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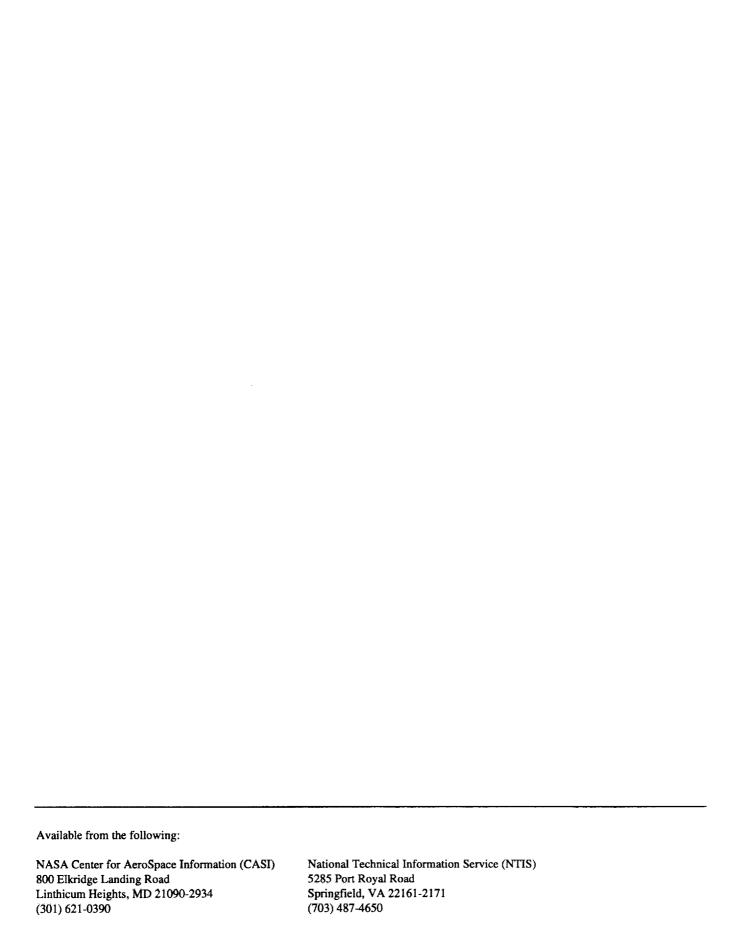*ICASE*

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A KINEMATICALLY CONSISTENT TWO-POINT CORRELATION FUNCTION

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Abstract. A simple kinematically consistent expression for the longitudinal two-point correlation function related to both the integral length scale and the Taylor microscale is obtained. On the inner scale, in a region of width inversely proportional to the turbulent Reynolds number, the function has the appropriate curvature at the origin. The expression for two-point correlation is related to the nonlinear cascade rate, or dissipation ε, a quantity that is carried as part of a typical single-point turbulence closure simulation. Constructing an expression for the two-point correlation whose curvature at the origin is the Taylor microscale incorporates one of the fundamental quantities characterizing turbulence, ε, into a model for the two-point correlation function. The integral of the function also gives, as is required, an outer integral length scale of the turbulence independent of viscosity. The proposed expression is obtained by kinematic arguments; the intention is to produce a practically applicable expression – in terms of simple elementary functions – that allow an analytical evaluation, by asymptotic methods, of diverse functionals relevant to single-point turbulence closures. Using the expression devised an example of the asymptotic method by which functionals of the two-point correlation can be evaluated is given.

Key words. two-point correlation functions, turbulence modeling, functionals

Subject classification. Fluid Mechanics, Aeroacoustics

1. Introduction. In single-point turbulence closures one is often in the position of needing to approximate functionals – integrals of functions of the two-point correlation function – Batchelor [1], Kraichnan [2], Ristorcelli [3]. In aeroacoustical developments in which acoustic radiation due to turbulence is related to the statistics of the turbulent source field one is in a similar position: the acoustic radiation is a function of diverse two-point integrals, Proudman [4], Ribner [5], Lilley [6].

