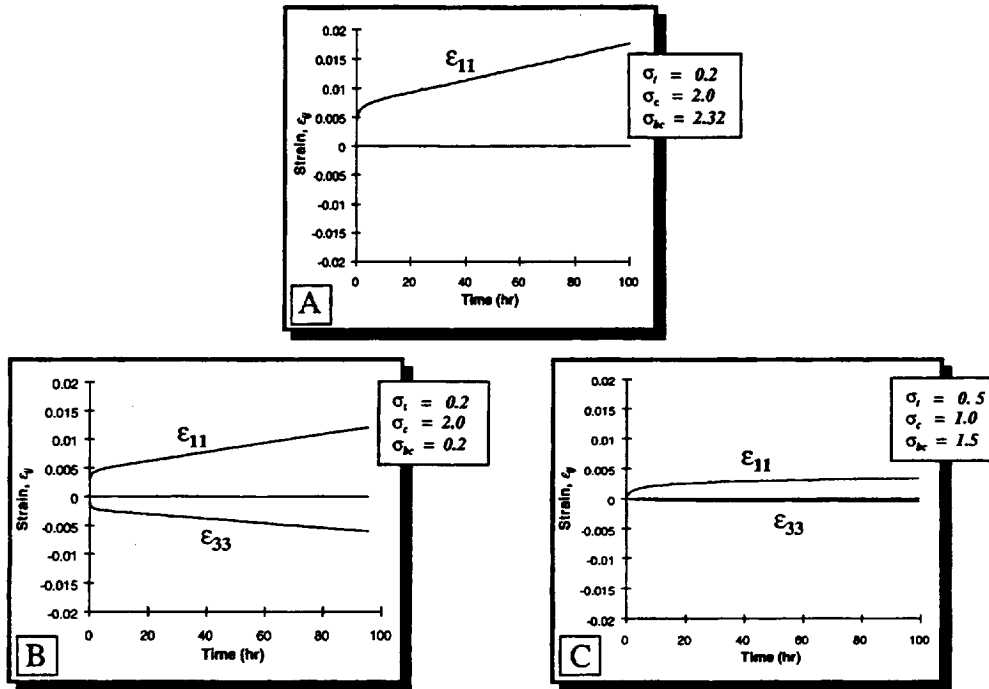


Fig. 13b

## **SUMMARY**

- *Fundamental concepts of viscoplasticity*
- *Specific form for the potential function*
- *Multiaxial derivation of viscoplastic constitutive model*
- *Preliminary applications of constitutive model*

## **FUTURE ENDEAVORS**

- *Incorporate model into computer algorithm*
- *Develop experimental test matrix*
- *Identify parameters applicable to ceramics*
- *Model phenomena such as creep, stress relaxation, strain-rate sensitivity, etc.*
- *Develop user-defined subroutines for commercial FEA software packages*

Fig. 14

