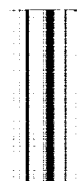# TECHNICAL REPORT R-45

EFFECT OF BODY PERTURBATIONS ON HYPERSONIC FLOW OVER SLENDER POWER LAW BODIES

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CONTENTS

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By Harold Mirels and Philip R. Thornton

SUMMARY

Hypersonic-slender-body theory, in the limit as the free-stream Mach number becomes infinite, is used to find the effect of slightly perturbing the surface of slender two-dimensional and axisymmetric power law bodies. The body perturbations are assumed to have a power law variation (with streamwise distance downstream of the nose of the body).

The perturbation equations formulated herein can be used for a variety of problems. In particular, the effect of boundary-layer development, very small angles of attack, and nose blunting can be found. Numerical results are presented for (1) the effect of boundary-layer development on the flow over two-dimensional and axisymmetric slender power law bodies, (2) the effect of very small angles of attack (on two-dimensional power law bodies), and (3) the effect of blunting the nose of very slender wedges and cones.

Differential equations for finding the effect of a power law lateral perturbation of the centerline of a slender power law body are formulated. No numerical results are given. Probably, the most important application of these equations is to determine the flow about axially symmetric power law bodies at very small angles of attack.

INTRODUCTION

Inviscid hypersonic flow over slender power law bodies was studied in references 1 to 3. These references assume $\delta^2 \ll 1$, $1/(M\delta)^2 \ll 1$ where M is the free-stream Mach number and δ is a characteristic streamline slope. In the limit $1/(M\delta)^2 = 0$, the shock shape and body shape are similar and the equations of motion can be reduced to a set of ordinary differential equations. Numerical solutions of these "zero-order" equations are tabulated

in references 1 and 2. Approximate analytical solutions are derived in reference 1. The first-order effect of small but nonvanishing values of $1/(M\delta)^2$ is also found in references 1 and 2 by expanding the equations of motion in terms of $1/(M\delta)^2$.

In the present paper attention is restricted to the limiting case $1/(M\delta)^2 = 0$. Two-dimensional and axisymmetric body shapes of the form [1] $r_b \sim x^m + \epsilon x^{N+m}$ are considered where r_b is the body ordinate, x is distance from the nose (in the free-stream direction), m, N, and ϵ are constants, and ϵ is small. For $\epsilon = 0$, the body is of the simple power law type considered in references 1 to 3. The flow corresponding to $\epsilon = 0$ is termed the "zero-order" flow herein and may be found from the zero-order solutions presented in references 1 and 2. For ϵ small, but not zero, the additional term ϵx^{N+m} represents a small power law perturbation of the zero-order body shape. The solution for the resulting flow field perturbations is termed the "perturbation solution" and is the subject of the present report. The equations developed herein can be used to find the effect of boundary-layer development, very small angles of attack, and nose blunting on slender power law bodies at hypersonic speeds. Numerical results are presented for a variety of cases.

It should be recalled (e.g., ref. 1) that the assumptions incorporated in hypersonic slender body theory are violated in the immediate vicinity of the nose ($x = 0$) of slender power law bodies (except for wedges and cones). Hence, the solutions found herein are not expected to be valid at the nose. However, they are expected to be valid downstream of this region.

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[1] A more specific representation of the body shapes considered herein is given by eqs. (13) and (G3a).

ANALYSIS

The equations of motion for hypersonic flow over slender bodies are utilized to obtain the zero-order and perturbation equations for hypersonic flow over slightly perturbed power law bodies. Expressions for shock shape, pressure distribution, and drag are then noted. Finally, an analytic solution of the perturbation equations is given for one class of perturbations. A numerical solution of the perturbation equations is required, in general. Symbols are listed in appendix A. Many of the details of the analysis are relegated to appendixes B to H.

HYPERSONIC SLENDER BODY THEORY

The equations of motion for hypersonic flow over slender bodies (e.g., ref. 4) are summarized in this section. The equations are applicable provided that $\delta^2 \ll 1$ and $1/(M\delta) \leq 0(1)$. The summary is the same as that given in reference 1 and is repeated here for convenience.

Dimensional variables are barred herein ($\bar{u}$, $\bar{v}$, $\bar{x}$, $\bar{r}$, etc.). See figure 1 for some of these quantities.
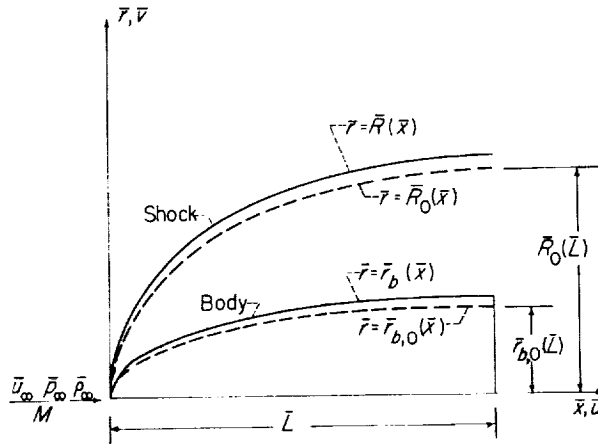

FIGURE 1. –Physical quantities for study of hypersonic flow over slightly perturbed power law bodies. Subscript zero refers to unperturbed power law body. Note, $\delta = \bar{R}_0(\bar{L})/\bar{L}$.

