# Testing and Implementation of Advanced Reynolds Stress Models

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**Proposed Work**

A research program was proposed for the testing and implementation of advanced turbulence models for non-equilibrium turbulent flows of aerodynamic importance that are of interest to NASA. Turbulence models that are being developed in connection with the Office of Naval Research *ARI on Nonequilibrium Turbulence are* provided for implementation and testing in aerodynamic flows at NASA Langley Research Center. Close interactions were established with researchers at NASA Langley RC and refinements to the models were made based on the results of these tests. The models that have been considered include two-equation models with an anisotropic eddy viscosity as well as full second-order closures. Three types of non-equilibrium corrections to the models have been considered in connection with the *ARI on Nonequilibrium Turbulence*: conducted for ONR

(1) Anisotropies in the turbulent dissipation rate through an analysis of the transport equation for the tensor dissipation. The leading order contribution of this effect is through the addition of nonlinear strain dependent terms in the modeled scalar dissipation rate equation via the production coefficient $C_{\varepsilon 1}$. The traditional constant value chosen for this coefficient makes it impossible to describe both equilibrium flows with moderate strain rates and non-equilibrium flows with large strain rates.

(2) Non-equilibrium vortex stretching in the turbulent dissipation rate equation. The commonly used modeled transport equation for the turbulent dissipation rate is based on an equilibrium hypothesis whereby the production of dissipation by vortex stretching is exactly counter-balanced by the leading order part of the destruction of dissipation term. In order to describe departures from equilibrium, unbalanced vortex stretching will be allowed for which is described by a physically based relaxation model.

(3) Non-equilibrium pressure-strain effects. Terms that are nonlinear in the mean velocity gradients are introduced into the model for the pressure-strain correlation through the implementation of a relaxation time approximation to a non-equilibrium algebraic stress model that bridges the equilibrium solution to the RDT solution for shear flows via a Padé approximation (see Appendix A).

These models have the potential to lead to a new generation of Reynolds stress closures. While, as part of the Office of Naval Research *ARI on Nonequilibrium Turbulence,* the models

will be tested in practical Naval Hydrodynamics flows, it would also be useful to test them in high speed aerodynamic flows that are of interest to NASA. This forms the raison d'être of the present research.

## Research Accomplished

The research focused on two central issues:

(a) The development of a more robust regularization scheme for explicit algebraic stress models which form a cornerstone of the models being developed. The previously derived regularization scheme allowed the eddy viscosity to get too low when the mean strain rates became large. The new regularization procedure allows the eddy viscosity to approach a sufficiently large enough finite lower bound for numerical robustness. This has led to the better calculation of aerodynamic flows (see Appendix A).

(b) The systematic incorporation of the effects of anisotropic dissipation into explicit algebraic stress models. By using the algebraic anisotropic dissipation rate model of Speziale and Gatski (1995), the explicit algebraic stress model approximation was repeated via integrity bases methods. It led to an explicit algebraic stress model where the coefficients simply assumed readjusted values (see Appendix B). Thus the effects of anisotropic dissipation can be now be systematically implemented within the framework of a model that is only slightly more computationally expensive than the $K - \varepsilon$ Model.

This research has great promise for future aerodynamic computations. Additional tests are currently underway.

**APPENDIX A**

# Prediction of Complex Aerodynamic Flows with Explicit Algebraic Stress Models

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## Abstract

An explicit algebraic stress equation, developed by Gatski and Speziale, is used in the framework of the $K - \varepsilon$ formulation to predict complex aerodynamic turbulent flows. The nonequilibrium effects are modeled through coefficients that depend nonlinearly on both rotational and irrotational strains. The proposed model was implemented in the ISAAC Navier-Stokes code. Comparisons with the experimental data are presented which clearly demonstrate that explicit algebraic stress models can predict the correct response to nonequilibrium flows.

## I. Introduction

Computational fluid dynamics has become an increasingly powerful tool in the aerodynamic design of aerospace vehicles as a result of improvements in numerical algorithms and computer capabilities (e.g., speed, storage). Major future gains in efficiency are expected to come about as massively parallel supercomputer technology matures. However, some critical pacing items limit the effectiveness of computational fluid dynamics in engineering. Chief among these items is turbulence modeling. Numerous turbulence models of varying degrees of complexity, which can be classified as either eddy viscosity or full Reynolds stress models, have been proposed. Excellent reviews of turbulence models have been recently provided by both Speziale[1] and Wilcox.[2]

Eddy viscosity models use the Boussinesq isotropic effective viscosity concept, which assumes that the turbulent stresses in the mean momentum equation are equal to the product of an eddy viscosity and a mean strain rate. Zero-, one-, and two-equation models are among the most popular eddy viscosity models for engineering applications because of their ease of implementation in computational fluid dynamics codes. Algebraic or zero-equation models, which assume local equilibrium of the turbulent and mean flow, have provided reasonable predictions for simple flows. When the turbulent transport is important or the mean conditions change abruptly, these models do not work well. One-equation models improve the predictions for simple near-equilibrium flows but do not account for more complex effects on turbulence. Two-equation models are developed to take explicit account of the history of the turbulence through two transport equations for combinations of the turbulent length and time scales. These models offer good predictions of

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the characteristics and physics of simple separated flows and flows with gradual changes in boundary conditions. However, basic two-equation models fail in many practical flows because they cannot properly account for streamline curvature, rotational strains and buoyancy; they provide an incorrect response to strong adverse pressure gradients; and they cannot describe the anisotropy of turbulence. As a result, various ad hoc modifications to these models have been proposed to achieve the proper response (see Lakshminarayana[3]). In these modifications, effects on turbulence, such as those due to streamline curvature, have been directly accounted for in the eddy viscosity expression or have been reflected indirectly in the turbulence-model equations by modifying the dissipation-rate equation. The improved two-equation models predict a wider range of flows; however, they still fail to properly capture the physics in a broad class of flows. To overcome some of these deficiencies, two-equation turbulence models that are nonlinear in the mean strain rate were proposed by Speziale[4] and Rubinstein and Barton.[5] These models have provided accurate predictions of turbulence intensities. However, these models are not consistent with full Reynolds stress models because they have constant coefficients.

Full Reynolds stress models represent the highest level of closure that is currently feasible for practical calculations. These models are superior to the two-equation models in that they eliminate the assumption that the turbulent stresses respond immediately to changes in the mean strain rate. Also, they account for the anisotropy of turbulence and body force effects on turbulence (e.g., due to streamline curvature and rotation) through extra production terms that explicitly appear in the Reynolds stress transport equation. However, models for many unknown turbulent quantities are required. This need is generally met by assuming that the turbulence is locally homogeneous and in equilibrium. Existing Reynolds stress models have been shown to give good descriptions of two-dimensional mean turbulent flows that are near equilibrium. However, computer costs and numerical stability problems that arise from the absence of a turbulent viscosity make assessments of the limitations of these models in predicting complex flows difficult. However, second-order closure models could be used to derive better two-equation models because fundamentally they are constructed on a stronger theoretical basis than the lower level models.

