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AN EXPLORATORY STATISTICAL ANALYSIS OF A PLANET APPROACH-PHASE GUIDANCE SCHEME USING ANGULAR MEASUREMENTS WITH SIGNIFICANT ERROR

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SUMMARY

An exploratory analysis of vehicle guidance during the approach to a target planet is presented. The objective of the guidance maneuver is to guide the vehicle to a specific perigee distance with a high degree of accuracy and minimum corrective velocity expenditure. The guidance maneuver is simulated by considering the random sampling of real measurements with significant error and reducing this information to prescribe appropriate corrective action. The instrumentation system assumed includes optical and/or infrared devices to indicate range and a reference angle in the trajectory plane. Statistical results are obtained by Monte-Carlo techniques and are shown as the expectation of guidance accuracy and velocity-increment requirements. Results are nondimensional and applicable to any planet within limits of two-body assumptions.

The problem of determining how many corrections to make and when to make them is a consequence of the conflicting requirement of accurate trajectory determination and propulsion. Optimum values were found for a vehicle approaching a planet along a parabolic trajectory with an initial perigee distance of 5 radii and a target perigee of 1.02 radii. In this example measurement errors were less than 1 minute of arc. Results indicate that four corrections applied in the vicinity of 50, 16, 5, and 1.5 radii, respectively, yield minimum velocity-increment requirements. Thrust devices capable of producing a large variation of velocity-increment size are required. For a vehicle approaching the earth, miss distances within 32 miles are obtained with 90-percent probability. Total velocity increments used in guidance are less than 3300 feet per second with 90-percent probability. It is noted that the above representative results are valid only for the particular guidance scheme hypothesized in this analysis.

A parametric study is presented which indicates the effects of measurement error size, initial perigee, and initial energy on the guidance requirements. Measurement error size significantly affects both guidance accuracy and velocity-increment expenditure. The initial trajectory, as given by its perigee and energy, affects the velocity-increment expenditure but not final guidance accuracy.

INTRODUCTION

The current literature contains many reports on the subject of interplanetary travel and vehicle systems with relation to guidance requirements. Studies have been made of the accuracy requirements at cutoff during the initial launch phase (ref. 1), and it is generally accepted that if most mission objectives are to be attained a space vehicle must be equipped with a guidance system allowing trajectory corrections enroute. The function of midcourse guidance is to assure a successful rendezvous with the target planet at the proper time and place. However, if close tolerance maneuvers in the vicinity of the target are called for, some form of terminal or approach-phase guidance becomes necessary. One particular example that has been given much attention is the use of atmospheric-drag decelerations. E-1740

This report is concerned with the guidance of a space vehicle as it approaches a target planet. The analysis considers the random sampling of real measurements, with significant error, and a multiple-correction (but not continuous) guidance scheme.

A previous study by the authors (ref. 2) contains an investigation of an approach-phase guidance scheme using range, range-rate, and angular-rate measurements. The present study hypothesizes a navigation scheme utilizing self-contained optical and/or infrared instrumentation to measure angles. Range is determined from the planet's apparent disk, and angular position is found by planet-star observation. Trajectory parameters, knowledge of which is required for control action, are determined by the simultaneous solution of equations corresponding to three successive position fixes. The details of instrumentation or data smoothing are not considered in this study.

It is desired to guide the vehicle so that its perigee (minimum range) is tangent to an arbitrary target sphere. The point of tangency is not considered; that is, the inclination of the plane of motion and the orientation of the perigee in that plane are not specified. In addition, the rotation of the vehicle about the planet (relative to the planet's direction of rotation) is not specified. A two-dimensional polar representation is thus sufficient for analysis. Thrust application is assumed impulsive in effect and perfectly executed. Emphasis is placed on high accuracy guidance; that is, miss distances of the order of tens of miles. The objective is to perform the necessary maneuvers with minimum velocity expenditure.

The method used in studying the guidance problem is based on standard Monte-Carlo techniques and consists of repeated calculation of random trajectory "runs" where the random variable is the measurement error. Statistical results are developed from a finite sample size which reasonably approximates the infinite sample. Results are analyzed primarily on the basis of the probability of error in final perigee and the probability of requiring total velocity increment for control capability.

This report presents a method of obtaining the statistical results associated with the guidance problem and illustrates the nature of the results with an example of a reasonable guidance scheme. It should be mentioned that certain classes of results are peculiar to the particular guidance scheme hypothesized herein.

SYMBOLS

E	dimensionless total energy per unit mass, $2\mathcal{E}/v_c^2$
$\mathcal{E}$	total energy per unit mass, (miles/sec)^2
$errP_{ind}$	distribution of indicated perigee
$err\theta$	error distribution of angle θ
$err\omega$	error distribution of angle ω
G	universal gravitational constant, miles^3/lb-sec^2
H	dimensionless angular momentum per unit mass, $h/v_c r_0$
h	angular momentum per unit mass, miles^2/sec
M	mass of target body, lb
n	number of corrective velocity impulses
P	dimensionless perigee, r_p/r_0
R	dimensionless radial distance (range), r/r_0
R_i	dimensionless radial distance at which first correction is made
r	radial distance (range), miles
r_p	perigee of trajectory, miles
S	sample size

S.E.	standard error
V	dimensionless velocity, v/v_e
ΔV	dimensionless velocity impulse, $\Delta v/v_e$
v	velocity, miles/sec
Δv	velocity impulse, ft/sec
v_e	surface escape velocity, miles/sec
α	trajectory angle, measured between local horizontal and trajectory tangent, deg (radians)
β	velocity impulse angle with respect to initial velocity vector, deg (radians)
γ	perigee argument in plane of motion, deg (radians)
ϵ	eccentricity
θ	angular position in plane or motion measured counterclockwise from reference axis, deg (radians)
ω	apparent angular diameter, deg (radians)

Subscripts:

err	error
f	final
i	initial
id	with perfect measurements
ind	indicated by measurements with errors
max	maximum
meas	measured
min	minimum
t	total

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tar	target
true	true (actual)
a,b,c	first, second, and third position fix of a given set
0	conditions at surface of planet
1	conditions before corrective thrust
2	conditions after corrective thrust

ANALYSIS

This exploratory analysis concerns the problem of guiding a space vehicle during the approach to a target planet. The approach phase is defined here as that region in the planet's vicinity, but above its atmosphere, where the predominant influence on the vehicle's motion is the planet's own attracting force. An inverse-square, symmetric, central force field is assumed. This definition leads to the use of two-body conic trajectories.

The target of guidance is defined in terms of perigee distance. The inclination of the trajectory plane, the orientation of the perigee in that plane, and the vehicle's direction of rotation about the planet are not considered. Hence, a two-dimensional polar representation is used in the analysis. It is recognized that some or all of the above factors may be of importance in an actual mission.

Trajectory corrections will be governed by the following: If the vehicle determines (from measurements) that it is off course but approaching the planet in a clockwise (counterclockwise) reference direction, then it will apply thrust in such a manner to correct its perigee distance while continuing to approach the planet is a clockwise (counterclockwise) direction. High-thrust devices are assumed. Thrust is then associated with negligible burning time relative to trajectory time scales and consequently is considered impulsive in effect. The impulsive correction is assumed to be applied in a constant direction in the plane of motion and perfectly executed.

The following analysis presents the conic trajectory relations in a nondimensional form. The velocity requirement corresponding to desired perigee corrections is then derived. A means of determining the trajectory parameters from position measurements is developed, and the effect of measurement errors on the accuracy of such determination is discussed. The questions concerning guidance logic are discussed, and a guidance scheme is hypothesized. Finally, the method by which statistical results are obtained is presented.

Trajectory Relations

Normalized trajectory equations. - Since the classical two-body problem is assumed, the governing equations can be expressed by the conservation of both energy and angular momentum (ref. 3). With the notation of sketch (a),

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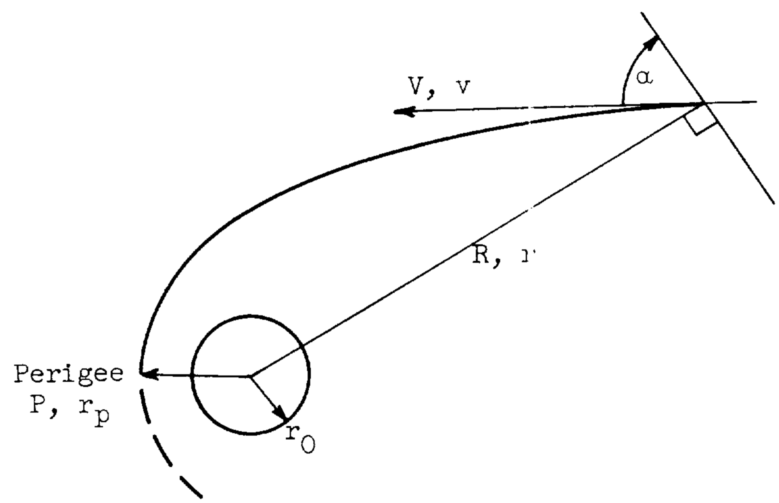

(a)

$$\mathscr{E} = \frac{v^2}{2} - \frac{GM}{r} \qquad \text{(energy per unit mass)} \tag{1}$$

$$h = vr \cos \alpha \qquad \text{(angular momentum per unit mass)} \tag{2}$$

These relations are nondimensionalized so as to be applicable to any target planet or moon. A convenient reference is the parabolic escape velocity at the surface of the planet. From equation (1), when $\mathscr{E} = 0$, the escape velocity is a constant given by

$$v_e^2 = \frac{2GM}{r_0} \tag{3}$$

The defining equations of the normalization are

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$$\left.\begin{aligned} V &\equiv \frac{v}{v_e} \\ R &\equiv \frac{r}{r_0} \\ E &\equiv \frac{2\mathcal{E}}{v_e^2} \\ H &\equiv \frac{h}{v_e r_0} \end{aligned}\right\} \tag{4}$$

The range R is now measured in planet surface radii and velocity V in surface escape velocities. Equations (1) and (2) may now be divided by v_e^2 and $v_e r_0$, respectively, and rewritten in dimensionless form:

$$E = V^2 - \frac{1}{R} \tag{5}$$

$$H = VR \cos \alpha \tag{6}$$

The trajectory angle at the perigee of any approach path is identically zero. Equations (5) and (6) may be combined for conditions at the perigee, and angular momentum is thus shown to be a function of energy and perigee:

$$H^2 = P^2E + P \tag{7}$$

The trajectory angle is now given as

$$\cos \alpha = \sqrt{\frac{P^2E + P}{R^2E + R}} \tag{8}$$

Another useful expression is the relation of angular momentum, eccentricity, and energy:

$$H^2 = \frac{\epsilon^2 - 1}{4E} \tag{9}$$

<u>Corrective thrust</u>. - In the vicinity of the target planet the vehicle coasts along a conic approach trajectory relative to the planet. With reference to figure 1(a), assume that the perigee P_1 (distance of closest approach) of the initial trajectory differs from a desired or target perigee P_{tar}. Therefore, control action in the form of thrust application

must be initiated which will act to guide the vehicle to P_{tar}. Guidance requirements will be measured by the velocity increments due to thrust which, throughout the analysis, will be designated by the terms "velocity increment," "velocity impulse," or ΔV. As previously mentioned, the ΔV used for guidance is assumed to be impulsive in effect.

From the trigonometric relations of figure 1(b),

$$\Delta V^2 = V_1^2 + V_2^2 - 2V_1V_2 \cos(\alpha_2 - \alpha_1) \tag{10}$$

$$\sin \beta = \frac{V_2 \sin(\alpha_2 - \alpha_1)}{\Delta V} \tag{11}$$

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It is possible to minimize ΔV by orienting the thrust vector in the proper direction. Such an analysis was the objective of an earlier study by the authors (ref. 4), where it is shown that an iterative solution is required. Since the present report is statistical in nature and therefore time-consuming, the optimum calculation becomes lengthy. A reasonable approximation to ΔV_{min} is obtained by a "zero-energy-change" correction; that is, thrust applied in such a direction that the energy and velocity magnitude remain the same before and after burning (ref. 4). For this condition $E_1 = E_2$, $V_1 = V_2$. Equations (10) and (11) reduce to

$$\Delta V = 2V_1 \sin \left|\frac{\alpha_2 - \alpha_1}{2}\right| \tag{12}$$

$$\sin \beta = \frac{\sin(\alpha_2 - \alpha_1)}{2 \sin \frac{|\alpha_2 - \alpha_1|}{2}} \tag{13}$$

The velocity impulse ΔV may be found using equations (5), (8), and (12) and is seen to be a function of initial trajectory parameters (E_1, P_1), range, and target perigee.

The magnitude of the corrective velocity impulse (eq. (12)) is plotted as a function of range and initial perigee in figure 2. The approach trajectory is parabolic, and the target perigee is 1.02 radii. For a given initial perigee, ΔV decreases as the range at which thrust is applied increases. Obviously then, from the standpoint of propulsion it is desirable to execute the corrective maneuver as far from the planet as possible.

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Determination of trajectory parameters. - As seen from the preceding section, equations (12) and (13) define the corrective velocity increment and may be evaluated for any condition if the parameters (E_1, P_1, R) are known. One means of determining these parameters is now developed in its simplest form. The details of measurement, smoothing techniques (if employed), and computing equipment are not considered here.

Suppose that navigation equipment is available which allows the measurement of angles between celestial bodies, together with the angular diameter of the planet. The result of a set of such measurements is a position fix for the vehicle, and it will be shown that a minimum of three successive fixes is required to determine the trajectory parameters for this particular measurement scheme.

Angular position can be obtained by measuring a series of planet-star angles. Since this analysis is limited to two dimensions, the simplifying assumption is made that a reference direction (i.e., a star) in the plane of motion is available and can be so determined. Thus, the number of planet-star measurements is reduced to one. The plane of motion is considered constant during fixes, and, since the star is essentially infinitely far away, the reference axis has insignificant motion during the interval between fixes.

Range can be obtained by measuring the angle subtended by the planet's apparent disk. This may be accomplished with optical or infrared scanning-type instrumentation. The details are not considered in this analysis; however, the geometry of disk scanning is fully treated in reference 5.

Figure 3 illustrates the angles used in the measurement scheme. The angle $(\pi - \theta)$ is measured from the planet-star observation and gives the angular position θ of the vehicle in the plane of motion with respect to the reference axis. Range is easily found using the notation of figure 3:

$$\sin \frac{\omega}{2} = \frac{r_0}{r} \equiv \frac{1}{R}$$

or

$$R = \csc \frac{\omega}{2} \tag{14}$$

It is now shown how the basic measurements are used to compute the desired trajectory parameters. The polar equation of a conic trajectory can be expressed in dimensionless form as

$$R = \frac{2H^2}{1 + \epsilon \cos(\theta - \gamma)} \tag{15}$$

The angle γ is the perigee argument and serves to define the orientation of the trajectory in the plane. The parameters (H, ϵ, γ) are constant and completely determine a coasting trajectory in the plane of motion. At least three successive position fixes $(R, \theta)_{a,b,c}$ are required to determine these constants. The simultaneous solution of (15) for fixes a,b,c gives

$$\tan \gamma = \frac{R_a(R_c - R_b)\cos \theta_a + R_b(R_a - R_c)\cos \theta_b + R_c(R_b - R_a)\cos \theta_c}{R_a(R_b - R_c)\sin \theta_a + R_b(R_c - R_a)\sin \theta_b + R_c(R_a - R_b)\sin \theta_c} \tag{16}$$

$$\epsilon = \frac{R_b - R_a}{R_a \cos(\theta_a - \gamma) - R_b \cos(\theta_b - \gamma)} \tag{17}$$

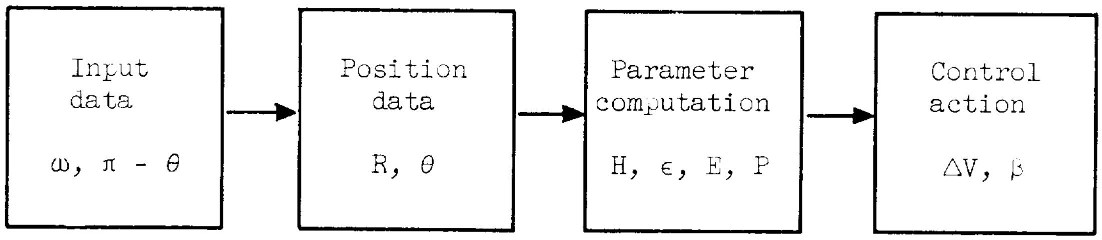
$$H^2 = \frac{R_a}{2} [1 + \epsilon \cos(\theta_a - \gamma)] \tag{18}$$

The trajectory parameters E and P are related to H and ϵ through equations (7) and (9).

It is important to mention one inherent fault of this measurement scheme. If the measurement errors are sufficiently large, the possibility of an indeterminate solution exists; specifically, H^2 may be negative. Geometrically interpreted, this means that no conic section, whose focus is situated at the force center, can be passed through the three positions. Analysis of the results to be presented has shown that this situation occurs infrequently. When it does occur, the set of measurements is disregarded and replaced. It is felt that this is the proper interpretation for the purpose of analysis since in a real flight situation a set of indeterminate data would not be tolerated as a final result and additional measurements would be taken to obtain information. Furthermore, smoothing techniques, which have not been considered, would tend to eliminate this possibility.

The measurement scheme and data reduction as described in this section are based upon the minimum number of position fixes and do not take into account past trajectory knowledge. There may be good reason to take more than three fixes, thereby taking advantage of the redundancy of data. This could be accomplished by means of a least-squares fit to the observed data to obtain a more accurate estimate of the trajectory parameters (refs. 5 and 6). In addition, the proper weighting of past knowledge would improve the trajectory determination.

At this point of the analysis the information flow can be represented by the following block diagram:



Effect of Measurement Errors

The accuracy with which a vehicle can be guided to a target and the ΔV requirements can be greatly affected by measurement errors. It is of interest to determine the effect of measurement errors on certain parameters for two reasons. First, such an analysis is helpful to the formulation of good guidance logic. Second, it serves to explain some subsequent results.

Range error coefficient. - The error coefficient involved in range determination is easily found by differentiating equation (14) with respect to ω. Thus,

$$\frac{dR}{d\omega} = -\frac{1}{2} R\sqrt{R^2 - 1} \tag{19}$$

Equation (19) is plotted as a function of R in figure 4. The error coefficient is seen to increase with the square of R for large R ($R^2 \gg 1$). Although the error coefficient is linearized, it adequately represents the true range error when the measurement error is not too large. For example, 1 minute of arc error (0.000291 radian) at a distance of 100 radii causes an error of about 1.5 radii or 1.5 percent. In contrast, if the measurement error can be reduced to 0.2 second of arc, range is determined to 0.005 percent.

Perigee error coefficient. - Perigee can be expressed in terms of angular momentum and eccentricity. When $R = P$, equation (15) becomes

$$P = \frac{2H^2}{1 + \epsilon} \tag{20}$$

differentiating

$$dP = \frac{\partial P}{\partial H} dH + \frac{\partial P}{\partial \epsilon} d\epsilon = \frac{4H}{1 + \epsilon} dH - \frac{2H^2}{(1 + \epsilon)^2} d\epsilon$$

$$dP = 2P \frac{dH}{H} - P\left(\frac{\epsilon}{1 + \epsilon}\right)\frac{d\epsilon}{\epsilon} \tag{21}$$

Note that $2P > P[\epsilon/(1 + \epsilon)]$; in particular, if the approach trajectory is parabolic ($\epsilon = 1$) and dH/H and $d\epsilon/\epsilon$ are assumed of the same order magnitude, then the first term of equation (21) is four times as large as the second. Actually, data obtained in this analysis have shown that $dH/H > d\epsilon/\epsilon$. Therefore, in order to simplify this discussion assume that the perigee is influenced mainly by the angular momentum

$$dP \cong 2P \frac{dH}{H} = \frac{P}{H^2} d(H^2) \tag{22}$$

The solution for $d(H^2)$ is treated in appendix A and is shown to be a function of positions $(R, \theta)_{a,b,c}$ and the error in the measured angles.