Let δ represent a characteristic body or streamline slope and $\bar{L}$ represent a characteristic streamwise length. Two-dimensional and axisymmetric flows are considered, with $(\bar{x},\bar{r})$ and $(\bar{u},\bar{v})$ being the streamwise and transverse coordinates and velocities, respectively. In order to obtain the hypersonic slender body equations of motion, the following nondimensional quantities are introduced (following ref. 4):

$$\left. \begin{aligned} x &= \bar{x}/\bar{L} & u &= (\bar{u}-\bar{u}_\infty)/\bar{u}_\infty \delta^2 & p &= \bar{p}/\gamma M^2 \delta^2 \bar{p}_\infty \\ r &= \bar{r}/\bar{L}\delta & v &= \bar{v}/\bar{u}_\infty \delta & \rho &= \bar{\rho}/\bar{\rho}_\infty \end{aligned} \right\} \quad (1)$$

The body shape and shock shape are denoted by $\bar{r}_b = \bar{r}_b(\bar{x})$ and $\bar{R} = \bar{R}(\bar{x})$, respectively, so that

$$r_b = \bar{r}_b/\bar{L}\delta \qquad R = \bar{R}/\bar{L}\delta \qquad (2)$$

If these quantities are introduced into the equations of motion, and terms of order δ^2 are neglected (compared with 1), the hypersonic slender body equations are obtained. These are (ref. 4):

Continuity:

$$\frac{\partial \rho}{\partial x} + \frac{\partial \rho v}{\partial r} + \sigma \frac{\rho v}{r} = 0 \qquad (3a)$$

r-Momentum:

$$\rho \left(\frac{\partial v}{\partial x} + v \frac{\partial v}{\partial r} \right) + \frac{\partial p}{\partial r} = 0 \qquad (3b)$$

Energy:

$$\frac{\partial (p/\rho^\gamma)}{\partial x} + v \frac{\partial (p/\rho^\gamma)}{\partial r} = 0 \qquad (3c)$$

The boundary conditions are:
At body surface:

$$v_b = \frac{dr_b}{dx} \qquad (4a)$$

Upstream of shock:

$$u_\infty = v_\infty = 0 \qquad (4b)$$

$$p_\infty = 1/(\gamma M^2 \delta^2) \qquad (4c)$$

$$\rho_\infty = 1 \qquad (4d)$$

Downstream side of shock:

$$v_s = \frac{2}{\gamma+1} \frac{dR}{dx} \left[1 - 1 \Big/ \left(\frac{dR}{dx} M\delta \right)^2 \right] \qquad (4e)$$

$$p_s = \frac{2}{\gamma+1} \left(\frac{dR}{dx} \right)^2 \left[1 - \left(\frac{\gamma-1}{2\gamma} \right) \Big/ \left(\frac{dR}{dx} M\delta \right)^2 \right] \qquad (4f)$$

$$\rho_s = \left(\frac{\gamma+1}{\gamma-1} \right) \Big/ \left[1 + \left(\frac{2}{\gamma-1} \right) \Big/ \left(\frac{dR}{dx} M\delta \right)^2 \right] \qquad (4g)$$

Here $\sigma = 0$ for two-dimensional flows and $\sigma = 1$ for axisymmetric flows. This system of equations can be solved independently of the x-momentum equation, and therefore the latter is neglected.

In the present report it is further assumed that $1/(M\delta)^2 \to 0$ so that equations (4c) and (4e) to (4g) become

$$p_\infty = 0 \tag{5a}$$

$$v_s = \frac{2}{\gamma+1} \frac{dR}{dx} \tag{5b}$$

$$p_s = \frac{2}{\gamma+1} \left(\frac{dR}{dx}\right)^2 \tag{5c}$$

$$\rho_s = \left(\frac{\gamma+1}{\gamma-1}\right) \tag{5d}$$

FLOW ABOUT SLIGHTLY PERTURBED POWER LAW BODIES

The hypersonic slender body equations of the previous section are now used to find the hypersonic flow over slightly perturbed slender power law bodies. The limiting case $1/(M\delta)^2 \to 0$ is assumed. Note that either the assumption that $\delta^2 \ll 1$ or that $1/(M\delta)^2 \to 0$ is violated at the nose $(x=0)$ of a slender power law body (except for wedges or cones) so that the solutions found herein are not expected to be valid in the immediate vicinity of the nose. However, they are expected to be valid downstream of this region.

