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APPLICATION OF CONSIDER COVARIANCE TO THE EXTENDED KALMAN FILTER

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INTRODUCTION

The extended Kalman filter (EKF) is the basis for many applications of filtering theory to real-time problems where estimates of the state of a dynamical system are to be computed based upon some set of observations. The form of the EKF may vary somewhat from one application to another, but the fundamental principles are typically unchanged among these various applications. As is the case in many filtering applications, models of the dynamical system (differential equations describing the state variables) and models of the relationship between the observations and the state variables are created. These models typically employ a set of constants whose values are established my means of theory or experimental procedure. Since the estimates of the state are formed assuming that the models are perfect, any modeling errors will affect the accuracy of the computed estimates. Note that the modeling errors may be errors of commission (errors in terms included in the model) or omission (errors in terms excluded from the model). Consequently, it becomes imperative when evaluating the performance of real-time filters to evaluate the effect of modeling errors on the estimates of the state.

EKF WITH CONSIDER COVARIANCE

Assume that the EKF is to be applied to a system described by the (possibly) nonlinear differential equations

$$\dot{X} = F(t,X,C) \tag{1}$$

using the (possibly nonlinear) observation-state equation

$$Y = G(t,X,C) + \varepsilon \tag{2}$$

where X is the state whose values are to be estimated and C is the set of consider parameters. Note that C encompasses parameters in the equations of motion and the observation equation although it will be rare that any single parameter will be in both sets of equations. Note also that the values of C are not perfectly know, i.e., the true values of C are not available; only some nominal values of C (which will be referred to as C_N) and the uncertainties are known. The variable ε represents the measurement error which is assumed to white noise and have the following statistics

$$E\left[\varepsilon_i\right] = 0 \quad ; \quad E\left[\varepsilon_i \varepsilon_i^T\right] = R_i \tag{3}$$

To apply the EKF, this system is converted into a linear system by expanding Eq.s (1) and (2) about a reference trajectory $X_N(t)$ that is known with the corresponding observations which are described by

$$\dot{X}_N = F(t,X_N,C_N)$$

$$Y_N = G(t,X_N,C_N)$$

The resulting system and observation equations are given by

$$\dot{x} = A(t,X_N,C_N)\, x + B(t,X_N,C_N)\, c \tag{4}$$

$$y = H_x(t,X_N,C_N)\, x + H_c(t,X_N,C_N)\, c + \varepsilon \tag{5}$$

where $x = X - X_N$, $c = C - C_N$, $A = \dfrac{\partial F}{\partial X}$, $B = \dfrac{\partial F}{\partial C}$, $H_x = \dfrac{\partial G}{\partial X}$, and $H_c = \dfrac{\partial G}{\partial C}$. Note that the solution for x(t) in Eq. (4) can be written as

$$x(t) = \Phi(t,t_k)\, x(t_k) + \Theta(t,t_k)\, c \tag{6}$$

where Φ and Θ satisfy

$$\dot{\Phi}(t,t_o) = A(t,X_N,C_N)\, \Phi(t,t_o) \quad ; \quad \Phi(t_o,t_o) = I$$

$$\dot{\Theta}(t,t_o) = A(t,X_N,C_N)\, \Theta(t,t_o) + B(t,X_N,C_N) \quad ; \quad \Theta(t_o,t_o) = 0$$

The filter state vector is x(t), the estimate of x(t) is $\hat{x}(t)$, and the estimate of X(t) is recovered using $\hat{X}(t) = X_N(t) + \hat{x}(t)$.

To initialize the filter, assume that initial values for $\hat{x}_{k-1} = \hat{x}(t_{k-1})$ and $P_{k-1} = E\left[(x_{k-1}-\hat{x}_{k-1})(x_{k-1}-\hat{x}_{k-1})^T\right]$ as well as $X_N(t_{k-1})$ are given. This information is to be propagated to t_k and combined with the observation $y_k = y(t_k)$ to form the estimate $\hat{x}(t_k)$. Once the values for $\Phi(t_k,t_{k-1})$ and $\Theta(t_k,t_{k-1})$ are computed, the propagation of the filter state vector is carried out using Eq. (6) in the form

$$\bar{x}_k = \Phi(t_k,t_{k-1})\, \hat{x}_{k-1} + \Theta(t_k,t_{k-1})\, c \tag{7}$$