The results of the error analysis are shown in figures 5. The root-mean-square error coefficient (eqs. (22) and (A4)) is plotted in figure 5(a) for a parabolic approach trajectory and three values of initial perigee. The independent variable R_c is the range at which the third position fix is taken. In this example the first position fix is taken at $R_a = 100$ radii and the second midway between the first and third.

Two results are immediately apparent: (1) The accuracy in determining perigee increases rapidly as the spacing between position fixes is increased, and (2) the perigee is better determined (in terms of absolute error) if the trajectory passes close to the planet. The relative error coefficient dP/P is approximately equal for the three values of perigee. As an estimate of the numerical pergiee error, consider a root-mean-square error in angle measurement of 30 seconds of arc. Within the assumption of linearized errors, the errors in determining the perigee for P equal to 5 and 1.0 radii are 1.00 and 0.22 radius, respectively, when $R_c = 50$ radii.

Figure 5(b) shows the effect of energy on the root-mean-square perigee error coefficient. For example, if $R_c = 50$ radii, increasing the energy from parabolic to one-tenth unit hyperbolic is associated with a tenfold increase in error sensitivity. This characteristic is a result of the particular measurement scheme used and is not generally true. As a consequence of the error sensitivity it can be expected that the Δv requirements attributed to imperfect guidance will increase with trajectory energy.

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Guidance Considerations

During the approach phase, the function of guidance is to direct the space vehicle to a specified target perigee and to do so with a minimum ΔV expenditure. The opposing requirements of propulsion and accuracy have been illustrated in figures 2 and 5. It is this inherent interaction that raises the question as to how optimum guidance should be performed.

If knowledge of the initial trajectory and thrust execution were perfect, a single correction made far from the planet would suffice to achieve the desired trajectory and would be relatively inexpensive. In the actual case, however, control action is taken based on information subject to error. The effect of thrust application is a new trajectory, possibly significantly improved, but still not on target. The difference between the perigee after correction and the target will be called the "miss distance." A target perigee and an acceptable miss distance corresponding to a particular mission objective will most likely be specified in advance. In this analysis, emphasis is placed on miss distances of the order of 10 miles; (i.e., as required for atmospheric drag decelerations, ref. 7). The achievement of a successful approach trajectory will, therefore, necessitate either highly accurate instrumentation or, in the case of less accurate instrumentation, a more sophisticated multicorrection guidance scheme.

The approach in this report is to hypothesize a multicorrection scheme for current state of the art equipment and in addition to note the effect of more accurate equipment. In formulating good guidance logic, the following questions are to be considered:

(1) When should guidance action be initiated?

(2) When should guidance action be cut off?

(3) How many corrective impulses are necessary, and what should be the interval between them?

(4) What part of the indicated error should control action attempt to correct?

Certain factors of the guidance logic will be arbitrarily chosen. A framework for analysis will be set up, and the effects of major parameters therein will be investigated in order to determine an optimum-type solution.

The random numbers are obtained from a method commonly in use, the details of which are presented in reference 9. Corrective action is taken to guide the vehicle to the target within the framework of the guidance scheme. Results include (1) size of individual velocity increment used, (2) total velocity increment used during approach, (3) accuracy, or miss distance, after each correction, and so forth.

The Monte-Carlo method consists of repeated calculation of random trajectory runs. The statistical results are developed from a practical sample size which reasonably characterizes the infinite sample. Average or probable events can be obtained with a fairly high degree of accuracy from a relatively small sample. The disadvantage of a small sample is that unlikely occurrences may not be represented correctly. Consequently, measures of unlikely occurrences are in question as to their significance, and care must be excersized so that erroneous interpretation of data is not drawn.

The problem considered in this report was programmed for an IBM 653 computer. A complete calculation of a single trajectory correction required 11 seconds. Statistics developed from 200 samples for a guidance scheme using four corrections required approximately $2\frac{1}{2}$ hours of computing time.

RESULTS AND DISCUSSION

The method of statistical analysis will now be applied to the particular example of the guidance scheme hypothesized. In order to fix attention on the characteristics of the results, a reference solution is presented first. The reference solution represents optimum-type guidance in the sense that guidance logic is chosen which minimizes the ΔV expenditure.

Following discussion of the reference solution, the results which led to its choice are presented parametrically. In addition, a parametric study is presented which illustrates the effects of initial perigee, measurement errors, and initial energy upon guidance requirements. Computing time considerations required that numerical results for the parametric study be developed from a smaller statistical sample than that for the reference solution.

Results of Reference Solution

As a baseline for presenting results certain input parameters and guidance logic are prescribed. These are summarized in table I, and the values for an earth approach are given when appropriate. Henceforth, values in parentheses will correspond to an earth approach.

The initial approach trajectory is parabolic with a perigee of 5 radii (20,000 miles). The target, chosen just above the planet's surface, is 1.02 radii (80-mile altitude); thus, the initial trajectory error is 3.98 radii. Angular measurements are initiated at a range of 100 radii. Four corrective impulses (based on the range-variant scheme described) are used. They are applied at 50, 15.57, 4.85, and 1.5 radii, respectively. Control action attempts to correct the entire trajectory error which is indicated by measurements. The statistical measurement error distribution is assumed rectangular in shape (see appendix A) with a maximum error of ±1 minute of arc. Two hundred random samples are used to develop statistical results.

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A typical trajectory run. - The following table shows the sequence of events for one typical random sample:

Condition	Energy, E	Perigee argument, γ, deg	Perigee, P, radii	Velocity increment, ΔV (escape velocity)
Initial	0	225	5.0000	------------
After 1st correction	-2.3×10^{-5}	242	1.4877	2.0957×10^{-2}
After 2nd	-2.4×10^{-5}	248	1.0610	1.2741×10^{-2}
After 3rd	-2.4×10^{-5}	249	1.0159	0.51280×10^{-2}
After 4th	-1.2×10^{-5}	248	1.0198	0.24640×10^{-2} $\Delta V_t = \Sigma\Delta V = 4.1290\times10^{-2}$

The deviation between initial and final energy is a very small value. This is to be expected since no deviation would exist if guidance were perfect. In terms of an earth approach this would correspond to a velocity deviation at the perigee of about 1 foot per second. The change in perigee argument is 23°; however, the point of tangency at the target sphere was not considered in the target specification. As seen from the P column, the largest part of the trajectory error is corrected by the first ΔV impulse; the final error is only 2×10^{-4} radius (0.8 mile). In this example, the propulsion system must be capable of producing an 8.5 to 1 range of ΔV, and a total velocity increment of 4.129×10^{-2} (1520 ft/sec).

<u>Guidance accuracy</u>. - The extent to which a vehicle was guided to the target is shown in figure 6(a). The final distribution about the target perigee is given in the form of a rectangular frequency polygon. Probability of a positive miss distance is 57.5 percent while that of a negative is 42.5 percent.

The results of guidance accuracy may be presented in another form, namely, the integrated frequency polygon commonly called the cumulative probability distribution. Figure 6(b) shows the probability of hitting a given size target band. The median miss distance (50 percentile) is 1.35×10^{-3} radius (5.4 miles), while guidance is accurate to within 10^{-2} radius (40 miles) with 90.5-percent probability. This type of plot is more useful in illustrating probability of success; however, it does not give as complete a picture as does the frequency polygon.

<u>Total velocity increment</u>. - The cost of guidance in terms of total velocity expenditure is shown by figure 6(c) in frequency polygon form. The ideal requirement is 0.0122 v_e (450 ft/sec) and arbitrarily represents a single corrective impulse applied at 100 radii assuming zero measurement error (see fig. 2). Observe that the distribution is skewed considerably toward high ΔV_t; this is characteristic of the guidance scheme. The smallest velocity expenditure indicated by the sample is 0.0279 v_e. Velocity increments exceeding 0.2 v_e (7360 ft/sec) are possible but with very little likelihood. The modal class interval, corresponding to the most probable requirement, is given by the maximum ordinate. Thus, 32 percent of the time-velocity increments between 0.04 and 0.05 v_e are used.

Figure 6(d) illustrates the cumulative probability distribution. The difference between the ideal requirement and the probability curve is the excess ΔV due to guidance in the presence of measurement errors. The median ΔV_t corresponds to a 285-percent excess over perfect guidance. This plot may be interpreted in two ways. First, it shows the probability of requiring ΔV_t less than a given amount and is therefore indicative of necessary guidance propulsion. Second, if a given ΔV_t is available, say 0.2 v_e, the probability of successfully completing the guidance maneuver is obtained; in this case about 99-percent expectation. Values of ΔV_t that have been exemplified may be considered high, but it should be recalled that the assumed values of initial trajectory error and measurement error were significantly large. The effect of reducing these errors on ΔV requirements will subsequently be shown.

<u>Individual velocity-impulse size</u>. - Figure 6(e) shows the cumulative probability distributions of each of the four corrective impulses. Note the dispersion of the last three impulses as compared to that of the first.

The first impulse acts to correct a vehicle whose particular trajectory error is 3.98 radii, and the possible range of ΔV required (about 0.01 to 0.065) is a function of the measurement error distribution. However, the statistical requirement of the remaining corrections is not only a function of measurement errors, but also of the true perigee distribution resulting from each previous thrust application. This magnifying effect accounts for the large range (about 10^{-4} to 10^{-1}) associated with the remaining impulses. The arithmetic means of each of the four velocity impulses are 0.0256, 0.0172, 0.00614, and 0.0119, respectively. In this example, the fourth correction requires a larger ΔV than the third since the range at which the correction was applied had a larger effect than the errors to be corrected.

Of importance to engine design is both the range of ΔV anticipated and the frequency of a given ΔV increment. The velocity requirement of each of the four corrections is shown in frequency polygon form in figure 6(f). The distribution of the first impulse is essentially symmetric with most frequently used increments (64 percent of the time) between 0.02 and 0.03 v_e (mean ±20 percent). A value of ΔV less than 0.01 or greater than 0.04 is called for very infrequently. The frequency distributions of the second, third, and fourth velocity impulses are similar with the characteristic of decreasing frequency with increasing ΔV, although the rate of decrease is considerably slower for the second impulse. In each case most frequently used increments are in the region less than 0.005 v_e.

Cutoff effects. - In hypothesizing guidance logic, four trajectory corrections were specified with cutoff at 1.5 radii. Heretofore, evaluation of the guidance scheme was presented after the final correction; however, it is of interest to note the effects of terminating control action after each correction. Probability of total velocity expenditure at cutoff is shown in figure 6(g). It is noted that additional corrections cause the probability curves to approach 100 percent at a slower rate. The median ΔV_t cost for four corrections is about twice that for one correction. At the 95-percent probability level, this increased requirement is almost a factor of 3.

It is necessary to compare the velocity expenditure to guidance accuracy attained at cutoff. Figure 6(h) shows the probability of hitting a given size target band. If a single correction is made, a miss distance greater than 0.9 radius (3600 miles) will result 10 percent of the time. This is an intolerable error since the target perigee is only 0.02 radii above the planet's surface. Additional corrections are seen to reduce this error by factors of approximately 9, 26, and 90, respectively. A comparison of figures 6(g) and (h) indicates that the increase in guidance accuracy, resulting from continuing trajectory control, appears to be far greater than the increase in velocity expenditure when compared on a percentage basis.

Correlation plots. - It is often desirable to show the relation between two statistical parameters. This can be accomplished with use of a scatter diagram wherein the variables are plotted on the x- and y-axes, respectively. The degree of correlation can be estimated by the pattern of the points plotted.

The correlation between the error in determining the perigee and the perigee error remaining after the correction is of interest. The latter parameter is not uniquely determined by the former, but on the contrary depends upon various other trajectory parameters and the particular random measurement errors chosen. A statistical correlation may be used to advantage in relating these parameters.

Figure 6(i) is a result of plotting the data points corresponding to the final correction, namely at $R \equiv 1.5$ radii. The absolute final miss distance $|P_f - P_{tar}|$ is plotted against the absolute error in determining perigee $|P_{ind} - P_{true}|$. The most interesting characteristic is the grouping of points about the 45° line, representing a 1:1 correlation. In a statistical sense, it can be concluded that errors in determining the perigee at the final correction point will result in a final miss distance of the same order.

Figure 6(j) illustrates a similar correlation; however, the data are not limited to the final correction but rather are taken from each of the four corrections. The 1:1 correlation is again indicated except in the region of large error. In this region, which is representative of the first correction, the characteristic appears linear but biased from the 1:1 correlation. An error in determining perigee of a given size is reflected in a miss distance of about one-half the size. The correlation characteristics illustrated in figures 6(i) and (j) were substantiated (in an approximate sense) by a linearized error analysis.

The overall success of a guidance program is measured in terms of velocity requirement and guidance accuracy probabilities. The two probability distributions have been presented separately. However, the question of combination arises; that is, will a vehicle which is considered successful in terms of final miss distance also be successful in terms of propellant expenditure, or do some other criteria exist? As a means of explanation, a scatter diagram is shown in figure 6(k) in which ΔV_t is plotted against final miss distance. The pattern of points appears truly scattered, thus indicating small or zero correlation between the two variables. The density of points in a given region provides a measure of overall success. For example, consider the rectangle formed

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by $0 < \Delta V_t < 0.06$ and $0 < |P_f - P_{tar}| < 0.002$. There is a 46-percent probability that a vehicle will guide to within 0.002 radius and expend no more than 0.06 escape velocity in doing so. Now, upon examining the separate probability distributions (figs. 6(b) and (d)) the probabilities are 64.5 and 70 percent, respectively. If the two variables are uncorrelated, the product of the separate probabilities should give the overall, or combined, probability. This procedure yields approximately 45.2 percent, which is sufficiently close to the previous value of 46 percent. Thus, for this particular guidance scheme, the uncorrelated characteristic proves useful in estimating the overall success of the guidance maneuver. It is realized that a high probability of combined success necessitates very high probability in both ΔV_t and miss distance.

A more useful illustration of combined success is shown in figure 6(l). A word of caution is necessary before interpreting the results. The guidance computation was performed under the assumption that the vehicle possessed an unlimited ΔV capability, this being necessary in order to obtain the velocity requirement. Therefore, no information is available on the guidance of those vehicles which may have begun with a limited propellant supply and subsequently run short before completing the guidance maneuver. That is to say, in reading figure 6(l) one may not choose a given ΔV_t availability and find the probability of guiding to within a given size miss distance. With this in mind we proceed with an example. If the objective is to guide to a perigee of 1.02 radii with a miss distance no greater than 0.006 radius (24 miles), the probability of doing so using less than 0.050 v_e (1840 ft/sec) is 50 percent. The probability of expending less than 0.098 (3610 ft/sec) is 80 percent. The shape of the curve is descriptive in that it approaches asymptotes parallel to the axes, thereby defining the limiting conditions. The numerical values are easily found from figures 6(b) and (d). For example, if 80-percent combined success is desired, the allowable miss distance must be no less than 0.00365 radius and the ΔV capability no less than 0.072 escape velocity.

Accuracy of sample size. - Results of the reference solution may be used to determine the accuracy to which a given sample size approximates the infinite distribution from which it was drawn. The 200 samples were subdivided into four groups of 50 samples each. A comparison is shown in figure 6(m). The deviation among sample groups, particularly at high probability, is not especially welcome and indicates that statistical inference from 50 samples may be considerably in error unless limited to average-type values such as the median.

It is frequently convenient, when testing the significance of a sample group, to determine a measure of the error in the computed arithmetic mean. This may be accomplished with the use of a simple formula, the assumption being that the means of an infinitely large number of sample groups will be asymptotically normally distributed (refs. 10 and 11). Denoting the standard error of sample mean by S.E. and the standard deviation of the sample by σ,

$$\text{S.E.} = \frac{\sigma}{\sqrt{S}}$$

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where S is the sample size. The standard deviation is a measure of the dispersion about the mean. By definition, if X_i denotes the difference between the i^{th} sample value and the arithmetic mean calculated for the sample, then

$$\sigma = \sqrt{\frac{\sum_i X_i^2}{S}}$$

The standard error is customarily interpreted as follows: The probability that the sample mean is less than 1 S.E. from the true mean is about 68 percent, while the probability that it is less than 3 S.E. from the true mean is 99.7 percent.

The following table compares the four sample groups (from fig. 6(m)) in terms of the arithmetic mean, standard deviation, and standard error:

Sample group	Arithmetic mean	Standard deviation	Standard error of mean	
Total velocity increment				
I	0.0478	0.0146	0.00207	(4.3%)
II	.0567	.0289	.00410	(7.2%)
III	.0575	.0209	.00296	(5.2%)
IV	.0689	.0443	.00627	(9.1%)
Absolute final miss distance				
I	1.40×10^{-3}	1.03×10^{-3}	0.146×10^{-3}	(10.4%)
II	17.3	68.6	9.70	(56%)
III	1.93	3.86	.546	(28%)
IV	9.96	33.3	4.71	(47%)

The first table compares the statistical results for the total velocity increment. The arithmetic mean varies between the four sample groups from 0.0478 to 0.0689. The largest standard error of the mean (9.1 percent) was indicated for sample group IV.

The second table compares the statistical results for the absolute final miss distance. The arithmetic mean varies between 0.0014 and 0.0173, with the largest standard error of the mean (56 percent) indicated for sample group II.

It is noted that the effect of sample size on accuracy of results is more critical in the case of the miss distance. One might expect the true dispersion of miss distance distributions to be relatively greater than that of ΔV distributions. Since the standard error varies directly with the standard deviation (measure of dispersion), this criticality would also be expected. Based upon this example, one might conclude that 50 samples are too small for high statistical significance, but may adequately be employed for first approximations in evaluating guidance performance.

Guidance Considerations

The choice of the reference solution, within the framework of the guidance scheme hypothesized, was based on the following results of a parametric study. With arbitrary values assigned to measurement errors and initial trajectory parameters, the major guidance parameters to be investigated are (1) the number of corrections and (2) the range at which corrections are initiated. As will be the general procedure herein, all parameters not specifically varied will be those of the reference solution as summarized in table I. Also, the sample size is reduced to 50 because of computing time limitations.

Number of corrections. - The starting and final correction points (50 and 1.5 radii, respectively) are held constant, and the effects on guidance due to variation of the number of corrections are investigated. A comparison of total ΔV requirements for various values of n is illustrated in figure 7(a). Of interest is the minimum ΔV_t corresponding to a given probability, that is, the curve farthest removed to the left. The minimum ΔV_t results for $n = 4$, while the velocity requirement for nine corrections is approximately twice the minimum. Since the sample size is small, the question of statistical significance arises and is treated in appendix B.

Figure 7(b) illustrates the effect of the number of corrections on the guidance accuracy. A curve for $n = 2$ is not shown since the corresponding ΔV requirement is excessive. The characteristic of increasing accuracy with decreasing n follows from prior considerations of the spacing between measurements.

A comparison of figures 7(a) and (b) indicates that either three or four corrections result in reasonably good guidance. If the miss distance is within mission requirements in both cases, emphasis should be placed on conserving velocity increment.

Guidance initiation. - The value of range R_i at which the first corrective impulse is applied is varied so as to determine its effect on guidance. Four corrections are used in each case. (A simultaneous variation of n and R_i was performed, indicating that $n = 4$ was a good choice for the values of R_i studied.)

The effects of guidance initiation on the ΔV requirement are shown in figure 8(a) for R_i of 70, 50, and 30 radii. The dispersion of the distribution decreases with decreasing R_i. This characteristic is to be expected since the dispersion of the distribution of P_{ind} decreases with small R_i and consequently affects the dispersion of the ΔV_t distribution. Considering the median probability of occurrence, an optimum R_i is anticipated between 70 and 30 radii. However, at the 90-percent probability level the ΔV requirements for $R_i = 30$ and 50 are about equal. In view of the entire probability distribution, an initial correction applied at 50 radii is considered to result in reasonably good guidance. The above characteristics were tested for statistical significance in appendix B. Figure 8(b) shows the probability of hitting a given size target band. The distributions are essentially alike with no appreciable effect indicated.