Batchelor [1], for example, in order to finish his development for the pressure variance in an isotopic turbulence needed to evaluate the two functionals $I(f) = \int x(f')^2 dx$ and $I(f) = \int x^{-1}(f')^2 dx$. The function $f(x)$ is the two-point longitudinal correlation function. Lilley [6] in an investigation of noise radiated from isotropic turbulence required an evaluation of $I(f) = \int x^4 (f')^2 dx$. Proudman [4] in his statistical application of Lighthill's acoustic analogy required approximations to integrals such as $I(f) = \int f(f''' + 4x^{-1}f'' - 4r^{-2}f')$. Ristorcelli [3] in a treatment of weakly compressible turbulence required integrals of the form $I(f) = \int x f dx$, and $I(f) = \int x f'(ff''' + 4x^{-1}ff'' + 8x^{-1}f'f' - 4x^{-2}ff')dx$. These functionals typically involve higher-order derivatives of the two-point function. Practical estimates of these functionals are usually made by assuming simple exponential or Gaussian behavior for the two-point correlation. It is argued that the functional is an integral and therefore only nominally sensitive to form of the assumed two-point function. Unfortunately this is not the case when the higher order derivatives appear in the functionals.

Given the practical need of evaluating these functionals a few attempts have been made to devise, in terms

of elementary functions, a useful approximation to the two-point correlation. Several examples of attempts are given in Hinze [7], Frenkiel [8], [9], Townsend [10]. Hinze [7] has several amplifications on these issues. Attempts have also been made in spectral space: one assumes a spectral energy function shape consistent with a Kolmogorov spectrum and transforms to physical space to get the two-point correlation. One obtains fractional K_n type Bessel functions that are consistent with the Kolmogorov inertial range scalings for the structure functions, von Karman [11], and Sirovich, Smith and Yakhot [12]. Such developments are focussed at reproducing the intermediate field scalings consistent with the ideal Kolmogorov spectrum: they accurately capture the $r^{2/3}$ Kolmogorov inertial range result, Batchelor [15], Monin and Yaglom [16].

Our interest, however, is in non-ideal, anisotropic, strained, inhomogeneous, engineering turbulence for which an inertial range with an ideal Kolmogorov scaling is unlikely. Such flows are typically computed with single-point moment closures in which one parameterizes the turbulence in terms of the Reynolds stresses, $\langle v_i v_j \rangle$, and the dissipation, ε. For calculations of such engineering problems, to which such a two parameter characterization is applicable, the details of the spectrum are not relevant. Pertinent to this is the fact that in physical flows the two-point function is notoriously insensitive to the flow type, Townsend [10], Frenkiel [13], [14]. All that appears necessary is that the two-point correlation capture the integral and the dissipation length scales. Associated with a characterization of the turbulence by $\langle v_i v_j \rangle$, and ε are two important length scales: the integral length scale, ℓ, and the Taylor microscale, λ.

The Taylor microscale is the radius of curvature of the two-point correlation at the origin. The Taylor microscale can be related to the viscosity and the dissipation. For a stationary turbulence the dissipation rate is the spectral cascade rate. Thus constructing an expression for the two-point correlation whose curvature at the origin is the Taylor microscale incorporates one of the fundamental quantities, the cascade rate, characterizing turbulence into a model for the two-point correlation function.

The Taylor microscale is also related to the turbulent Reynolds number, Tennekes and Lumley [19], and as consequence the Reynolds number dependence of the curvature at the origin is built into the expression for the two-point correlation. Thus the functionals are no longer pure numbers but functions of the turbulent Reynolds numbers, $I = I(f(x; R_t))$, as one might expect. Any satisfactory approximation of the higher-order derivatives in the vicinity of the origin, where they are large *and a function of the turbulent Reynolds number* is crucial for any useful evaluation of such functionals. What is desired is a simple analytical expression that will be a good approximation for the two-point correlation and its higher derivatives. It must also be analytically tractable in terms of elementary functions so that functionals as complicated as those given above can be evaluated. Several attempts at such an empirical function are known: Frenkiel [8], Townsend [10], von Karman [11], Hinze [7]. None of these attempts satisfy the requirements that allow application to situations described in Ristorcelli [3], Ribner [5] Proudman [4], Lilley [6] in which the behavior to the two-point correlation near the origin is necessary.