The zero-order body shape and shock shape can be expressed as (from ref. 1)

$$r_{b,0}(\bar{x}) = \eta_{b,0} C \bar{x}^m \tag{6a}$$

$$\bar{R}_0(\bar{x}) = C \bar{x}^m \tag{6b}$$

where $\eta_{b,0}$, C, and m are constants. For the remainder of the report the characteristic length $\bar{L}$ is taken to be the streamwise length of the body, while the characteristic slope is taken to be the zero-order shock ordinate at $\bar{x} = \bar{L}$ divided by $\bar{L}$, or

$$\delta \equiv \frac{\bar{R}_0(\bar{L})}{\bar{L}} = \frac{\bar{r}_{b,0}(\bar{L})}{\bar{L}\eta_{b,0}} \tag{7}$$

Equations (6) can then be written, in nondimensional form,

$$r_{b,0} = \eta_{b,0} x^m \tag{8a}$$

$$R_0 = x^m \tag{8b}$$

The zero-order flow field is the same as that of references 1 and 2 and is considered known. The constant $\eta_{b,0}$, which is the ratio of the zero-order body ordinate to the zero-order shock ordinate, is denoted by the symbol η_b in reference 1 and is tabulated therein for various values of γ, σ, and m.

The present problem may be viewed as that of finding the flow field associated with shock shapes of the form

$$R = x^m(1 + \epsilon a_2 x^N) \tag{9}$$

where ϵ, a_2, and N are constants and ϵ is small. To effect a solution, new independent variables are introduced according to the relations

$$\left. \begin{array}{l} \xi = x \\ \eta = \dfrac{r}{R_0} = \dfrac{r}{x^m} \end{array} \right\} \tag{10}$$

so that

$$\left. \begin{array}{l} \dfrac{\partial}{\partial x} = \dfrac{\partial}{\partial \xi} - \dfrac{m\eta}{\xi} \dfrac{\partial}{\partial \eta} \\[2mm] \dfrac{\partial}{\partial r} = \dfrac{1}{\xi^m} \dfrac{\partial}{\partial \eta} \end{array} \right\} \tag{11}$$

These are the same independent variables used in reference 1. The zero-order shock location corresponds to $\eta = 1$ while the zero-order body location corresponds to $\eta = \eta_{b,0}$. In the new notation, equation (9) becomes

$$R = \xi^m(1 + \epsilon a_2 \xi^N) \tag{12a}$$

The boundary conditions (eqs. (5b) to (5d)) suggest the following forms for the dependent variables:

$$r = m\xi^{m-1}(\varphi_0 + \epsilon\xi^N \varphi_2) \tag{12b}$$

$$p = m^2 \xi^{2(m-1)}(F_0 + \epsilon\xi^N F_2) \tag{12c}$$

$$\rho = \psi_0 + \epsilon\xi^N \psi_2 \tag{12d}$$

where φ, F, and ψ are functions of η and the subscript 0 indicates the zero-order solution. The subscript 2 is used for the perturbation solution to avoid confusion with the first-order solution of reference 1.

The body shape, consistent with equations (12), is

$$r_b = x^m(\eta_{b,0} + \epsilon x^N) \qquad \beta < 1 \tag{13a}$$

$$r_b = x^m(\epsilon x^N)^{1/(\sigma+1)} \qquad \beta = 1 \tag{13b}$$

where

$$\beta \equiv \frac{2}{\sigma+1} \left(\frac{1}{m} - 1\right)$$

Equations (13) follow from equation (4a) and the asymptotic form of equation (12b) near $\eta_{b,0}$. (See appendix B for further discussion.)

For problems wherein the body is presented, equations (13) are used to determine ϵ and N. For these problems the constant a_2 in equation (12a) is initially unknown and is found as a consequence of the solution.

Substituting equations (12) into equations (3) and collecting terms of order ϵ^0 and ϵ^1 yield the zero-order and perturbation equations, respectively, which are summarized as follows:

Zero-order equations:

Continuity:

$$(\varphi_0-\eta)\psi_0'+\psi_0\varphi_0'+\sigma\,\frac{\psi_0\varphi_0}{\eta}=0 \qquad (14a)$$

η-Momentum:

$$(\varphi_0-\eta)\varphi_0'+\frac{F_0'}{\psi_0}-\binom{\sigma+1}{2}\beta\varphi_0=0 \qquad (14b)$$