Effect of Error Assumptions and Initial Energy

The guidance requirements have been investigated for a vehicle approaching a planet along a particular trajectory; namely, parabolic with an initial perigee equal to 5 planet radii. It is recalled that the target is 1.02 radii. Furthermore, the maximum error in measuring angles was taken as 1 minute of arc. The initial perigee, measurement error, and initial energy will now be varied and their effects on guidance noted.

Initial trajectory error. - As previously defined, the initial trajectory error is the difference between the initial perigee and the target perigee. The effect of initial perigee on velocity expenditure is illustrated in figure 9(a). All parameters not varied are those from the reference solution as listed in table I. The total velocity increment increases with the initial trajectory error $(P_i - P_{tar})$. Each curve in the family has essentially the same shape; however, as P_i increases, the skewness towards larger ΔV_t increases slightly. Figure 9(b) illustrates the effect of initial perigee on the efficiency of guidance. The excess velocity increment $(\Delta V_t - \Delta V_{id})$ is plotted against P_i for constant values of probability. As before, ΔV_{id} was calculated for a single correction applied at $R = 100$ assuming perfect measurements (see fig. 2). The excess velocity increment is seen to increase approximately linearly with P_i. For example, the median excess at $P_i = 1.02$ is 0.015 escape velocity, and at $P_i = 10$, 0.048 escape velocity. The figure also shows that the rate of linear variation increases with the probability level.

There was no significant effect on guidance accuracy due to variation of P_i. If only a single correction were made, an appreciable difference in accuracy would result (see figs. 5(a) and 6(j)); however, successive corrections act to bring all vehicles into a common probability distribution about the target.

Size of measurement error distribution. - The magnitude of the measurement error is one of the most important factors affecting guidance. Results are now presented for a variation of error from 1 to 600 seconds of arc. As in the reference solution a rectangular distribution is assumed, the maximum error being representative of the size.

Results of guidance accuracy are shown in figure 10(a) where the miss distance is a function of measurement error size for constant values of probability. Miss distance increases approximately linearly with the measurement error for errors less than about 5 minutes of arc. Thus, an error reduction from 1 minute to 1 second results in a sixtyfold increase in accuracy. Measurement errors of 1 second permit guidance to 2×10^{-5} radius (422 ft) with 50-percent probability, and to about 5×10^{-5} radius (1050 ft) with 90-percent probability. Note: The small sample size reduces the numerical significance for high probability levels.

The ΔV requirement is shown in figure 10(b) as a function of the maximum measurement error. The total velocity increment ΔV_t increases rapidly for very large errors (greater than 20 sec arc). The median (50-percent probability) velocity expenditure is about 0.025 v_e for 1 second of arc error, 0.032 v_e for 20 seconds of arc, and 0.13 v_e for 400 seconds of arc. The statistical dispersion, as exemplified by the difference between 90- and 10-percent probability, also increases with measurement error.

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As an example of using figures 10, assume that a vehicle is to be guided to within ±0.001 radius of the target with a 90-percent probability. From figure 10(a) it is found that the maximum measurement error must be no greater than 20 seconds of arc. Using this error in figure 10(b) it is found that a total velocity increment less than 0.039 escape velocity is expended 90 percent of the time. This is 0.0268 escape velocity over the ideal requirement. If miss distance and ΔV_t are assumed uncorrelated, there exists an 81-percent probability that the vehicle will guide to within ±0.001 radius and in so doing require no more than 0.039 escape velocity.

Initial energy. - A comparison of ΔV requirements for initial energies of 0, 0.10, and 0.20 is shown in figure 11(a). Results are given in terms of excess ΔV so that the effect of energy on guidance is clearly indicated. There are two reasons for the increase of excess velocity increment with initial energy. First, the ΔV required to correct a given trajectory error increases with energy (this may be shown from eqs. (5), (8), and (12)). Second, as discussed in the error analysis, accurate knowledge of trajectory parameters decreases with increasing energy. Note: This latter effect is peculiar to the particular measurement scheme considered. Figure 11(a) also illustrates that higher energies increase the dispersion of the statistical distribution. For example, the difference in excess ΔV (comparing E_i of 0 and 0.10) at 90-percent probability is about $3\frac{1}{2}$ times greater than the difference at 50 percent.

Figure 11(b) illustrates the effect of initial energy on final guidance accuracy. Results show that there is no significant effect, although the miss distance was slightly greater for higher energies. One might have expected a larger difference in accuracy because of the discussion of figure 5(b). However, the final accuracy is dependent upon the conditions of the set of measurements corresponding to the final correction; figure 5(b) is not in the range of the final correction.

CONCLUDING REMARKS

An exploratory analysis of vehicle guidance during the approach to a planet was presented, with the target of guidance being defined in terms of a perigee distance. The analysis assumed two-body conic trajectories, a two-dimensional polar representation, and impulsive velocity corrections. A simplified navigation scheme was hypothesized utilizing optical or infrared instrumentation (with significant errors) to obtain measurements of range and a reference angle in the trajectory plane. Trajectory parameters were determined from the minimum information available by this scheme, namely, three successive position fixes. The method used in studying the guidance problem was based on the Monte-Carlo technique, which consisted of repeated calculation of random trajectory runs where the random variable was the measurement error. Statistical results for the reference solution were developed from 200 samples; however, the sample size was reduced to 50 for the parametric study. Results were analyzed primarily on the basis of the probability of error in final perigee distance and the probability of velocity-increment requirements as a measure of propellant expenditure.

The major objectives of this report were as follows: (1) to present a method of obtaining statistical results associated with the guidance problem, and to indicate the type of results, along with their interpretation, which may be obtained, and (2) to illustrate the method by a reasonable and comprehensive example of a guidance scheme. Obviously, certain classes of results are peculiar to the initial conditions assumed and to the particular guidance scheme hypothesized. For instance, numerical results depend upon the energy level of the mission, residual trajectory errors incurred during midcourse or launch guidance, and the size of measurement errors among other factors. Also, the choice of an optimum-type solution (specifically, values of guidance initiation and frequency of corrections) may be greatly affected by the measurement scheme and guidance logic assumed.

In an earlier report by the authors (ref. 2) an investigation of guidance requirements was made within a framework of measurement and guidance logic differing substantially from that considered here. The factors governing the difference in results are those mentioned previously. However, it would be of interest to note certain areas of agreement; the implication being that some results are basic to the problem and relatively independent of the guidance scheme. Of considerable importance are the conflicting requirements of guidance accuracy and propulsion. As range to the planet decreases, the ability to determine proper corrective action improves rapidly; however, the ΔV required increases. Unless instrumentation is extremely accurate, a single trajectory correction far from the planet will not suffice for highly accurate guidance as would be required for atmospheric drag entries. The manner in which corrective maneuvers are executed can have an appreciable effect on ΔV requirement; consequently, optimum guidance logic may be important. Guidance accuracy is sensitive to the desired target perigee since this is the minimum range at which corrections can be made. For a given allowable miss distance a preference for low-altitude targets is indicated. It is realized, of course, that the effect of miss distance is more critical for such targets. The thrust devices needed for optimum guidance should be capable of producing a large variation of ΔV increment.

Use of the Monte-Carlo technique for obtaining statistical results offers a number of advantages for vehicle guidance analysis. The number of random variables that could be considered is unlimited and may all be taken into account simultaneously. For example, a complete analysis might include randomness in initial trajectory parameters, measurement errors, and errors in the application of ΔV. A running statistical record of guidance performance is obtained, and the identity of a given vehicle need not be lost. A drawback of the method is that results for high success probability are liable to be inaccurate unless a large sample size is used. The sample size required is most affected by the dispersion (range) of the random variables.

Representative results for the particular measurement scheme and guidance logic hypothesized are as follows: Consider a parabolic approach trajectory and the first position fix taken at 100 radii. Optimum-type guidance results when four corrections are made and executed in the vicinity of 50, 16, 5, and 1.5 radii, respectively. With measurement errors less than 1 minute of arc, the probability is 90 percent that the miss distance will be less than ± 0.008 radius (32 miles). The ΔV_t requirement depends upon the initial trajectory error. A vehicle which is initially on target requires a median ΔV_t of 0.016 v_e (590 ft/sec) and has a 98-percent probability of using less than 0.04 v_e (1470 ft/sec). In comparison, a vehicle having an initial trajectory error of about 4 radii would require a median ΔV_t of 0.047 v_e (1730 ft/sec) and have a 98-percent probability of using less than 0.14 v_e (5150 ft/sec). A reduction of the measurement error to 20 seconds of arc causes these miss distances to be reduced by a factor of 3, and ΔV_t to be reduced by a factor of 2.

Lewis Research Center
National Aeronautics and Space Administration
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APPENDIX A

ANGULAR MOMENTUM ERROR ANALYSIS

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An expression is to be derived which gives the error in angular momentum as a function of errors in measured position. A solution of angular momentum is found from equation (15) for position fixes (a,b,c):

$$2H^2 = \frac{\sin(\theta_b - \theta_c) + \sin(\theta_c - \theta_a) + \sin(\theta_a - \theta_b)}{\frac{1}{R_a}\sin(\theta_b - \theta_c) + \frac{1}{R_b}\sin(\theta_c - \theta_a) + \frac{1}{R_c}\sin(\theta_a - \theta_b)} \equiv \frac{N}{D} \qquad \text{(A1)}$$

Equation (A1) is differentiated with respect to each of the six variables:

$$d(H^2) = \frac{1}{2}\sum_{k=a}^{c} \frac{\partial(2H^2)}{\partial\theta_k}\, d\theta_k + \frac{1}{2}\sum_{k=a}^{c} \frac{\partial(2H^2)}{\partial R_k}\, dR_k$$

recalling that dR is given by equation (19). After performing the partial differentiation and expanding we obtain $d(H^2)$ as the sum of six independent error terms. Substituting the following notation:
$d(H^2) = (H^2)_{err}$, $d\omega \equiv (\omega)_{err}$; $d\theta \equiv (\theta)_{err}$,

$$(H^2)_{err} = T_1(\omega_a)_{err} + T_2(\omega_b)_{err} + T_3(\omega_c)_{err} + T_4(\theta_a)_{err} + T_5(\theta_b)_{err} + T_6(\theta_c)_{err} \qquad \text{(A2)}$$

where

$$
\left.
\begin{aligned}
T_1 &= -\frac{N}{4D^2}\,\frac{\sqrt{R_a^2 - 1}}{R_a}\,\sin(\theta_b - \epsilon_c) \\
T_2 &= -\frac{N}{4D^2}\,\frac{\sqrt{R_b^2 - 1}}{R_b}\,\sin(\theta_c - \epsilon_a) \\
T_3 &= -\frac{N}{4D^2}\,\frac{\sqrt{R_c^2 - 1}}{R_c}\,\sin(\theta_a - \epsilon_b) \\
T_4 &= \frac{1}{2D^2}\left\{D\left[\cos(\theta_a - \theta_b) - \cos(\theta_c - \theta_a)\right]\right. \\
&\qquad \left. - N\left[\frac{1}{R_c}\cos(\theta_a - \theta_b) - \frac{1}{R_b}\cos(\theta_c - \theta_a)\right]\right\} \\
T_5 &= \frac{1}{2D^2}\left\{D\left[\cos(\theta_b - \theta_c) - \cos(\theta_a - \theta_b)\right]\right. \\
&\qquad \left. - N\left[\frac{1}{R_a}\cos(\theta_b - \theta_c) - \frac{1}{R_c}\cos(\theta_a - \theta_b)\right]\right\} \\
T_6 &= \frac{1}{2D^2}\left\{D\left[\cos(\theta_c - \theta_a) - \cos(\theta_b - \theta_c)\right]\right. \\
&\qquad \left. - N\left[\frac{1}{R_b}\cos(\theta_c - \theta_a) - \frac{1}{R_a}\cos(\theta_b - \theta_c)\right]\right\}
\end{aligned}
\right\} \tag{A3}
$$

Assume the errors in measured angles to arise from the same error distribution (identical size and shape). For convenience, consider two measures of $(H^2)_{err}$:

Root-mean-square error

$$
\left[(H^2)_{err}\right]_{rms} \equiv \left(\sqrt{\sum_{i=1}^{6} T_i^2}\right)(\psi_{err})_{rms} \tag{A4}
$$

where ψ_{err} is the common angular error.

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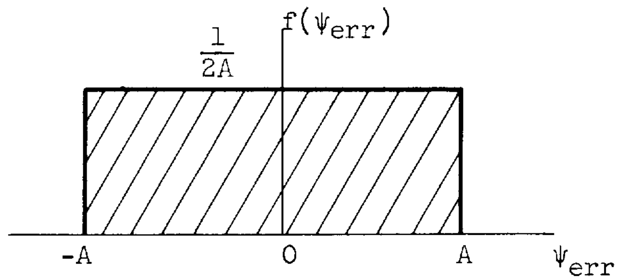
Maximum absolute error

$$\left[(H^2)_{err}\right]_{max} \equiv \left(\sum_{i=1}^{6} |T_i|\right) (\psi_{err})_{max} \tag{A5}$$

As an example, consider an instrument possessing a rectangular (uniform) error distribution.



(c)

Sketch (c) shows the error density function $f(\psi_{err})$ to be constant over the interval $-A$ to A. Thus, any error is equally likely between $-A$ and A, the maximum absolute error being A.

The rms error, or standard deviation σ, is defined as the square root of the second moment of $f(\psi_{err})$. For this example $(\psi_{err})_{rms} \equiv \sigma = A/(\sqrt{3})$ with A to be expressed in radians.

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APPENDIX B

SIGNIFICANCE TEST

When comparing two statistical results which are obtained from some sampling method, a measure of statistical significance must be included in order that correct inference be drawn from the data. There are various methods, such as the chi-square test, which are used to determine how well a particular sample fits some assumed parent (infinite) distribution. In our case, however, it is not apparent whether the data correspond to a standard type distribution. A simple "test of proportions" is subsituted for a more complicated technique. This method is presented in reference 10.

Let two random samples be given in which the number of items less than a certain value (C) is N_1 for sample 1 and N_2 for sample 2. The sizes of the samples are S_1 and S_2; p_1 and p_2 are the respective probabilities of being less than C, where $p_1 = N_1/S_1$ and $p_2 = N_2/S_2$. Let p denote the total probability $(N_1 + N_2)/(S_1 + S_2)$, and $q = 1 - p$.

The purpose of the test is to show whether the difference between p_1 and p_2 is significant, or whether it is a chance difference due to sampling fluctuation. The standard error of the difference may be calculated by the formula

$$\text{S.E.} = \sqrt{pq\left(\frac{1}{S_1} + \frac{1}{S_2}\right)}$$

The number of standard errors in the difference is thus $(p_1 - p_2)/\text{S.E.}$ A measure of significance is obtained from the property of a normal probability distribution commonly available in table form. For example, if the difference contains three standard errors, the probability that it is significant and did not arise due to chance is 99.7 percent.

Figures 7 and 8 show the results of a parametric study undertaken to indicate the effects of the number of corrections (n) and the range at which the first correction is made (R_i) upon guidance performance. The guidance performance is measured by total velocity requirement (ΔV_t) and absolute miss distance $|P_f - P_{tar}|$. Results are obtained from a sample size of 50; therefore, some measure of statistical significance is indicated.

The significance test is applied to the statistical data shown in figures 7(a) and (b).

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A. - Total velocity requirements (variation of n)

ΔV_t	Probability of using less than ΔV_t, %					Significance of difference, %			
	n=4	n=5	n=3	n=7	n=9	4-5	4-3	4-7	4-9
0.04	26	8	20	2	0	99	53	100	100
.05	72	63	42	14	0	65	100	100	100
.06	86	78	64	48	2	70	99	100	100
.07	91	86	78	74	12	57	94	98	100
.08	94	92	86	82	30	30	82	94	100

B. - Absolute miss distance (variation of n)

$\lvert P_f - P_{tar}\rvert$	Probability, percent within $\lvert P_f - P_{tar}\rvert$					Significance of difference, %			
	n=3	n=4	n=5	n=7	n=9	3-4	3-5	3-7	3-9
0.001	60	35	38	20	22	99	97	100	100
.002	92	76	62	42	54	97	100	100	100
.003	95	90	80	68	66	66	98	100	100

Figure 7(a) illustrates that minimum velocity expenditure results for a four-correction guidance scheme. Table A shows the significance of this conclusion over a range of the probability distribution. A comparison with values of $n = 7, 9$ shows very high significance while that with $n = 3$ is significant except at low probability levels, which are of little interest anyway. A comparison between $n = 4$ and $n = 5$ shows poor significance; thus, it is likely that the difference in statistical results arose because of sample fluctuation. Either value may be considered to result in minimum velocity expenditure.

By a similar argument based on table B, it is concluded that most accurate guidance corresponds to $n = 3$ (with reservation at high probability levels).

Application of the significance test to data from figures 8(a) and (b) gives the following results:

C. - Total velocity requirement (variation of R_i)

ΔV_t	Probability of using less than ΔV_t, %			Significance of difference, %	
	R_i=50	R_i=30	R_i=70	50-30	50-70
0.05	72	10	42	100	100
.06	86	70	68	95	97
.07	92	98	80	84	92
.08	94	100	90	92	55

D. - Absolute miss distance (variation of R_i)

$\lvert P_f-P_{tar}\rvert$	Probability, percent within $\lvert P_f - P_{tar}\rvert$			Significance of difference, %	
	R_i=30	R_i=50	R_i=70	30-50	30-70
0.001	46	36	41	69	37
.002	80	77	74	27	52
.003	98	90	90	91	91

A comparison of the effect of guidance initiation on ΔV requirement, as given in figure 8(a), shows that $R_i = 50$ results in smaller velocity expenditure if the probability level is below 90 percent. The significance of this result is verified in table C. The probability of requiring less than 0.07 escape velocity is 98 and 92 percent for $R_i = 30$ and 50, respectively. The significance of this difference is about 84 percent. Although statistics were developed from a small sample size, the characteristic of figure 8(a) is felt to be significant.

On the other hand, as a result of this significance test little value is placed on the comparison shown in figure 8(b). However, since the results are so similar, it could be said that no essential effect on guidance accuracy is indicated because of variation of guidance initiation.

