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Homotopy Algorithm for Fixed Order Mixed H_2/H_∞ Design

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Recent developments in the field of robust multivariable control have merged the theories of H_∞ and H_2 control. This mixed H_2/H_∞ compensator formulation allows design for nominal performance by H_2 norm minimization while guaranteeing robust stability to unstructured uncertainties by constraining the H_∞ norm. A key difficulty associated with mixed H_2/H_∞ compensation is compensator synthesis. A homotopy algorithm is presented for synthesis of fixed order mixed H_2/H_∞ compensators. Numerical results are presented for a four disk flexible structure to evaluate the efficiency of the algorithm.

Introduction

MODERN control theory has revolutionized control system design, with H_2 and H_∞ methods gaining widespread recognition and application in controller synthesis for single-input/single-output (SISO) and multi-input/multi-output (MIMO) problems. Early work in multivariable control synthesis utilized a quadratic cost functional to minimize the 2 norm of a system response to white noise inputs. Although the H_2 procedure is well suited to many systems that specify performance in terms of rms quantities such as minimizing line-of-sight errors or control energy, it is well known that stability and performance cannot be guaranteed in the presence of model uncertainties. Robustness is addressed in H_∞ control theory, which guarantees stability and performance (when defined by an ∞-norm measure) in the presence of unstructured uncertainty models, albeit often resulting in overly conservative designs. A significant disadvantage of these modern control techniques is that the resulting compensator is the same order as the generalized plant, which is often larger than the original plant due to the inclusion of frequency dependent weights to achieve the desired performance and robustness characteristics.

The consequential large controller order can be indirectly alleviated by reducing the order of the controller or alternatively by reducing the order of the design plant. In either case, indirect methods are suboptimal in performance and do not guarantee closed-loop stability. However, direct methods may be employed that impose constraints on controller order or architecture in the optimization procedure and, hence, provide stability and performance guarantees. In an optimization-based synthesis procedure, necessary conditions are formulated for the constrained closed-loop system that ensure internal stability. The optimal projection approach[1] is an H_2 procedure whereby order constraints are imposed on the controller and the necessary conditions for minimizing a quadratic cost functional with respect to the fixed-order controller are derived. The resulting necessary conditions consist of two modified Riccati equations and two modified Lyapunov equations coupled by an oblique projection matrix. However, solution of the necessary conditions for realistic large-order systems is a difficult task. Homotopy methods have been employed to solve the optimal projection equations.[2]

As a means of providing robustness and performance, the mixed H_2/H_∞ methodology has been developed. Much work has been done with variations of the mixed H_2/H_∞ problem. (For a summary, see Ref. 3.) The earliest refereed work was done by Bernstein and Haddad,[4] who extended the optimal projection approach to linear quadratic Gaussian (LQG) control with an H_∞ norm constraint. Their formulation minimized an overbound on the H_2 norm from a disturbance input to one output while satisfying an H_∞ norm overbound from the same disturbance input to a second output. The difficulty with this approach is the potential conservatism resulting from minimizing an overbound on the H_2 norm. The first attempt at solving the general mixed H_2/H_∞ problem was by Rotea and Khargonekar,[5] who allowed independent inputs and outputs for the two transfer functions and minimized the actual H_2 norm based on full state feedback. Ridgely and Walker[3] extended the formulation to output feedback including the fixed-order case with either regular or singular H_∞ constraints, and Canfield et at.[6] provided a numerical solution. Another approach to the general mixed H_2/H_∞ problem was developed by Sweriduk and Calise,[7] who used a differential games formulation to obtain fixed-order controllers. A numerical solution of this formulation using homotopy was developed by Whorton et al.,[8] an extension of which is the subject of this paper. Davis et al.[9] also recently developed an algorithm for mixed H_2/H_∞ design.

Ridgely and Walker[3] showed the mixed H_2/H_∞ to be a strictly convex optimization problem with a unique solution when the controller is of the order equal to or larger than the underlying H_2 problem. The solution is shown to lie on the boundary of the infinity norm constraint when active (for $\gamma < \bar{\gamma}$, where $\bar{\gamma}$ is the H_∞ norm when the optimal H_2 controller is used) and is just the H_2 controller when $\gamma \geq \bar{\gamma}$. For controllers with order less than the underlying H_2 problem, the possibility of local minima in the unconstrained H_2 problem (optimal projection) exists and the solution of the mixed H_2/H_∞ problem may or may not lie on the boundary of the H_∞ norm constraint. As will be shown in the next section, Sweriduk and Calise[7] begin with the fixed-order H_∞ cost functional and append the fixed-order H_2 cost functional. Reference 3 takes the converse approach by appending the H_∞ norm constraint to the H_2 cost functional. As a consequence, the formulation in Ref. 3 can handle both regular and singular H_∞ constraints and can be specialized to the H_2 problem. Conversely, the Ref. 7 formulation can handle both regular and singular H_2 constraints and can be specialized to the H_∞ problem. However, the ability to handle singular constraints on either cost exists as long as one of the cost constraints is regular. Whereas the Ref. 7 formulation assumes that the plant dynamics (A matrix) of the two transfer functions are the same, Ref. 3 allows different dynamics in the two separate underlying H_∞ and H_2

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problems. Using the canonical compensator with a static gain output formulation presented in following section, the Ref. 7 formulation requires the simultaneous solution of five coupled nonlinear matrix equations. Reference 3 presents the necessary conditions in the form of seven coupled, nonlinear matrix equations.

The objective is to build on the results of Ref. 7 by presenting a homotopy algorithm that solves the mixed H_2/H_∞ fixed-order compensator synthesis problem. Fixed-order H_2 and H_∞ controllers are obtained as special cases of the algorithm. The paper is organized as follows. First, a formulation of the problem with the compensator in controller canonical form is presented and the necessary conditions for the fixed-order H_2 controller are developed. These results are then extended to the fixed-order H_∞ and mixed H_2/H_∞ controller design using the differential game results of Ref. 7. (The approach and algorithms of this paper also follow Refs. 10 and 11 for H_2 controller synthesis.) Second, homotopy methods are introduced and a homotopy algorithm is developed to synthesize mixed H_2/H_∞ compensators. Finally, numerical results are presented for evaluation of these homotopy algorithms followed by a discussion and concluding remarks.

Problem Formulation

The generalized plant of a standard control problem is given by

$$\dot{x} = Ax + B_1 w + B_2 u \tag{1}$$

$$z = C_1 x + D_{12} u \tag{2}$$

$$y = C_2 x + D_{21} w + D_{22} u \tag{3}$$

where $x \in R^n$ is the state vector, $w \in R^{nw}$ is the disturbance vector, $u \in R^{nu}$ is the control vector, $z \in R^{nz}$ is the performance vector, and $y \in R^{ny}$ is the measurement vector. The following is assumed.

1) (A, B_1, C_1) is stabilizable and detectable.
2) (A, B_p, C_p) is stabilizable and detectable.
3) D_{12} has full column rank.
4) D_{21} has full row rank.