Recently, a methodology for deriving a general nonlinear constitutive relation (or an explicit algebraic stress equation) for the Reynolds stress tensor from second-order closures, has been proposed by Gatski and Speziale,[6] based on the ideas of Pope.[7] This derivation is based on the assumptions that the net convection of the turbulent stresses is proportional to the net convection of the turbulent kinetic energy and that the structural parameters of the turbulence are constant along a streamline. As a result, a new generation of non-linear two-equation models is obtained with coefficients that depend on rotational and irrotational strains. This new feature extends the range of applicability of the standard two-equation models.

Abid et al.[8] used the explicit algebraic stress relation within the context of the $K-\omega$ and $K-\varepsilon$ two-equation format to predict separated airfoil flows. The Launder, Reece and Rodi[9] pressure-strain correlation model was considered in the above study. Comparisons with the experimental data have shown that this new nonlinear turbulence model improves the ability of two-equation models to account for nonequilibrium effects. However, the Reynolds stress anisotropies were not well predicted.

In this paper, the algebraic stress relation is applied within the context of the $K-\varepsilon$ two-equation format using the Speziale, Sarkar and Gatski[10] pressure-strain correlation model. The ability of the proposed model to predict complex flows which include nonequilibrium and anisotropic effects is assessed. Transonic flows over two airfoils and a wing are considered in this study. The ISAAC Navier-Stokes code is used to compute the three test cases.

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## II. Theoretical Analysis

For a weakly compressible turbulent flow at high Reynolds numbers, the Reynolds stress tensor $\tau_{ij} = \overline{u_i u_j}$ is a solution of the transport equation[11]

$$\frac{D\tau_{ij}}{Dt} = -\tau_{ik}\frac{\partial \overline{u}_j}{\partial x_k} - \tau_{jk}\frac{\partial \overline{u}_i}{\partial x_k} + \frac{\Pi_{ij}}{\overline{\rho}} - \frac{2}{3}\varepsilon\delta_{ij}$$

$$+ D_{ij}^T + v\nabla^2\tau_{ij} \qquad (1)$$

given that $\Pi_{ij}$ is the pressure-strain correlation, $D_{ij}^T$ is the turbulent transport term, $\varepsilon$ is the turbulent dissipation-rate, $v$ is the kinematic viscosity, $\overline{u}_i$ is the mean-velocity component, and $\overline{\rho}$ is the mean density. Explicit compressibility effects are neglected in Eq. (1) due to the applicability of Markovin's hypothesis in these weakly compressible flows.

If we contract the indices in (1), then we obtain the transport equation for the turbulent kinetic energy $K = \overline{u_i u_i}/2$:

$$\frac{DK}{Dt} = P - \varepsilon + D_K^T + v\nabla^2 K \qquad (2)$$

given that $P = -\tau_{ij}\left(\partial \overline{u}_i/\partial x_j\right)$ is the turbulence production term and $D_K^T$ is the turbulent transport term.

Rodi[12] proposed the idea of algebraic stress closure, which provides algebraic equations without solving differential equations for the Reynolds stresses. He assumed that

$$\frac{D\tau_{ij}}{Dt} - v\nabla^2\tau_{ij} - D_{ij}^T = \frac{\tau_{ij}}{K}\left(\frac{DK}{Dt} - D_K^T - v\nabla^2 K\right) \quad (3)$$

and

$$\frac{Db_{ij}}{Dt} = 0 \qquad (4)$$

where

$$b_{ij} = \frac{\tau_{ij} - \frac{2}{3}K\delta_{ij}}{2K} \qquad (5)$$

is the Reynolds stress anisotropy. Physically, two assumptions are made in the algebraic

Reynolds stress closures: the convection term minus the diffusion term in the Reynolds stress equation is proportional to the convection term minus the diffusion term in the turbulent kinetic energy equation and the Reynolds stress anisotropy $b_{ij}$ is constant along a streamline.

The substitution of (3) and (4) into (1) yields the following algebraic stress equation:

$$(P - \varepsilon)b_{ij} = -\frac{2}{3}KS_{ij} - K\left(b_{ik}S_{jk} + b_{jk}S_{ik}\right.$$

$$\left. - \frac{2}{3}b_{mn}S_{mn}\delta_{ij}\right) - K\left(b_{ik}W_{jk} + b_{jk}W_{ik}\right) + \frac{\Pi_{ij}}{2\overline{\rho}} \quad (6)$$

where

$$S_{ij} = \frac{1}{2}\left(\frac{\partial \overline{u}_i}{\partial x_j} + \frac{\partial \overline{u}_j}{\partial x_i}\right) \qquad (7)$$

and

$$W_{ij} = \frac{1}{2}\left(\frac{\partial \overline{u}_i}{\partial x_j} - \frac{\partial \overline{u}_j}{\partial x_i}\right) \qquad (8)$$

are the mean-rate-of-strain tensor and mean-vorticity tensor, respectively.

Given a pressure-strain-correlation model, (6) provides an implicit algebraic equation for the determination of the Reynolds stress $\tau_{ij}$. Computations that use this model have shown that stable numerical solutions can be difficult to obtain. Hence, an explicit algebraic stress equation which is a mathematically consistent representation of (6) is preferable.

Pope[7] developed a methodology for obtaining explicit algebraic stress equations by using a tensorial polynomial expansion in the integrity basis.[5] Gatski and Speziale[6] used this method to derive an explicit algebraic stress equation for two- and three-dimensional turbulent flows. In order to generalize their results, they applied their algebraic stress representation to the general class of pressure-strain correlation models for $\Pi_{ij}$ which are linear in the anisotropic tensor $b_{ij}$. The general linear form of $\Pi_{ij}$ is

$$\frac{\Pi_{ij}}{\overline{\rho}} = -C_1 \epsilon b_{ij} + C_2 K S_{ij} + C_3 K \left( b_{ik} S_{jk} + b_{jk} S_{ik} \right.$$

$$\left. -\frac{2}{3} b_{mn} S_{mn} \delta_{ij} \right) + C_4 K \left( b_{ik} W_{jk} + b_{jk} W_{ik} \right) \quad (9)$$