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TABLE I. - PARAMETERS ASSUMED IN REFERENCE SOLUTION

Parameter	Assumed value	Approx. equivalent for earth approach
Trajectory:		
E_i energy	0 (parabolic)	0
P_i perigee	5 radii	20,000 miles
P_{tar} target perigee	1.02 radii	80-mile altitude
γ perigee argument	225°	
Measurement error distribution:		
err^ω and err^θ		
Shape	Rectangular (uniform)	
Max. error size	±1 minute of arc	
Guidance scheme:		
Measurements initiated	100 radii	400,000 miles
R_i range of first correction	50 radii	200,000 miles
R_f range of final correction	1.5 radii	6,000 miles
n number of corrections	4	
Zero restraint		
Statistics:		
S sample size	200 cases	

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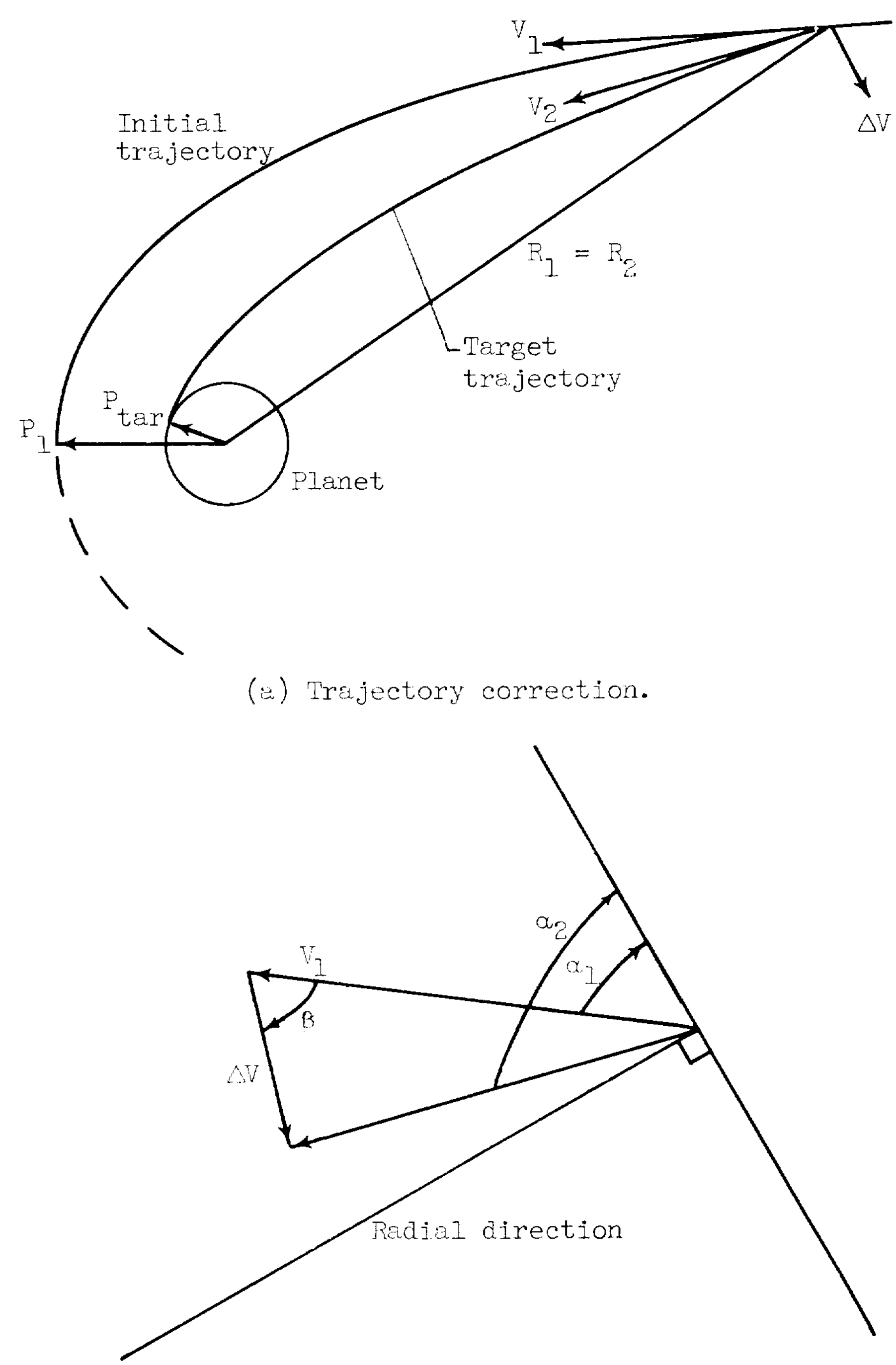

(a) Trajectory correction.

(b) Velocity relations.

Figure 1. - Notation used in analysis.

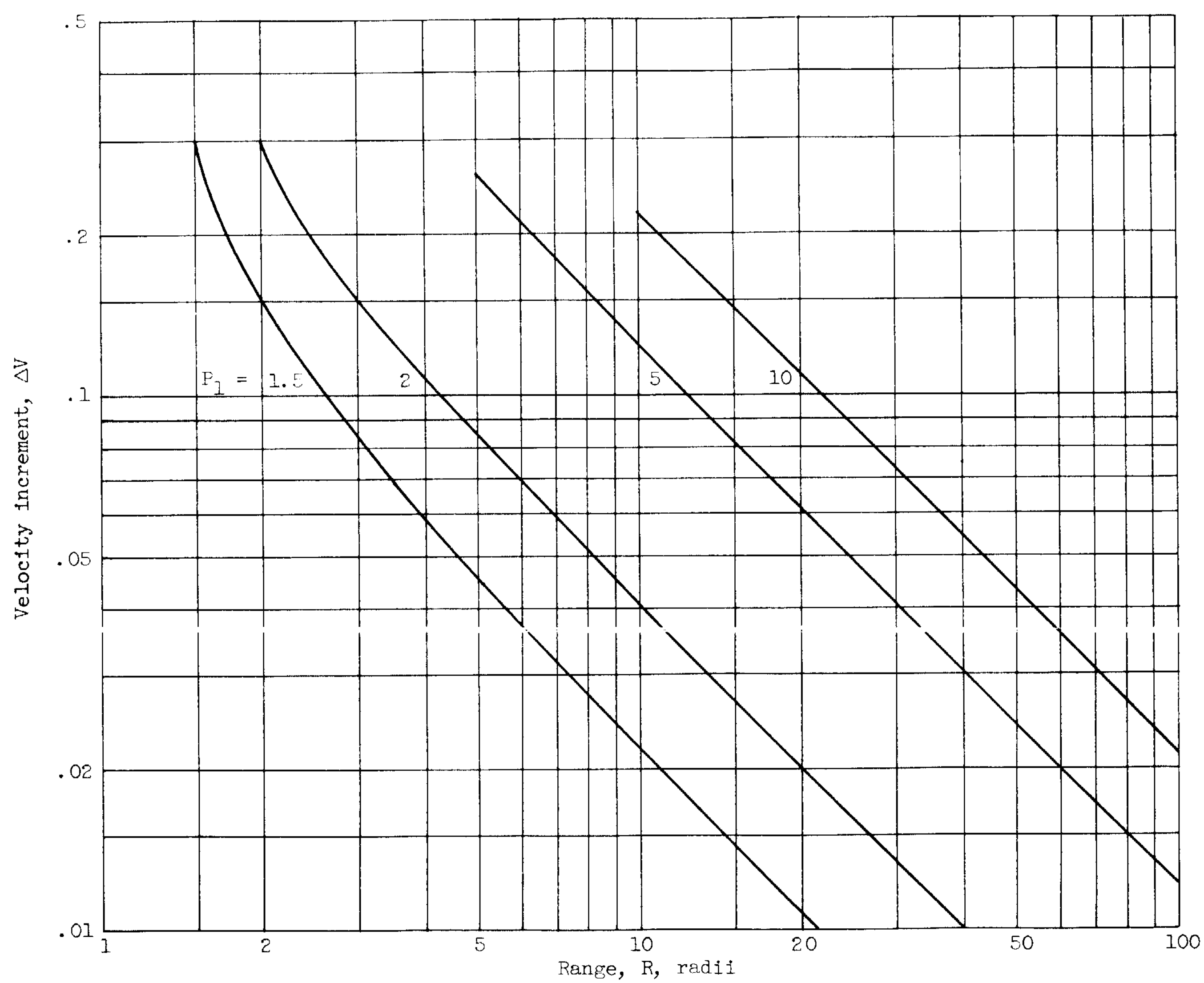

Figure 2. - Velocity requirement of perigee corrections. $E_1 = 0$; $P_{tar} = 1.02$ radii.

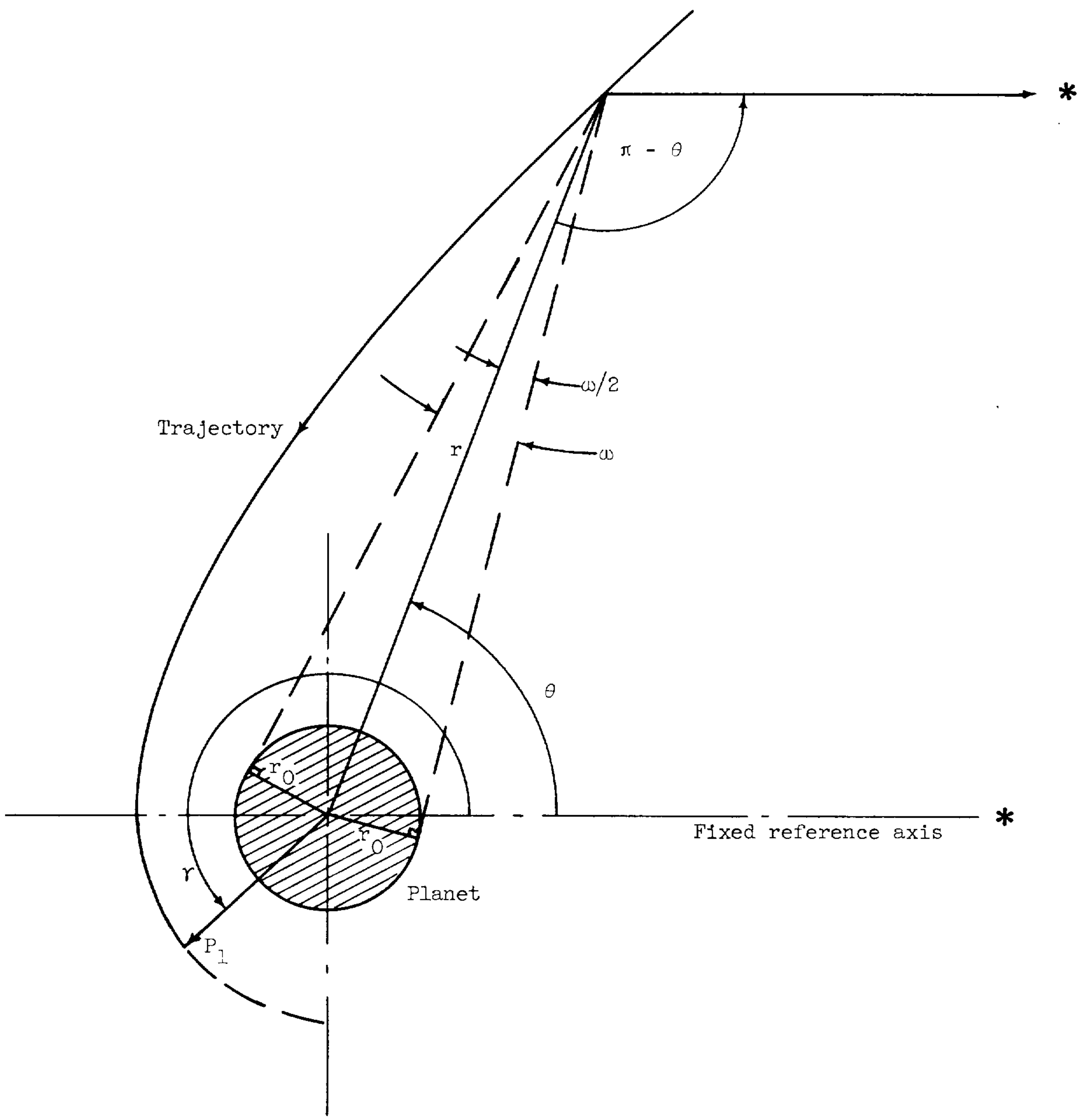

Figure 3. - Measurement scheme.

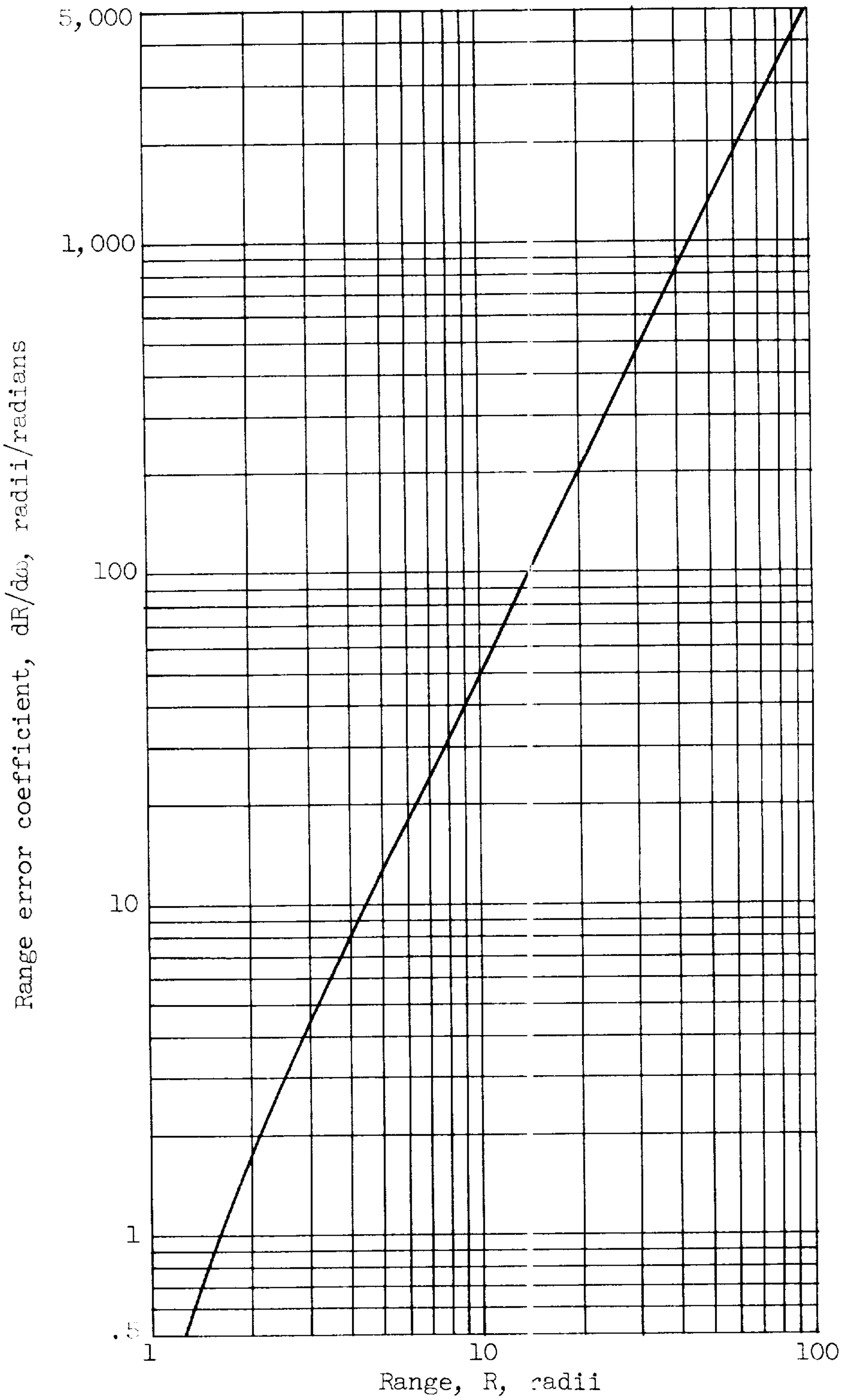

Figure 4. - Error sensitivity in range determination.

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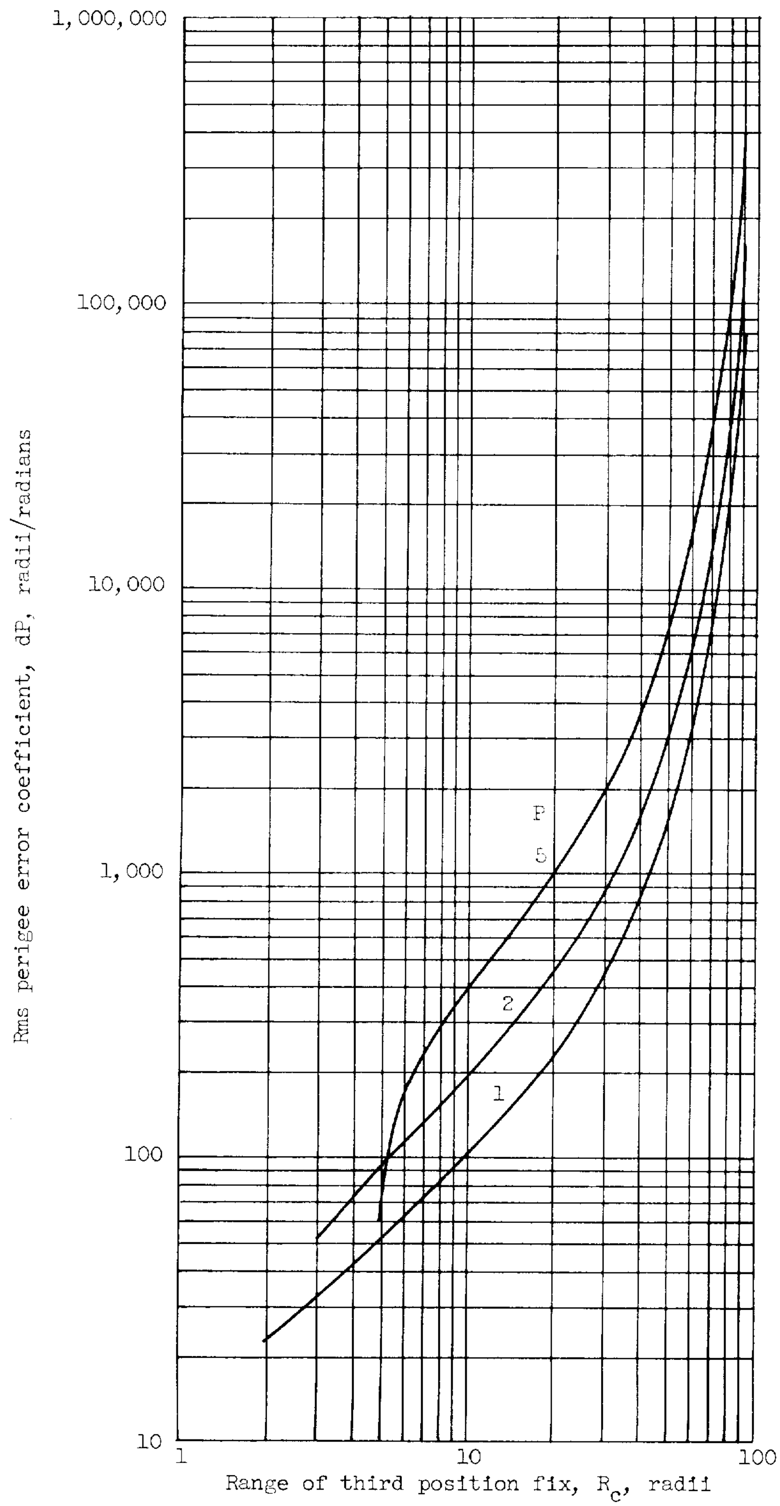
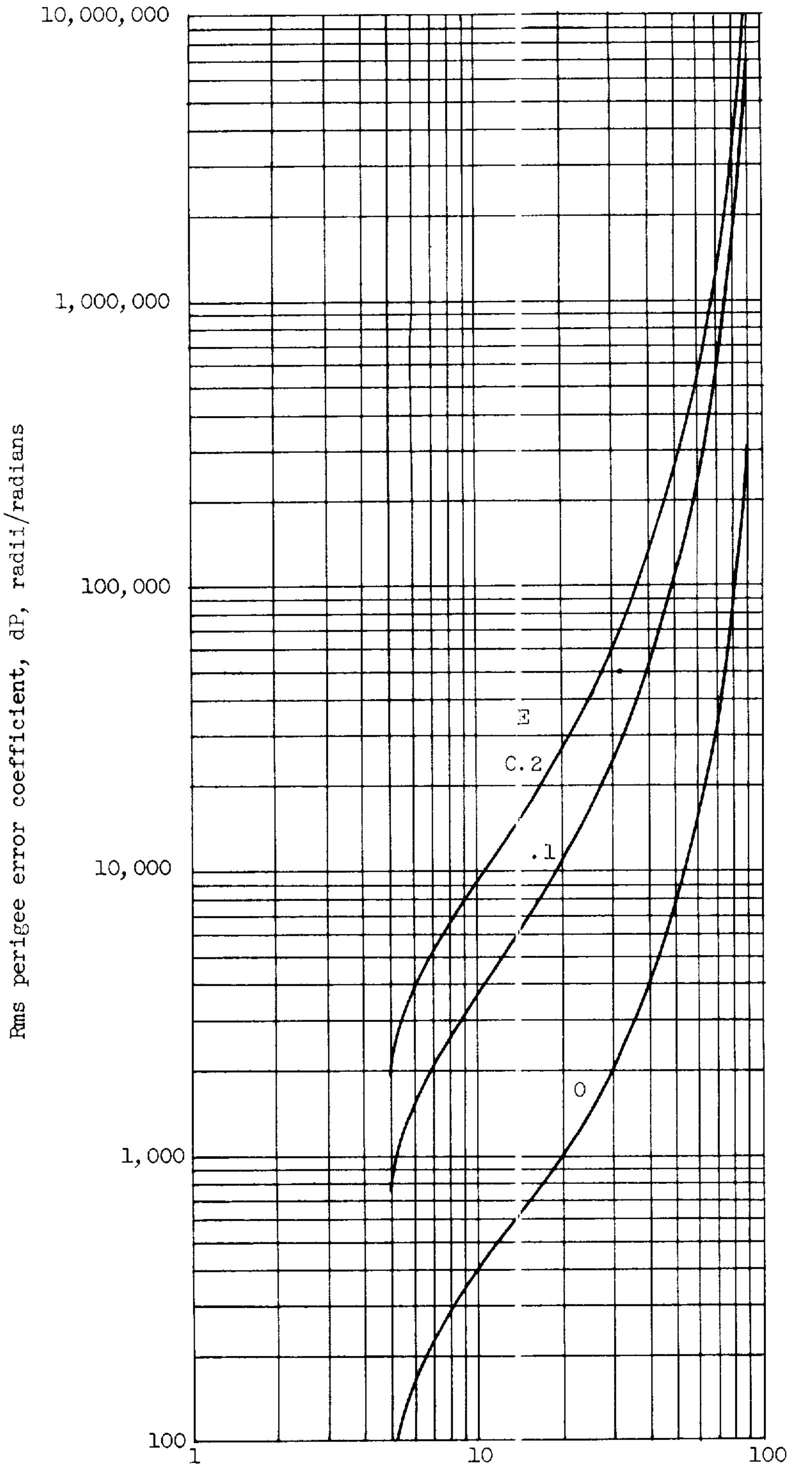