A general compensator for this system is

$$\dot{x}_c = A_c x_c + B_c y \tag{4}$$

$$u = C_c x_c \tag{5}$$

where $x_c \in R^{nc}$ is the state vector of the controller the dimension of which can be specified. Closing the loop using negative feedback yields the closed-loop system dynamics

$$\dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{B}w \tag{6}$$

$$z = \tilde{C}\tilde{x} \tag{7}$$

where

$$\tilde{x} = \begin{bmatrix} x \\ x_c \end{bmatrix} \tag{8}$$

$$\tilde{A} = \begin{bmatrix} A & -B_2 C_c \\ B_c C_2 & A_c - B_c D_{22} C_c \end{bmatrix} \tag{9}$$

$$\tilde{B} = \begin{bmatrix} B_1 \\ B_c D_{21} \end{bmatrix} \tag{10}$$

$$\tilde{C} = [C_1 \quad -D_{12} C_c] \tag{11}$$

The set of all internally stabilizing compensators is defined as

$$S_c = \{(A_c, B_c, C_c) : \tilde{A} \text{ is asymptotically stable}\} \tag{12}$$

For an H_2 problem, the objective is to minimize the H_2 norm on the closed-loop transfer function from disturbance inputs to performance outputs

$$T_{zw} = \tilde{C}(sI - \tilde{A})^{-1}\tilde{B} \tag{13}$$

where the disturbances are confined to the set of signals with bounded power and fixed spectra. If the disturbance is modeled as white noise, the objective is

$$\min_{S_c}\left\{ J(A_c, B_c, C_c) = \lim_{t\to\infty} E\{z(t)^T z(t)\}\right\} \tag{14}$$

It can be shown that the cost can be expressed as

$$J(A_c, B_c, C_c) = \text{tr}\{Q\tilde{B}\tilde{B}^T\} = \text{tr}\{P\tilde{C}^T\tilde{C}\} \tag{15}$$

where

$$\tilde{A}P + P\tilde{A}^T + \tilde{B}\tilde{B}^T = 0 \tag{16}$$

$$\tilde{A}^T Q + Q\tilde{A} + \tilde{C}^T\tilde{C} = 0 \tag{17}$$

P is the controllability grammian of $(\tilde{A}, \tilde{B})$, and Q is the observability grammian of $(\tilde{C}, \tilde{A})$. An equivalent cost functional also arises for the case of impulsive inputs.

To obtain the H_2 optimal compensator, the Lagrangian is defined as

$$\mathcal{L}(Q, L, A_c, B_c, C_c) = \text{tr}\{Q\tilde{B}\tilde{B}^T + (\tilde{A}^T Q + Q\tilde{A} + \tilde{C}^T\tilde{C})L\} \tag{18}$$

where L is a symmetric matrix of multipliers. Matrix gradients are taken to determine the first-order necessary conditions,

$$\frac{\partial \mathcal{L}}{\partial Q} = 0, \qquad \frac{\partial \mathcal{L}}{\partial L} = 0, \qquad \frac{\partial \mathcal{L}}{\partial A_c} = 0$$
$$\frac{\partial \mathcal{L}}{\partial B_c} = 0, \qquad \frac{\partial \mathcal{L}}{\partial C_c} = 0 \tag{19}$$

Hence, computing an H_2 optimal controller of fixed order $nc < n$ for the general controller structure given in Eqs. (4) and (5) requires the simultaneous solution of five coupled equations. This is not only computationally expensive but is also further complicated by the fact that the problem is overparametrized with such a compensator.

A controller form architecture[12] is imposed on the compensator dynamics. This minimal realization avoids the problem of overparametrization and is a canonical form under mild conditions.[13] The internal structure of the compensator is prespecified by assigning a set of feedback invariant indices ν_i. In controller canonical form the compensator is defined as

$$\dot{x}_c = P^0 x_c + N^0 u_c - N^0 y \tag{20}$$

$$u_c = -P x_c \tag{21}$$

$$u = -H x_c \tag{22}$$

where $x_c \in R^{nc}$ and $u_c \in R^{ny}$. P and H are free-parameter matrices, and P^0 and N^0 are fixed matrices of zeros and ones determined by the choice of controllability indices. Similarly, a compensator in observer canonical form can be constructed. Only the controller canonical form is employed, which imposes the lower bound $nc \geq ny$ on the order of the compensator.

Let

$$\bar{u} = \begin{bmatrix} u \\ u_c \end{bmatrix} \tag{23}$$

The augmented system may be expressed as

$$\begin{aligned} \dot{\bar{x}} &= \begin{bmatrix} A & 0 \\ -N^0 C_2 & P^0 \end{bmatrix}\bar{x} + \begin{bmatrix} B_1 \\ -N^0 D_{21} \end{bmatrix} w + \begin{bmatrix} B_2 & 0 \\ -N^0 D_{22} & N^0 \end{bmatrix}\bar{u} \\ &= \bar{A}\bar{x} + \bar{B}_1 w + \bar{B}_2 \bar{u} \end{aligned} \tag{24}$$

$$\begin{aligned} z &= [C_1 \quad 0]\bar{x} + [D_{12} \quad 0]\bar{u} \\ &= \bar{C}_1 \bar{x} + \bar{D}_{12}\bar{u} \end{aligned} \tag{25}$$

$$\bar{y} = [0 \quad I]\bar{x} = \bar{C}_2\bar{x} \tag{26}$$

$$\bar{u} = -\begin{bmatrix} H \\ P \end{bmatrix}\bar{y} = -G\bar{y} \tag{27}$$

Equations (24–27) define a static gain output feedback problem where the compensator is represented by a minimal number of free parameters in the design matrix G. The closed-loop system is given by

$$\begin{aligned} \dot{\bar{x}} &= (\bar{A} - \bar{B}_2 G \bar{C}_2)\bar{x} + \bar{B}_1 w \\ &= \tilde{A}\bar{x} + \tilde{B}w \end{aligned} \tag{28}$$

$$z = (\bar{C}_1 - \bar{D}_{12} G \bar{C}_2)\bar{x} = \tilde{C}\bar{x} \tag{29}$$

Minimizing the H_2 norm of $T_{zw} = \tilde{C}(sI - \tilde{A})^{-1}\tilde{B}$ utilizes the same Lagrangian as given in Eq. (18), but now $\mathcal{L}$ is only a function of three parameter matrices, i.e., $\mathcal{L}(Q, L, G)$. Thus, only three first-order necessary conditions result:

$$\frac{\partial \mathcal{L}}{\partial Q} = \tilde{A}L + L\tilde{A}^T + \tilde{B}\tilde{B}^T = 0 \tag{30}$$

$$\frac{\partial \mathcal{L}}{\partial L} = \tilde{A}^T Q + Q\tilde{A} + \tilde{C}^T\tilde{C} = 0 \tag{31}$$

$$\frac{\partial \mathcal{L}}{\partial G} = 2\left(\bar{D}_{12}^T\bar{D}_{12}G\bar{C}_2 - \bar{D}_{12}^T\bar{C}_1 - \bar{B}_2^T Q\right)L\bar{C}_2^T = 0 \tag{32}$$

Controller canonical forms can also be used to solve the H_∞ problem. The objective is now to minimize the ∞ norm of the transfer function from disturbance inputs w to performance outputs z given in Eq. (13). In this case the necessary conditions for an H_∞ suboptimal fixed-order compensator gain G are[7]

$$\frac{\partial \mathcal{L}}{\partial Q_\infty} = \left(\tilde{A} + \gamma^{-2}\tilde{B}\tilde{B}^T Q_\infty\right)L + L\left(\tilde{A} + \gamma^{-2}\tilde{B}\tilde{B}^T Q_\infty\right)^T + \tilde{B}\tilde{B}^T = 0 \tag{33}$$

$$\frac{\partial \mathcal{L}}{\partial L} = \tilde{A}^T Q_\infty + Q_\infty\tilde{A} + \tilde{C}^T\tilde{C} + \gamma^{-2}Q_\infty\tilde{B}\tilde{B}^T Q_\infty = 0 \tag{34}$$

$$\frac{\partial \mathcal{L}}{\partial G} = 2\left(\bar{D}_{12}^T\bar{D}_{12}G\bar{C}_2 - \bar{D}_{12}^T\bar{C}_1 - \bar{B}_2^T Q_\infty\right)L\bar{C}_2^T = 0 \tag{35}$$

where

$$\mathcal{L}(Q_\infty, L, G) = \mathrm{tr}\left\{Q_\infty\tilde{B}\tilde{B}^T + \left(\tilde{A}^T Q_\infty + Q_\infty\tilde{A} + \tilde{C}^T\tilde{C} + \gamma^{-2}Q_\infty\tilde{B}\tilde{B}^T Q_\infty\right)L\right\} \tag{36}$$

As in the H_2 problem, three coupled equations have to be solved to obtain a fixed-order compensator that satisfies the constraint $\|T_{zw}\|_\infty < \gamma$.