The explicit nonlinear constitutive equation, derived by Gatski and Speziale,[6] is then given after regularization by

$$\overline{\rho} \tau_{ij} = \frac{2}{3} \overline{\rho} K \delta_{ij} - 2\mu_t \left[ \left( S_{ij} - \frac{1}{3} S_{kk} \delta_{ij} \right) + \frac{\alpha_4}{\omega} \left( S_{ik} W_{kj} \right. \right.$$

$$\left. \left. + S_{jk} W_{ki} \right) - \frac{\alpha_5}{\omega} \left( S_{ik} S_{kj} - \frac{1}{3} S_{k\ell} S_{k\ell} \delta_{ij} \right) \right] \quad (10)$$

with

$$\mu_t = \overline{\rho} C_\mu^* \frac{K}{\omega} \quad (11)$$

$$C_\mu^* = \frac{3\left(1 + \eta^2\right) \alpha_1}{3 + \eta^2 + 6\eta^2 \xi^2 + 6\xi^2} \quad (12)$$

$$\eta^2 = \frac{\alpha_2}{\omega^2} \left( S_{ij} S_{ij} \right), \quad \xi^2 = \frac{\alpha_3}{\omega^2} \left( W_{ij} W_{ij} \right) \quad (13)$$

where $\overline{\rho}$ is the mean density and $\omega = \epsilon / K$ is the specific dissipation rate. The constants in (11)–(13) are given by

$$\alpha_1 = \left( \frac{4}{3} - C_2 \right) \frac{g}{2}, \quad \alpha_2 = \left(2 - C_3\right)^2 \frac{g^2}{4} \quad (14)$$

$$\alpha_3 = \left(2 - C_4\right)^2 \frac{g^2}{4}, \quad \alpha_4 = \left( \frac{2 - C_4}{2} \right) g \quad (15)$$

$$\alpha_5 = \left(2 - C_3\right) g, \quad g = \frac{1}{\frac{C_1}{2} + C_5 - 1} \quad (16)$$

To avoid numerical problems in the initial stages of the computation or in the free-stream region, a modified form of $C_\mu^*$ is used

$$C_\mu^* = \alpha_1 \frac{3\left(1 + \eta^2\right) + 0.2\left(\eta^6 + \xi^6\right)}{3 + \eta^2 + 6\eta^2 \xi^2 + 6\xi^2 + \eta^6 + \xi^6} \quad (17)$$

which is equivalent to Eq. (12) to order $\eta^4$ and $\xi^4$. Relation (17) does not change the value of $C_\mu^*$ near equilibrium conditions, but limits $C_\mu^*$ to a small non-zero value ($= 0.2\alpha_1$) for high values of $\eta$ or $\xi$ to avoid numerical instabilities. In the present study, the pressure-strain-correlation model of Speziale, Sarkar, and Gatski[10] is considered; the coefficients are:

$$C_1 = 6.8, \quad C_2 = 0.36, \quad C_3 = 1.25,$$

$$C_4 = 0.40, \quad C_5 = 1.88 \quad (18)$$

The nonlinear constitutive equation (10) must be solved in conjunction with the following modeled transport equations.

$$\overline{\rho} \frac{DK}{Dt} = \overline{\rho} P - \overline{\rho} \epsilon + \frac{\partial}{\partial x_j} \left( \left( \mu_\ell + \frac{\mu_{t\ell}}{\sigma_k} \right) \frac{\partial K}{\partial x_j} \right) \quad (19)$$

and

$$\overline{\rho} \frac{D\epsilon}{Dt} = C_{\epsilon_1} \overline{\rho} \frac{\epsilon}{K} P - C_{\epsilon_2} \overline{\rho} f \frac{\epsilon^2}{K}$$

$$+ \frac{\partial}{\partial x_j} \left( \left( \mu_\ell + \frac{\mu_{t\ell}}{\sigma_\epsilon} \right) \frac{\partial \epsilon}{\partial x_j} \right) \quad (20)$$

given that $\mu_{t\ell} = C_{\mu\ell}^* \overline{\rho} \frac{K^2}{\epsilon}$ and $C_{\mu\ell}^* (= 0.081)$ is the value of $C_\mu^*$ in the logarithmic layer. The coefficients of the model are

$$\sigma_k = 1.0, \quad \kappa = 0.40, \quad C_{\epsilon_2} = 1.83, \quad C_{\epsilon_1} = 1.44$$

$$\sigma_\epsilon = \frac{\kappa^2}{\left( C_{\epsilon_2} - C_{\epsilon_1} \right) \sqrt{C_{\mu\ell}^*}} \quad (21)$$

and

$$f = \left[ 1 - \exp\left( -\frac{y^+}{5.5} \right) \right]^2, \quad y^+ = \frac{\overline{\rho} y u_\tau}{\mu} \quad (22)$$

given that $u_\tau$ is the shear velocity and $y$ is normal to the wall. Note that new model can be integrated directly to the wall without adding a damping to the eddy viscosity. The function $f$ is introduced to remove the singularity in the dissipation rate equation at the wall.

At the wall, the boundary conditions for $K$ and $\epsilon$ are

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$$K = 0, \quad \varepsilon = 2\nu \left( \frac{\partial \sqrt{K}}{\partial y} \right)^2 \qquad (23)$$

## III. Results and Discussion

The calculations to be presented were done with the three-dimensional Navier-Stokes ISAAC code,[13] which uses a second-order accurate finite-volume scheme. The convective terms are discretized with an upwind scheme that is based on Roe's flux-difference splitting method. All viscous terms are centrally differenced. The equations are integrated in time with an implicit, spatially split approximate-factorization scheme.

The performance of the explicit algebraic turbulence model (hereafter referred to as EASM) was evaluated for the flat-plate turbulent boundary layer at a zero-pressure gradient. As expected (the results are not shown here), the turbulence model yielded good predictions for the mean-velocity profiles and skin-friction coefficients. Although some turbulence properties near the wall are not captured (i.e., the peak of the turbulent kinetic energy), the algebraic stress model does give accurate results away from the buffer layer (i.e., $y^+ > 30$). Remember that the algebraic stress model can be integrated directly to a solid boundary with no damping function in the turbulent eddy viscosity.

The first two test cases to be considered are the RAE 2822 airfoil flows (cases 9 and 10), which were tested by Cooke et al.[14] The airfoil has a maximum thickness of 12.1 percent $c$ and a leading-edge radius of 0.827 percent $c$ ($c$ is the chord of the airfoil). The grid used is a 257×97 $C$ mesh with 177 points on the airfoil, and a minimum spacing at the wall of $0.932 \times 10^{-6}c$. The outer boundary extent is approximately 18$c$, and transition is assumed at 3 percent $c$. For the case 9, the conditions include a Mach number $M_\infty = 0.73$, an angle of attack $\alpha = 2.8°$, and a Reynolds number $Re = 6.5 \times 10^6$. This case contains no separated flow. For the case 10, the conditions include a Mach number $M_\infty = 0.75$, an angle of attack $\alpha = 2.72$, and a Reynolds

number $Re = 6.2 \times 10^6$. This case involves separation based on visual surface streamline patterns. However, there are no skin-friction coefficient data indicating separation. Hence, case 10 is considered as an incipiently separated flow and, therefore, is more challenging than the previous case.