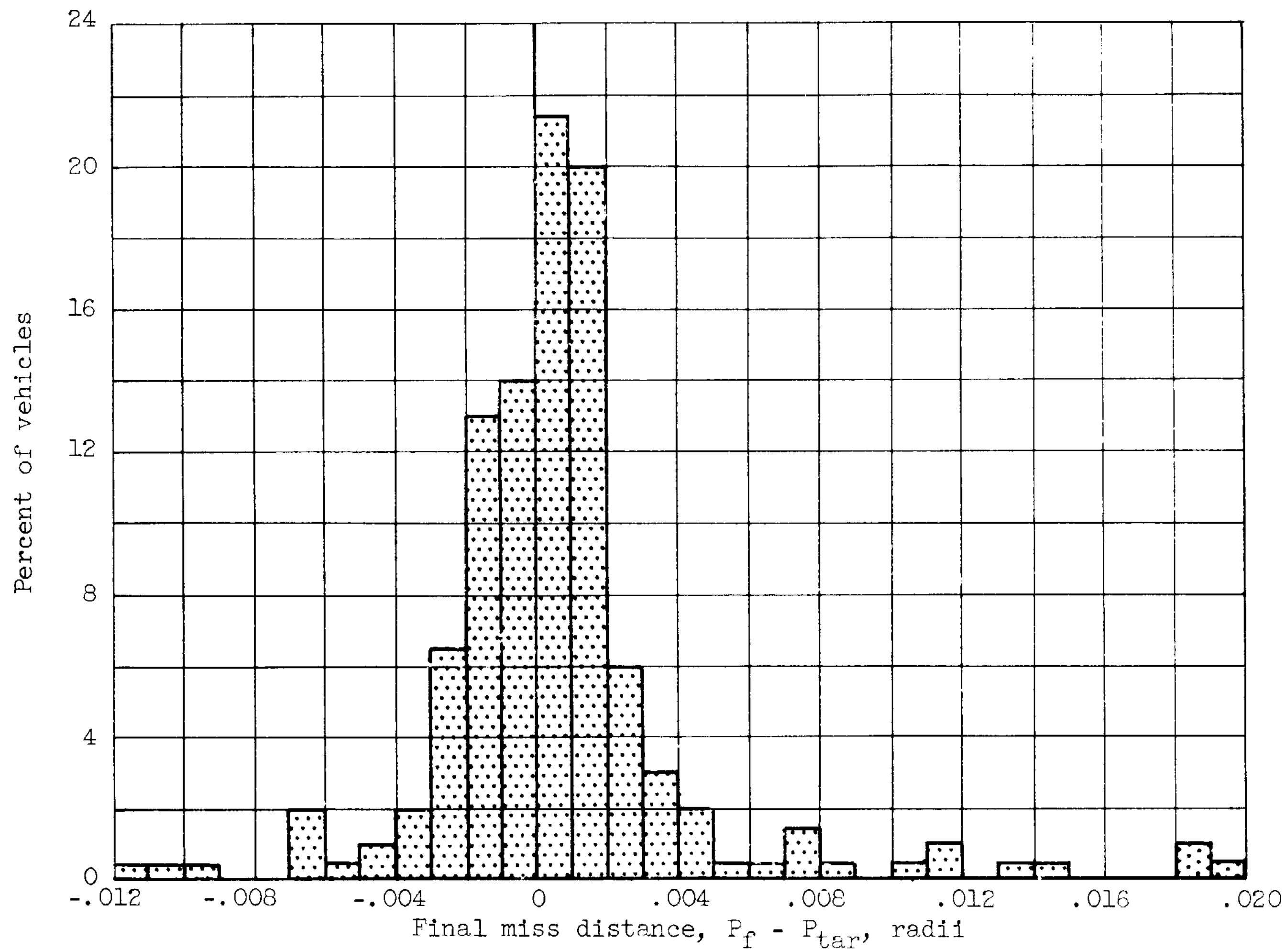
(a) Effect of perigee. E = 0; R_a = 100.

Figure 5. - Error sensitivity in perigee determination.



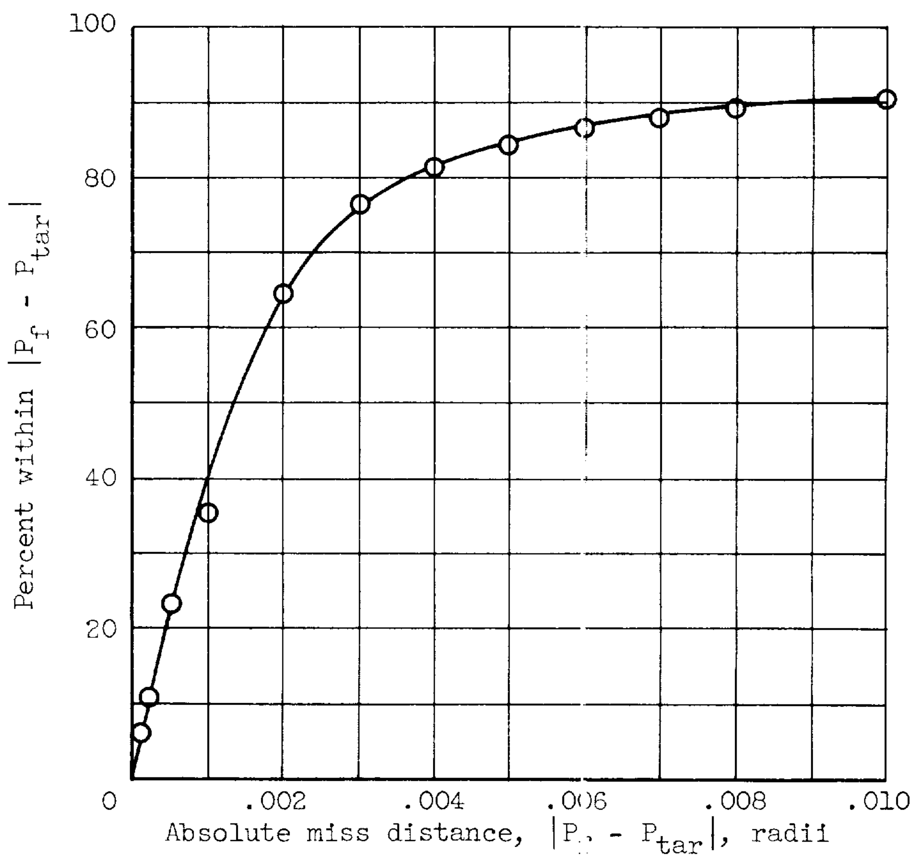
(b) Effect of energy. P = 5; R_a = 100.

Figure 5. - Concluded. Error sensitivity in perigee determination.



(a) Frequency distribution of vehicles about target after guidance.

Figure 6. - Results of reference solution.



(b) Probability of hitting a given size target band.

Figure 6. - Continued. Results of reference solution.

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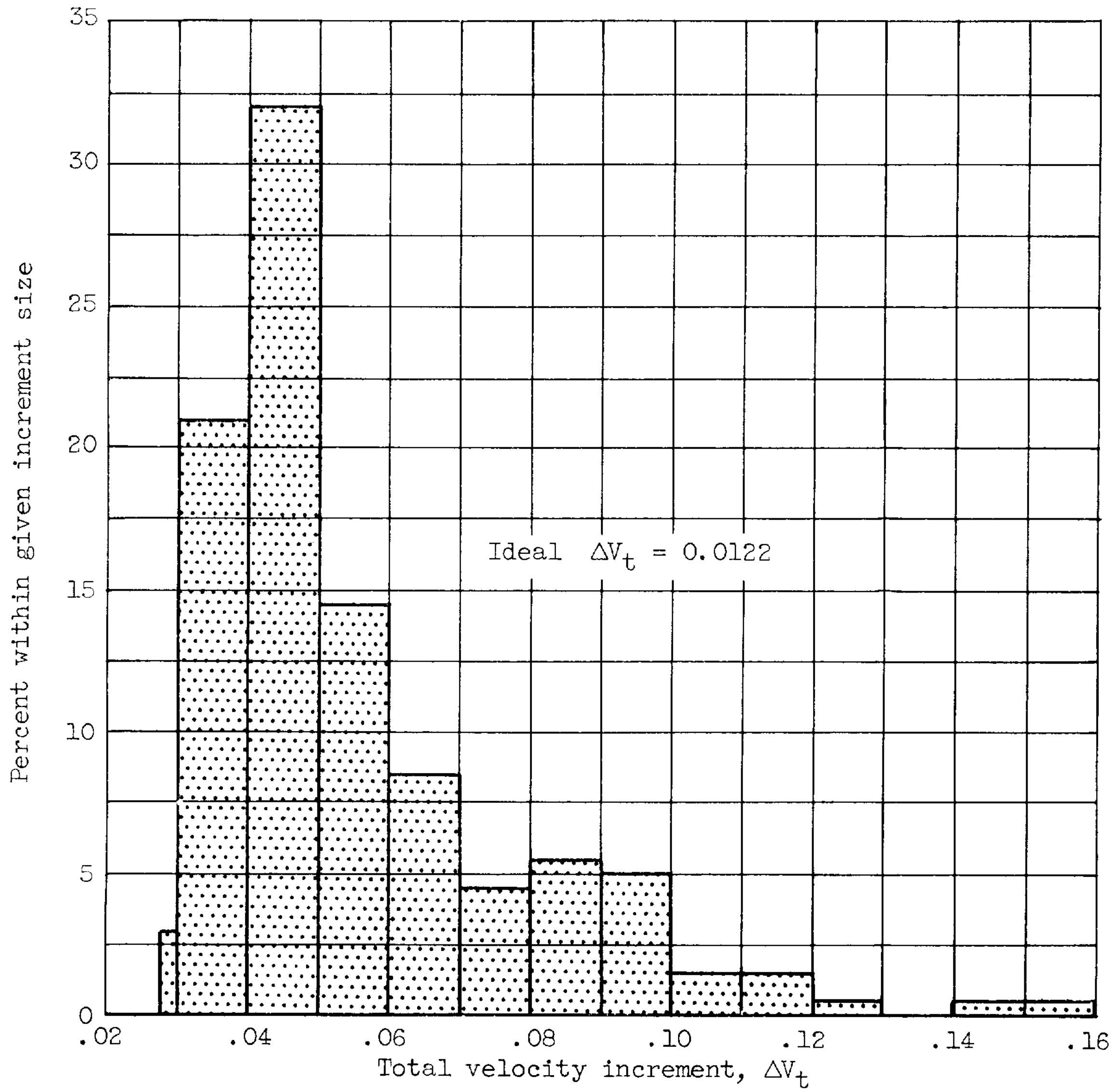
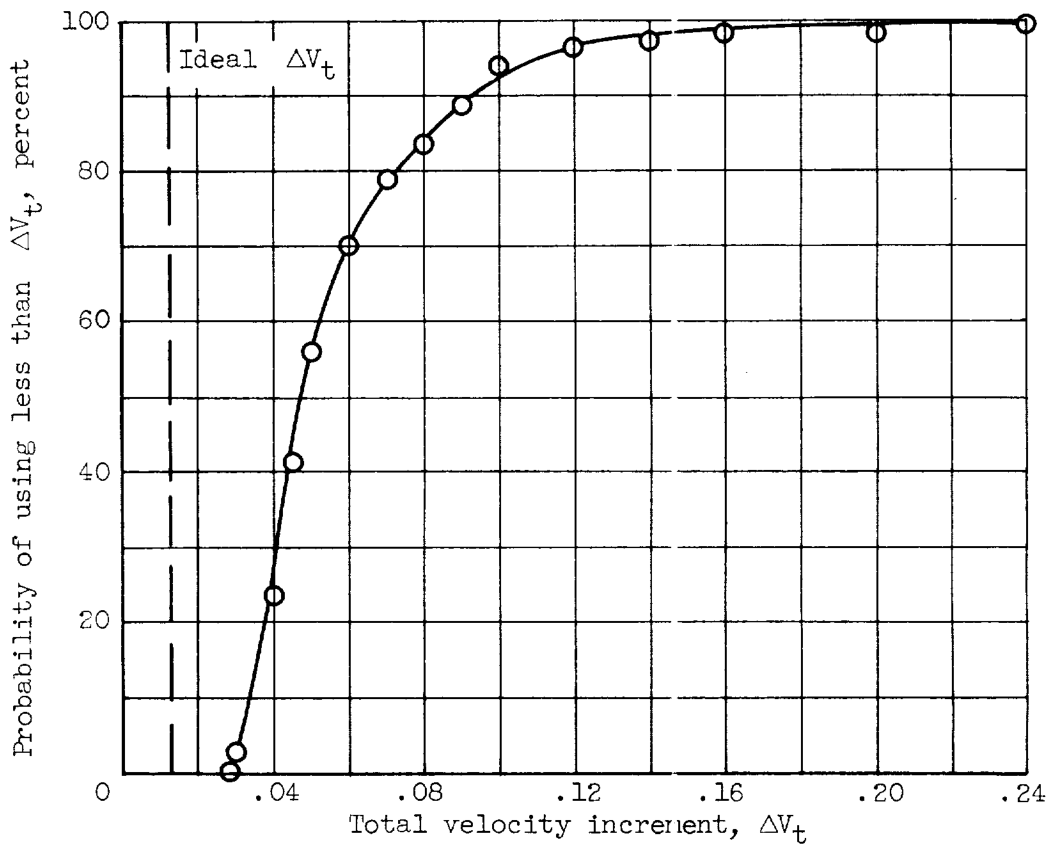


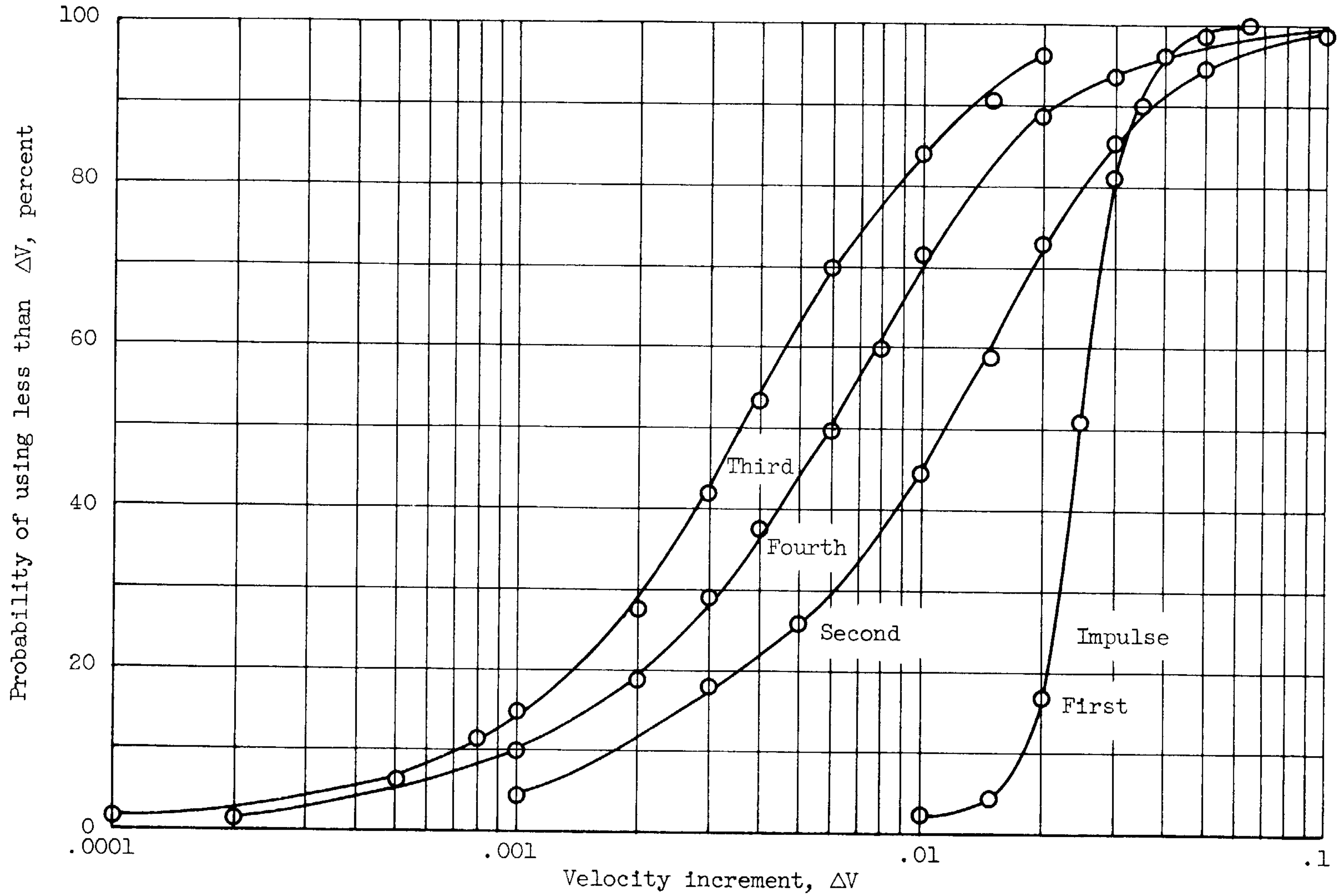
(c) Frequency distribution of total velocity expenditure.

Figure 6. - Continued. Results of reference solution.