Fixed-order H_∞ design has also been extended to fixed-order μ synthesis.[14,15] Because H_∞ controller design is a subproblem when designing for robust performance with structured uncertainty, the fixed-order technique just introduced has the potential to constrain the order of the controller that is normally subject to significant increases in the μ-synthesis procedure.

The mixed H_2/H_∞ problem can be approached in a similar fashion.[7] In this case, the generalized plant has additional inputs and outputs w_p and z_p, respectively, which define the H_2 performance criterion. The plant dynamics are illustrated in Fig. 1. The inputs w and outputs z are used to define the H_∞ performance criterion. Using the controller canonical form for the compensator, the augmented system for the mixed problem is

$$\dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}_1 w + \bar{B}_p w_p + \bar{B}_2\bar{u} \tag{37}$$

$$z_p = \bar{C}_p\bar{x} + \bar{D}_{1p}\bar{u} \tag{38}$$

$$z = \bar{C}_1\bar{x} + \bar{D}_{12}\bar{u} \tag{39}$$

$$\bar{y} = \bar{C}_2\bar{x} \tag{40}$$

$$\bar{u} = -G\bar{y} \tag{41}$$

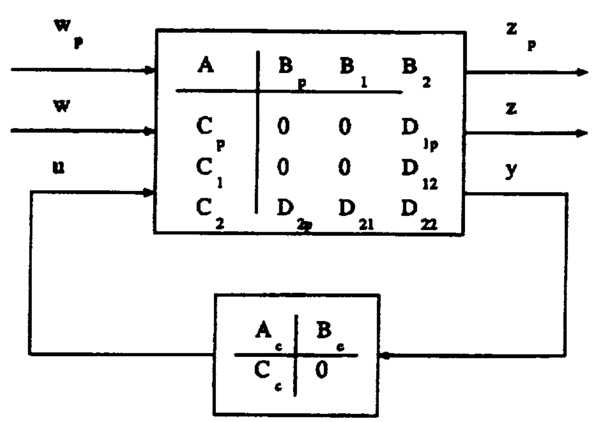

Fig. 1 Mixed H_2/H_∞ problem.

where

$$\bar{B}_p = \begin{bmatrix} B_p \\ -N^0 D_{2p} \end{bmatrix} \tag{42}$$

$$\bar{C}_p = [C_p \quad 0] \tag{43}$$

$$\bar{D}_{1p} = [D_{1p} \quad 0] \tag{44}$$

The other expressions are the same as in Eqs. (24–27). Consequently, the closed-loop system is given by

$$\begin{aligned} \dot{\bar{x}} &= (\bar{A} - \bar{B}_2 G\bar{C}_2)\bar{x} + \bar{B}_p w_p + \bar{B}_1 w \\ &= \tilde{A}\bar{x} + \bar{B}_p w_p + \tilde{B}w \end{aligned} \tag{45}$$

$$z_p = (\bar{C}_p - \bar{D}_{1p}G\bar{C}_2)\bar{x} = \tilde{C}_p\bar{x} \tag{46}$$

$$z = (\bar{C}_1 - \bar{D}_{12}G\bar{C}_2)\bar{x} = \tilde{C}\bar{x} \tag{47}$$

To formulate the performance index of the mixed problem, the Lagrangian for the H_2 problem in Eq. (18) is adjoined to the Lagrangian for the H_∞ problem in Eq. (36) by a scalar weight λ:

$$\mathcal{L} = \mathrm{tr}\left\{Q_\infty\tilde{B}\tilde{B}^T + \left(\tilde{A}^T Q_\infty + Q_\infty\tilde{A} + \tilde{C}^T\tilde{C} + \gamma^{-2}Q_\infty\tilde{B}\tilde{B}^T Q_\infty\right)L + \lambda X\tilde{C}_p^T\tilde{C}_p + \left(\tilde{A}X + X\tilde{A}^T + \bar{B}_p\bar{B}_p^T\right)L_p\right\} \tag{48}$$

The weight λ on the H_2 norm allows a tradeoff between performance (H_2 norm) and robustness (H_∞ norm). The first-order necessary conditions are

$$\frac{\partial \mathcal{L}}{\partial Q_\infty} = \left(\tilde{A} + \gamma^{-2}\tilde{B}\tilde{B}^T Q_\infty\right)L + L\left(\tilde{A} + \gamma^{-2}\tilde{B}\tilde{B}^T Q_\infty\right)^T + \tilde{B}\tilde{B}^T = 0 \tag{49}$$

$$\frac{\partial \mathcal{L}}{\partial L} = \tilde{A}^T Q_\infty + Q_\infty\tilde{A} + \tilde{C}^T\tilde{C} + \gamma^{-2}Q_\infty\tilde{B}\tilde{B}^T Q_\infty = 0 \tag{50}$$

$$\frac{\partial \mathcal{L}}{\partial X} = \tilde{A}^T L_p + L_p\tilde{A} + \lambda\tilde{C}_p^T\tilde{C}_p = 0 \tag{51}$$

$$\frac{\partial \mathcal{L}}{\partial L_p} = \tilde{A}X + X\tilde{A}^T + \bar{B}_p\bar{B}_p^T = 0 \tag{52}$$

$$\frac{\partial \mathcal{L}}{\partial G} = 2\left[\bar{D}_{12}^T\bar{D}_{12}G\bar{C}_2 L\bar{C}_2^T - \bar{D}_{12}^T\bar{C}_1 L\bar{C}_2^T - \bar{B}_2^T Q_\infty L\bar{C}_2^T + \lambda\bar{D}_{1p}^T\bar{D}_{1p}G\bar{C}_2 X\bar{C}_2^T - \lambda\bar{D}_{1p}^T\bar{C}_p X\bar{C}_2^T - \bar{B}_2^T L_p X\bar{C}_2^T\right] = 0 \tag{53}$$

Promising results have been obtained for the H_∞ and the mixed problem where a conjugate gradient method was used in the computation.[7] A disadvantage of this method is that convergence slows down near the optimum. Also, an initial starting guess for the compensator gain G has to be provided that stabilizes the closed-loop system. In this paper, a homotopy method is used to continuously deform the solution of a simple problem formulation to the solution of the desired problem formulation.

Homotopy Methods

Homotopy methods offer an attractive alternative to more standard approaches of optimal controller synthesis such as sequential and conjugate gradient methods. The basic philosophy of homotopy methods is to deform a problem that is relatively easily solved into the problem for which a solution is desired.