The purpose of this note is to 1) develop a kinematically appropriate form of a two-point correlation function, 2) that incorporates the effect of the turbulent Reynolds number and 3) that is analytically tractable in terms of elementary functions. This is accomplished in the following sequence: In §2 the properties of the two-point correlation are reviewed. Relationships from statistical fluid mechanics indicating the dependence of the function on the Reynolds number and dissipation are summarized. The problem is recognized as a two-length scale problem. In as much as there is a large disparity in the two length scales the problem is further recognized as one in which there exists a small parameter. Given the existence of a small parameter a form of the two-point correlation satisfying all the properties delineated in §2 is proposed and investigated in §3. In §3 the boundary layer nature of the problem is used to produce a relationship between the parameters

that yields a viscosity independent first integral. The details of the calculations are provided as an example of the asymptotic method of evaluating similar functionals.

2. Mathematical and observed properties. The diverse kinematic and observed properties of the two-point correlation are now given. The relationship between curvature at the origin, Taylor microscale, dissipation, and Reynolds number is summarized. The definition for the correlation function is obtained from $< v_i(x)v_j(x+r) >=< v_i v'_j >= \frac{2k}{3} R_{ij}(r)$, where $k = \frac{1}{2} < v_j v_j >$. The isotropic portion of the two-point correlation can be written as

$$(1) \qquad R_{ij}(r) = -\frac{r_i r_j}{2r} f' + (f + \frac{1}{2} r f') \delta_{ij}$$

where $f = f(r)$ the longitudinal correlation is defined:

$$(2) \qquad < v_1(0)v_1(r) >=< v_1 v_1 > f(r) = \frac{2k}{3} f(r) = \frac{2k}{3} R_{11},$$

von Karman and Howarth [17], Batchelor [18], Monin and Yaglom [16]. Attention will be restricted to the isotropic component of the longitudinal two-point correlation correlation. The correlation function, $f(r)$, has the following properties

$$f(0) = 1$$
$$f'(0) = 0$$
$$f''(0) = -1/\lambda^2$$
$$f(\infty) \rightarrow 0.$$

The function is also even: $f^{2n+1}(0) = 0$. It is typical to work in nondimensional variables: one rescales according to $x = r/\ell$ where ℓ is the integral length scale of the turbulence. The integral length scale is defined as $\frac{2}{3} k \ell = \int_0^\infty < v_1 v'_1 > dr$ and thus

$$(3) \qquad \int_0^\infty f(x) \, dx = 1.$$

The curvature at the origin, in nondimensional coordinates, is then specified as $f''(x = 0) = -\ell^2/\lambda^2$. The Taylor microscale, λ, can be shown to be equal to the spectrally weighted length scale

$$(4) \qquad \frac{1}{\lambda^2} = \frac{\int \kappa^2 E(\kappa) d\kappa}{\int E(\kappa) d\kappa}.$$

The Taylor microscale is an intermediate length scale; smaller then the outer integral length scale and larger than the inner Kolmogorov scale, Tennekes and Lumley [19].

