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Abstract. A temperature gradient across a thick (≥ .1 mm) film selective emitter will produce a significant reduction in the spectral emittance from the no temperature gradient case. Thick film selective emitters of rare earth doped host materials such as yttrium-aluminum-garnet (YAG) are examples where temperature gradient effects are important. In this paper a model is developed for the spectral emittance assuming a linear temperature gradient across the film. Results of the model indicate that temperature gradients will result in reductions the order of 20% or more in the spectral emittance.

INTRODUCTION

Emission from thick films is not a surface phenomenon as is usually assumed when discussing emissive materials. It depends on the geometry of the material, which for the film emitters means the film thickness. Thus radiation leaving the film originates at various depths within the film.

To model these film emitters we use a macroscopic approach. That is we solve the radiative transfer equation that applies for Boltzmann equilibrium of excited state densities and includes stimulated emission and absorption, as well as, spontaneous emission and scattering of radiation. These atomic processes manifest themselves on the macroscopic scale through the extinction coefficient, α_x.

The product of the extinction coefficient, α_x, and the film thickness, d, α_x d = K_d, which is usually called the optical depth, will determine the spectral emittance if the temperature is a constant through the film. However, for thick films (≥ .1 mm) the temperature gradients are not negligible (> 100°K) so the emittance model must include a variable temperature through the film. In the analysis to follow we assume a linear temperature variation across the film. This is the result that will
occur if thermal conduction dominates radiative energy transfer. In the case where 
\( d \leq 1 \text{ mm} \) this is a good assumption for the rare-earth selective emitters we are 
considering (3).

In the following section the emittance model will be developed. Following that, 
two approximate expressions for the spectral emittance, \( \varepsilon_\lambda \), that apply when 
scattering is neglected and the temperature gradient is small will be presented. The 
first approximation is applicable for large optical depth, \( K_d \), and the second 
approximation applies for small optical depth. Both of these approximations are 
compared to the exact result for \( \varepsilon_\lambda \), neglecting scattering but for any temperature 
gradient, obtained by a numerical solution of the governing equations. Following 
that a discussion of the optimum film thickness to obtain maximum emittance will 
be presented. Finally, spectral emittance results will be compared to experimental 
results obtained for an erbium oxide (Er2O3) selective emitter that has an emission 
band centered at a photon wavelength, \( \lambda = 1.5 \mu\text{m} \).

THICK FILM EMISSANCE MODEL

The emittance model for the thick film emitter has been previously developed 
for the case of no temperature gradient (1, 4). This model can be extended to 
include a temperature gradient across the film. The model is based on the radiative 
transfer equation (5), which is macroscopic in nature. Thus the emissive, 
absorptive and scattering properties of the material, which depend on the atomic 
structure, are expressed through the extinction coefficient, \( \alpha_\lambda \). The key parameter 
in determining the spectral emittance, \( \varepsilon_\lambda \), is the optical depth, \( K = \alpha_\lambda d \).

Consider Figure 1 which is a schematic drawing of a thick film emitter. 
Thermal energy enters through the film substrate. Part or all of the thermal input 
leaves the film at \( x = d \) as radiation flux, \( Q_\lambda(K_d) \). To determine \( \varepsilon_\lambda, Q_\lambda(K_d) \) must be 
calculated since \( \varepsilon_\lambda \) is defined as follows.

\[
\varepsilon_\lambda = \frac{Q_\lambda(K_d)}{e_{bs}(\lambda, T_s)} \quad (1)
\]

Where \( e_{bs}(\lambda, T_s) \) is the blackbody emissive power and \( T_s \) is the substrate 
temperature.

\[
e_{bs} = \pi i_{bs} = \frac{2\pi h c^2}{\lambda^5 \left[ \exp(hc_\lambda / \lambda kT_s) - 1 \right]} \quad (2)
\]

Here \( h \) is Plank’s constant, \( k \) is Boltzmann’s constant, \( c_\circ \) is the vacuum speed of 
light, and \( i_{bs} \) is the blackbody intensity. Notice that \( \varepsilon_\lambda \) has been defined in terms of 
the substrate temperatures, \( T_s \). The spectral emittance could be defined in terms of 
the film surface temperature, \( T_f \), or some combination of \( T_f \) and \( T_s \). However,
defining $e_\lambda$ in terms of $T_s$ means $e_\lambda \leq 1$ in all cases since $e_{m_\lambda}(\lambda, T_s) \geq Q_\lambda(K_d)$. This definition agrees with the usual concept of emittance.