Similarly, the true state vector would be propagated using

$$x_k = \Phi(t_k,t_{k-1})\, x_{k-1} \tag{8}$$

since for the true state vector there is no error nor uncertainty in the consider parameters. Using Eq.s (7) and (8), the covariance matrix corresponding to $\bar{x}_k$ is written as

$$\begin{aligned}
\bar{P}_k &= E\left[(x_k-\bar{x}_k)(x_k-\bar{x}_k)^T\right] \\
&= \Phi_{k,k-1} E\left[(x_{k-1}-\hat{x}_{k-1})\,(x_{k-1}-\hat{x}_{k-1})^T\right] \Phi^T_{k,k-1} + \Phi_{k,k-1} E\left[(x_{k-1}-\hat{x}_{k-1})\, c^T\right] \Theta^T_{k,k-1} \\
&\quad + \Theta_{k,k-1} E\left[c\,(x_{k-1}-\hat{x}_{k-1})^T\right] \Phi^T_{k,k-1} + \Theta_{k,k-1} E\left[c\, c^T\right] \Theta^T_{k,k-1}
\end{aligned}$$

For now let, $W_{k-1} = E\left[(x_{k-1}-\hat{x}_{k-1})\, c^T\right]$ which will be discussed in more detail in the following sections. The expected value expression in the last term represents the uncertainties in the consider parameters which is assumed known and can be represented as

$$E\left[c\, c^T\right] = \Pi$$

Thus, for now, the propagated covariance matrix becomes

$$\bar{P}_k = \Phi_{k,k-1} P_{k-1} \Phi_{k,k-1}^T + \Phi_{k,k-1} W_{k-1} \Theta_{k,k-1}^T$$
$$+ \Theta_{k,k-1} W_{k-1} \Phi_{k,k-1}^T + \Theta_{k,k-1} \Pi \Theta_{k,k-1}^T \qquad (9)$$

Note that the first term in this equation represents the propagate covariance matrix if there were no consider parameter effects and the remaining three terms are the contributions of the consider terms.

Once the filter state vector estimate and associated covariance matrix have been propagated to t_k, it can be treated as an observation and combined with y_k to write

$$\begin{bmatrix} y_k - H_{c\,k}\, c \\ \bar{x}_k \end{bmatrix} = \begin{bmatrix} H_{x\,k} \\ I \end{bmatrix} x_k + \begin{bmatrix} \varepsilon_k \\ \eta_k \end{bmatrix}$$

where $E\left[\eta_k\right] = 0$, $E\left[\eta_k \eta_k^T\right] = \bar{P}_k$, and $E\left[\varepsilon_k \eta_k^T\right] = 0$. The solution for $\hat{x}_k$ which satisfies these equations in a least squares or minimum variance sense is

$$\hat{x}_k = \left(H_{x\,k}^T R_k^{-1} H_{x\,k} + \bar{P}_k^{-1} \right)^{-1} \left(H_{x\,k}^T R_k^{-1} \left(y_k - H_{c\,k}\, c\right) + \bar{P}_k^{-1} \bar{x}_k \right) \qquad (10)$$

For convenience, let

$$M_k = \left(H_{x\,k}^T R_k^{-1} H_{x\,k} + \bar{P}_k^{-1} \right)^{-1} \qquad (11)$$

which would be the state vector covariance if there were no consider parameters.

The covariance matrix associated with $\hat{x}_k$ is formally defined by

$$P_k = E\left[(x_k-\hat{x}_k)\,(x_k-\hat{x}_k)^T\right]$$

From Eq.s (10) and (11),

$$x_k - \hat{x}_k = x_k - M_k H_{x\,k}^T R_k^{-1} \left(y_k - H_{c\,k}\, c\right) - M_k \bar{P}_k^{-1} \bar{x}_k \qquad (12a)$$

Using Eq. (5) to express the observation in terms of the true state and true consider parameters (c=0) to write

$$y_k = H_{xk} x_k + \varepsilon_k$$

then Eq. (12a) can be written as

$$x_k - \hat{x}_k = \left[I - M_k H_{xk}^T R_k^{-1} H_{xk} \right] x_k - M_k H_{xk}^T R_k^{-1} \varepsilon_k$$

$$- M_k H_{xk}^T R_k^{-1} H_{ck}\, c - M_k \bar{P}_k^{-1} \bar{x}_k \tag{12b}$$

Note that from Eq. (11), it can be shown that

$$\left[I - M_k H_{xk}^T R_k^{-1} H_{xk} \right] = M_k \bar{P}_k^{-1} \tag{13}$$

Consequently, Eq. (12b) can now be written as

$$x_k - \hat{x}_k = M_k \bar{P}_k^{-1} (x_k - \bar{x}_k) - M_k H_{xk}^T R_k^{-1} \varepsilon_k - M_k H_{xk}^T R_k^{-1} H_{ck}\, c \tag{12c}$$