Figures 1 and 2 compare the surface pressure and skin-friction coefficients computed along the airfoil surface with the experimental data for case 9. It is clear that the explicit algebraic stress model provides a good representation of the pressure over most of the airfoil. However, the turbulence model over predicts the skin-friction coefficient downstream of the shock. This deficiency results from the tendency of the models based on $K-\varepsilon$ formulation to predict excessive near-wall levels of turbulent length scale in the presence of an adverse pressure gradient, which leads to high values of the eddy-viscosity. A modification of the dissipation equation is required in order to improve the response of the algebraic stress model to adverse pressure-gradient effects.

In order to demonstrate the improvement resulting from the use of the EASM model for non-equilibrium flows, comparisons between the results obtained by the EASM model and the Speziale, Abid and Anderson $K-\varepsilon$ model[15] (hereafter referred to as SAA) were performed (Figures 3–10). From Figure 4, it appears clearly that neither turbulence model predicts separation. This is reflected by the high level of the skin-friction coefficient, downstream from the shock. This probably is a result of the inability of the length scale equation to provide proper response to adverse pressure gradients. To date, several modifications to the dissipation equation for separation do not seem to be successful. On the other hand, the EASM model predicts the shock location better than the SAA model, although slightly downstream of the experimental shock location (see Figure 3). This results from the prediction by the EASM of lower values of eddy viscosity in the inner part of the boundary layer, therefore, lower values of the turbulent kinetic energy (see Figure 7). Comparison of the computed and measured

velocity profiles further support the latter observation. An additional finding that can be inferred from the above comparison is that the EASM model gives a realistic representation of the normal stresses (see Figures 8–10).

The third test case to be considered is the ONERA M6 wing at Mach number of 0.8447, an angle of attack $\alpha$ of 5.06 and a Reynolds number of $11.7 \times 10^6$ based on the mean aerodynamic chord.[16] A C–O grid, used in this study has $193 \times 49 \times 33$ points in the streamwise, normal and spanwise direction. The minimum normal spacing over the wing of 0.000015 $c_{root}$ and a distance from the wing to the outer boundary of at least 7.95 $c_{root}$. No wind tunnel test corrections are employed for this case.

Figure 11 shows a comparison of the surface pressure distributions with the experimental data at four different spanwise locations $2y/B$. It is clear from this figure, that the predicted shock location and the surface pressure distributions by the EASM model are in good agreement with the experimental data, and similar to the results reported in [17] for the Johnson-King model, which has been highly tuned for airfoil flows.

## Conclusions

A study of an explicit algebraic stress model, used in the framework of the $K - \varepsilon$ formulation for separated turbulent flows, has been conducted. This new generation of two-equation models, which is derived from second-order closures, has been tested against three test cases, two of which involve separation. Two major findings have been made in this study: explicit algebraic stress models have shown some improvement over the standard two-equation models because of their ability to account for nonequilibrium effects and to give a realistic representation of the anisotropy of the turbulence. However, this improvement is still limited by the dissipation rate equation which fails to respond properly to adverse pressure gradients. A major research effort to correct this deficiency is currently underway.

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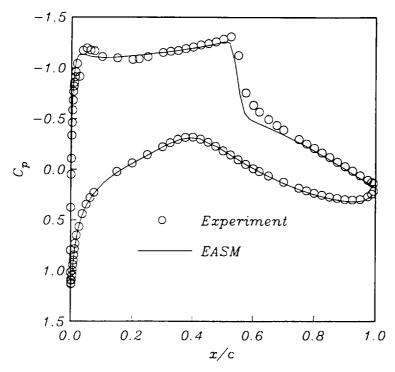
Figure 1: Surface pressure distributions for RAE 2822 airfoil (Case 9)
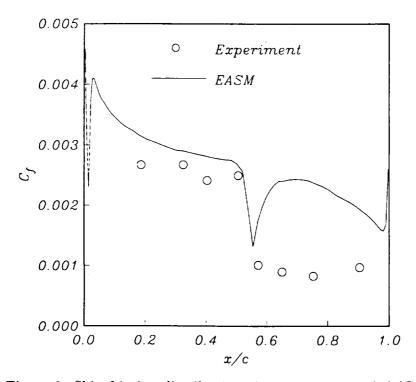


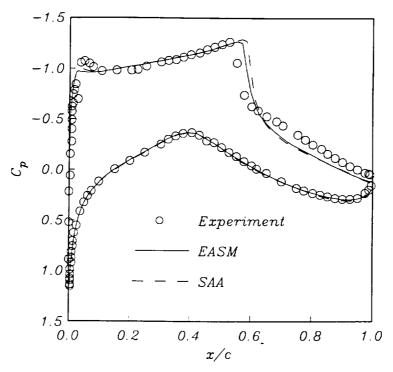Figure 2: Skin friction distributions for RAE 2822 airfoil (Case 9)

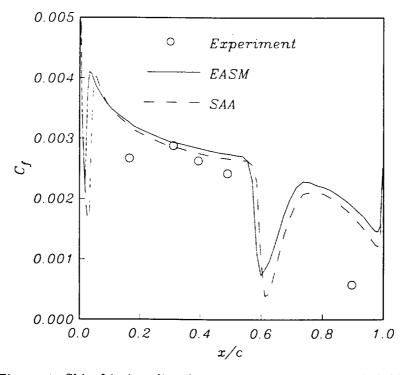Figure 3: Surface pressure distributions for RAE 2822 airfoil (Case 10)



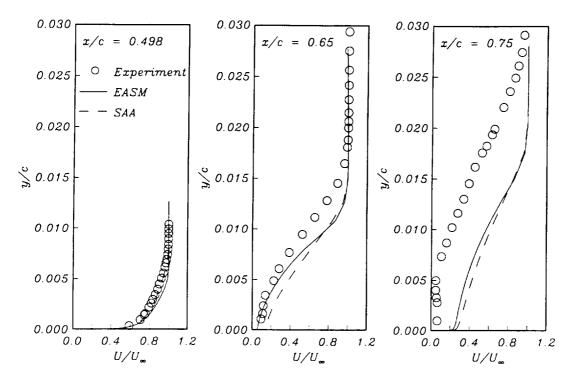Figure 4: Skin friction distributions for RAE 2822 airfoil (Case 10)