Energy:

$$(\varphi_0-\eta)\left(\frac{F_0'}{F_0}-\gamma\,\frac{\psi_0'}{\psi_0}\right)-(\sigma+1)\beta=0 \qquad (14c)$$

where primes indicate differentiation with respect to η. The boundary conditions at $\eta=1$ are

$$\left.\begin{array}{l}\varphi_0(1)=F_0(1)=2/(\gamma+1)\\ \psi_0(1)=(\gamma+1)/(\gamma-1)\end{array}\right\} \qquad (15)$$

Equations (14) and (15) completely define the zero-order flow field. The body location is found from the tangency condition (eq. (4a)) which, for the zero-order solution, becomes

$$\varphi_0(\eta_{b,0})=\eta_{b,0} \qquad (16)$$

Equation (16) is, in fact, the basis for determining $\eta_{b,0}$. The zero-order solution is discussed in references 1 and 2.

Perturbation equations:

$$\frac{\varphi_2'}{\eta-\varphi_0}-\frac{\psi_2'}{\psi_0}+\left(\frac{\psi_0'}{\psi_0}+\frac{\sigma}{\eta}\right)\frac{\varphi_2}{\eta-\varphi_0}+\left[\frac{\frac{\sigma}{\eta}\varphi_0+\varphi_0'+(\sigma+1)\mu}{\eta-\varphi_0}\right]$$
$$\frac{\psi_2}{\psi_0}=0 \qquad (17a)$$

$$\frac{\varphi_2'}{\eta-\varphi_0}-\frac{1}{(\eta-\varphi_0)^2}\frac{F_0}{\psi_0}\frac{F_2'}{F_0}+\left[\frac{(\sigma+1)\left(\frac{\beta}{2}-\mu\right)-\varphi_0'}{\eta-\varphi_0}\right]\frac{\varphi_2}{\eta-\varphi_0}$$
$$+\left[\frac{F_0'}{(\eta-\varphi_0)^2\psi_0}\right]\frac{\psi_2}{\psi_0}=0 \qquad (17b)$$

$$\frac{F_2'}{F_0}-\gamma\frac{\psi_2'}{\psi_0}+\left(\gamma\frac{\psi_0'}{\psi_0}-\frac{F_0'}{F_0}\right)\frac{\varphi_2}{\eta-\varphi_0}+\gamma\left[\frac{\psi_0'}{\psi_0}+\frac{(\sigma+1)\mu}{\eta-\varphi_0}\right]\frac{\psi_2}{\psi_0}$$
$$-\left[\gamma\frac{\psi_0'}{\psi_0}+\frac{(\sigma+1)\,(\mu-\beta)}{\eta-\varphi_0}\right]\frac{F_2}{F_0}=0 \qquad (17c)$$

where

$$\mu\equiv\frac{1}{\sigma+1}\frac{N}{m}$$

The boundary conditions at $\eta=1$ are (eqs. (B5) and (B6))

$$\frac{\varphi_2(1)}{a_2}=\frac{2}{\gamma+1}\left[1+(\sigma+1)\mu\right]-\varphi_0'(1) \qquad (18a)$$

$$\frac{F_2(1)}{a_2}=\frac{4}{\gamma+1}\left[1+(\sigma+1)\mu\right]-F_0'(1) \qquad (18b)$$

$$\frac{\psi_2(1)}{a_2}=-\psi_0'(1) \qquad (18c)$$

where, from reference 1,

$$\varphi_0'=\frac{1}{(\gamma+1)^2}\left[3(\gamma+1)(\sigma+1)\beta-4\sigma\gamma\right]$$

$$F_0'(1)=\frac{2}{(\gamma-1)(\gamma+1)^2}\left[(2\gamma-1)(\gamma+1)(\sigma+1)\beta\right.$$
$$\left.-2\sigma\gamma(\gamma-1)\right]$$

$$\psi_0'(1)=\frac{1}{(\gamma-1)^2}\left[3(\gamma+1)(\sigma+1)\beta-2\sigma(\gamma-1)\right]$$

When a body shape is specified, the constant a_2 is initially unknown. Its value must be such that the tangent flow boundary condition at the body surface, namely (eqs. (B11) and (B16)),

$$\varphi_2(\eta_{b,0})=(\sigma+1)(\gamma+\gamma\mu-\beta)/\gamma \qquad \beta<1 \quad (19a)$$

$$\lim_{\eta\to0}(\eta^\sigma\varphi_2)=(\gamma+\gamma\mu-1)/\gamma \qquad \beta=1 \quad (19b)$$

is satisfied.