Homotopy (or continuation) methods, arising from algebraic and differential topology, embed a given problem in a parametrized family of problems. More specifically, consider sets U and $Y \in \Re^n$ and a mapping $F: U \rightarrow Y$, where solutions of the problem

$$F(\boldsymbol{u}) = 0 \tag{54}$$

are desired with $\boldsymbol{u} \in U$ and $F(\boldsymbol{u}) \in Y$. The homotopy function is defined by the mapping $H: Ux[0, 1] \rightarrow \Re^n$ such that

$$H(\boldsymbol{u}_1, 1) = F(\boldsymbol{u}) \tag{55}$$

and there exists a known (or easily calculated) solution $\boldsymbol{u}_0$, such that

$$H(\boldsymbol{u}_0, 0) = 0 \tag{56}$$

The homotopy function is a continuously differentiable function given by

$$H[\boldsymbol{u}(\alpha), \alpha] = 0, \qquad \forall \alpha \in [0, 1] \tag{57}$$

Thus, the homotopy begins with a simple problem with a known solution, Eq. (56), which is deformed by continuously varying the parameter until the solution of the original problem, Eq. (54), is obtained.[16] The power of homotopy methods is that minimization is not strongly dependent on the starting solution but depends on local, small variations in the solution. Theoretically, these methods are globally convergent for a wide range of complex optimization problems, but in actuality, finite wordlength computation often introduces numerical ill-conditioning, resulting in difficulties with convergence. In light of these numerical limitations, a judicious choice of the initial problem and the associated initial stabilizing compensator is necessary for convergence and efficient computation. However, the ability to select an initial problem with a simple solution renders homotopy methods more widely applicable than sequential or gradient-based methods, which have a stringent requirement for an initial stabilizing solution.

Both discrete and continuous methods are used to solve the homotopy. Discrete methods simply partition the interval [0, 1] to obtain a finite chain of problems

$$H(\boldsymbol{u}, \alpha_n) = 0, \qquad 0 = \alpha_0 < \alpha_1 < \cdots < \alpha_N = 1 \tag{58}$$

Starting with a known solution at α_n, the solution for $H(\boldsymbol{u}, \alpha_{n+1})$ is computed by a local iteration scheme. Continuous methods involve integration of Davidenko's differential equation, which is obtained by differentiating Eq. (57) with respect to α, yielding

$$\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}\alpha} = -\left(\frac{\partial H}{\partial \boldsymbol{u}}\right)^{-1} \frac{\partial H}{\partial \alpha} \tag{59}$$

Given $\boldsymbol{u}(0) = \boldsymbol{u}_0$, this initial value problem may be numerically integrated to obtain the solution at $\alpha = 1$ if the solution exists and is uniquely defined.

Research remains to be done in the application of homotopy algorithms. Efficient application of homotopy methods depends on the initial problem, the final problem, and the deformation undertaken. Given a good initial solution, the key to convergence is the ability to accurately track the solution curve, which is determined by the deformation undertaken. The ability to predict the solution along the homotopy path via Davidenko's differential equation makes continuous homotopy methods superior to discrete methods. Efficient computation of the Hessian is the primary issue for practical implementation of continuous homotopy algorithms. In the following sections, a continuous homotopy algorithm is presented for fixed-order mixed H_2/H_∞ compensator design. The set of homotopy algorithms for synthesizing fixed-order H_2, H_∞, and mixed H_2/H_∞ compensators has been organized into a MATLAB™ toolbox called fixed-order compensation of uncertain systems (FOCUS).

Homotopy Algorithm

This section describes the algorithm used for implementing the continuous homotopy. In essence, a mixed discrete and continuous approach is employed where Davidenko's differential equation, Eq. (59), is integrated along the homotopy path, and at discrete points along the trajectory, a local optimization is used to remove integration error. The optimization scheme is a hybrid Newton/conjugate direction search method developed for this application.[17] Local optimization at discrete points along the homotopy trajectory allows a crude integration procedure with large step sizes to be employed for efficiently tracking the solution curve. This approach follows closely that of Refs. 10 and 11 and is employed in the following algorithm.

1) Find initial solution ($\alpha = 0$).

2) Advance α

$$\alpha_{1,k} = \alpha_0 + \Delta\alpha_{0,k}$$

3) Predict $\boldsymbol{\theta}$

$$\boldsymbol{\theta}(\alpha_{1,k}) = \boldsymbol{\theta}(\alpha_0) + \Delta\alpha_{0,k}\boldsymbol{\theta}'(\alpha_0)$$

where

$$\boldsymbol{\theta}'(\alpha) = \frac{\mathrm{d}\boldsymbol{\theta}}{\mathrm{d}\alpha} = -\left(\frac{\partial H}{\partial \boldsymbol{\theta}}\right)^{-1} \frac{\partial H}{\partial \alpha}$$

4) Check prediction error

$$e_k(\boldsymbol{\theta}, \alpha) = \|J_\theta[\boldsymbol{\theta}(\alpha_{1,k})]\| < \epsilon$$

a) if error less than tolerance, continue; b) if not, $0.5\Delta\alpha_{0,k} \rightarrow \Delta\alpha_{0,k+1}$; and c) increment k and repeat steps 2–4.

5) Correct with hybrid Newton/conjugate direction method to compute local minimum.

6) If $\alpha = 1$, stop. Otherwise, go to step 2.

Various approaches may be taken when selecting the deformation, but the general procedure applied in this effort is outlined as follows.

1) Synthesize a low-authority H_2 (full-order) compensator.

2) Reduce the compensator to desired order and transform to canonical form.[12]

3) Set γ large enough so that the H_2 and H_∞ compensators are approximately equivalent.

4) Use homotopy to deform the initial low-authority, reduced-order H_2 compensator into a full-authority reduced-order H_2 compensator (H_2 homotopy).

5) Deform the full-authority H_2 compensator into a nearly optimal H_∞ compensator with γ approaching its infimum (H_∞ homotopy).

6) At discrete values of λ, fix γ and deform the compensator into the mixed H_2/H_∞ compensator by varying λ (mixed H_2/H_∞ homotopy).

This procedure was chosen because it has been observed numerically that order reduction techniques tend to work best for low-authority LQG controllers.[10,18] A canonical form is imposed on the compensator structure to minimize the number of free parameters, which in some cases can also lead to numerical ill conditioning. A balancing transformation, which does not affect the controller characteristics, relaxes the strict structure in the P^0 and N^0 matrices in Eq. (20) and improves the conditioning of the problem.

The procedure outlined separates the compensator synthesis into distinct phases. The initial reduced-order full-authority compensator is synthesized using the H_2 homotopy, which is then deformed into the reduced-order H_∞ compensator. During the H_∞ phase, the scalar H_2 norm weight λ is fixed (as are the plant matrices) and only the H_∞ norm overbound γ is varied. At discrete values, γ is fixed and λ is varied to perform the H_2 norm minimization. Thus, the procedure alternates between the H_∞ and H_2 norm minimization.

During the homotopy, both the predicted and corrected gains are checked to ensure closed-loop stability. After each correction step, the cost gradient is checked to verify descent. During the H_∞ homotopy, the solvability of the Riccati equation using predicted or corrected gains must also be checked. If any of these conditions are violated during correction, the correction step size is scaled and the condition is checked again. If scaling the correction step size is ineffective, the prediction step size is decreased and the prediction phase is repeated. This process continues until the homotopy is completed or until the prediction step size is decreased below a prespecified tolerance.