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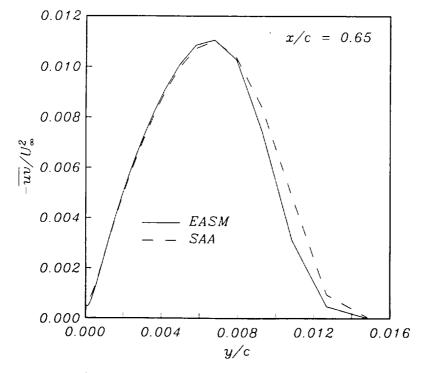Figure 5: Comparison of mean velocity profiles for RAE 2822 airfoil (Case 10)



Figure 6: Comparison of turbulent shear stress distributions for RAE 2822 airfoil (Case 10)
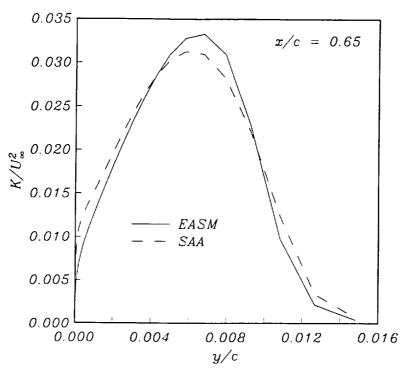
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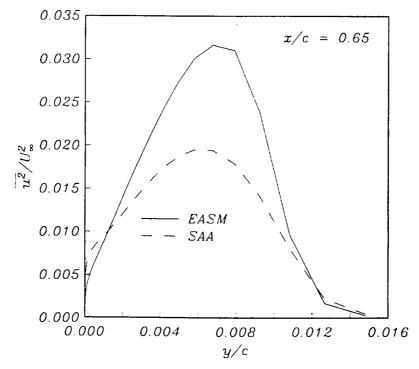Figure 7: Comparison of turbulent kinetic energy distributions for RAE 2822 airfoil (Case 10)



Figure 8: Comparison of $\overline{u^2}$ normal stress distributions for RAE 2822 airfoil (Case 10)
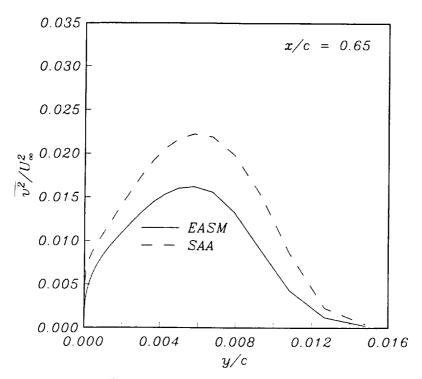
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Figure 9: Comparison of $\overline{v^2}$ normal stress distributions for RAE 2822 airfoil (Case 10)
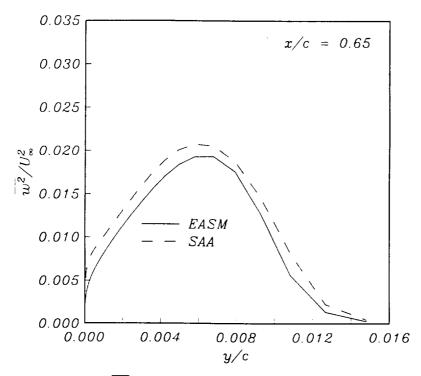


Figure 10: Comparison of $\overline{w^2}$ normal stress distributions for RAE 2822 airfoil (Case 10)
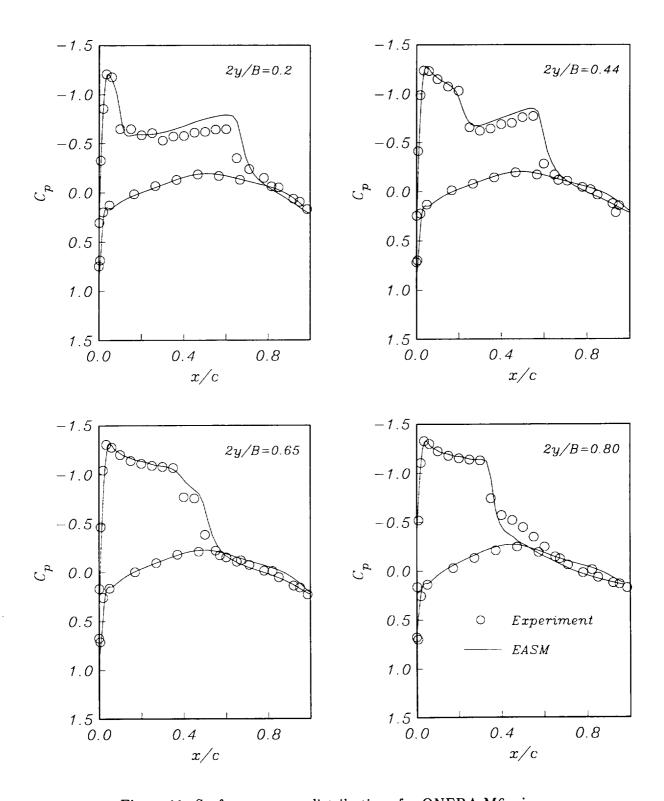
Figure 11: Surface pressure distributions for ONERA M6 wing

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# APPENDIX B

# AN EXPLICIT ALGEBRAIC STRESS MODEL OF TURBULENCE WITH ANISOTROPIC DISSIPATION

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## 1. INTRODUCTION

Turbulent flows near solid boundaries – or at low turbulence Reynolds numbers – can exhibit significant anisotropies in the turbulent dissipation rate[1]. Nevertheless, Reynolds stress turbulence closures are routinely formulated that neglect such effects by invoking the Kolmogorov assumption of local isotropy[2]. Recently, however, attempts have been made to extend full Reynolds stress turbulence closures to incorporate the effects of anisotropic dissipation (see Speziale, Raj and Gatski[3], Speziale and Gatski[4], Oberlack[5] and Hallbäck *et al.*[6]). These more sophisticated Reynolds stress turbulence closures can involve the solution of up to twelve additional transport equations. As such, most of these models are not currently feasible for the solution of complex turbulent flows in an engineering setting.

During the past few years, explicit algebraic stress models have been developed that are formally consistent with full second-order closures in the limit of homogeneous turbulence in equilibrium (see Gatski and Speziale[7]). These models allow for the solution of complex turbulent flows with a substantially reduced level of computation compared to full second-order closures, since they constitute two-equation models[7,8]. The purpose of the present note is to show how the effects of anisotropic dissipation can be systematically incorporated into these explicit algebraic stress models by a simple readjustment of the coefficients. For homogeneous turbulent flows that are close to equilibrium, it will be shown that the results obtained from such models are virtually indistinguishable from those obtained from a full second-order closure model with the anisotropic dissipation rate model of Speziale and Gatski[4]. All of this extra turbulence physics is incorporated within the framework of a two-equation model that is not much more computationally expensive to implement than the standard $K - \epsilon$ model.