Since the unknown constant a_2 appears in the boundary conditions at $\eta=1$ (eqs. (18)), it is not possible numerically to integrate equations (17), starting at $\eta=1$. However, if equations (17) are each divided by a_2, the quantities φ_2/a_2, F_2/a_2, and ψ_2/a_2 can be considered as the dependent variables, and a numerical integration of equations (17), starting from $\eta=1$, is then possible (using the boundary conditions listed in eqs. (18)). The numerical integration cannot proceed all the way

to $\eta_{b,0}$ since the differential equations are singular at that point. Instead, the numerical integration is terminated at some point near $\eta_{b,0}$ for which the asymptotic solutions derived in appendix D are valid. If the values of φ_2/a_2, F_2/a_2, ψ_2/a_2, $\eta-\varphi_0$, ψ_0, F_0, and θ_0 (defined in ref. 1 and appendix D) are known at this point (from the numerical integration), equations (D7), (D12), or (D17) (depending on whether $0<\beta<1$, $\beta=0$, or $\beta=1$, respectively) can be solved [2] for the constants a_2, B, and D. (The constant a_2 is introduced into equations (D7), (D12), or (D17) by replacing φ_2, ψ_2, and F_2 by $a_2(\varphi_2/a_2)$, $a_2(\psi_2/a_2)$, and $a_2(F_2/a_2)$, respectively.) When B and D are known, equations (D7), (D12), and (D17) completely define the flow in the vicinity of $\eta_{b,0}$. When a_2 is known, the dependent variables of the numerical integrations can be converted back to φ_2, ψ_2, and F_2 and the solution is complete.

EXPRESSIONS FOR SHOCK SHAPE, PRESSURE DISTRIBUTION, AND DRAG

Expressions for shock shape, pressure distribution, and drag are summarized in the present section in terms of dimensional quantities. The cases $\beta<1$ and $\beta=1$ are treated separately.

Case $\beta<1$.—The zero-order body is of the form

$$\bar{r}_{b,0}(\bar{x}) = \eta_{b,0}C\bar{x}^m \qquad (20a)$$

which defines C for a specified body. For a power law perturbation

$$\frac{\bar{r}_b(\bar{x})}{\bar{r}_{b,0}(\bar{x})} = 1 + \frac{\epsilon}{\eta_{b,0}}\left(\frac{\bar{x}}{L}\right)^N \qquad (20b)$$

which defines ϵ for a specified body perturbation. The corresponding shock shape is

$$\frac{\bar{R}(\bar{x})}{\bar{r}_{b,0}(\bar{x})} = \frac{1}{\eta_{b,0}}\left[1 + \epsilon a_2\left(\frac{\bar{x}}{L}\right)^N\right] \qquad (21)$$

[2] If the leading term in B is the only term retained in the expression for F_2 and if the leading terms in A and B are the only terms retained in the expression for φ_2, there results

$$a_2 \approx \frac{(\sigma+1)(\gamma+\gamma\mu-\beta)/\gamma}{\frac{\varphi_2}{a_2}+\frac{\mu(\eta-\varphi_0)}{(\gamma-\beta)F_0}\left(\frac{F_2}{a_2}\right)} \qquad \beta<1$$

$$\approx \frac{(\gamma+\gamma\mu-1)/\gamma}{\eta^\sigma\left[\frac{\varphi_2}{a_2}+\frac{\mu(\eta-\varphi_0)}{(\gamma-1)F_0}\frac{F_2}{a_2}\right]} \qquad \beta=1$$

$$B \approx \frac{a_2}{(\gamma-\beta)F_0}\frac{F_2}{a_2} \qquad \beta\leq 1$$

for η near $\eta_{b,0}$.

where the zero-order shock shape is $\bar{R}_0(\bar{x}) = C\bar{x}^m = \bar{r}_{b,0}(\bar{x})/\eta_{b,0}$.

The local pressure coefficient, at any point, is (from eqs. (1), (5a), and (12c))

$$c_p \equiv \frac{\bar{p}-\bar{p}_\infty}{q} = 2\delta^2 m^2\left(\frac{\bar{x}}{L}\right)^{2(m-1)}\left[F_0+\epsilon F_2\left(\frac{\bar{x}}{L}\right)^N\right] \qquad (22a)$$

$$= 2\left(\frac{1}{\eta_{b,0}}\frac{d\bar{r}_{b,0}}{d\bar{x}}\right)^2\left[F_0+\epsilon F_2\left(\frac{\bar{x}}{L}\right)^N\right] \qquad (22b)$$

$$= 2\left(\frac{d\bar{R}_0}{d\bar{x}}\right)^2\left[F_0+\epsilon F_2\left(\frac{\bar{x}}{L}\right)^N\right] \qquad (22c)$$

In appendix C it is shown that

$$F_0(\eta_b) = F_0(\eta_{b,0})$$

$$F_2(\eta_b) = F_2(\eta_{b,0}) \qquad (23)$$

so that the pressure at $\eta_{b,0}$ is the same as that at η_b for a given $\bar{x}$. Using equations (22b) and (23) gives the pressure coefficient on the perturbed body:

$$\frac{c_{p,b}}{(d\bar{r}_{b,0}/d\bar{x})^2} = \frac{2F_0(\eta_{b,0})}{\eta_{b,0}^2}\left[1+\epsilon\frac{F_2(\eta_{b,0})}{F_0(\eta_{b,0})}\left(\frac{\bar{x}}{L}\right)^N\right] \qquad (24)$$

Alternative forms can also be deduced.