The following sections detail the derivations employed in the homotopy algorithm for mixed H_2/H_∞ design. A complete development of the H_2, H_∞, and mixed H_2/H_∞ algorithms is given in Ref. 8.

Mixed H_2/H_∞ Development

The homotopy function as well as the gradient and Hessian matrices are determined from the first-order necessary conditions for an optimal mixed H_2/H_∞ compensator given by Eqs. (49–53). Define θ to be a vector comprising the free compensator parameters

$$\theta = \mathrm{vec}(G) \tag{60}$$

where G is the output feedback gain matrix defined in Eq. (27). The gradient of the cost is

$$f(\theta) = \frac{\partial \mathcal{L}}{\partial \theta} = \mathrm{vec}\left(\frac{\partial \mathcal{L}}{\partial G}\right) = 0 \tag{61}$$

where $\partial\mathcal{L}/\partial G$ is given by Eq. (53).

The homotopy function is defined as

$$H(\theta,\alpha) = \frac{\partial \mathcal{L}(\theta,\alpha)}{\partial \theta} = \mathrm{vec}\left(\frac{\partial \mathcal{L}(\theta,\alpha)}{\partial G}\right) = 0 \tag{62}$$

Note that $\mathcal{L}$ is now a function of the homotopy parameter α since the system matrices are now functions of α. The gradient of the homotopy function is

$$\nabla[H^T(\theta,\alpha)] = \begin{bmatrix}\nabla_\theta H^T & \nabla_\alpha H^T\end{bmatrix} \tag{63}$$

Computation of Hessian

The derivative of the $N \times 1$ vector valued homotopy function, $H^T(\theta) = [h_1(\theta), h_2(\theta), \ldots, h_N(\theta)]$, with respect to the N parameter vector θ is the $N \times N$ Hessian matrix given by

$$\nabla_\theta H = \begin{bmatrix}\dfrac{\partial H}{\partial \theta_1} & \dfrac{\partial H}{\partial \theta_2} & \cdots & \dfrac{\partial H}{\partial \theta_N}\end{bmatrix} \tag{64}$$

where

$$\frac{\partial H}{\partial \theta_j} = \mathrm{vec}\left[\frac{\partial(\partial\mathcal{L}/\partial G)}{\partial \theta_j}\right] \tag{65}$$

and using Eq. (53),

$$\frac{\partial H}{\partial \theta_j} = \mathrm{vec}\left(\frac{\partial}{\partial \theta_j}\{2[(\bar D_{12}^T \bar D_{12} G \bar C_2 - \bar D_{12}^T \bar C_1 - \bar B_2^T Q_\infty) L \bar C_2^T + (\lambda \bar D_{1p}^T \bar D_{1p} G \bar C_2 - \lambda \bar D_{1p}^T \bar C_p - \bar B_2^T L_p) X \bar C_2^T]\}\right) \tag{66}$$

$$\begin{aligned} = \mathrm{vec}\{2[&(\bar D_{12}^T \bar D_{12} G^{(j)} \bar C_2 - \bar B_2^T Q_\infty^{(j)}) L \bar C_2^T \\ &+ (\bar D_{12}^T \bar D_{12} G \bar C_2 - \bar D_{12}^T \bar C_1 - \bar B_2^T Q_\infty) L^{(j)} \bar C_2^T \\ &+ (\lambda \bar D_{1p}^T \bar D_{1p} G^{(j)} \bar C_2 - \bar B_2^T L_p^{(j)}) X \bar C_2^T \\ &+ (\lambda \bar D_{1p}^T \bar D_{1p} G \bar C_2 - \lambda \bar D_{1p}^T \bar C_p - \bar B_2^T L_p) X^{(j)} \bar C_2^T]\}\end{aligned} \tag{67}$$

The derivatives with respect to θ are denoted

$$(*)^{(j)} = \frac{\partial(*)}{\partial \theta_j} \tag{68}$$

For $\theta_j = g_{ik}$,

$$G^{(j)} = \frac{\partial G}{\partial \theta_j} = E_{ik} \tag{69}$$

which is a matrix of zeros except for a one in the ik element.

To obtain expressions for $L^{(j)}$, $Q_\infty^{(j)}$, $L_p^{(j)}$, and $X^{(j)}$, differentiate Eqs. (49–52) with respect to θ_j to obtain

$$\begin{aligned} 0 = {} & (\tilde A + \gamma^{-2}\bar B \bar B^T Q_\infty) L^{(j)} + L^{(j)} (\tilde A + \gamma^{-2}\bar B \bar B^T Q_\infty)^T \\ & + [(\tilde A^{(j)} + \gamma^{-2}\bar B \bar B^T Q_\infty^{(j)}) L + L (\tilde A^{(j)} + \gamma^{-2}\bar B \bar B^T Q_\infty^{(j)})^T]\end{aligned} \tag{70}$$

$$\begin{aligned} 0 = {} & (\tilde A + \gamma^{-2}\bar B \bar B^T Q_\infty)^T Q_\infty^{(j)} + Q_\infty^{(j)} (\tilde A + \gamma^{-2}\bar B \bar B^T Q_\infty) \\ & + [\tilde A^{(j)T} Q_\infty + Q_\infty \tilde A^{(j)} + (\tilde C^T \tilde C)^{(j)}]\end{aligned} \tag{71}$$

$$0 = \tilde A^T L_p^{(j)} + L_p^{(j)} \tilde A + [\tilde A^{(j)T} L_p + L_p \tilde A^{(j)} + \lambda(\tilde C_p^T \tilde C_p)^{(j)}] \tag{72}$$

$$0 = \tilde A X^{(j)} + X^{(j)} \tilde A^T + [\tilde A^{(j)} X + X \tilde A^{(j)T} + (\tilde B_p \tilde B_p^T)^{(j)}] \tag{73}$$

Derivatives of the closed-loop matrices are obtained from Eqs. (45–47) and are given by

$$\tilde A^{(j)} = -\bar B_2 G^{(j)} \bar C_2 \tag{74}$$

$$(\tilde B \tilde B^T)^{(j)} = (\tilde B_p \tilde B_p^T)^{(j)} = 0 \tag{75}$$

$$\begin{aligned}(\tilde C^T \tilde C)^{(j)} = {} & (\bar C_2^T G^{(j)T} \bar D_{12}^T \bar D_{12} G \bar C_2) + (\bar C_2^T G^{(j)T} \bar D_{12}^T \bar D_{12} G \bar C_2)^T \\ & - (\bar C_1^T \bar D_{12} G^{(j)} \bar C_2) - (\bar C_1^T \bar D_{12} G^{(j)} \bar C_2)^T\end{aligned} \tag{76}$$

$$\begin{aligned}(\tilde C_p^T \tilde C_p)^{(j)} = {} & (-\bar D_{1p} G^{(j)} \bar C_2)^T (\bar C_p - \bar D_{1p} G \bar C_2) \\ & + (\bar C_p - \bar D_{1p} G \bar C_2)^T (-\bar D_{1p} G^{(j)} \bar C_2)\end{aligned} \tag{77}$$