To calculate $Q_\lambda$ we require the radiative transfer equations for radiation intensity moving in the $+x$ direction, $i_{\lambda}^+(K, \cos \theta)$, and intensity in the $-x$ direction, $i_{\lambda}^-(K, \cos \theta)$, (5).

\begin{align*}
i_{\lambda}^+(K, \mu) &= i_{\lambda}^+(0, \mu) \exp \left[ -\frac{K}{\mu} \right] + \int_0^K S(K^*, \mu) \exp \left[ \frac{K - K^*}{\mu} \right] \frac{dK^*}{\mu} \\
0 &\leq \mu = \cos \theta \leq 1
\end{align*}

\begin{align*}
i_{\lambda}^-(K, \mu) &= i_{\lambda}^-(K_d, \mu) \exp \left[ -\frac{K_d - K}{\mu} \right] - \int_0^K S(K^*, \mu) \exp \left[ \frac{K^* - K}{\mu} \right] \frac{dK^*}{\mu} \\
-1 &\leq \mu = \cos \theta \leq 0
\end{align*}

**FIGURE 1.** Schematic Diagram of Thick Film Emittance Model
In using these equations we are assuming that y and z variation of intensity can be neglected. Appearing in equations (3) and (4) is the so-called source function, \( S(K,\mu) \), which in the case of isotopic scattering \( (S(K,\mu) = S(K)) \) satisfies the following equation (5).

\[
S(K) = n_f^2 (1 - \Omega) i_{ab}(T, \lambda) + \Omega \frac{1}{2} \left\{ \int_0^l i_\perp^*(0,\mu) \exp \left[ -\frac{K}{\mu} \right] d\mu + \int_0^l i_\parallel^*(K_\perp, -\mu) \exp \left[ \frac{K_\parallel - K}{-\mu} \right] d\mu \right\} + \frac{\Omega}{2} \int_0^{K_s} S(K^*) E_1(|K^*-K|) dK^*
\]

(5)

Appearing in equation (5) is the scattering albedo.

\[
\Omega_\perp = \frac{\sigma_\perp}{\sigma_\perp + \alpha_\perp} = \frac{\sigma_\perp}{\alpha_\perp}
\]

(6)

Where \( \sigma_\perp \) is the scattering coefficient and \( \alpha_\perp \) is the absorption coefficient, which have the dimensions, \( \text{cm}^{-1} \). The sum of \( \sigma_\perp \) and \( \alpha_\perp \) is the extinction coefficient, \( \alpha_\perp \). Also appearing in equation (5) is the film index of refraction, \( n_{\text{film}} \), and the exponential integral, \( E_1(x) \).

The general exponential integral, \( E_n(x) \), is defined as follows.

\[
E_n(x) = \int_0^x v^{n-2} \exp \left[ -\frac{x}{v} \right] dv
\]

(7)

Note that we are assuming isotropic scattering. As a result, \( S \) is independent of \( \mu = \cos \theta \). Therefore, assuming diffuse boundary intensities, \( i_\perp^*(0,\mu) = i_\perp^*(0) \) and \( i_\parallel^*(K_\perp,\mu) = i_\parallel^*(K_\perp) \) we see from equations (3) and (4) that \( i_\perp^* \) and \( i_\parallel^* \) are also independent of \( \mu \).

The diffuse (independent of \( \mu \)) boundary conditions at \( K = K_\perp \) and \( K = 0 \) are the following.

\[
i_\perp^*(K_\perp) = \rho_{\text{abs}} i_\perp^*(K_\perp)
\]

at \( K = K_\perp \) \hspace{1cm} (8a)

\[
i_\parallel^*(0) = \rho_{\text{abs}} i_\parallel^*(0) + (1 - \rho_{\text{abs}}) \varepsilon_{\text{abs}} (n_f / n_{\text{film}})^2 i_{\text{em}}(\lambda, T_\parallel)
\]

at \( K = 0 \) \hspace{1cm} (8b)

Equation (8a) states that the radiation leaving the film-vacuum interface in the -x direction is equal to the reflected radiation at that interface. For the film-substrate interface equation (8b) states that \( i_\parallel^*(0) \) is the sum of the reflected radiation and the radiation emitted by the substrate that is transmitted \( (1 - \rho_{\text{abs}}) \) through that interface.
The \((n_f/n_s)^2\) term accounts for refraction at the interface (5, pg. 738). The reflectance at the film-vacuum interface is \(\rho_{\lambda o}\) and the reflectance at the film-substrate interface is \(\rho_{\lambda s}\). In the previous studies (1,3,4) the transmittance, \((1-\rho_{\lambda s})\), at the film-substrate interface was assumed to be 1 and the refraction term \((n_f/n_s)^2\) was neglected. We approximate \(\rho_{\lambda o}\) and \(\rho_{\lambda s}\) by the reflectance for normal incidence, (5)

\[
\rho_{\lambda o} = \left(\frac{n_f - 1}{n_f + 1}\right)^2
\]

\[
\rho_{\lambda s} = \left(\frac{n_s - n_f}{n_s + n_f}\right)^2
\]

Where, \(n_s\) is the substrate index of refraction.