The forebody drag up to station $\bar{x}$ can be found by integrating the pressure distribution along the body surface. This is done in appendix E and the result is (from eq. (E4))

$$\frac{D(\bar{x})}{2\pi^\sigma m^3 q\delta^2[\bar{r}_{b,0}(L)]^{\sigma+1}} = \frac{2F_0(\eta_{b,0})}{m(\sigma+3)-2}\left\{\left(\frac{\bar{x}}{L}\right)^{m(\sigma+3)-2}\right.$$

$$+\frac{\epsilon[m(\sigma+3)-2]}{m(\sigma+3)-2+N}\left[\frac{F_2(\eta_{b,0})}{F_0(\eta_{b,0})}\right.$$

$$\left.+\frac{(\sigma+1)(1+\mu)}{\eta_{b,0}}\right]\left(\frac{\bar{x}}{L}\right)^{m(\sigma+3)-2+N}\right\} \qquad (25)$$

Equation (25) is valid provided that

$$m > 2/(\sigma+3) \qquad (26a)$$

$$N > 2-m(\sigma+3) \qquad (26b)$$

Equation (26a) corresponds to $\beta<1$, which is the case under consideration in the present section. For $\sigma=0$, equation (25) assumes that the body is symmetric about the $\bar{x}$-axis. The overall fore-

body drag coefficient, referenced to the cross-sectional area of the base of the zero-order body, is

$$C_D \equiv \frac{D(\bar{L})}{2^{1-\sigma}\pi^\sigma q[\bar{r}_{b,0}(\bar{L})]^{\sigma+1}}$$

$$= \frac{2^{\sigma+1} m^3 \delta^2 F_0(\eta_{b,0})}{m(\sigma+3)-2} \left\{ 1 + \frac{\epsilon[m(\sigma+3)-2]}{m(\sigma+3)-2+N} \right.$$

$$\left. \left[\frac{F_2(\eta_{b,0})}{F_0(\eta_{b,0})} + \frac{(\sigma+1)(1+\mu)}{\eta_{b,0}} \right] \right\} \quad (27)$$

When the perturbations are due to boundary-layer development on the zero-order body, the pressure drag on the zero-order body can be found from equations (25) and (27) by omitting the term $(\sigma+1)(1+\mu)/\eta_{b,0}$.

The drag can also be found from consideration of the energy of the transverse flow field (refs. 1 and 2). This is done in appendix E. The resulting expressions for drag (eqs. (E11) to (E17)) are more general than equations (25) and (27) since the former can be made to apply for all values of m and N.

Case $\beta=1$.—For $\beta=1$, the zero-order flow corresponds to flow over a flat plate ($\sigma=0$) or circular cylinder ($\sigma=1$) of semithickness, or radius, equal to $\bar{r}_N$. If the nose drag at $\bar{x}=0$ is known, and is denoted by D_N, the zero-order shock shape is given by (ref. 1)

$$\frac{\bar{R}_0(\bar{x})}{\bar{r}_N} = \frac{1}{2} \left[\frac{(\sigma+3)^2 C_{D_N}}{I} \right]^{1/(\sigma+3)} \left(\frac{\bar{x}}{\bar{r}_N} \right)^{2/(\sigma+3)} \quad (28)$$

where $C_{D_N} \equiv D_N / 2^{1-\sigma}\pi^\sigma q(\bar{r}_N)^{\sigma+1}$ is the nose drag coefficient and I is tabulated in reference 1 as a

(a)
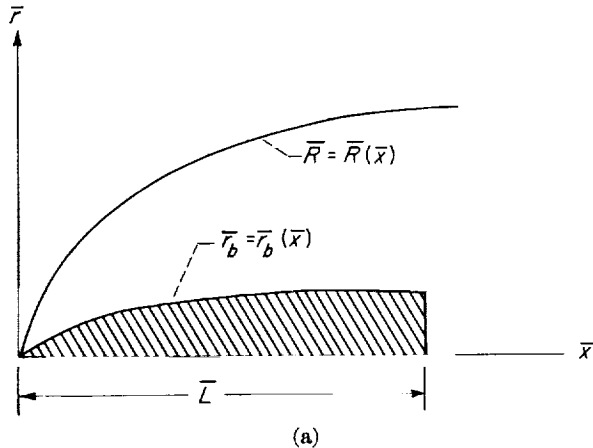
Perturbed constant-energy flow. (Flow due to drag impulse at $\bar{x}=0$ plus perturbations associated with $\bar{r}_b(\bar{x})$).