Computation of H_α

Similarly, the derivative of the homotopy function with respect to the homotopy parameter α is the $N \times 1$ vector

$$\nabla_\alpha H^T = \frac{\partial H^T}{\partial \alpha} \tag{78}$$

and

$$\begin{aligned}\frac{\partial H}{\partial \alpha} &= \mathrm{vec}\left[\frac{\partial(\partial\mathcal{L}/\partial G)}{\partial \alpha}\right] \\ &= \mathrm{vec}\left(\frac{\partial}{\partial \alpha}\{2[(\bar D_{12}^T \bar D_{12} G \bar C_2 - \bar D_{12}^T \bar C_1 - \bar B_2^T Q_\infty) L \bar C_2^T + (\lambda \bar D_{1p}^T \bar D_{1p} G \bar C_2 - \lambda \bar D_{1p}^T \bar C_p - \bar B_2^T L_p) X \bar C_2^T]\}\right) \\ &= \mathrm{vec}\{2[(\dot{\bar D}_{12}^T \bar D_{12} G \bar C_2 + \bar D_{12}^T \dot{\bar D}_{12} G \bar C_2 - \dot{\bar D}_{12}^T \bar C_1 \\ &\quad - \bar B_2^T \dot Q_\infty) L \bar C_2^T + (\bar D_{12}^T \bar D_{12} G \bar C_2 - \bar D_{12}^T \bar C_1 - \bar B_2^T Q_\infty) \dot L \bar C_2^T \\ &\quad + (\dot\lambda \bar D_{1p}^T \bar D_{1p} G \bar C_2 - \dot\lambda \bar D_{1p}^T \bar C_p - \bar B_2^T \dot L_p) X \bar C_2^T]\}\end{aligned} \tag{79}$$

where

$$\dot{(*)} = \frac{\partial(*)}{\partial \alpha} \tag{80}$$

Implicit in these equations is the assumption that all system matrices are fixed with the exception of $\bar{D}_{12}$ and $\bar{D}_{1p}$ and the parameters λ and γ. In general, the homotopy can be performed with any/all system matrices deformed, but the methodology for mixed H_2/H_∞ employed is to vary only one parameter at a time.

The derivative terms in the third equality in Eq. (79) depend on the deformation undertaken in the specified problem, i.e., the initial and final problem. Suppose that the deformation of the matrix $\bar{D}_{12}$ is prescribed to be

$$\bar{D}_{12}(\alpha) = \bar{D}_{12,0}(\alpha) + \alpha[\bar{D}_{12,f}(\alpha) - \bar{D}_{12,0}(\alpha)] \tag{81}$$

where the 0 and f subscripts indicate the initial and final system matrices, respectively. It follows that

$$\dot{\bar{D}}_{12} = \bar{D}_{12,f} - \bar{D}_{12,0} \tag{82}$$

The derivative of other plant matrices and the parameters λ and γ are determined accordingly. To obtain expressions for the derivatives of L, Q_∞, L_p, and X with respect to α, Eqs. (49–52) are differentiated, resulting in the following:

$$0 = \left(\tilde{A} + \gamma^{-2}\tilde{B}\tilde{B}^T Q_\infty\right)\dot{L} + \dot{L}\left(\tilde{A} + \gamma^{-2}\tilde{B}\tilde{B}^T Q_\infty\right)^T + (\Gamma L + L\Gamma^T) \tag{83}$$

$$0 = \left(\tilde{A} + \gamma^{-2}\tilde{B}\tilde{B}^T Q_\infty\right)^T \dot{Q}_\infty + \dot{Q}_\infty\left(\tilde{A} + \gamma^{-2}\tilde{B}\tilde{B}^T Q_\infty\right) - \left(2\gamma^{-3}\dot{\gamma} Q_\infty \tilde{B}\tilde{B}^T Q_\infty\right) \tag{84}$$

$$0 = \tilde{A}^T \dot{L}_p + \dot{L}_p \tilde{A} + \left(\dot{\tilde{A}}^T L_p + L_p \dot{\tilde{A}} + \lambda \dot{\tilde{C}}_p^T \tilde{C}_p + \lambda \tilde{C}_p^T \dot{\tilde{C}}_p + \dot{\lambda} \tilde{C}_p^T \tilde{C}_p\right) \tag{85}$$

$$0 = \tilde{A}\dot{X} + \dot{X}\tilde{A}^T + \left(\dot{\tilde{A}}X + X\dot{\tilde{A}}^T + \dot{\tilde{B}}_p \tilde{B}_p^T + \tilde{B}_p \dot{\tilde{B}}_p^T\right) \tag{86}$$

where

$$\Gamma = \dot{\tilde{A}} - 2\gamma^{-3}\dot{\gamma}\tilde{B}\tilde{B}^T Q_\infty + \gamma^{-2}\left(\dot{\tilde{B}}\tilde{B}^T Q_\infty + \tilde{B}\dot{\tilde{B}}^T Q_\infty + \tilde{B}\tilde{B}^T \dot{Q}_\infty\right) \tag{87}$$

and, from Eqs. (45–47),

$$\dot{\tilde{A}} = \dot{\bar{A}} - \dot{\bar{B}}_2 G \bar{C}_2 - \bar{B}_2 G \dot{\bar{C}}_2 \tag{88}$$

$$\dot{\tilde{B}}_p = \dot{\bar{B}}_p \tag{89}$$

$$\dot{\tilde{B}} = \dot{\bar{B}}_1 \tag{90}$$

$$\dot{\tilde{C}}_p = \dot{\bar{C}}_p - \dot{\bar{D}}_{1p} G \bar{C}_2 - \bar{D}_{1p} G \dot{\bar{C}}_2 \tag{91}$$

$$\dot{\tilde{C}} = \dot{\bar{C}}_1 - \dot{\bar{D}}_{12} G \bar{C}_2 - \bar{D}_{12} G \dot{\bar{C}}_2 \tag{92}$$

The use of the canonical compensator formulation in the augmented system dynamics simplifies not only the necessary conditions but also the derivative expressions of the system matrices. From Eqs. (37–40), the augmented matrix derivatives reduce to

$$\dot{\bar{A}} = \begin{bmatrix} A_f - A_0 & 0 \\ -N^0(C_{2,f} - C_{2,0}) & 0 \end{bmatrix} \tag{93}$$

$$\dot{\bar{B}}_p = \begin{bmatrix} B_{p,f} - B_{p,0} \\ -N^0(D_{2p,f} - D_{2p,0}) \end{bmatrix} \tag{94}$$

$$\dot{\bar{B}}_1 = \begin{bmatrix} B_{1,f} - B_{1,0} \\ -N^0(D_{21,f} - D_{21,0}) \end{bmatrix} \tag{95}$$

$$\dot{\bar{B}}_2 = \begin{bmatrix} B_{2,f} - B_{2,0} & 0 \\ -N^0(D_{22,f} - D_{22,0}) & 0 \end{bmatrix} \tag{96}$$

$$\dot{\bar{C}}_p = [C_{p,f} - C_{p,0} \quad 0] \tag{97}$$

$$\dot{\bar{C}}_1 = [C_{1,f} - C_{1,0} \quad 0] \tag{98}$$

$$\dot{\bar{C}}_2 = [0 \quad 0] \tag{99}$$

$$\dot{\bar{D}}_{1p} = [D_{1p,f} - D_{1p,0} \quad 0] \tag{100}$$

$$\dot{\bar{D}}_{12} = [D_{12,f} - D_{12,0} \quad 0] \tag{101}$$

The presence of the zero subblocks significantly enhances the computational efficiency of this approach. When implementing the procedure described at the beginning of this section, the preceding equations may be further specialized. In the initial H_2 homotopy procedure the initial and final plant matrices are the same and the homotopy is performed only on the measurement and process noise intensities, D_{12} and D_{21}. Hence $\dot{\bar{A}}$, $\dot{\bar{B}}_2$, $\dot{\bar{C}}_1$, and $\dot{\bar{C}}_2$ are identically zero. For the mixed H_2/H_∞ homotopy, the H_∞ and H_2 homotopies are performed distinctly, which simplifies the computations significantly because the plant matrices remain fixed and only γ or λ are varied at one time.