The turbulent Reynolds number is defined as $R_t = u_c \ell / \nu$ where u_c is a characteristic fluctuating velocity, $u_c = (\frac{2}{3} k)^{1/2}$. The Kolmogorov scaling $\varepsilon = \alpha_k u_c^3/\ell$, [19], is used to eliminate the length scale: thus $R_t = \alpha_k \frac{4}{9} k^2/(\nu \varepsilon)$. Note that this is the traditional Reynolds number, Tennekes and Lumley [19]; the Reynolds number definition used in contemporary DNS is a factor 9 larger and does not realistically reflect the relative magnitude of inertial to viscous forces. The isotropic portion of the dissipation tensor, $\varepsilon_{ij} = \nu < u_{i,k} u_{j,k} >$, can be written in terms of the two-point correlation tensor:

$$(5) \qquad \varepsilon_{jj} = 2\varepsilon = \nu < u_{i,k} u_{j,k} >= \frac{2}{3} k\nu R_{jj,kk} |_0 = -\frac{2}{3} k\nu 15 f''(0) = \frac{2}{3} k\nu 15/\lambda^2,$$

Batchelor [15]. The curvature, λ, has been related to the spectral cascade rate of energy, ε:

$$(6) \qquad \varepsilon = 5k\nu/\lambda^2,$$

Tennekes and Lumley [19]. The cascade rate, ε, is a measure of one of the most fundamental characteristics of turbulence – its nonlinear decorrelating effect; any model for a two-point correlation must be dependent on this quantity. The curvature, λ, is also related to the Reynolds number. The Reynolds number, $R_t = \frac{4}{9}k^2/(\nu\varepsilon)$, with $\alpha_k = 1$ can be used to eliminate ε. One finds, Tennekes and Lumley [19], that

$$(7) \qquad \ell^2/\lambda^2 = \frac{1}{15}R_t = f''(x=0).$$

The dependence of the curvature at origin to turbulent Reynolds number needs to be embedded in any empirical expression of the correlation function. Moreover any model for the two-point correlation, $f = f(x; R_t)$, must have

$$(8) \qquad \int_0^\infty f(x; R_t)\, dx = 1,$$

a first integral independent of Reynolds number (or, equivalently, viscosity).

An additional constraint for any model of f can be found from the problem of the final period of the decay. In the final period of the decay, first treated by von Karman and Howarth [17], (see also [18], [7]), the two-point correlation is found to be a Gaussian: as $R_t \to 0$

$$(9) \qquad f \sim e^{-\frac{r^2}{8\nu t}}.$$

Note that in the linear decay problem $8\nu t = \lambda^2 \sim 1/R_t$ and $8\nu t \to \infty$ corresponds to $R_t \to 0$. This function has the appropriate properties at the origin: it is an even (differentiable) function with finite curvature specified by the Reynolds number. The Gaussian, however, cannot satisfy $\int_0^\infty f(x)\, dx = 1$ in a way that is independent of Reynolds number.

It is empirically observed that the decay of the correlation function in high Reynolds number turbulence can be usefully approximated as $f \sim e^{-x}$, Hinze [7], Frenkiel [8], Townsend [10]. A similar behavior for the two-time correlation is discussed in Pope [20]. In as much as $e^{-|x|}$ is an easily integrable function it is used in estimates of integrals of the two-point correlation. It does not have any of the required properties near the origin. This, nonetheless, has not stopped its application as an estimate in a number of situations, [2], [7], [10]. Its application being understood to be limited to issues related to larger scales of the motion, [20]. The form $e^{-|x|}$ cannot, of course, be used in any functionals that feature a dependence on higher order derivatives which are very large near the origin and cannot be represented by $e^{-|x|}$, which is non-differentiable at the origin.

3. Mathematical representation. Many of the empirical functions chosen to approximate the two-point correlation in the estimation of functionals of f are single parameter curves like $e^{-a|x|}$ or e^{-ax^2} which while satisfying one constraint do not reflect the two-length scale nature the correlation function. For example, near the origin, $f \sim e^{-|x|}$ has positive curvature while the real correlation function has negative curvature. The use of $f \sim e^{-ax^2}$ which has proper sign for the curvature will not have the proper magnitude of curvature at the origin *and* produce, as is required, a viscosity or Reynolds number independent integral length scale. Furthermore $f \sim e^{-ax^2}$ is inconsistent with the experimentally the observed exponential decay, $e^{-a|x|}$, [8]. Yet the choice $f \sim e^{-|x|}$ while a useful approximation to the experimental data, Frenkiel [8], Townsend [10], has finite (discontinuous) slope at the origin and is an odd function. In addition the positive curvature of $f \sim e^{-ax}$ at the origin which is unacceptably inconsistent with the dissipative nature of the small scales. Any single parameter two-point correlation model cannot capture both of the length scale properties of a turbulence correlation: the integral length scale, ℓ, and the Taylor microscale, λ.