At the film-substrate and film-vacuum interfaces there is the possibility of total reflection occurring. At an interface between a material with an index of refraction, \(n_e\), and a material with index of refraction \(n_m\), where \(n_e > n_m\), radiation moving from \(e\) into \(m\) with an angle of incidence, \(\theta > \theta_{\text{cm}}\), where \(\theta_{\text{cm}}\) is given by Snell’s law will be totally reflected. This will be taken into account when calculating \(Q_{\lambda}(K_d)\). At the film-substrate interface refraction has been taken into account by including the \((n_f/n_s)^2\) term in equation (8b). However, the possibility of total reflection is not included. Therefore, by using equation (8b) as the boundary condition we are assuming that \(n_f > n_s\) so that total reflection does not occur for radiation entering the film from the substrate.

Now consider \(Q_{\lambda}(K_d)\), which is the radiation flux leaving the film. Since \(n_f > 1\) the radiation leaving the film will be refracted and some of the radiation that reaches the film-vacuum interface will be totally reflected at the interface. Therefore,

\[
Q_{\lambda}(K_d) = 2\pi \int_{\theta=0}^{\theta_f} \left[ i_\lambda(K_d, \cos \theta) - i_\lambda(K_d, \cos \theta) \right] \cos \theta \sin \theta \, d\theta
\]

(11a)

and using equation (8a) and letting \(\mu = \cos \theta\) this becomes the following.

\[
Q_{\lambda}(K_d) = 2\pi \left(1 - \rho_{\lambda o}\right) \int_{\mu_m}^{1} i_\lambda(K_d, \mu) \mu \, d\mu
\]

(11b)

Where \(\mu_m\) is given by Snell’s Law.
\[ \mu_M^2 = \cos^2 \mu_M = 1 - n^2 \lambda f \]  

(12)

Substituting (3) in (11b) yields the following.

\[ Q_\lambda(K_d) = \left(1 - \rho_{\lambda_0}\right) \left[2 \pi i^*_\lambda(0) h_- + \Phi_+ - \Phi_M\right] \]  

(13)

Where,

\[ h_- = E_3(K_d) - \mu_M^2 E_3 \left(\frac{K_d}{\mu_M}\right) \]  

(14)

\[ \Phi_+ = 2\pi \int_0^K S(K) E_2(K_d - K) dK \]  

(15)

\[ \Phi_M = 2\pi \mu_M \int_0^K S(K) E_2 \left(\frac{K_d - K}{\mu_M}\right) dK \]  

(16)

Equation (13) gives \( Q_\lambda(K_d) \) in terms of the source function \( S(K) \) and \( i^*_\lambda(0) \). The \( i^*_\lambda(0) \) intensity is obtained by using equations (3) and (4) to get two simultaneous equations for \( i^*_\lambda(K_d) \) and \( i_\lambda(0) \). These can then be solved for \( i^*_\lambda(0) \) and the result used in equation (8b) to obtain \( i_\lambda(0) \). (4).

\[ \pi i^*_\lambda(0) = q^*(0) = \frac{1}{D} \left[ \left(\frac{n_\lambda}{n_{\lambda_0}}\right)^2 (1 - \rho_{\lambda_0}) e_{\lambda_0} e_{\lambda_0}(\lambda, T_s) \right. \]  

\[ + 2 \rho_{\lambda_0} \rho_{\lambda_\lambda} E_1(K_d) \Phi_+ + \rho_{\lambda_\lambda} \Phi_- \]  

(17)

Where,

\[ D = 1 - 4 \rho_{\lambda_0} \rho_{\lambda_\lambda} E_3^2(K_d) \]  

(18)

\[ \Phi_- = 2\pi \int_0^K S(K) E_3(K) dK \]  

(19)

Now substitute equation (17) in (13).
\[ Q_{\lambda}(K_d) = \frac{1 - \rho_{\lambda,0}}{D} \left( 2 \left( \frac{n_{\lambda,0}}{n_{\lambda}} \right)^2 \left( 1 - \rho_{\lambda,0} \right) e_{\lambda}(\lambda, T) + \rho_{\lambda} \Phi - h_{\lambda} \right) \Phi_{\lambda} d - \Phi_{\lambda} \Phi_{M} d \] \tag{20}

Where,

\[ h_{\lambda} = 1 - 4 \rho_{\lambda,0} \rho_{\lambda} \mu_{M}^2 E_{3}(K_d) E_{3}\left( \frac{K_d}{\mu_{M}} \right) \] \tag{21}

Equation (20) can be substituted in equation (1) to obtain the spectral emittance, \( \varepsilon_{\lambda} \), in terms of the source function, \( S(K) \). In the general case where scattering exists the source function must be obtained by solving equation (5). In the case of no scattering, \( \Omega_{\lambda} = 0 \), and equation (5) reduces to the following.

\[ S(K) = n_{\lambda}^2 i_{\lambda}(\lambda, T) \] \tag{22}

If we also assume \( T \) is a constant through the film, \( T = T_s \), then the integrations in \( \Phi_{\lambda}, \Phi_{\lambda}, \) and \( \Phi_{M} \), can be carried out to yield the following.