Design Example

To demonstrate the homotopy algorithm applied to optimal controller synthesis, the four disk example originally described in Ref. 19 and more recently by numerous others[10] will be used. The four disk model used in the example problem was derived from a laboratory experiment and represents an apparatus developed for testing of pointing control systems for flexible space structures with noncollocated sensors and actuators. Four disks are rigidly attached to a flexible axial shaft with control torque applied to selected disks and the angular displacement of selected disks measured. The plant parameters are taken from Ref. 4.

H_2 Case

To demonstrate the homotopy algorithm applied to H_2 controller synthesis, a direct comparison between the homotopy algorithm of Ref. 10 called HAO and the FOCUS algorithm (called H2HOM for the H_2 case) will be presented first. The main distinction between the two homotopy algorithms for the H_2 case is the compensator architecture. HAO employs a general architecture that may be restricted to various parametrizations including the controllability canonical form, which is similar to the controller canonical form used in H2HOM for this SISO example problem. The HAO code has been highly optimized for efficient computation with the result that superfluous computations are not evaluated. The homotopy algorithm of this paper is patterned after the general approach of HAO and utilizes some of the more efficient computational aspects of the HAO code.

The control design philosophy for this example is to scale the nominal control weight and the nominal sensor noise intensity by the parameter q. As q is reduced, the control authority is increased. For comparison with the published results in Ref. 10, a full- (eighth-) order compensator was synthesized. Although the results can be directly obtained from the LQG Riccati equations, the full-order compensator was chosen to tax the H2HOM algorithm, which must optimize over a greater number of parameters with increasing compensator order.

Table 1 shows a comparison of the results from the H2HOM and HAO algorithms for the full-order compensator along with results from the H2HOM algorithm for sixth- and second-order compensators. All pertinent parameters as well as logic for step-size scaling and the computation of the prediction and correction errors are identical in both algorithms, which are implemented in MATLAB on a 486 66-MHz computer. Whereas the HAO code required a minimum step size of 1.907e-7, the H2HOM solution was much better conditioned and required a minimum step size of 0.025. As a consequence of the smaller step sizes with HAO, 2504 Hessian computations were required, as opposed to only 63 Hessian computations with H2HOM. The HAO code has been tuned extensively for efficient computation, as is reflected in the small number of floating point operations per second (flops) required. In spite of the significantly smaller number of flops with HAO, the H2HOM code required significantly less clock time for convergence to the same final compensator. (The results generated by the authors using the

Table 1 Comparison of H2HOM and HAO algorithms

Algorithm	HAO	H2HOM	H2HOM	H2HOM
Compensator order	8	8	6	2
Number of Hessian computations	2504	63	60	30
Min. step size	1.907e-7	0.025	0.05	0.1
Max. step size	0.1	0.1	0.1	0.2
Max. number of correction iterations	9	7	8	5
Mflops	287	936	455	37
Time, s	5104	883	488	73

Table 2 Comparison of FOCUS and Ref. 4 results

γ	$\|T_{zw}\|_{\infty,\mathrm{BH}}$	$\|T_{zw}\|_{\infty,F}$	$\|T_{zpwp}\|_{2,\mathrm{BH}}$	$\|T_{zpwp}\|_{2,F}$
2.0	1.18	—	0.382	—
1.5	1.06	—	0.389	—
1.0	0.855	0.947	0.410	0.398
0.9	0.797	0.889	0.420	0.403
0.8	0.732	0.776	0.432	0.421
0.7	0.661	0.668	0.451	0.438
0.6	—	0.589	—	0.460
0.52	0.511	0.520	0.512	0.508
0.5	—	0.5	—	0.518
0.4	—	0.398	—	0.572
0.3	—	0.300	—	0.799

HAO code differ slightly from the published results,[10] although the parameters in the HAO algorithm are the same. It is likely that the published results were generated with an earlier version of the HAO code. The qualitative trends remain the same.)

In Ref. 10, the controllability canonical form is assessed as poorly conditioned because of the small minimum step size. However, Table 1 indicates that the static gain formulation in H2HOM yields a substantial improvement in conditioning along the homotopy path over the HAO implementation of the canonical compensator. Although the static gain formulation is better conditioned for this example problem, this may not be the case in general. An even more significant benefit of this formulation is the straightforward extension to the H_∞ and mixed H_2/H_∞ problems.

Mixed H_2/H_∞ Case

One-Input/Two-Output Case

The seminal paper[4] dealing with mixed H_2/H_∞ design addressed the case where $w = w_p$ in Eq. (37) with results from the four disk problem given for the full-order case. In this section, the one-input/two-output case will be repeated with the FOCUS algorithm as well as a two-input/two-output case with full- and fixed-order compensators to demonstrate the capabilities of the homotopy algorithm.

Table 2 presents a comparison of the results from the FOCUS algorithm and the results published in Ref. 4. In Table 2, the BH subscript indicates results from Bernstein and Haddad[4] and the F subscript indicates results from FOCUS. Gaps in the columns of Table 2 denoted BH correspond to values of γ for which results were not published. The absence of data generated by FOCUS for $\gamma = 2$ and 1.5 is due to the fact that γ is larger than the maximum ∞ norm, as described in the next section. A key distinction between the formulations of the mixed H_2/H_∞ optimization problem used in this paper and that of Ref. 4 is that whereas their formulation minimizes an overbound on the H_2 norm, the formulation utilized in FOCUS minimizes the actual H_2 norm. Consequently, as shown in Fig. 2, the FOCUS algorithm results in smaller H_2 norms for a given γ with the gap in the ∞ norm overbound smaller than the Ref. 4 results in the meaningful region for γ, as described in the next section. Whereas $\gamma = 0.52$ was the smallest γ value reported in Ref. 4, γ values of 0.5, 0.4, and 0.3 are utilized with FOCUS, as indicated in Table 2. This example also demonstrates the synthesis of fixed-order mixed controllers with singular H_∞ constraints. In practice, compensators representing the extreme values of γ or λ typically will not be used because either performance or robustness would be severely diminished. A compromise value in the elbow

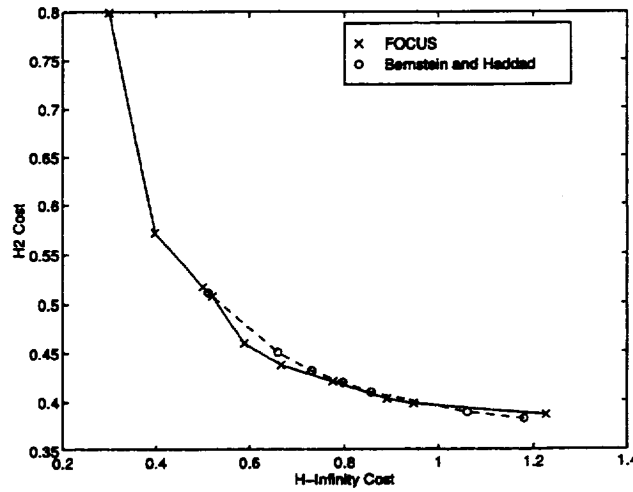

Fig. 2 H_∞ vs H_2 cost for single-input/multi-output example.

of Fig. 2 typically would be chosen. Although the FOCUS results in Table 2 are not significantly better than the results of Ref. 4, the example demonstrates that the FOCUS code performs satisfactorily and lessens the gap between the overbound and the ∞ norm.