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A simple two parameter expression for $f(x; R_t)$ in terms of elementary functions satisfying the λ and ℓ properties is now considered. The properties that any model for the two-point correlation must satisfy are:

- The two-point function has the properties: $f(0) = 1, f'(0) = 0, f(\infty) \to 0$ and $f^{2n+1}(0) = 0$.
- The curvature at the origin is specified $f''(0) = -1/\lambda^2$.
- Its outer length scale is specified by the normalization $\int_0^\infty f(x)dx = 1$.
- It satisfies the exact result for the low Reynolds number limit, $f \sim e^{-\frac{x^2}{\lambda^2}}$.
- It has, as is observed for large Reynolds number, an exponential decay for large x, $f \sim e^{-x}$.

We are now in the position of proposing a function that has all these properties:

$$(10) \qquad f(x; \epsilon, b) = e^{-\frac{x^2}{(\epsilon^2 + b^2 x^2)^{1/2}}}.$$

The function has two scale parameters, ϵ and b, that will be used to satisfy the two length scales constraints. The two asymptotic forms of f, in as much as they highlight the two scale parameters, are worth considering. For small x

$$(11) \qquad x \to 0, \qquad f \sim e^{-\frac{x^2}{\epsilon}\frac{1}{(1+b^2 x^2/\epsilon^2)^{1/2}}} \to e^{-\frac{x^2}{\epsilon}}$$

For a high Reynolds number turbulence, the quantity ϵ, will be seen to be a small parameter: $\epsilon \sim 1/R_t$ number. The small parameter, ϵ, forms a boundary layer in the vicinity of the origin, $x << \epsilon^2$, and the function has the required large and positive curvature. For low Reynolds number, $\epsilon \sim 1$ and the Gaussian behavior of the von Karman and Howarth result is obtained. For large x the function is written as

$$(12) \qquad x \to \infty, \qquad f \sim e^{-\frac{x}{b}\frac{1}{(1+\epsilon^2/(b^2 x^2))^{1/2}}} \to e^{-\frac{x}{b}}.$$

The values of the two scale parameters are now related to quantities describing the turbulence.

It was shown above that the curvature of the correlation function at the origin must satisfies $f''(0) = -\frac{1}{15}R_t$. Differentiating f twice, $f''(0) = -\frac{2}{\epsilon} = -\frac{1}{15}R_t$ and the small parameter is determined by the turbulent Reynolds number:

$$(13) \qquad \epsilon = \frac{15}{2}\frac{1}{R_t}.$$

The two-point correlation also satisfies the normalization condition,

$$(14) \qquad I_1(f(\epsilon, b)) = \int_0^\infty f(x; \epsilon, b)\, dx = 1$$

which will serve to specify b in terms of R_t in a way that I_1 is independent of R_t. The integral I_1 is not tractable but an asymptotic analysis produces simple analytical results.