\[ \varepsilon_{\lambda} = \frac{n_{\lambda}^2 (1 - \rho_{\lambda,0})}{D} \left\{ 2 h_{\lambda} \left[ \frac{\varepsilon_{\lambda}(1 - \rho_{\lambda,0})}{n_{\lambda}^2} + \rho_{\lambda} \left( 1 - 2 E_{3}(K_d) \right) \right] + h_{\lambda} \left[ 1 - 2 E_{3}(K_d) \right] \right\} \]

\[ -\mu_{M}^2 D \left[ 1 - 2 E_{3}\left( \frac{K_d}{\mu_{M}} \right) \right] \}

constant temperature, no scattering \tag{23}

Thus \( \varepsilon_{\lambda} \) is determined by the optical depth, \( K_d \), the indices of refraction, \( n_{\lambda} \) and \( n_{\lambda} \), and the substrate emittance, \( \varepsilon_{\lambda} \). In the case when scattering is important \( \varepsilon_{\lambda} \) will also be a function of the scattering albedo, \( \Omega_{\lambda} \).

Now consider the case where a temperature gradient exists. To demonstrate the temperature gradient effects in the simplest manner we consider the no scattering case since in that case the source function has the simple solution given by equation (22). We also assume a linear temperature gradient across the film. As discussed in the introduction this is a good approximation for the rare earth selective emitters. As a result, the temperature across the film is given by the following expression.

\[ \frac{T}{T_s} = 1 - \Delta T \left( \frac{x}{d} \right) = 1 - \Delta T \left( \frac{K}{K_d} \right) \] \tag{24}

Where, the temperature gradient is defined as follows.

\[ \Delta T = \frac{T_s - T}{T_s} \]
Using equations (24), (22) and (2) in the expressions for \( \Phi_+ \), \( \Phi_- \), and \( \Phi_M \), yields the following.

\[
\Phi_+ = \frac{\Phi_+}{2n^2e_b\left(\lambda, T_s\right)} = (e^u - 1)K_d \int_{0}^{1} \frac{E_2[K_d(1-v)]}{\exp \left[ \frac{u}{1-v\Delta T} \right] - 1} \, dv
\]

\[
\Phi_- = \frac{\Phi_-}{2n^2e_b\left(\lambda, T_s\right)} = (e^u - 1)K_d \int_{0}^{1} \frac{E_2(K_dv)}{\exp \left[ \frac{u}{1-v\Delta T} \right] - 1} \, dv
\]

\[
\Phi'_M = \frac{\Phi'_M}{2n^2e_b\left(\lambda, T_s\right)} = \mu_M(e^u - 1)K_d \int_{0}^{1} \frac{E_2[K_d(1-v)]}{\frac{\mu_M}{K_d}(1-v)} \, dv
\]

Where,

\[
u = \frac{hc}{\lambda kT_s}
\]

\[
u = \frac{K}{K_d}
\]

Equations (26) - (28) can be used in equations (20) and (1) to obtain \( \varepsilon_\lambda \).

\[
\varepsilon_\lambda = \frac{2n^2(1-\rho_{\lambda\nu})}{D} \left\{ \frac{\varepsilon_\lambda(1-\rho_{\lambda\nu})}{n^2_{\lambda\nu}} + 2\rho_{\lambda\nu}\Phi'_- \right\} h_- + \Phi'_+ h_- + \Phi'_M D
\]

no scattering, with temperature gradient

As equations (26) - (28) indicate \( \Phi'_+ \), \( \Phi'_- \), and \( \Phi'_M \), are functions of \( \Delta T \). The integrations in equations (26) - (28) must be carried out numerically. However, for small \( \Delta T \) approximations to the integrals can be made. In most cases of interest for selective emitters, \( \lambda \leq 7 \mu m, T_s \leq 2000 K \), the dimensionless photon energy, \( u \), is greater than 1. Therefore, the following approximations can be made.
\[ \left[ \exp\left( \frac{u}{1 - u\Delta T} \right) - 1 \right] = \exp\left( \frac{-u}{1 - u\Delta T} \right) \quad e^u \gg 1 \quad (32a) \]

\[ e^u - 1 = e^u \quad e^u \gg 1 \quad (32b) \]

In addition for \( \Delta T \ll 1 \) and \( 0 \leq u \leq 1; \)
\[ \exp\left( \frac{-u}{1 - u\Delta T} \right) \approx e^{-u} e^{-u\Delta T} \quad e^u \gg 1, \Delta T \ll 1 \quad (33) \]

With the approximations given by equations (32) and (33) equations (26) - (28) become the following after a change in the integration variables.

\[ \Phi_+ = e^{-u\Delta T} \int_0^{K_d} \exp\left( \frac{Ku\Delta T}{K_d} \right) E_2(K) dK \quad (34) \]

\[ \Phi_- = -e^{-u\Delta T} \int_0^{K_d} \exp\left( \frac{-Ku\Delta T}{K_d} \right) E_2(K) dK \quad (35) \]

\[ \Phi_M = \mu^2 e^{-u\Delta T} \int_0^{K_d} \exp\left( \frac{K\mu\Delta T}{K_d} \right) E_2(K) dK \quad (36) \]