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## 2. THEORETICAL BACKGROUND

We will consider incompressible turbulent flows where the velocity $v_i$ and kinematic pressure $P$ are decomposed, respectively, into ensemble mean and fluctuating parts as follows:

$$v_i = \overline{v}_i + u_i, \quad P = \overline{P} + p. \tag{1}$$

In homogeneous turbulence, where all higher-order correlations are spatially uniform, the Reynolds stress tensor $\tau_{ij} \equiv \overline{u_i u_j}$ satisfies the transport equation[9]

$$\dot{\tau}_{ij} = -\tau_{ik}\frac{\partial \overline{v}_j}{\partial x_k} - \tau_{jk}\frac{\partial \overline{v}_i}{\partial x_k} + \Pi_{ij} - \varepsilon_{ij}, \tag{2}$$

where

$$\Pi_{ij} \equiv \overline{p\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)},$$

$$\varepsilon_{ij} \equiv 2\nu\overline{\frac{\partial u_i}{\partial x_k}\frac{\partial u_j}{\partial x_k}}$$

are, respectively, the pressure-strain correlation and the dissipation rate tensor. Thus, in homogeneous turbulence, only $\Pi_{ij}$ and $\varepsilon_{ij}$ need to be modeled in order to achieve closure.

Speziale, Sarkar and Gatski[10] showed that, for two-dimensional mean turbulent flows in equilibrium, the commonly used hierarchy of pressure-strain models simplifies to:

$$\Pi_{ij} = -C_1\varepsilon b_{ij} + C_2\varepsilon(b_{ik}b_{kj} - \tfrac{1}{3}b_{kl}b_{kl}\delta_{ij})$$

$$+C_3 K\overline{S}_{ij} + C_4 K(b_{ik}\overline{S}_{jk} + b_{jk}\overline{S}_{ik} \tag{3}$$

$$-\tfrac{2}{3}b_{kl}\overline{S}_{kl}\delta_{ij}) + C_5 K(b_{ik}\overline{\omega}_{jk} + b_{jk}\overline{\omega}_{ik})$$

where

$$\overline{S}_{ij} = \frac{1}{2}\left(\frac{\partial \overline{v}_i}{\partial x_j} + \frac{\partial \overline{v}_j}{\partial x_i}\right), \quad \overline{\omega}_{ij} = \frac{1}{2}\left(\frac{\partial \overline{v}_i}{\partial x_j} - \frac{\partial \overline{v}_j}{\partial x_i}\right), \quad b_{ij} = \frac{\tau_{ij} - \tfrac{2}{3}K\delta_{ij}}{2K}$$

are, respectively, the mean rate of strain tensor, the mean vorticity tensor, and the Reynolds stress anisotropy tensor; $K \equiv \tfrac{1}{2}\tau_{ii}$ is the turbulent kinetic energy. The SSG model is a simple extension of (3) that is valid for moderate departures from equilibrium. It has been found that the nonlinear return term containing $C_2$ can be neglected without introducing an appreciable error.[7,10] With the choice of constants

$$C_1 = 6.80, \quad C_2 = 0, \quad C_3 = 0.36, \quad C_4 = 1.25, \quad C_5 = 0.40,$$

in (3), excellent equilibrium values are obtained for the benchmark case of homogeneous shear flow. We refer to this as the linearized, equilibrium form of the SSG model.

The Kolmogorov assumption of local isotropy is typically invoked wherein it is assumed that[2,9]

$$\epsilon_{ij} = \frac{2}{3}\epsilon\delta_{ij} \tag{4}$$

where $\epsilon \equiv \frac{1}{2}\epsilon_{ii}$ is the (scalar) turbulent dissipation rate. It is generally accepted that homogeneous turbulent flows, with constant mean velocity gradients, achieve equilibrium values for $b_{ij}$ that are largely independent of the initial conditions. This is characterized by

$$\dot{b}_{ij} = 0$$

or, equivalently,

$$\dot{\tau}_{ij} = 2(\mathcal{P} - \epsilon)b_{ij} + \frac{2}{3}(\mathcal{P} - \epsilon)\delta_{ij} \tag{5}$$

where $\mathcal{P} \equiv -\tau_{ij}\partial\overline{v}_i/\partial x_j$ is the turbulence production. The substitution of (3) – (5) into (2) yields an implicit algebraic system that can be solved by integrity bases methods. This solution – which has come to be referred to as an explicit algebraic stress model (ASM) – is given in the equilibrium limit of homogeneous turbulence by:[7]

$$\begin{aligned}
\tau_{ij} &= \frac{2}{3}K\delta_{ij} - \frac{3}{3 - 2\eta^2 + 6\xi^2}\left[\alpha_0\frac{K^2}{\epsilon}\overline{S}_{ij}\right.\\
&\quad + \alpha_1\frac{K^3}{\epsilon^2}(\overline{S}_{ik}\overline{\omega}_{kj} + \overline{S}_{jk}\overline{\omega}_{ki})\\
&\quad \left. - \alpha_2\frac{K^3}{\epsilon^2}\left(\overline{S}_{ik}\overline{S}_{kj} - \frac{1}{3}\overline{S}_{kl}\overline{S}_{kl}\delta_{ij}\right)\right]
\end{aligned} \tag{6}$$

where

$$\alpha_0 = \left(\frac{4}{3} - C_3\right)g, \quad \alpha_1 = \frac{1}{2}\left(\frac{4}{3} - C_3\right)(2 - C_5)g^2$$

$$\alpha_2 = \left(\frac{4}{3} - C_3\right)(2 - C_4)g^2, \quad g = \left(\frac{1}{2}C_1 + \frac{\mathcal{P}}{\epsilon} - 1\right)^{-1}$$

$$\eta = \frac{1}{2}\frac{\alpha_2}{\alpha_0}\frac{K}{\epsilon}(\overline{S}_{ij}\overline{S}_{ij})^{1/2}, \quad \xi = \frac{\alpha_1}{\alpha_0}\frac{K}{\epsilon}(\overline{\omega}_{ij}\overline{\omega}_{ij})^{1/2} \tag{7}$$

and $\mathcal{P}/\epsilon$ assumes its constant equilibrium value. When far from equilibrium, a singularity may arise since the denominator $(3 - 2\eta^2 + 6\xi^2)$ in (6) can vanish for sufficiently high strain rates $\eta$. Gatski and Speziale[7] introduced a regularized expression for $3/(3 - 2\eta^2 + 6\xi^2)$ which eliminated the singular behavior. However, that model is not formally valid for non-equilibrium turbulence — particularly in the rapid distortion limit.