The integral I_1 is an example of integral that has local and global contributions. The usual Laplace method of evaluating such integrals with large parameters is not applicable, however, the boundary layer of the expression for $f(x; R_t)$ can be exploited. Near origin for $x < \epsilon$ there is a boundary layer whose width scales with $1/R_t$. In the boundary layer region f does not change appreciably and the local contribution to the integral I_1 scales with the width of the region, of $\mathcal{O}(\epsilon)$. The global contribution to I_1 occurs over a region $\mathcal{O}(b)$ for $x > \epsilon$. In this region, $f \sim e^{-x}$ and $\int_0^\infty e^{-x}\, dx = 1$ and the global contribution is $\mathcal{O}(1)$. With these ideas in mind the the interval of integration is subdivided into local and global regions in which the two-point function has, respectively, the near and far field behavior given by (11) and (12):

$$(15) \qquad I_{1\epsilon} + I_{1\infty} = \int_0^\delta f(x)\, dx + \int_\delta^\infty f(x)\, dx = 1.$$

Here δ satisfies $\epsilon << \delta << b << \infty$ but is otherwise arbitrary. The fact of its arbitrariness can be used to validate the success of the asymptotic evaluation of the two integrals $I_{1\epsilon}$ and $I_{1\infty}$; the combination $I_{1\epsilon} + I_{1\infty}$ must be independent of δ. As the contribution to each integral is proportional to the their intervals one might expect expect $I_{1\epsilon} \sim \mathcal{O}(\delta)$ and $I_{1\infty} \sim \mathcal{O}(b)$; since $\frac{\delta}{b} << 1$ the major contribution to I_1 is from $I_{1\infty}$ and one might expect $b \sim 1$ to leading order. The local and global contributions to the integral I_1 are now evaluated.

Inner integral. For the inner region the change of variable $x = \epsilon\eta$ is made. A Taylor series expansion for the exponential, since $\epsilon << 1$, is used to produce the following expression valid for $x \sim \mathcal{O}(\epsilon)$,

$$(16) \qquad f_0 = e^{-\frac{\epsilon\eta^2}{(1+b^2\eta^2))^{1/2}}} \approx 1 - \epsilon\frac{\eta^2}{(1 + b^2\eta^2))^{1/2}} + \mathcal{O}(\epsilon^2)$$

where $0 < \eta < \delta/\epsilon$. The integral $I_{1\epsilon}$ is rewritten

$$I_{1\epsilon} = \epsilon \int_0^{\delta/\epsilon} [\, 1 - \epsilon\frac{\eta^2}{(1 + b^2\eta^2)^{1/2}} + \mathcal{O}(\epsilon^2)\,]\, d\eta$$

$$(17) \qquad = \delta - \frac{1}{2}\frac{\epsilon\delta}{b^2}(\frac{\delta^2 b^2}{\epsilon^2} + 1)^{1/2} + \frac{1}{2}\frac{\epsilon^2}{b^3} ln[\frac{b\delta}{\epsilon} + (\frac{\delta^2 b^2}{\epsilon^2} + 1)^{1/2}] + \dots .$$

Re-expressing the terms as a function of $\frac{\epsilon}{b\delta}$ and realizing that both $\epsilon << 1$ and $\frac{\epsilon}{\delta} << 1$ and, using the Taylor series expansions for the quadratic and the logarithm, one obtains

$$(18) \qquad I_{1\epsilon} = \delta - \frac{1}{2}\frac{\delta^2}{b}(1 + \frac{1}{2}\frac{\epsilon^2}{\delta^2 b^2}) + \frac{1}{2}\frac{\epsilon^2}{b^2} ln[\frac{2b\delta}{\epsilon}] + \mathcal{O}(\epsilon^4).$$

Outer integral. For large x the two-point function is rewritten

$$(19) \qquad f = e^{-\frac{x}{b}\frac{1}{(1+\epsilon^2/(b^2 x^2))^{1/2}}} = e^{-\frac{x}{b}[1-\frac{1}{2}\frac{\epsilon^2}{b^2 x^2} + \frac{3}{8}(\frac{\epsilon^2}{b^2 x^2})^2 + \dots]} = e^{-\frac{x}{b}} e^{\frac{1}{2}\frac{\epsilon^2}{b^3 x}} e^{-\frac{3x}{8b}(\frac{\epsilon^2}{b^2 x^2})^2}$$

where we have used the binomial expansion since $x > \delta$ and therefore $\frac{\epsilon^2}{b^2 x^2} << 1$. The last two factors are expanded using the Taylor series for the exponential. To $\mathcal{O}(\epsilon^2)$ one obtains

$$(20) \qquad I_{1\infty} = \int_\delta^\infty e^{-\frac{x}{b}} dx + \frac{1}{2}\frac{\epsilon^2}{b^2} \int_\delta^\infty \frac{e^{-\frac{x}{b}}}{x} dx.$$