For a selective emitter the optical depth, \( K_d \), will be large \((K_d > 1)\) in the emission band and small \((K_d \ll 1)\) outside the emission band. Therefore, consider the two limiting cases; \( \frac{u\Delta T}{K_d} \ll 1 \) and \( \frac{K_d}{u\Delta T} \ll 1 \). For the case where \( \frac{u\Delta T}{K_d} \ll 1 \), integration by parts using
\[ E_{n-1}(x) = -\frac{dE_n(x)}{dx} \quad (37) \]
results in the following to first order in \( \frac{u\Delta T}{K_d} \).

\[ \Phi_+ = \frac{1}{2} e^{-u\Delta T} - E_3(K_d) - E_3(K_d) - \frac{1}{3} e^{-u\Delta T} \left( \frac{u\Delta T}{K_d} \right) \quad (38) \]

\[ \Phi_- = \frac{1}{2} - e^{-u\Delta T} E_3(K_d) + e^{-u\Delta T} E_4(K_d) - \frac{1}{3} \left( \frac{u\Delta T}{K_d} \right) \quad \frac{u\Delta T}{K_d} \ll 1 \quad (39) \]
\[
\Phi'_M = \mu'_M \left\{ \frac{1}{2} e^{-u\Delta T} - \mu \left( K_d - \frac{1}{\mu_M} \right) \right\} \left\{ \left[ \left( \left( K_d - \frac{1}{\mu_M} \right) - \frac{1}{3} e^{-u\Delta T} \right) \frac{u\Delta T}{K_d} \right] \right\} \tag{40}
\]

Since we are interested in showing the effect of temperature gradient on \( \varepsilon_\lambda \) we define the following quantity.

\[\Delta \varepsilon_\lambda = \varepsilon_{\lambda 0} - \varepsilon_\lambda \tag{41}\]

Where \( \varepsilon_{\lambda 0} \) is the emittance for no temperature gradient and is given by equation (23). By using \( \Delta \varepsilon_\lambda \) to demonstrate the temperature gradient effect the dependence on substrate emittance, \( \varepsilon_{\lambda s} \), is removed. Therefore, using equations (38) - (40) in (31) and equation (23) for \( \varepsilon_{\lambda 0} \) results in the following.

\[
\Delta \varepsilon_\lambda = \frac{2n^2_s (1-n) e^{-u\Delta T}}{D} \left\{ \frac{1}{2} \left[ h_e - 4p_s h_e E_4(K_d) - \mu_M D \right] \left[ 1 - e^{-u\Delta T} \right] \right. \\
+ \left. \left\{ 2p_s \left( 1 - e^{-u\Delta T} E_4(K_d) \right) h_e - \left( \frac{e^{-u\Delta T}}{3} - E_4(K_d) \right) h_e \right. \\
- \mu_M \left( \frac{e^{-u\Delta T}}{3} - E_4 \left( \frac{K_d}{\mu_M} \right) D \right) \right\} \frac{u\Delta T}{K_d} \right\}
\]

no scattering, \( \Delta T << 1 \), \( e^u \gg 1 \), \( \frac{u\Delta T}{K_d} \ll 1 \) \tag{42}

Notice that if \( u\Delta T << 1 \) then \( \Delta \varepsilon \) as given by equation (42) will be a linear function of \( \Delta T \). However, if \( u\Delta T \) is not small then \( \Delta \varepsilon_\lambda \sim \left( 1 - e^{-u\Delta T} \right) \) provided \( \frac{u\Delta T}{K_d} \ll 1 \).

Also, note by looking at equation (31) that \( \Delta \varepsilon_\lambda \) is independent of the substrate emittance.

Now consider \( \Delta \varepsilon_\lambda \) for the case where \( K_d << 1 \). In that case \( E_4(K) \) can be expanded in a power series and the integrations in equations (34) - (36) performed. To first order in \( \frac{K_d}{u\Delta T} \) the results are the following.

\[
\Phi'_e = \Phi'_e \approx \left( 1 - e^{-u\Delta T} \right) \frac{K_d}{u\Delta T} \tag{43}
\]

\[
\Phi'_M = \mu'_M \left( 1 - e^{-u\Delta T} \right) \frac{K_d}{\mu_M u\Delta T} \tag{44}
\]
If equations (43) and (44) are used in (31) and equation (23) for $\varepsilon_{\lambda 0}$ then $\Delta \varepsilon_{\lambda}$ becomes the following when the approximation $E_{3}(K_{d}) = \frac{1}{2} - K_{d}$ is used.