Speziale and Xu[11] later introduced a formal Padé approximation that built in some limited consistency with Rapid Distortion Theory (RDT) for homogeneous shear flow. They rewrote (6) in the form:

$$\tau_{ij} = \frac{2}{3}K\delta_{ij} - \alpha_0^*\frac{K^2}{\varepsilon}\overline{S}_{ij} - \alpha_1^*\frac{K^3}{\varepsilon^2}(\overline{S}_{ik}\overline{\omega}_{kj}+\overline{S}_{jk}\overline{\omega}_{ki}) + \alpha_2^*\frac{K^3}{\varepsilon^2}\left(\overline{S}_{ik}\overline{S}_{kj} - \frac{1}{3}\overline{S}_{kl}\overline{S}_{kl}\delta_{ij}\right) \quad (8)$$

and made use of the fact that in the short-time RDT solution ($\eta \to \infty$), $K/K_0$ remains of order one. This implies that

$$\alpha_0^* \sim \frac{1}{\eta}. \quad (9)$$

It is obvious that the equilibrium model (6) violates this constraint ($\alpha_0^* \sim 1/\eta^2$ instead). Speziale and Xu[11] then introduced a Padé approximation and obtained the expression:

$$\alpha_0^* = \frac{(1 + 2\xi^2)(1 + 6\eta^5) + \frac{5}{3}\eta^2}{(1 + 2\xi^2)(1 + 2\xi^2 + \eta^2 + 6\beta_0\eta^6)}\alpha_0 \quad (10)$$

(with the constant $\beta_0 \approx 7$) to ensure asymptotic consistency and the proper energy growth rate in line with the RDT data. By a comparable Padé approximation, they also derived the expression

$$\alpha_i^* = \frac{(1 + 2\xi^2)(1 + \eta^4) + \frac{2}{3}\eta^2}{(1 + 2\xi^2)(1 + 2\xi^2 + \beta_i\eta^6)}\alpha_i \quad (11)$$

for $i = 1, 2$ (with $\beta_1 \approx 6$ and $\beta_2 \approx 4$), in order to establish consistency with the approach to a one component turbulence in the RDT limit of homogeneous shear flow. For near-equilibrium turbulent flows, (8) with (10) – (11) yields results that are virtually indistinguishable from (6).

## 3. ANISOTROPIC DISSIPATION

In a recent study, Speziale and Gatski[4] derived a modeled transport equation for $\varepsilon_{ij}$ which is valid for homogeneous turbulence. By invoking the equilibrium limit where

$$\dot{d}_{ij} = 0 \quad (12)$$

for the anisotropy of dissipation $d_{ij}$, they obtained an algebraic system — analogous to that for algebraic stress models — which was solved by integrity bases methods. This ultimately led to the algebraic model:[4]

$$d_{ij} = -2C_{\mu\epsilon}\{\tau\overline{S}_{ij} - \frac{(\frac{7}{11}\alpha_3 + \frac{1}{11})\tau^2}{C_{\epsilon5} + \mathcal{P}/\varepsilon - 1}(\overline{S}_{ik}\overline{\omega}_{kj} + \overline{S}_{jk}\overline{\omega}_{ki})$$

$$+ \frac{(\frac{30}{11}\alpha_3 - \frac{2}{11})\tau^2}{C_{\epsilon5} + \mathcal{P}/\varepsilon - 1}(\overline{S}_{ik}\overline{S}_{kj} - \frac{1}{3}\overline{S}_{mn}\overline{S}_{mn}\delta_{ij})\} \quad (13)$$

where

$$d_{ij} = \frac{\varepsilon_{ij} - \frac{2}{3}\varepsilon\delta_{ij}}{2\varepsilon} \tag{14}$$

is the anisotropy of dissipation and $\tau \equiv K/\varepsilon$ is the turbulent time scale. Here,

$$C_{\mu\epsilon} = \frac{1}{15(C_{\epsilon5} + \mathcal{P}/\varepsilon - 1)}\{1 + 2\tau^2 \left(\frac{\frac{7}{11}\alpha_3 + \frac{1}{11}}{C_{\epsilon5} + \mathcal{P}/\varepsilon - 1}\right)^2 \overline{\omega}_{mn}\overline{\omega}_{mn}$$

$$- \frac{2}{3}\tau^2 \left(\frac{\frac{15}{11}\alpha_3 - \frac{1}{11}}{C_{\epsilon5} + \mathcal{P}/\varepsilon - 1}\right)^2 \overline{S}_{mn}\overline{S}_{mn}\}^{-1} \tag{15}$$

and

$$C_{\epsilon5} = 5.8, \quad \alpha_3 = 0.6$$

are constants. This then yields the full dissipation rate tensor since, by definition,

$$\varepsilon_{ij} = \frac{2}{3}\varepsilon\delta_{ij} + 2\varepsilon d_{ij}. \tag{16}$$

After introducing the anisotropic dissipation model for $d_{ij}$ in (2), we can derive an algebraic stress model with anisotropic dissipation. Again, making use of the fact that $\dot{b}_{ij} = 0$ for equilibrium turbulent flows, the Reynolds stress transport equation (2) then reduces to:

$$2(\mathcal{P} - \varepsilon)b_{ij} = -2K(b_{ik}\overline{S}_{jk} + b_{jk}\overline{S}_{ik} - \frac{2}{3}b_{mn}\overline{S}_{mn}\delta_{ij}) - \frac{4}{3}K\overline{S}_{ij}$$

$$-2K(b_{ik}\overline{\omega}_{jk} + b_{jk}\overline{\omega}_{ik}) + \Pi_{ij} - 2\varepsilon d_{ij}. \tag{17}$$