The first integral is $I_{1\infty} = be^{-\frac{\delta}{b}}$ and is to be expanded in powers of $\frac{\delta}{b}$ since $\frac{\delta}{b} << 1$. The second integral is the exponential integral. The exponential integral is also expanded in powers of $\frac{\delta}{b}$:

$$(21) \qquad \frac{1}{2}\frac{\epsilon^2}{b^2} \int_\delta^\infty \frac{e^{-\frac{x}{b}}}{x} dx = \frac{1}{2}\frac{\epsilon^2}{b^2}[\,-\gamma - ln(\frac{\delta}{b}) + \frac{\delta}{b} - \frac{1}{4}\frac{\delta^2}{b^2} + \frac{1}{18}\frac{\delta^3}{b^3}\dots]$$

where $\gamma \approx 0.57721$ is Euler's constant. The integral becomes

$$(22) \qquad I_{1\infty} = b[1 - \frac{\delta}{b} + \frac{1}{2}(\frac{\delta}{b})^2 + \dots] + \frac{1}{2}\frac{\epsilon^2}{b^2}[\,-\gamma - ln(\frac{\delta}{b}) + \frac{\delta}{b} - \frac{1}{4}\frac{\delta^2}{b^2} + \frac{1}{18}\frac{\delta^3}{b^3}\dots]$$

Using the expressions (18) and (22) in $I_{1\epsilon} + I_{1\infty} = 1$ produces, to leading order,

$$(23) \qquad b = 1 + \frac{1}{2}\frac{\epsilon^2}{b^2}[\gamma - \frac{1}{2b} - ln(2\frac{b^2}{\epsilon})\,].$$

Note that the expression does not depend on the arbitrary scale factor δ; this is a vindication of the procedure and can be used to check for errors. The nonlinear expression for b is solved iteratively. Only one iteration is required. Setting $b = 1$ in the right hand side produces

$$(24) \qquad b = 1 + \frac{1}{8}(\frac{15}{R_t})^2[\gamma - \frac{1}{2} - ln(\frac{4}{15}R_t)\,].$$

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To a very good approximation $b \approx 1$ for high Reynolds (as low as $R_t > 100$). The expression for the two-point correlation, with $b \approx 1$, becomes

$$(25) \qquad f(x; R_t) = e^{-\frac{R_t x^2}{((\frac{15}{2})^2 + R_t^2 x^2)^{1/2}}}.$$

This, (25), is the expression for a kinematically consistent two-point correlation whose curvature at the origin is given by the Taylor microscale. The small and large x limits are, respectively $f \rightarrow e^{-\frac{2}{15} R_t x^2}$ and $f \rightarrow e^{-x}$. The asymptotic evaluation of I_1 was used to obtain $b = b(R_t)$ such that I_1 was independent of R_t; $b \approx 1$ was indicated to be a good approximation. The asymptotic evaluation of the I_1 integral was described in some detail: it is an illustration of similar asymptotic procedures needed to evaluate the functionals described in Batchelor [1], Proudman [4], Ribner [5], Lilley [6], Ristorcelli [3].

The fact that $b \approx 1$ is a result of the fact that $I_{1\epsilon} \sim \mathcal{O}(\epsilon) << I_{1\infty}$. This will only be the case for functionals that do not include higher derivatives. Higher order derivatives and their products will scales with R_t^n, $n > 0$, near the origin and the fact that the interval scales with R_t^{-1} will mean that the local contribution will be larger than the global contribution, $I_\epsilon >> I_\infty$.