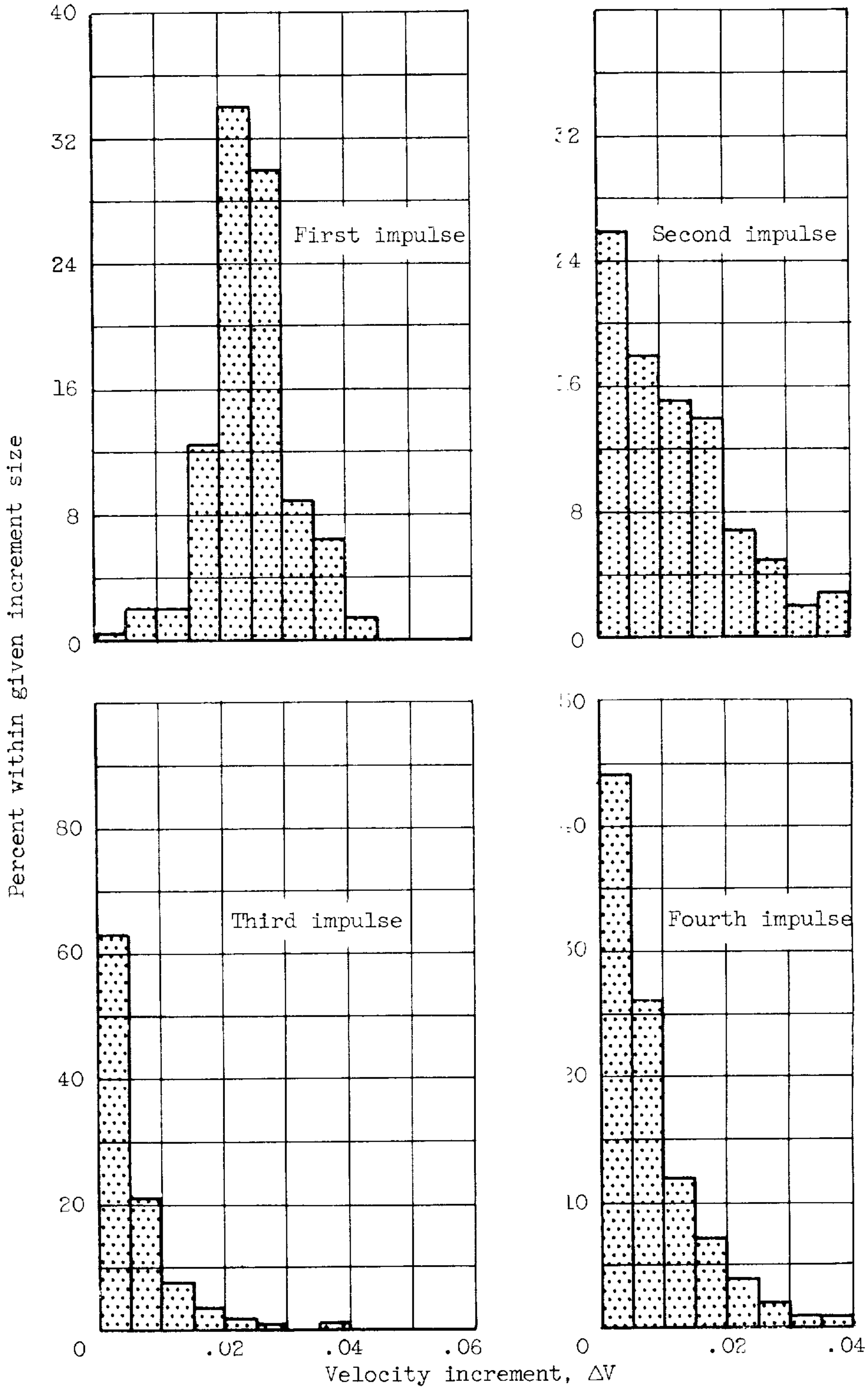
(d) Probability of total velocity increment requirements.

Figure 6. - Continued. Results of reference solution.



(e) Probability of individual velocity impulse requirements.

Figure 6. - Continued. Results of reference solution.



(f) Frequency distribution of individual velocity impulse.

Figure 6. - Continued. Results of reference solution.

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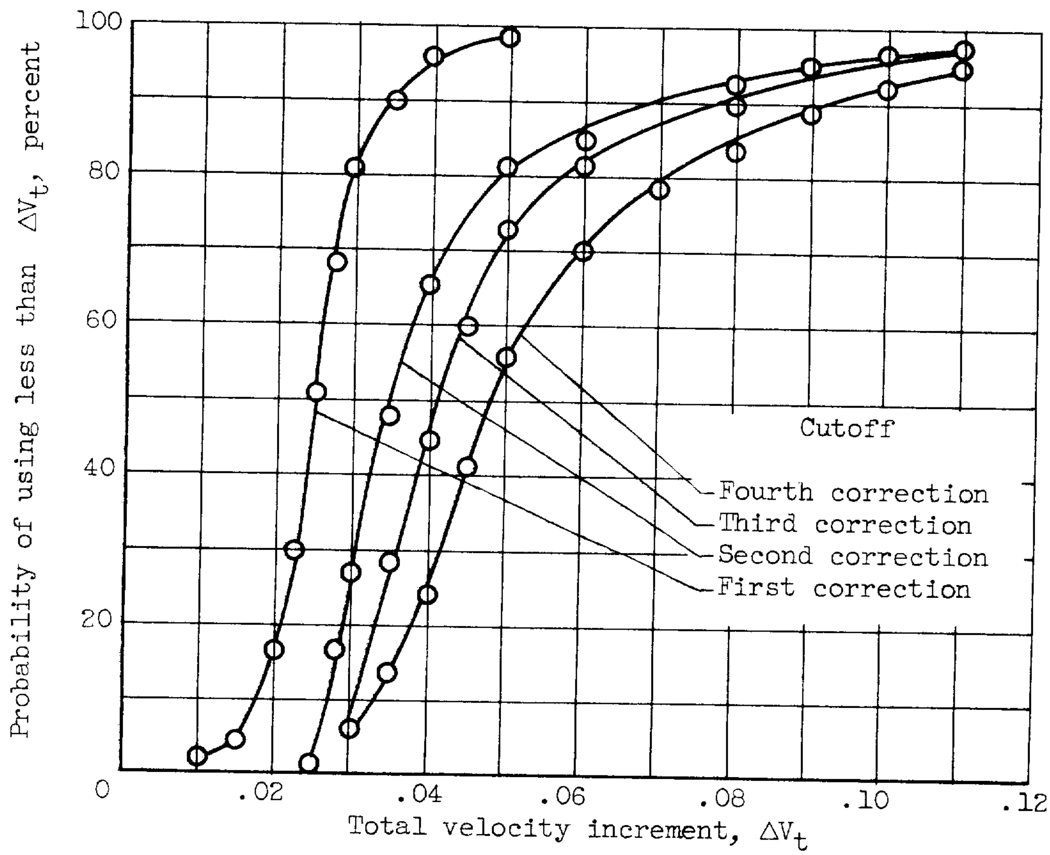
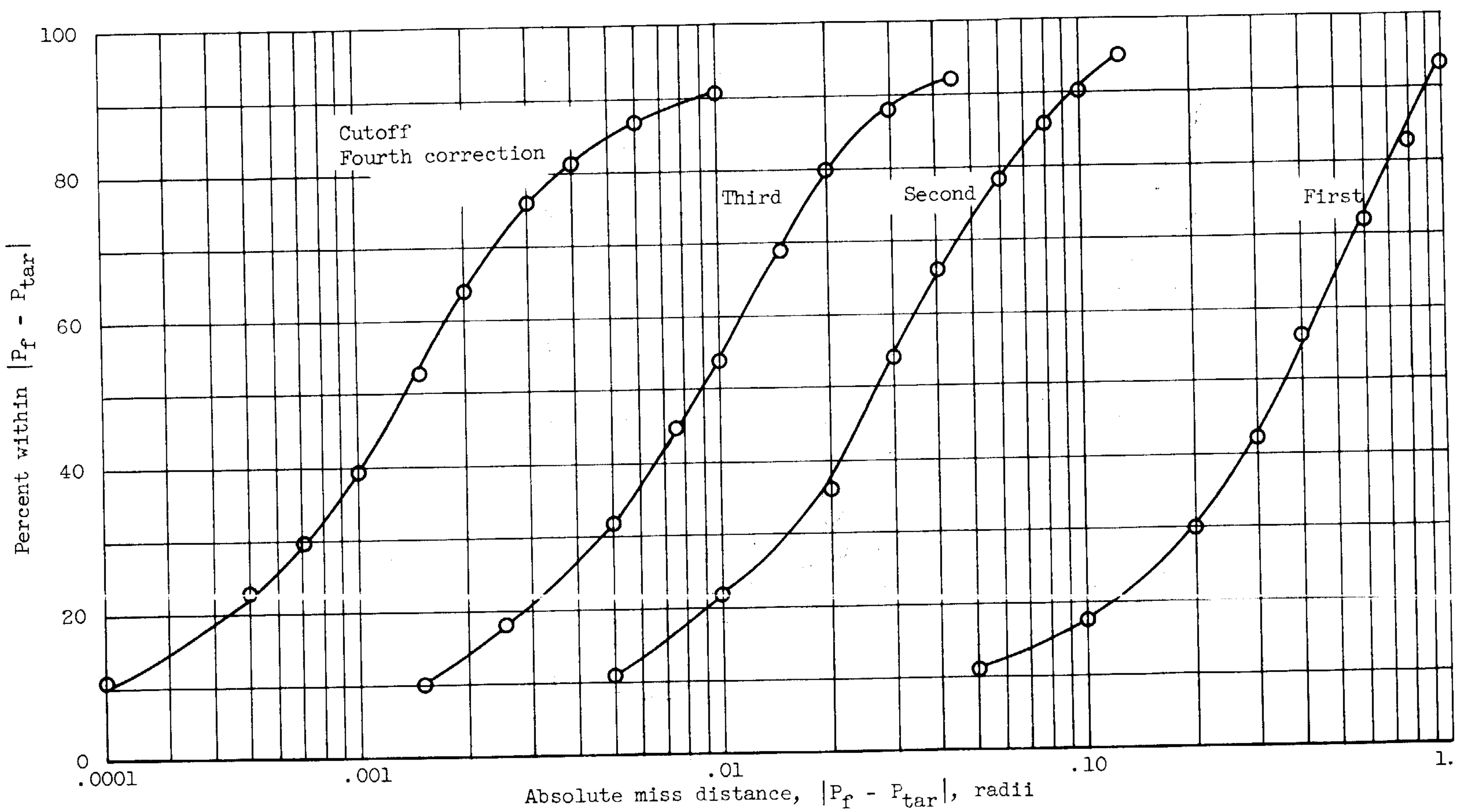


(g) Requirement of total velocity increment at guidance cutoff.

Figure 6. - Continued. Results of reference solution.



(h) Probability of hitting a given size target band at guidance cutoff.

Figure 6. - Continued. Results of reference solution.

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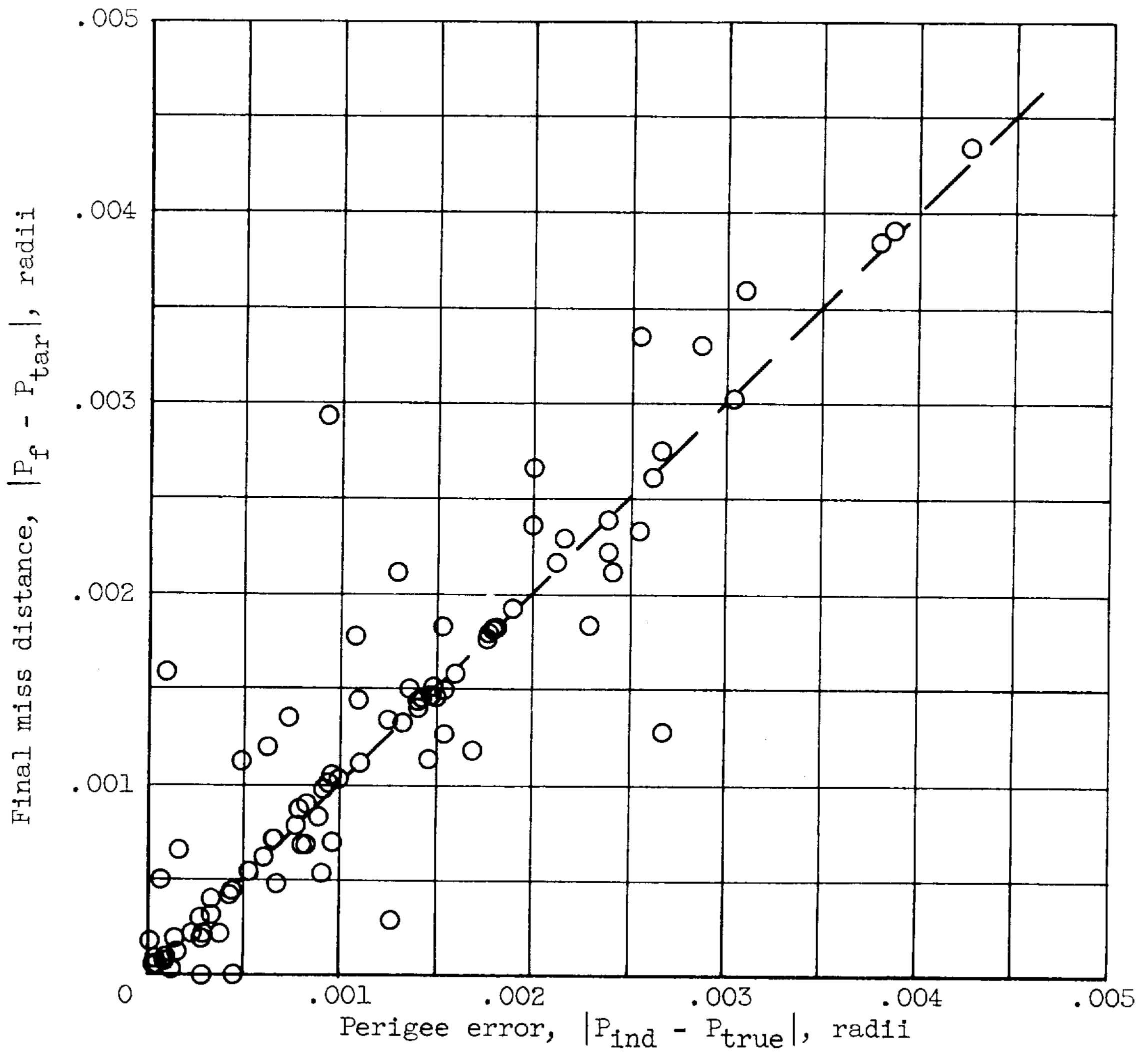
CE-7 back



(i) Correlation between perigee error and final miss distance.

Figure 6. - Continued. Results of reference solution.

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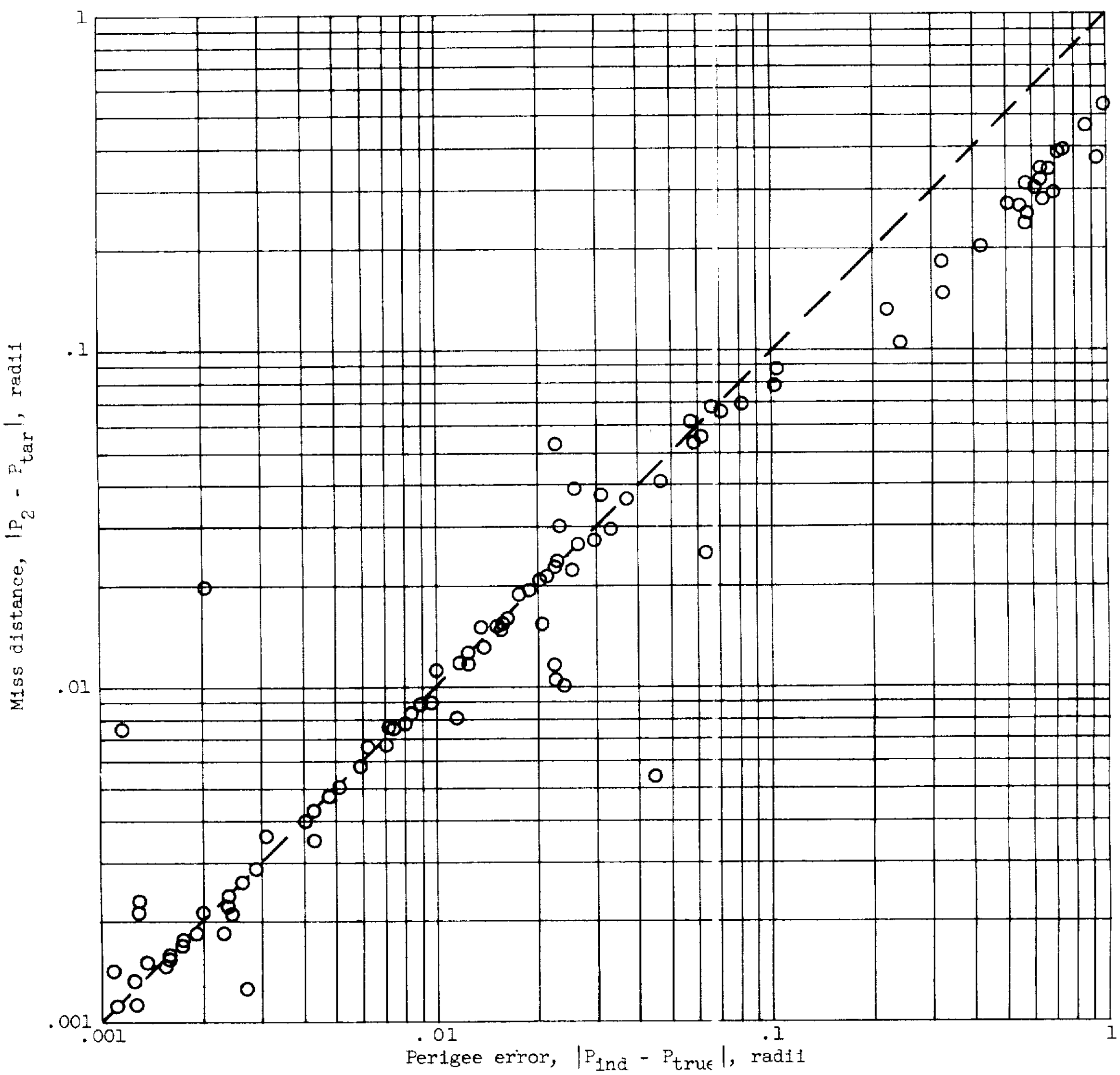

(j) Correlation between perigee error and miss distance.

Figure 6. - Continued. Results of reference solution.

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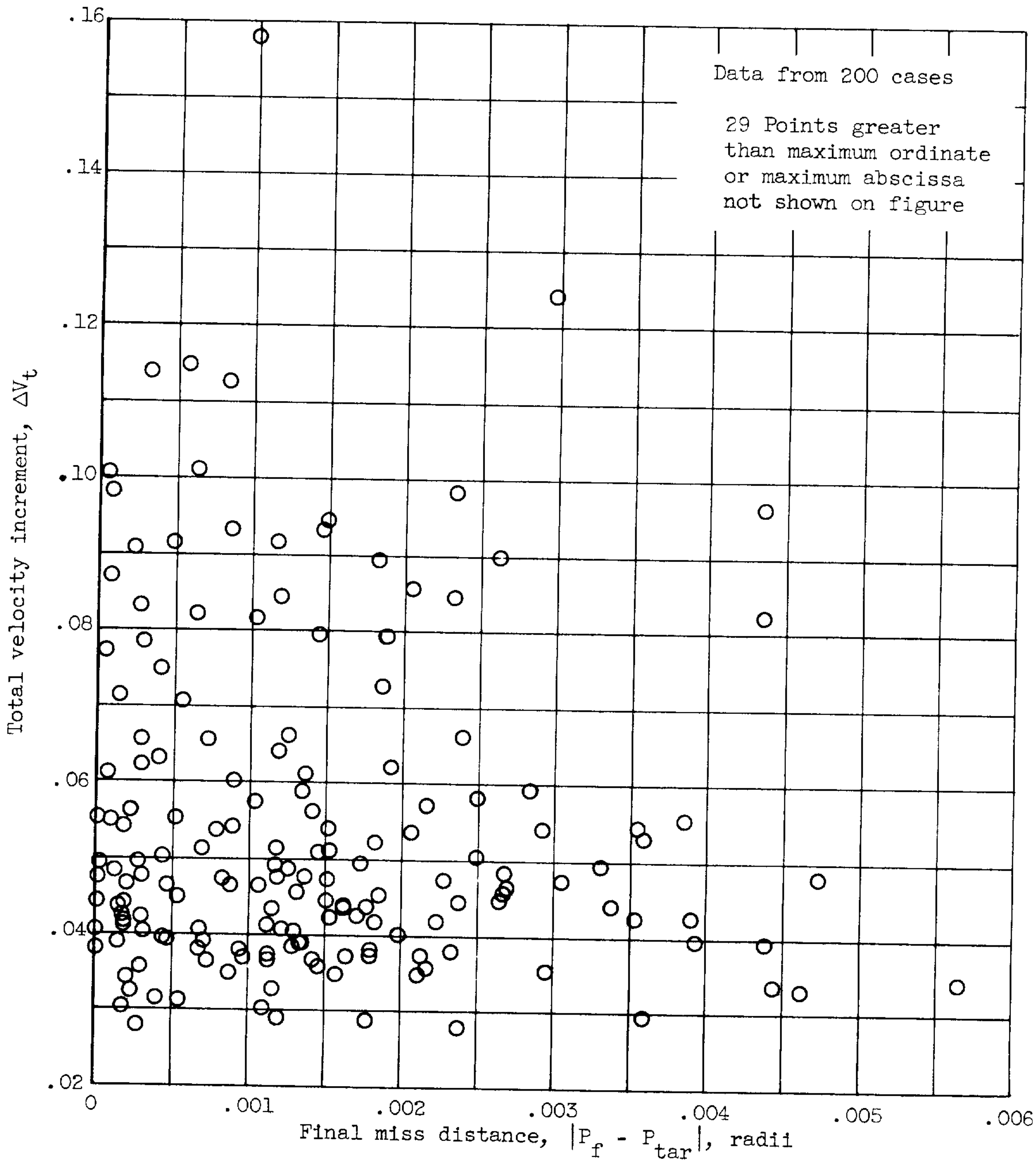

(k) Correlation between final miss distance and total velocity expended.