(b)
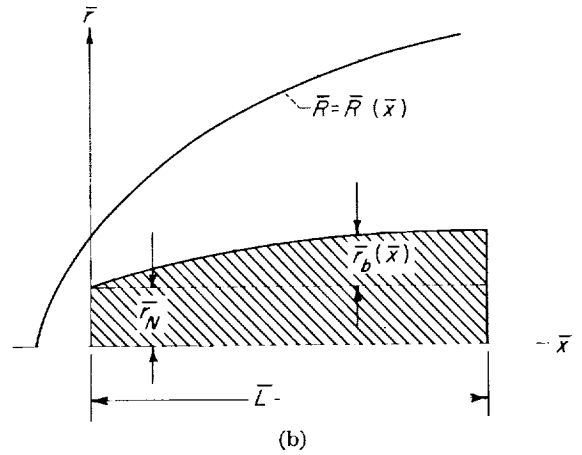
Equivalent physical flow.

function of σ and γ. The body shape can be expressed as (from eq. (13b))

$$\frac{\bar{r}_b(\bar{x})}{\bar{r}_N} = \frac{\bar{R}_0(\bar{x})}{\bar{r}_N} \left[\epsilon \left(\frac{\bar{r}_N}{\bar{L}} \frac{\bar{x}}{\bar{r}_N} \right)^N \right]^{1/(\sigma+1)} \quad (29)$$

which defines ϵ and N for a specified zero-order flow and body perturbation. The body ordinate $\bar{r}_b(\bar{x})$ is measured from the surface of the zero-order body (sketches (a) and (b)). Sketch (a) is the hypersonic-slender-body-theory idealization of the flow pictured in sketch (b). Hypersonic slender body theory gives a poor representation of the flow near $\bar{x}=0$, as previously noted.

From equation (29) it is seen that $\epsilon(\bar{x}/\bar{L})^N = [\bar{r}_b(\bar{x})/\bar{R}_0(\bar{x})]^{\sigma+1}$. The perturbed shock location can then be expressed as

$$\frac{\bar{R}(\bar{x})}{\bar{R}_0(\bar{x})} = 1 + a_2 \left(\frac{\bar{r}_b(\bar{x})}{\bar{R}_0(\bar{x})} \right)^{\sigma+1} \quad (30)$$

From equation (22c), the pressure coefficient on the body is

$$\frac{c_{p,b}}{(d\bar{R}_0/d\bar{x})^2} = 2F_0(0) \left[1 + \frac{F_2(0)}{F_0(0)} \left(\frac{\bar{r}_b(\bar{x})}{\bar{R}_0(\bar{x})} \right)^{\sigma+1} \right] \quad (31)$$

The forebody drag up to station $\bar{x}$ is, for $N>0$ (from eqs. (E7) and (28))

$$D(\bar{x}) = D_N \left[1 + \frac{(1+\mu)F_0(0)}{\mu(\sigma+1)I} \left(\frac{\bar{r}_b(\bar{x})}{\bar{R}_0(\bar{x})} \right)^{\sigma+1} \right] \quad (32)$$

The perturbation solution is valid provided $[\bar{r}_b(\bar{x})/\bar{R}_0(\bar{x})]^{\sigma+1} \ll 1$.

SOLUTION FOR $N=0$

When $N=0$, the solution of the perturbation equations can be found analytically in terms of the zero-order flow. The procedure is as follows.

For a shock shape given by

$$R = x^m(1+\epsilon a_2) \qquad (33a)$$

the perturbation equations and the boundary conditions at the shock are satisfied by

$$\frac{\varphi_2}{a_2} = \varphi_0 - \eta\varphi_0' \qquad (33b)$$

$$\frac{F_2}{a_2} = 2F_0 - \eta F_0' \qquad (33c)$$

$$\frac{\psi_2}{a_2} = -\eta\psi_0' \qquad (33d)$$

Note that $F_2(\eta_{b,0}) = 2a_2F_0(\eta_{b,0})$ since $F_0'(\eta_{b,0}) = 0$. For $\beta < 1$ the boundary condition on the body is satisfied if (eq. (B9))

$$\varphi_2(\eta_{b,0}) = 1 - \varphi_0'(\eta_{b,0}) \qquad (34)$$

Evaluating equation (33b) at $\eta_{b,0}$, noting $\varphi_0(\eta_{b,0}) = \eta_{b,0}$, and substituting into equation (34) then give

$$a_2 = 1/\eta_{b,0} \qquad \beta < 1 \qquad (35)$$

For $\beta = 1$, the shock perturbation $\Delta\bar{R}$ may be considered as due to a body perturbation $\Delta\bar{r}_N$. From equation (28)

$$\frac{\Delta\bar{R}}{\bar{R}} = \frac{\sigma+1}{\sigma+3}\frac{\Delta\bar{r}_N}{\bar{r}_N} \qquad (36)$$