The covariance matrix for $\hat{x}_k$ can now be expressed as

$$P_k = M_k \bar{P}_k^{-1} M_k^T - M_k \bar{P}_k^{-1} \overline{W}_k H_{ck}^T R_k^{-1} H_{xk} M_k^T$$

$$- M_k H_{xk}^T R_k^{-1} H_{ck} \overline{W}_k^T \bar{P}_k^{-T} M_k^T + M_k H_{xk}^T R_k^{-1} H_{xk} M_k^T$$

$$+ M_k H_{xk}^T R_k^{-1} H_{ck} \Pi H_{ck}^T R_k^{-T} H_{xk} M_k^T \tag{14}$$

where it has been assumed that

$$E\left[(x_k - \bar{x}_k)\, \varepsilon_k^T\right] = 0 \qquad E\left[c\, c^T\right] = \Pi \qquad E\left[\varepsilon_k\, \varepsilon_k^T\right] = R_k$$

$$E\left[c\, \varepsilon_k^T\right] = 0 \qquad E\left[(x_k - \bar{x}_k)\, c^T\right] = \overline{W}_k$$

Note that the first and fourth terms of Eq. (14) can be combined to write

$$M_k \left(\bar{P}_k^{-1} + H_{xk}^T R_k^{-1} H_{xk} \right) M_k^T = M_k$$

Also note that from Eq. (10), the sensitivity matrix can be defined as

$$S_k = \frac{\partial \hat{x}_k}{\partial c} = - M_k H_{xk}^T R_k^{-1} H_{ck} \tag{15}$$

Using this notation, the covariance matrix for can be written in the form

$$P_k = M_k + M_k \bar{P}_k^{-1} \overline{W}_k S_k^T + S_k \overline{W}_k^T \bar{P}_k^{-T} M_k^T + S_k \Pi S_k^T \tag{16}$$

As was the case with the expression for $\bar{P}_k$, the first term in Eq. (16) represents the covariance matrix if there were no consider parameters and the remaining three terms represent the contribution of the consider parameters.

The only remaining issues to resolve are the expressions for W_k from Eq. (9) and $\overline{W}_k$ from Eq. (16) which are defined as

$$W_k = E\left[(x_k-\hat{x}_k)\, c^T\right] \qquad \text{and} \qquad \overline{W}_k = E\left[(x_k-\bar{x}_k)\, c^T\right]$$

Using Eq. (12c), W_k can be written as

$$\begin{aligned} W_k &= M_k \bar{P}_k^{-1} \overline{W}_k - M_k H_{x\,k}^T R_k^{-1} H_{c\,k} \Pi \\ &= M_k \bar{P}_k^{-1} \overline{W}_k + S_k \Pi \end{aligned} \tag{17}$$

Using Eq.s (7) and (8), $\overline{W}_k$ can be written as

$$\overline{W}_k = \Phi_{k,k-1} W_{k-1} + \Theta_{k,k-1} \Pi \tag{18}$$

Eq.s (17) and (18) provide the necessary equations to propagate and update W along with the covariance matrices M_k and P_k. To initialize this part of the filter, it is assumed that

$$W_o = E\left[(x_o-\hat{x}_o)\, c^T\right] = 0$$

i.e., the initial estimate of the state vector is not correlated to the uncertainties in the consider parameters. Note that by using Eq. (17), Eq. (16) can also be written as

$$P_k = M_k + W_k S_k^T + S_k W_k^T - S_k \Pi S_k^T \tag{19}$$

Finally, note that Eq. (7) can be used to define the propagated sensitivity matrix as

$$\bar{S}_k = \frac{\partial \bar{x}_k}{\partial c} = \Phi_{k,k-1} S_{k-1} + \Theta_{k,k-1} \tag{20}$$

CONCLUSION

The consider covariance EKF can be used to evaluate the effect of uncertainties of dynamic and measurement modeling errors on the estimate of the state by adjustments made to the state covariance matrix. While the consider covariance EKF does not predict the actual errors since the actual model errors are assumed unknown, it does provide a means of identifying critical elements of the dynamic and observation models for real-time filtering applications.