Figure 6. - Continued. Results of reference solution.

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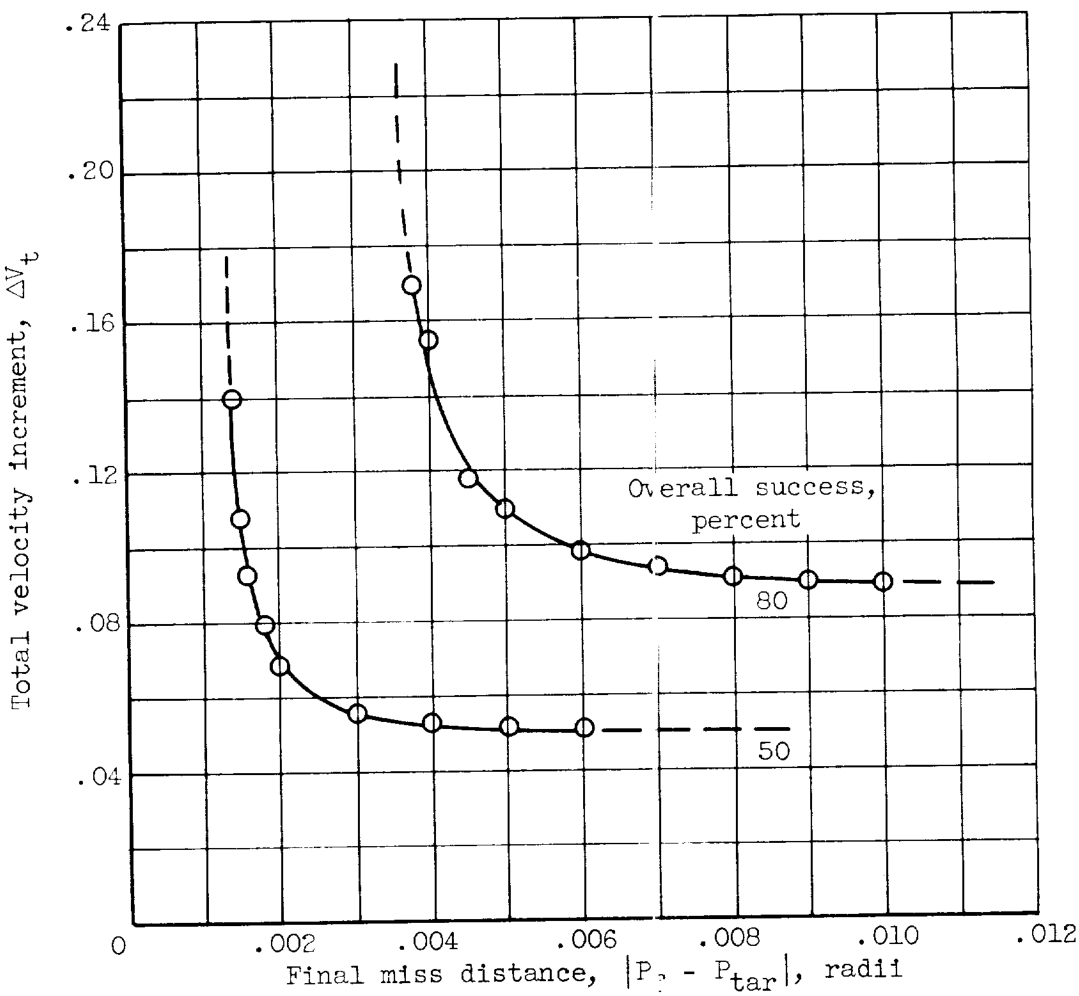

(l) Combined probability of success.

Figure 6. - Continued. Results of reference solution.

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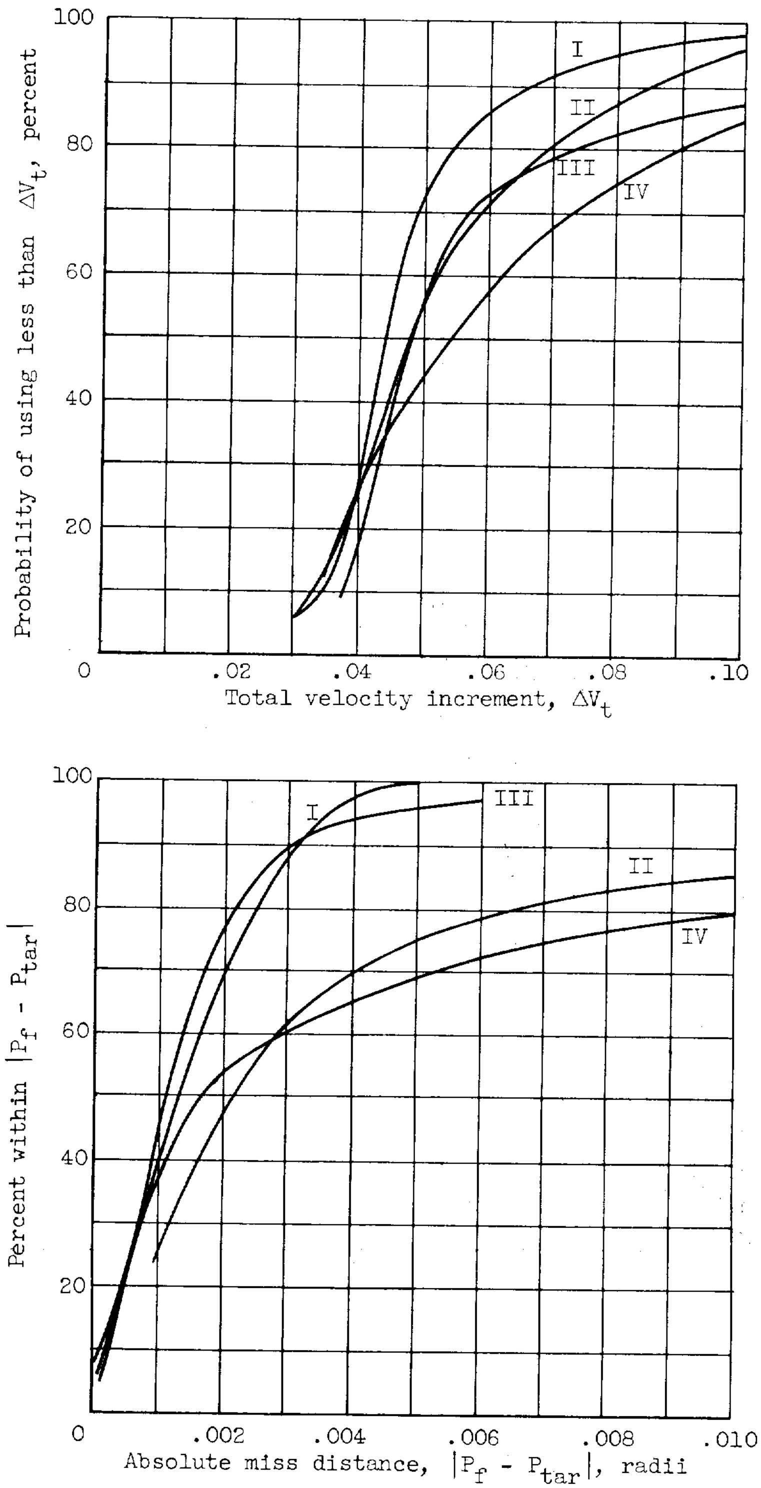
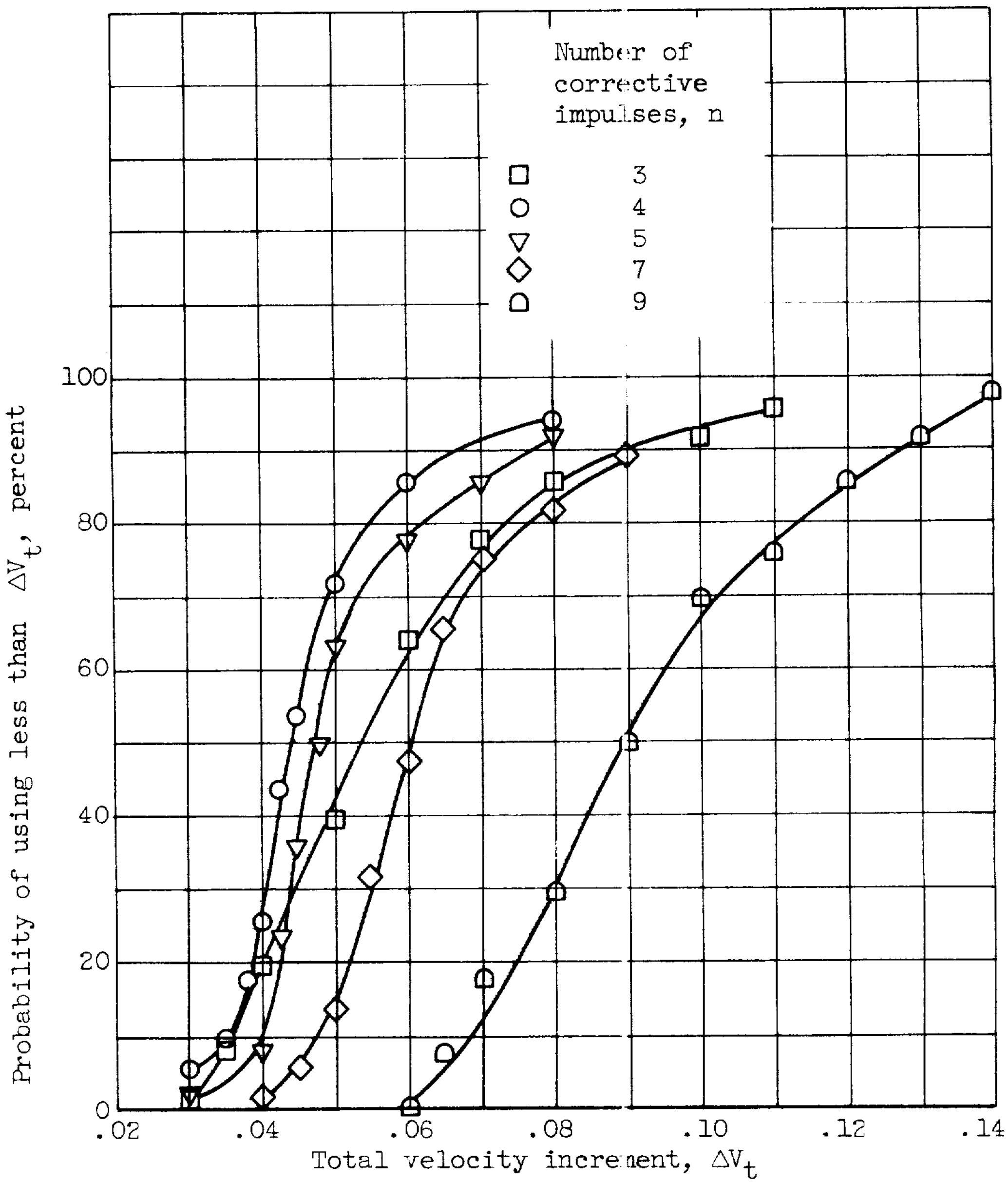


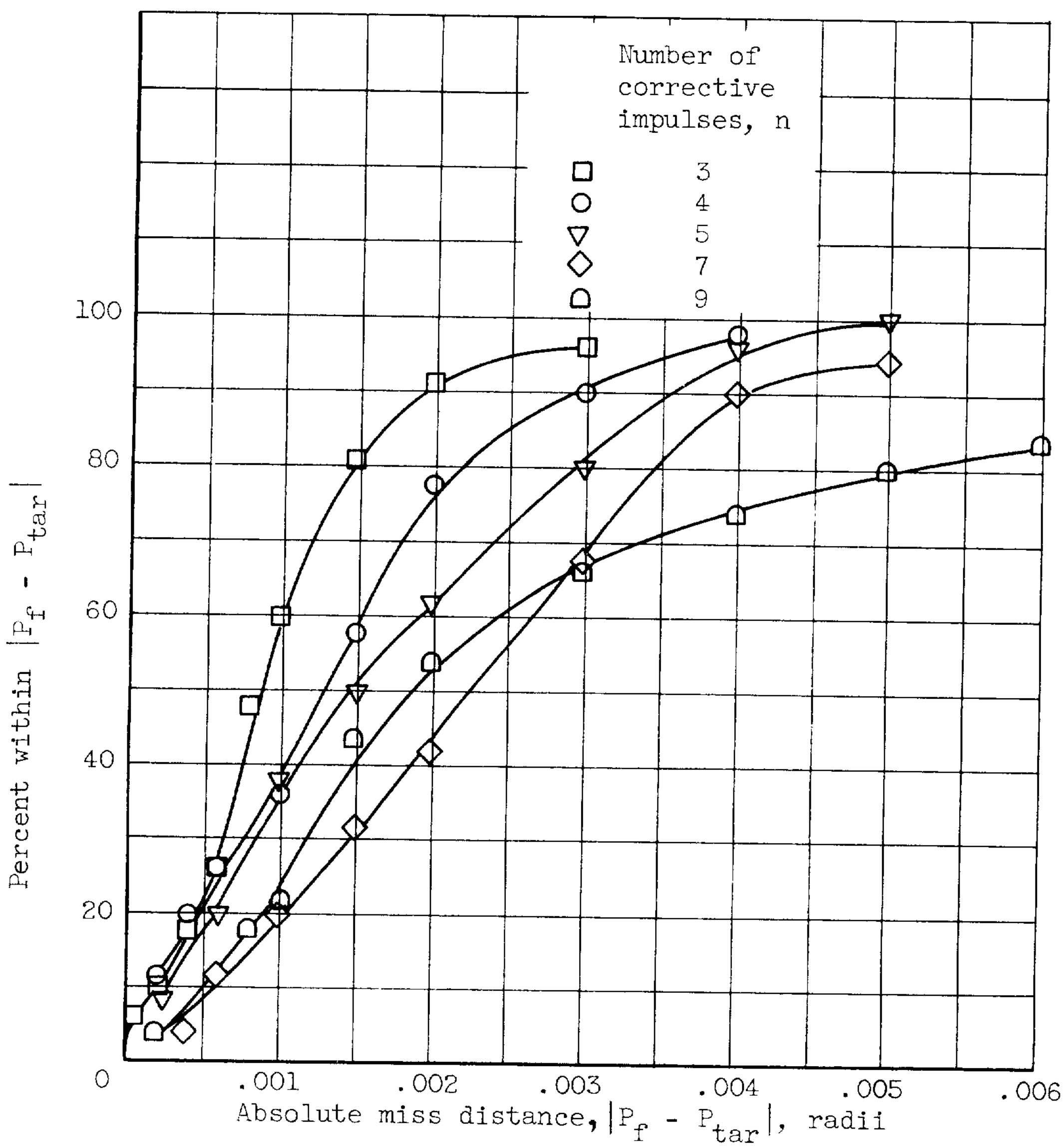
(m) Comparison of sample groups.

Figure 6. - Concluded. Results of reference solution.



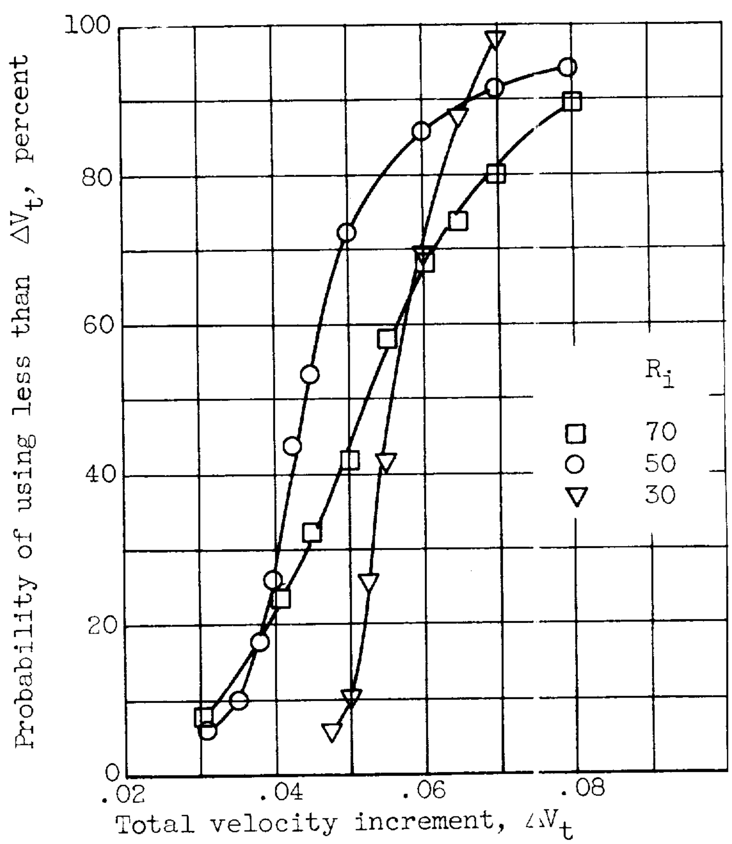
(a) Probability of total velocity increment requirements.

Figure 7. - Effects of variation in number of corrective impulses. Assumed values, table I.



(b) Probability of hitting a given size target band.

Figure 7. - Concluded. Effects of variation in number of corrective impulses. Assumed values, table I.



(a) Probability of total velocity increment requirements.

Figure 8. - Effects of guidance initiation. Assumed values, table I.