$$
\Delta \varepsilon_{\lambda} = \frac{2n_{d}^{2}(1 - \rho_{\lambda 0})}{1 - \rho_{\lambda 0}\rho_{\lambda 1}} \left[ 1 + \left( 1 - \mu_{M}^{2} \right) \rho_{\lambda f} - \mu_{M} \left[ 1 - \rho_{\lambda 0} \rho_{\lambda 1} (1 - \mu_{M}) \right] \right] K_{d} \left[ 1 - \frac{1 - e^{-u\Delta T}}{u\Delta T} \right]
$$

no scattering, $\Delta T \ll 1$, $e^{u} \gg 1$, $K_{d} \ll 1$. \hspace{1cm} (45)

Again, if $u\Delta T \ll 1$ then $\Delta \varepsilon_{\lambda}$ will be approximately a linear function of $\Delta T$ just as in the case of $\frac{u\Delta T}{K_{d}} \ll 1$. Also note that $\Delta \varepsilon_{\lambda}$ is a linear function of $K_{d}$ and that $\varepsilon_{\lambda 0}$ has no effect on $\Delta \varepsilon_{\lambda}$.

**TEMPERATURE GRADIENT EFFECT ON SPECTRAL EMITTANCE FOR NO SCATTERING**

**Comparison of Exact and Approximate Solutions for Spectral Emittance**

With the results developed in the previous section we can now illustrate the effect of $\Delta T$ on $\varepsilon_{\lambda}$. In Figure 2 $\Delta \varepsilon_{\lambda}$ is shown as a function of $\Delta T$ for large optical depth ($K_{d} = 2$) at several values of $u$. The exact result for $\Delta \varepsilon_{\lambda}$ is obtained using

![Figure 2](image_url)

**FIGURE 2.** Emittance change as a function of temperature gradient at large optical depth, $K_{d} = 2$, for several dimensionless photon energies, $u = h\nu / \lambda kT_{s}$ with $n_{s} = 10$ and $n_{f} = 1.9$. 

NASA TM-107523
equation (31) for $\varepsilon_\lambda$ and numerical integration to obtain $\Phi'_\nu, \Phi'_\nu$ and $\Phi'_M$. Also, the result for $\Delta \varepsilon_\lambda$ (equation (42)) is shown in Figure 2.

As Figure 2 indicates $\Delta \varepsilon_\lambda$ changes rapidly at small $\Delta T$ with the slope increasing for increasing $u$. Thus even for $\Delta T \leq 1$ there will be a significant reduction in the spectral emittance for $u \geq 5$. In most cases, for the emission bands of rare earth selective emitters where $K_d > 1$ the dimensionless photon energy, $u > 5$. Therefore, even a small temperature gradient will result in a significant reduction in the spectral emittance in the emittance band of the rare earth selective emitters. Obviously, making the emitter as thin as possible will reduce $\Delta T$. However, the optical depth will also be reduced, if the thickness, $d$, is reduced, resulting in decreased $\varepsilon_\lambda$. As a result, there will be an optimum thickness, $d$, to obtain maximum $\varepsilon_\lambda$. This will be discussed in the next section. Note also that the approximate solution (equation (42)) is in close agreement with the exact results when $\Delta T < 1$.

Results in Figure 2 are for large optical depth ($K_d = 2$). However, similar results occur for small optical depth and are illustrated in Figure 3 where $K_d = 1$. Again there is good agreement between the approximate solution (equation (45))

![Figure 3](attachment:image.png)

**FIGURE 3.** Emittance change as a function of temperature gradient at an optical depth, $K_d = 1$, for several dimensionless photon energies, $u = h c_0 / \Lambda kT_s$ with $n_{\lambda s} = 1.9$ and $n_{\lambda s} = 10$. 

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and the exact solution when $\Delta T < 1$. The range of values for $\Delta \varepsilon_\lambda$ is much smaller for the case where $K_d << 1$ than for $K_d > 1$. Thus the temperature gradient has only a small effect on $\varepsilon_\lambda$ when $K_d << 1$. Therefore, for a selective emitter the emittance outside the emission band will not be greatly effected by $\Delta T$.

### Optimum Thickness for Maximum Spectral Emittance

As already stated, the counteracting effects of increasing spectral emittance with optical depth and decreasing spectral emittance with increasing temperature gradient will result in an optimum film thickness for maximum spectral emittance. This can be demonstrated as follows. Neglecting any conductive or convective heat transfer at the film surface (which will occur if a vacuum exists at the film surface) then the total power/area leaving the film is the following.

$$Q_{\text{out}} = \int_0^{\lambda} Q_\lambda(K_d) d\lambda$$  \hspace{1cm} (46)

This same power/area must be supplied by thermal conduction and radiation at the film-substrate interface to maintain a steady state. Therefore, at $x = 0$, assuming conduction is much greater than radiation,

$$Q_{\text{out}} = -\beta_f \frac{dT}{dx} \bigg|_{x=0}$$  \hspace{1cm} (47)

Where $\beta_f$ is the film thermal conductivity. As stated earlier, energy transfer through the film is dominated by thermal conduction so that, equation (24) applies and

$$-\frac{dT}{dx} \bigg|_{x=0} = \left( \frac{T_s - T_f}{d} \right).$$

Therefore, from equations (46) and (47) the following is obtained.