Discussion

This example serves to illustrate some interesting features of the mixed H_2/H_∞ formulation implemented in FOCUS, as well as distinctions from other formulations. The formulation implemented in FOCUS is a method for generating suboptimal H_∞ controllers of fixed order that are subject to an H_2 constraint. The cost functional for the mixed H_2/H_∞ problem can be written as

$$J_{\mathrm{mix}} = J_\infty + \lambda J_2 \tag{102}$$

where

$$J_\infty = \mathrm{tr}\{Q_\infty \tilde{B}\tilde{B}^T\} \tag{103}$$

$$J_2 = \mathrm{tr}\{X\tilde{C}_p^T\tilde{C}_p\} \tag{104}$$

subject to the corresponding Riccati and Lyapunov equations. The resulting Lagrangian, also given by Eq. (48), is

$$\mathcal{L} = \mathrm{tr}\{Q_\infty\tilde{B}\tilde{B}^T + (\tilde{A}^TQ_\infty + Q_\infty\tilde{A} + \tilde{C}^T\tilde{C} + \gamma^{-2}Q_\infty\tilde{B}\tilde{B}^TQ_\infty)L + \lambda X\tilde{C}_p^T\tilde{C}_p + (\tilde{A}X + X\tilde{A}^T + \tilde{B}_p\tilde{B}_p^T)L_p\} \tag{105}$$

Minimization of J_∞ results in an H_∞ controller with an upper bound on the ∞ norm given by the ∞ norm of the H_2 controller (as $\gamma \to \infty$, the H_2 compensator for T_{zw} is recovered by minimizing J_∞). By successively lowering gamma, the minimum H_∞ norm controller for T_{zw} is obtained. Minimization of J_2 results in the optimal H_2 compensator for T_{zpw}. Thus, when nonzero λ is used in J_{mix}, the H_∞ cost functional imposes an additional constraint on the H_2 norm of T_{zpw}, and for large γ, the necessary conditions for the mixed problem yield the simultaneous solution of two H_2 problems. By increasing the H_2 weight, λ, for a fixed γ, $\|T_{zpw}\|_2$ is reduced while $\|T_{zw}\|_\infty$ approaches the gamma overbound. At that point, the minimum H_2 controller for T_{zpw} such that $\|T_{zw}\|_\infty \le \gamma$ is obtained.

For this example problem, the optimal H_2 controller for T_{zpw} results in $\|T_{zw}\|_\infty = 1.392$ and $\|T_{zw}\|_2 = 0.3786$. The γ values larger than 1.392 are not meaningful for our formulation because the optimal H_2 compensator for T_{zpw} satisfies the γ constraint on $\|T_{zw}\|_\infty$. In that case, the H_∞ norm constraint is inactive in the mixed norm optimization. However, since the Ref. 4 formulation seeks to minimize the H_2 norm overbound, it is possible for their formulation to generate meaningful solutions for $\gamma > 1.392$. When using FOCUS, the first step should be to establish an upper bound on γ by computing $\|T_{zw}\|_\infty$ resulting from the optimal H_2 compensator for T_{zpw}.

Table 3 Comparison of two-input/two-output results

γ	$\|T_{zw}\|_{\infty,8}$	$\|T_{zpwp}\|_{2,8}$	$\|T_{zw}\|_{\infty,4}$	$\|T_{zpwp}\|_{2,4}$
2	——	——	2	0.421
1.5	1.304	0.446	1.483	0.437
1.0	1.000	0.500	1.000	0.496
0.9	0.900	0.529	0.900	0.532
0.8	0.789	0.560	0.800	0.587
0.7	0.694	0.652	0.700	0.640
0.6	0.597	0.799	0.600	0.952
0.5	0.500	3.842	0.500	1.180
0.4	0.400	8.281	——	——
0.3	0.300	24.84	——	——

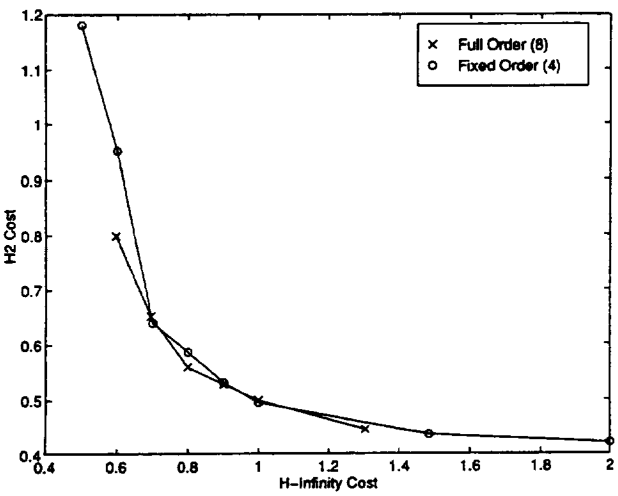

Fig. 3 H_∞ vs H_2 cost for MIMO example.

Two-Input/Two-Output Reduced-Order Case

To demonstrate the capability of the homotopy algorithm to synthesize fixed-order compensators for the two-input/two-output case, the four disk example was modified by creating a fictitious disturbance input for the H_∞ minimization subproblem. The new coefficient matrices are given by $B_1 = 2 \times B_p$ and $D_{21} = 2 \times D_{2p}$. Additionally, a small penalty was placed on the control in the H_∞ subproblem ($D_{12} = [0\ 0.001]^T$). Table 3 shows the results of the comparison for the full- and fixed-order two-input/two-output case. Figure 3 plots the intermediate values of Table 3 and illustrates the cost trade (extreme values are removed to preserve scale). In Table 3, the subscript 8 indicates the full-order design and the subscript 4 indicates the fixed-order design. In general, full-order compensator synthesis is more taxing on the numerical algorithm due to the larger number of parameters to be optimized, which is evident in both Table 3 and Fig. 3. The fixed-order design is able to achieve smaller H_2 norms at the smaller γ values. Also, the ∞ norms for the larger values of γ are closer to the overbound with the fixed-order designs. The reduced number of parameters to be optimized results in more accurate solution of the minimization problem. The closeness of the H_2/H_∞ cost curves in Fig. 3 indicates that the fourth-order compensator closely approximates the performance/robustness tradeoff of an eighth-order compensator except at the low end of the attainable H_∞ norm values. The infinity norm of T_{zw} resulting from closing the loop with the full-order optimal H_2 controller for T_{zpwp} is 1.845, so that the full-order case with $\gamma = 2$ was not a meaningful solution. However, a meaningful solution for $\gamma = 2$ was achieved with the fixed-order compensator.

Conclusions

A homotopy algorithm is developed to synthesize fixed-order H_2, H_∞, and mixed H_2/H_∞ compensators employing a controller canonical form, and a representative flexible structure is used to demonstrate the numerical results. Fixed-order mixed H_2/H_∞ compensators are synthesized for the case of one disturbance input set and two outputs sets, as well as the case of two disturbance input sets, two output sets. These numerical results indicate that for the example presented, the static output feedback formulation using the controller canonical form is a more efficient means of synthesizing dynamic compensators than employing a general compensator architecture. The synthesized reduced-order compensators perform well when compared to full-order controllers. The fixed-order mixed H_2/H_∞ formulation is shown to offer improved performance over standard H_∞ compensators by minimizing the H_2 norm while removing (or reducing) the gap between the actual H_∞ norm and the gamma overbound.

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