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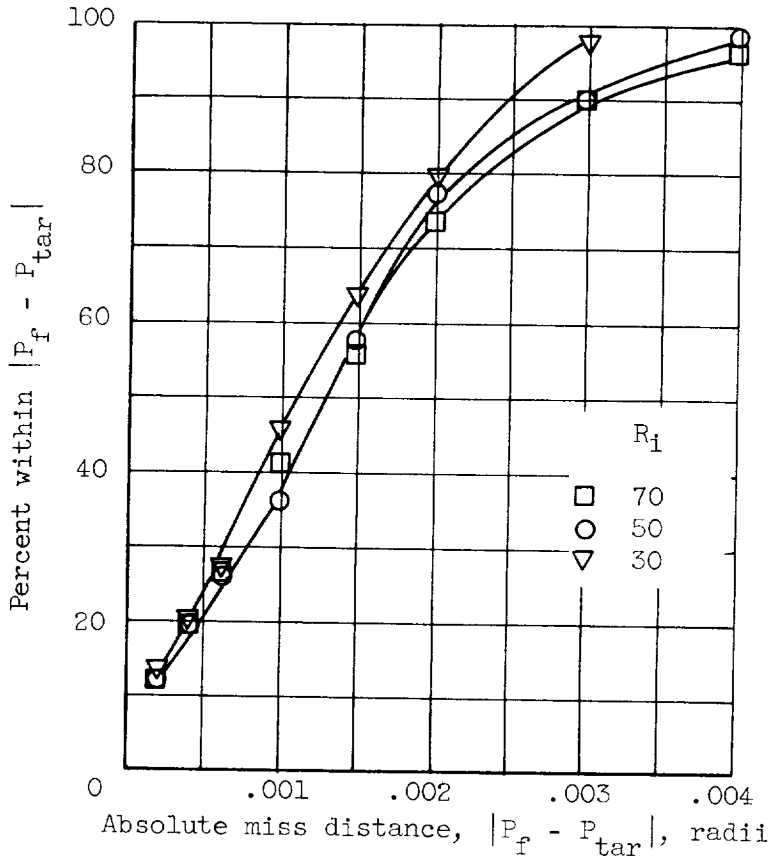
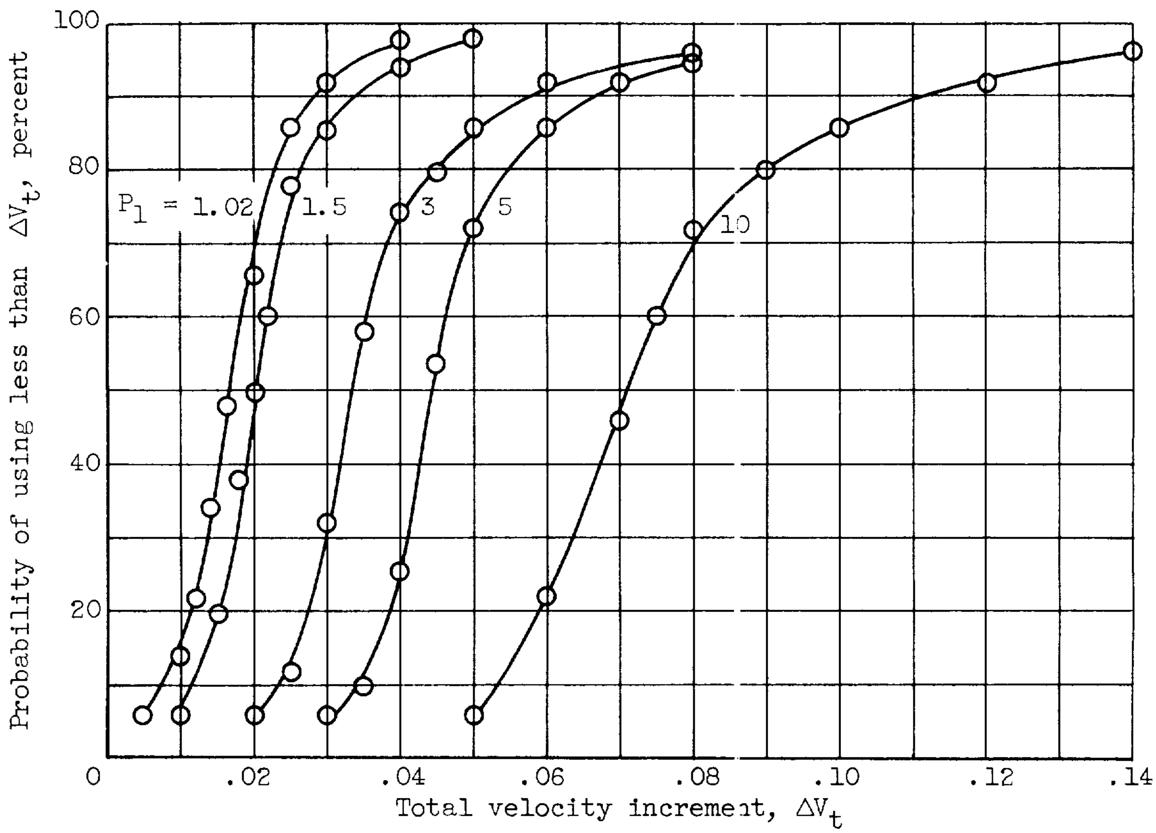
CE-8 back



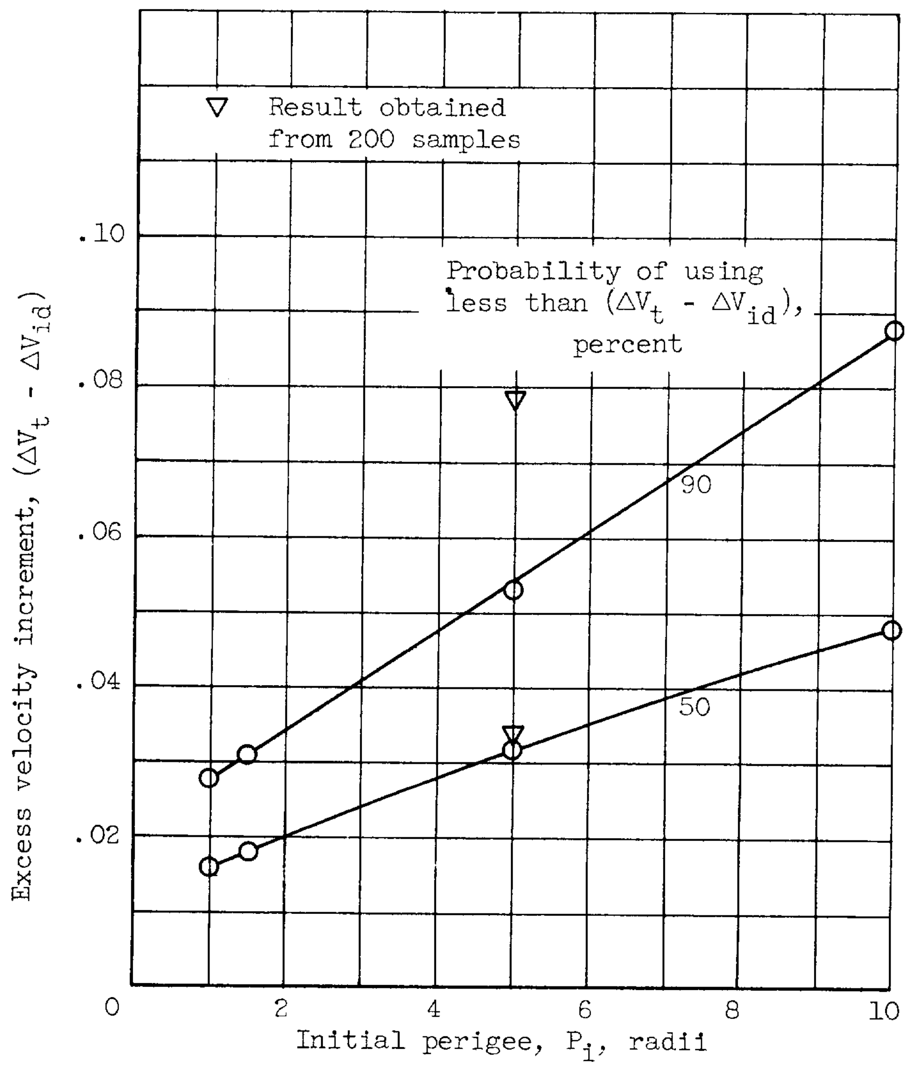
(b) Probability of hitting a given size target band.

Figure 8. - Concluded. Effects of guidance initiation. Assumed values, table I.



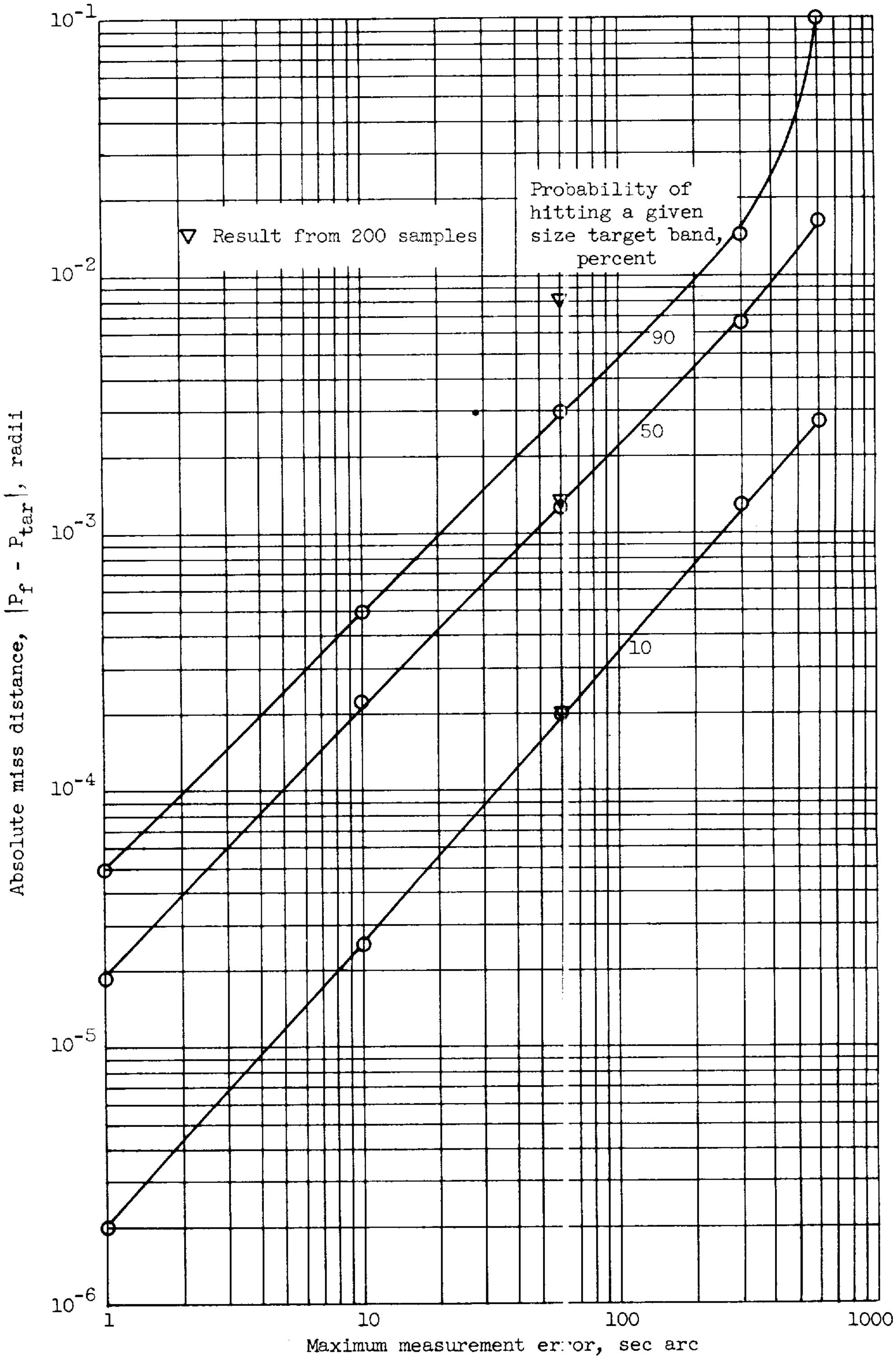
(a) Probability of total velocity-increment requirements.

Figure 9. - Effect of initial perigee. Assumed values, table I.



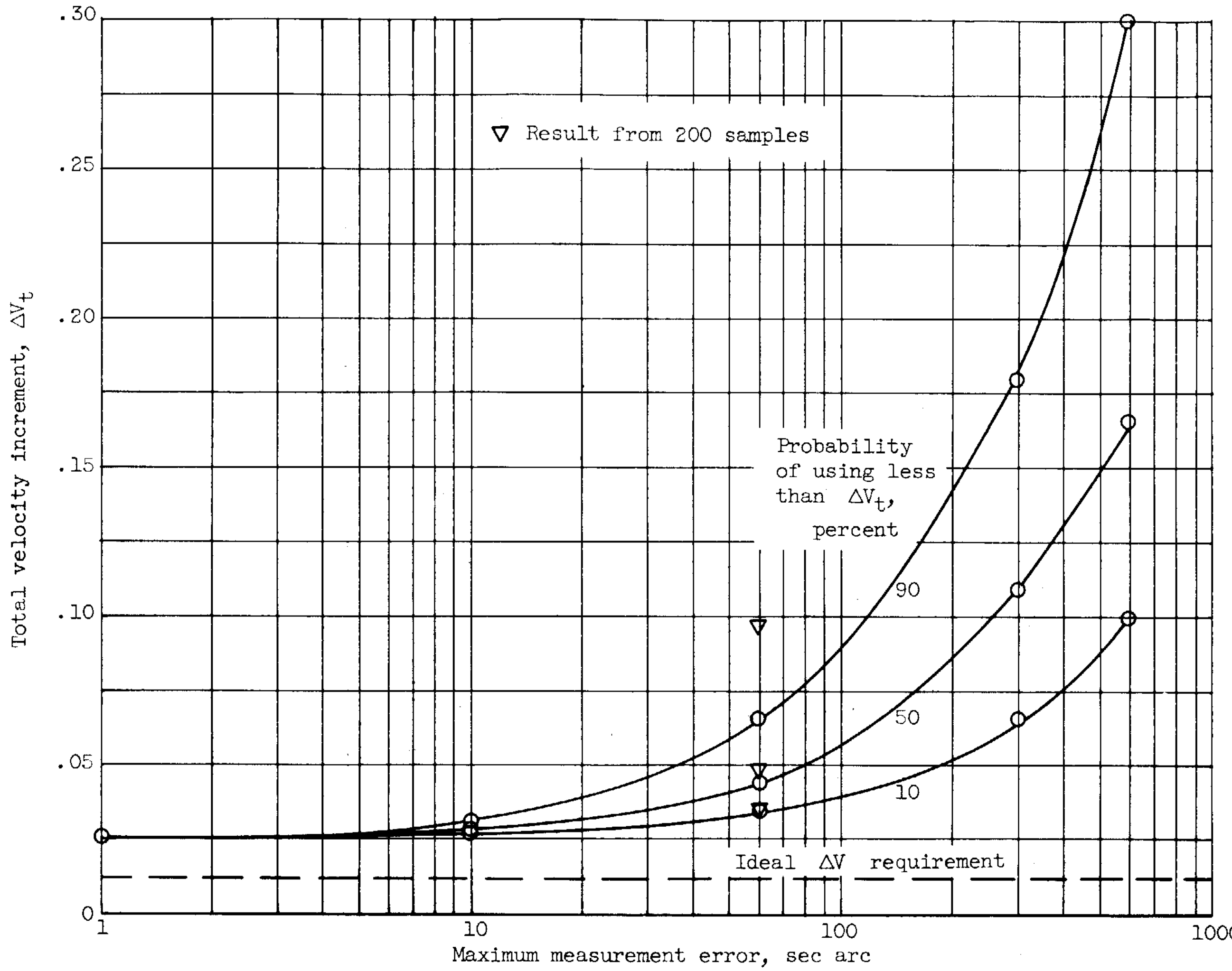
(b) Probability of excess velocity-increment requirements.

Figure 9. - Concluded. Effect of initial perigee. Assumed values, table I.



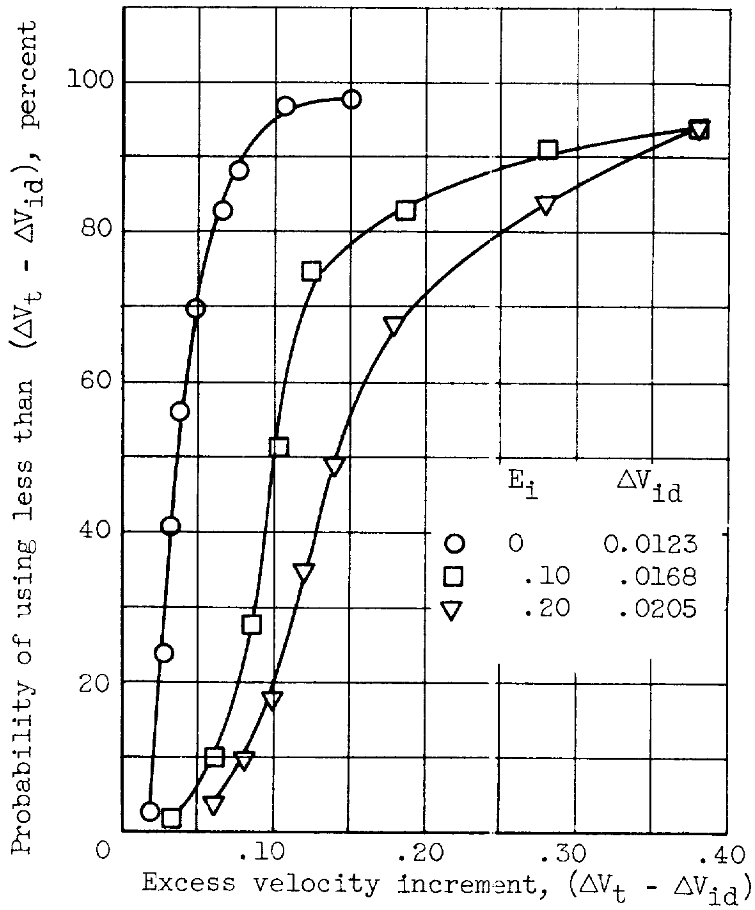
(a) Guidance accuracy.

Figure 10. - Effect of variation of measurement error size. Assumed values, table I.



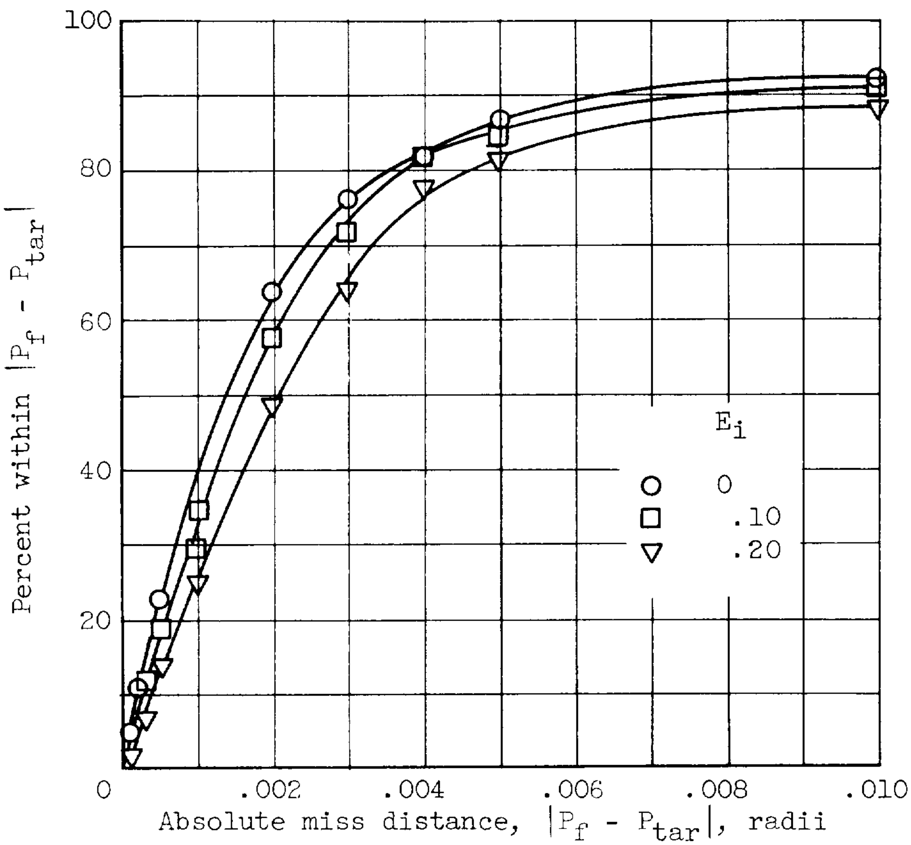
(b) Total velocity-increment requirements.

Figure 10. - Concluded. Effect of variation of measurement error size. Assumed values, table I.



(a) Probability of excess velocity-increment requirement.

Figure 11. - Effect of initial energy. Assumed values, table I.



(b) Probability of hitting a given size target band.

Figure 11. - Concluded. Effect of initial energy. Assumed values, table I.