4. Discussion. The kinematic two-point correlation has been obtained with the idea of applying it to functionals calculated for non-ideal flow situations in which a Kolmogorov spectrum is not expected. To this end we have not incorporated the $r^{2/3}$ scaling, [15], associated with an ideal Kolmogorov turbulence. It is, at the expense of convenience and simplicity, possible to incorporate the Kolmogorov behavior. [It involves another free parameter and a length scale.] Due to the complicated expression needed to build in Kolmogorov behavior, the fact that information necessary to specify the additional length scale can not be obtained from turbulence closures, and the nonideal nature of practical flows, consistency with the $r^{2/3}$ scaling has been foregone.

The result, (25), is more general than if we required consistency with the Kolmogorov behavior. It becomes an interesting kinematically consistent expression for any random velocity field with distinguished inner and outer [integral and dissipation] scales. It is straightforward to produce similar expressions for either Eulerian or Lagrangian two-time correlations, [7]. In a Langevin approaches to turbulence, Pope [20], two-time correlations of the form $\rho(s) = e^{-s/T}$ are used; higher order derivatives are again, unphysically discontinuous at the origin. The model, $e^{-s/T}$, is the correlation function of the Orhnstein-Uhlenbeck process (colored noise) with non-differentiable velocity increments for which Pope [20] has a nice discussion relevant to the turbulence problem. Should statistics to which the differentiability property or the low Reynolds number limit be important than an expression of the form (25) is straightforward. The boundary layer at the origin is a temporal layer and the same procedure with two relevant time scales, the Lagrangian integral time scale and the Lagrangian dissipation time scale (Hinze [7]), is applicable.

5. Closure. A simple kinematically consistent two-parameter expression for the two-point spatial correlation function has been obtained. The model two-point expression has curvature at the origin given by the Taylor microscale and satisfies the integral constraint associated with a Reynolds number independent integral length scale. In as much as the curvature at the origin is related to the dissipation the model constructed incorporates one of the fundamental quantities characterizing a stationary turbulence, ε. As the curvature at the origin can also be expressed in terms of the turbulent Reynolds number, a measure of nonlinearity, the model expression for the two-point correlation includes the decorrelating effects associated with the [nonlinear] cascade.

The expression is meant to be used to provide estimates, using an asymptotic procedure described, of functionals used in single-point moment models. The expression is purely kinematic; no appeal to the

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dynamical two-point equations has been made. Any development based on the dynamical two-point equations for a general anisotropic inhomogeneous turbulent flow undergoing strain is prohibitively complicated and unclosed. In as much as the shape of the two-point correlation is insensitive to flow situation, Townsend [10], a kinematically consistent expression will capture the two-length scale features seen isotropic Frenkiel [8], strained Rogers [21], and boundary layer Frenkiel and Klebanoff [13]. The expression is consistent with several kinematic properties including differentiability at the origin and allows one to express two-point behavior in terms of two quantities carried in a typical moment closure scheme, the kinetic energy, k, and dissipation, ε, of turbulence.

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A simple kinematically consistent expression for the longitudinal two-point correlation function related to both the integral length scale and the Taylor microscale is obtained. On the inner scale, in a region of width inversely proportional to the turbulent Reynolds number, the function has the appropriate curvature at the origin. The expression for two-point correlation is related to the nonlinear cascade rate, or dissipation ε, a quantity that is carried as part of a typical single-point turbulence closure simulation. Constructing an expression for the two-point correlation whose curvature at the origin is the Taylor microscale incorporates one of the fundamental quantities characterizing turbulence, ε, into a model for the two-point correlation function. The integral of the function also gives, as is required, an outer integral length scale of the turbulence independent of viscosity. The proposed expression is obtained by kinematic arguments; the intention is to produce a practically applicable expression – in terms of simple elementary functions – that allow an analytical evaluation, by asymptotic methods, of diverse functionals relevant to single-point turbulence closures. Using the expression devised an example of the asymptotic method by which functionals of the two-point correlation can be evaluated is given.

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