$$\Delta T = \frac{T_s - T_f}{T_s} = \frac{Q_{\text{out}}}{\beta_f T_s} d$$  \hspace{1cm} (48)

To calculate $Q_{\text{out}}$, equation (31) for $\varepsilon_\lambda$, which is a function of $\Delta T$ must be used to determine $Q_\lambda(K_d)$ (equation (1)). However, since $\varepsilon_\lambda$ is a function of $\Delta T$, equations (46) and (48) must be solved simultaneously in order to obtain $\Delta T$ as a function of $Q_{\text{out}}$. This has been done in ref. 3. But to illustrate how an optimum thickness occurs we can write $Q_{\text{out}}$ as follows.

$$Q_{\text{out}} = \varepsilon_\gamma \sigma_{sb} T_s^4$$  \hspace{1cm} (49)
Where $\varepsilon_\lambda$ is the total emittance of the film and will be a function of $T$, and $\sigma_{sb}$ is the Stefan-Boltzmann constant ($5.67 \times 10^{-12}$ w/cm$^2$ K$^4$). By using equation (49) in equation (48) the following results.

$$\Delta T = \tau_f d$$  \hspace{1cm} (50)

Where,

$$\tau_f = \frac{\varepsilon_\lambda \sigma_{sb} T_s^3}{\beta_f} \text{ cm}^{-1}$$  \hspace{1cm} (51)

The quantity $\tau_f d$ is the ratio of radiation to thermal conduction (3). Thus equation (50) shows that $\Delta T$ will be small as long as thermal conduction dominates.

For selective emitters of interest, $\varepsilon_\lambda < .2$, $\beta_f > .02$ w/cmK and $T_s < 2000$K, so that $0 < \tau_f < 5$mm$^{-1}$. If equation (50) is used for $\Delta T$ in equation (31) and since $K_d = \alpha_\lambda d$ the results for $\varepsilon_\lambda$ when $\alpha_\lambda = 100$ cm$^{-1}$ shown in figure 4 are obtained. An extinction coefficient $\alpha_\lambda = 100$ cm$^{-1}$ is representative of the emission band of a selective emitter. The first thing to note from figure 4 is that for $\Delta T > 0$ ($\tau_f > 0$) there is an optimum thickness for maximum $\varepsilon_\lambda$. For the case of no temperature gradient ($\tau_f = 0$) there is no optimum $d$. The larger the temperature gradient the more pronounced the optimum $d$ becomes. For small $\tau_f$ large values of $\varepsilon_\lambda$ occur over a broad range of thicknesses. Note that the curve for $\tau_f = 2$ mm$^{-1}$ and $\tau_f = 5$ mm$^{-1}$ have been truncated at $d = .5$ mm and $d = .2$ mm since $\Delta T = \tau_f d \leq 1$.

Also notice that the optimum $d$ becomes smaller as $\tau_f$ increases (larger $\Delta T$). Based on the results of figure 4 it appears that the optimum selective emitter thickness to obtain maximum emittance in the emission band for $\alpha_\lambda = 100$ cm$^{-1}$ is in the range $.15 \leq d \leq .4$ mm. For $\alpha_\lambda = 100$ cm$^{-1}$ this corresponds to an optical depth range, $1.5 \leq K_d \leq 4$. 

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FIGURE 4. Effect of temperature gradient on spectral emittance for large extinction coefficient, $\alpha_x = 100 \text{ cm}^{-1}$, at several values of the temperature gradient parameter, $\tau_t$. Also, $u=5., \epsilon_x=1, n_x=1.9, n_x=10$.

Now consider the case of small extinction coefficient, which is representative of the wavelength region outside the emission band of a selective emitter. Spectral emittance results for $\alpha_x = 1 \text{ cm}^{-1}$ are shown in figure 5. In this case, $\varepsilon_\lambda$ does not attain a maximum value even for thicknesses over 1 mm. Because $\alpha_x$ is small much larger thicknesses (1 cm to obtain $K_d=1$) are required before $\varepsilon_\lambda$ will approach its maximum value. For $d<.4 \text{ mm}$, the region where maximum $\varepsilon_\lambda$ occurs for large $\alpha_x$, the spectral emittance is nearly independent of $\tau_t$. 

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FIGURE 5. Effect of temperature gradient on spectral emittance for small extinction coefficient, $\alpha_\lambda = 1 \text{ cm}^{-1}$, at several values of the temperature gradient parameter, $\tau_t$. Also, $u=5$, $\varepsilon_{\lambda_0}=.1$, $n_{\lambda}=1.9$, $n_{\lambda_s}=10$.