The explicit ASM incorporating anisotropic dissipation – which is obtained from the solution of (17) after (3) and (13) are implemented – is of the same general tensorial form as (6):

$$b_{ij} = G^{(1)}T_{ij}^{(1)} + G^{(2)}T_{ij}^{(2)} + G^{(3)}T_{ij}^{(3)} \tag{18}$$

where

$$T_{ij}^{(1)} = \overline{S}_{ij}, \quad T_{ij}^{(2)} = \overline{S}_{ik}\overline{\omega}_{kj} + \overline{S}_{jk}\overline{\omega}_{ki}$$

$$T_{ij}^{(3)} = \overline{S}_{ik}\overline{S}_{kj} - \frac{1}{3}\overline{S}_{mn}\overline{S}_{mn}\delta_{ij}$$

are the integrity bases. The solution is given by

$$G^{(1)} = \frac{1}{2}\alpha_0\tau \left[\frac{-1 - A_1 + \frac{1}{3}\eta^2 A_3 + 2\xi^2 A_2}{1 - \frac{2}{3}\eta^2 + 2\xi^2}\right]$$

$$G^{(2)} = \frac{1}{2}\alpha_1\tau^2 \left[\frac{-1 - A_1 - A_2 + \frac{1}{3}\eta^2(A_3 + 2A_2)}{1 - \frac{2}{3}\eta^2 + 2\xi^2}\right]$$

$$G^{(3)} = \frac{1}{4}\alpha_2\tau^2 \left[\frac{2 - A_3 + 2A_1 - 2\xi^2(A_3 + 2A_2)}{1 - \frac{2}{3}\eta^2 + 2\xi^2}\right] \tag{19}$$

where

$$A_1 = -\frac{4C_{\mu\epsilon}}{\frac{4}{3} - C_3}$$

$$A_2 = \frac{2}{g(\frac{4}{3} - C_3)(2 - C_5)} \frac{4C_{\mu\epsilon}(\frac{7}{11}\alpha_3 + \frac{1}{11})}{C_{\epsilon 5} + \frac{P}{\epsilon} - 1}$$

$$A_3 = -\frac{2}{g(\frac{4}{3} - C_3)(2 - C_4)} \frac{4C_{\mu\epsilon}(\frac{30}{11}\alpha_3 - \frac{2}{11})}{C_{\epsilon 5} + \frac{P}{\epsilon} - 1}.$$

Equivalently, from (18) we have

$$\tau_{ij} = \frac{2}{3}K\delta_{ij} + 2K(G^{(1)}T_{ij}^{(1)} + G^{(2)}T_{ij}^{(2)} + G^{(3)}T_{ij}^{(3)}) \tag{20}$$

which is obviously of the same tensorial form as (6) — only the coefficients are different.

The factor $1/(1 - \frac{2}{3}\eta^2 + 2\xi^2)$ in (19) can be regularized in the same general way as discussed earlier to ensure the correct asymptotic behavior in the RDT limit. The standard explicit ASM given in (6) is then recovered in the limit as $C_{\mu\epsilon}$ (and hence, $A_1$, $A_2$, and $A_3$) $\rightarrow 0$. Anisotropies in the dissipation rate are then accounted for simply through a systematic readjustment of the coefficients.

## 4. RESULTS AND DISCUSSION

The anisotropic dissipation rate model has been tested in detail by Speziale and Gatski[4] within the context of a full second-order closure, which will not be repeated here. Our purpose in this note is to simply demonstrate that – for homogeneous turbulent flows close to equilibrium – the new explicit ASM with anisotropic dissipation yields results that are indistinguishable from the full second-order closure, with anisotropic dissipation, on which it is based.

The new explicit ASM with anisotropic dissipation derived herein is solved with a modeled transport equation for the turbulent dissipation rate that is of the form[4]

$$\dot{\epsilon} = C_{\epsilon 1}^* \frac{\epsilon}{K}\mathcal{P} - C_{\epsilon 2}\frac{\epsilon^2}{K} \tag{21}$$

for homogeneous turbulence. Here

$$C_{\epsilon 1}^* = 1.26 + \frac{2(1 + \alpha)}{15C_\mu} \left[\frac{C_{\epsilon 5} + 2C_\mu\eta^2 - 1}{(C_{\epsilon 5} + 2C_\mu\eta^2 - 1)^2 + 2\beta_1^2\xi^2 - \frac{2}{3}\beta_2^2\eta^2}\right] \tag{22}$$

where

$$\alpha = \frac{3}{4}(\frac{14}{11}\alpha_3 - \frac{16}{33})$$

$$\beta_1 = \frac{7}{11}\alpha_3 + \frac{1}{11}, \quad \beta_2 = \frac{15}{11}\alpha_3 - \frac{1}{11}$$

and $C_\mu$ and $C_{\epsilon2}$ are constants that assume the approximate values of 0.09 and 1.83, respectively. In (22), $\eta \equiv (\overline{S}_{ij}\overline{S}_{ij})^{1/2}\tau$ and $\xi \equiv (\overline{\omega}_{ij}\overline{\omega}_{ij})^{1/2}\tau$. The effects of anisotropic dissipation are rigorously accounted for in (21) through the variable coefficient $C_{\epsilon1}^*$. For inhomogeneous turbulent flows, a gradient transport term of the standard form

$$\frac{\partial}{\partial x_i}\left(\frac{\nu_T}{\sigma_\epsilon}\frac{\partial \varepsilon}{\partial x_i}\right)$$

is added to the r.h.s. of (21) to account for turbulent diffusion.

In order to demonstrate the ability of the new model to properly capture the physics of the more complicated full second-order closure with anisotropic dissipation, we will present a simple example of a benchmark turbulent flow. In Table 1, we compare the equilibrium results of the new explicit ASM for homogeneous shear flow with the predictions of the linearized SSG second-order closure incorporating the anisotropic dissipation rate model of Speziale and Gatski[4]. These calculations were conducted with a fourth-order accurate Runge-Kutta numerical integration scheme. It is clear from these calculations that the new explicit ASM yields results that are virtually identical to the results of the full second-order closure with anisotropic dissipation and compare favorably with the DNS results of Rogers, Moin and Reynolds[12]. This definitively demonstrates the power of the new model to yield results that are indistinguishable from a full second-order closure — with anisotropic dissipation — for turbulent flows that are close to equilibrium. All of this at a small fraction of the computer costs. While the effects of anisotropic dissipation are not that significant for this case, they can be important in other flows of engineering interest as alluded to earlier. We consider this to be a highly promising new approach for such flows that can have important engineering applications and, thus, warrants further study.

## ACKNOWLEDGMENTS

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| Equilibrium Values | Explicit ASM | Full Closure | DNS |
|---|---|---|---|
| $b_{11}$ | 0.204 | 0.205 | 0.215 |
| $b_{12}$ | -0.150 | -0.150 | -0.158 |
| $b_{22}$ | -0.148 | -0.147 | -0.153 |
| $b_{33}$ | -0.056 | -0.058 | -0.062 |
| $SK/\varepsilon$ | 5.98 | 5.96 | 5.70 |

Table 1. Equilibrium values for homogeneous shear flow. Comparison of the new explicit ASM incorporating anisotropic dissipation with the DNS of Rogers, Moin and Reynolds[12] and a full Reynolds stress closure with anisotropic dissipation (containing the linearized SSG second-order closure and the anisotropic dissipation rate model of Speziale and Gatski[4]).