Based on the results displayed in figures 4 and 5 several conclusions can be made about the efficiency of a thick film selective emitter. The emitter efficiency (1,3,4) depends on the ratio of the emittance within the emission band $\varepsilon_b$ to the emittance outside the emission band, $\varepsilon_e$. Obviously it is desirable for $\varepsilon_b/\varepsilon_e$ to be as large as possible. For the emission band, where $\alpha_\lambda$ is large, there will be an optimum thickness, $d_{\text{opt}}$, (corresponding to $1.5 \leq K_\lambda \leq 4$) to maximize $\varepsilon_b$. Outside the emission band, where $\alpha_\lambda$ is small, the spectral emittance increases at a much slower rate with $d$ than for the emission band for $d < d_{\text{opt}}$. For $d < 4\text{mm}$ figure 5 shows that $\varepsilon_b$ increases nearly at the same linear rate regardless of the temperature gradient. Thus it appears that maximum emitter efficiency will occur for the thickness, $d_{\text{opt}}$, corresponding to maximum emittance within the emission band. As stated earlier this thickness corresponds to $1.5 \leq K_\lambda \leq 4.0$ when $\alpha_\lambda = 100 \text{ cm}^{-1}$. 

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FIGURE 6. Extinction coefficient and index of refraction for Er₂O₃-Al₂O₃ selective emitter
FIGURE 7. Comparison of theoretical and experimental spectral emittance for Er₂O₃-Al₂O₃ thick film selective emitter. Film thickness, \( d = 0.36 \) mm, \( n_x = 1.0, \epsilon_x = 0.2, T_s = 1500 \text{K} \).

Comparison of Experimental and Theoretical Spectral Emittance

To complete this study we compare the measured spectral emittance of a selective emitter made of erbia (Er₂O₃) reinforced with alumina (Al₂O₃) with the spectral emittance calculated using equation (31). This emitter was fabricated at the Auburn Space Power Institute (6). The calculated \( \epsilon_\lambda \) is based on the extinction coefficient, \( \alpha_\lambda \), and index of refraction, \( n_x \), shown in figure 6. These quantities were obtained using measured transmittance and reflectance data (6).

Figure 7 shows the experimental and theoretical \( \epsilon_\lambda \) for an emitter of thickness, \( d = 0.36 \) mm. This emitter had a platinum foil substrate. A constant substrate emittance \( \epsilon_s = 0.2 \) was used for the platinum foil. However, since there is an air gap between the foil and the film the appropriate index of refraction for the film-substrate interface is \( n_x = 1.0 \), which was used in the calculation. The measured temperature gradient was \( \Delta T = 0.13 \) and the platinum foil substrate temperature was \( T_s = 1500 \text{K} \).

The first thing to notice is the considerable reduction in \( \epsilon_\lambda \) within the emission bands centered at \( \lambda \approx 1000 \text{ nm} \) and \( \lambda \approx 1500 \text{ nm} \) as a result of the temperature...
gradient. In the main emission band at $\lambda = 1500$ nm the theoretical maximum goes from $\varepsilon_\lambda = .8$ when $\Delta T = 0$ to $\varepsilon_\lambda = .35$ when $\Delta T = .13$. As discussed earlier (fig. 3), the spectral emittance outside the emission bands is not greatly affected by $\Delta T$.

The measured emission band is broader than the theoretical emission band. This occurs because the theoretical result is based on the extinction coefficient that was measured at room temperature. At high temperature broadening of the emission band will occur which will therefore not be accounted for in the theoretical results. Part of the difference between the theoretical and experimental $\varepsilon_\lambda$ for radiation outside the emission band is caused by experimental error. Outside the emission band where $\varepsilon_\lambda$ is small, background radiation coming from sources other than the emitting film result in the measured $\varepsilon_\lambda$ being larger than the actual value, (2).

CONCLUSION

The no scattering theoretical spectral emittance model shows the importance of even small ($\Delta T = .1$) temperature gradients on $\varepsilon_\lambda$. For both small ($K_t<<1$) and large ($K_t>>1$) optical depths, approximations for $\varepsilon_\lambda$ were developed that give good agreement with the exact results as long as $\Delta T \leq .1$.

Because of the opposite dependence of $\varepsilon_\lambda$ on temperature gradient and optical depth there will an optimum film thickness for maximum, $\varepsilon_\lambda$. The model predicts that the optimum optical depth has the range, $1.5 \leq K_t \leq 4.0$, depending on the temperature gradient.

Finally, there is good agreement between the theoretical spectral emittance and experimental spectral emittance for a Er$_2$O$_3$-Al$_2$O$_3$ selective emitter fabricated at the Auburn Space Power Institute.

REFERENCES


A temperature gradient across a thick (≥0.1 mm) film selective emitter will produce a significant reduction in the spectral emittance from the no temperature gradient case. Thick film selective emitters of rare earth doped host materials such as yttrium-aluminum-garnet (YAG) are examples where temperature gradient effects are important. In this paper a model is developed for the spectral emittance assuming a linear temperature gradient across the film. Results of the model indicate that temperature gradients will result in reductions the order of 20% or more in the spectral emittance.