But, from equation (33a), $\Delta R/R = \Delta\bar{R}/\bar{R} = \epsilon a_2$. Substitution into equation (36) then gives

$$\epsilon a_2 = \frac{\sigma+1}{\sigma+3}\frac{\Delta\bar{r}_N}{\bar{r}_N} \qquad \beta = 1 \qquad (37)$$

The present solution results from the fact that for $N=0$ the perturbed shock and body follow the same power law as does the zero-order flow. The resulting flow can in fact be treated as a zero-order problem. However, the solution is useful for making a partial check on calculating machine programs (when the latter are used to obtain numerical solutions of the perturbation equations) and for providing additional data when tabulating a_2 and $F_2(\eta_{b,0})$ as functions of μ, γ, σ, and β.

NUMERICAL RESULTS AND DISCUSSION

The equations of motion have been integrated numerically to determine the effect of (1) boundary-layer development, (2) very small angles of attack (for $\sigma=0$), and (3) blunting the nose of very slender wedges and cones. The results are discussed herein.

EFFECT OF BOUNDARY-LAYER DEVELOPMENT

The boundary-layer displacement thickness $\bar{\delta}^*$ on a slender power law body at hypersonic speeds is derived in appendix F. The result shows (recalling that the superscript bars have been omitted in eq. (F13))

$$\frac{\bar{\delta}^*}{\bar{L}} = \frac{\gamma-1}{\sqrt{2\gamma}}\sqrt{\frac{\omega}{R_{c,\bar{L}}}}\frac{M_\infty}{\delta\sqrt{F_0(\eta_{b,0})}}\frac{J_2\left(1+\dfrac{J_1+J_3}{J_2}\right)}{m\sqrt{2m(\sigma+1)-1}}\left(\frac{\bar{x}}{\bar{L}}\right)^{\frac{3}{2}-m} \qquad (38)$$

where $R_{c,\bar{L}} = \bar{\rho}_\infty\bar{u}_\infty\bar{L}/\bar{\mu}_\infty$, $\omega = \bar{T}_\infty\bar{\mu}_c/\bar{T}_c\bar{\mu}_\infty = $ constant, and $\bar{\mu}_\infty$ is free-stream viscosity. The quantities J_2 and $(J_1+J_3)/J_2$ can be found from table I for specified

$$\tilde{\beta} \equiv \frac{\gamma-1}{\gamma}\frac{2(1-m)}{2m(\sigma+1)-1} = \frac{\gamma-1}{\gamma}\frac{2(\sigma+1)\beta}{(\sigma+1)(4-\beta)-2} \qquad (39a)$$

$$g_w \equiv \frac{(h_{st})_w}{(h_{st})_\infty}$$

$$\equiv \frac{\text{stagnation enthalpy of fluid at wall}}{\text{stagnation enthalpy of fluid in free stream}} \qquad (39b)$$

For $0 \leq \beta \leq 1$ and $\sigma=0,1$, the quantity $\tilde{\beta}$ varies between the limits $0 \leq \tilde{\beta} \leq 2(\gamma-1)/\gamma$. Table I is based on the numerical results of references 6 to 8. The assumptions involved in the derivation of equations (38) and (39a) are noted in appendix F. Equation (38) is not valid for $\sigma=1$, $\beta=1$ as discussed in appendix F.

The effective body shape is $\bar{r}_b = \bar{r}_{b,0} + \bar{\delta}^*$. For $\bar{\delta}^*$ small compared with $\bar{r}_{b,0}$, the effective body shape can be expressed as

$$r_b = x^m\left(\eta_{b,0} + \epsilon x^{\frac{3}{2}-2m}\right) \qquad (40)$$

where ϵ is found from equation (38). Thus, for this case,

$$\mu = \frac{3}{4}\beta - \frac{1}{2(\sigma+1)} \qquad (41)$$

The perturbation equations have been integrated numerically for $\sigma=0,1$; $\gamma=1.15$, 1.4, 1.67; $\beta=0$, $\tfrac{1}{3}$, $\tfrac{1}{2}$, $\tfrac{3}{4}$, 1; and values of μ defined by equation (41). The case $\sigma=1$, $\beta=1$ is excluded. The results are given in table II.

For $m=\tfrac{3}{4}$ (i.e., $\mu=0$), the perturbed shock shape and body shape are similar, both following the $\tfrac{3}{4}$ power law variation of the unperturbed flow. This fact was used in references 9 and 10 to study boundary-layer development at hypersonic speeds. For this case, the perturbation solution is given by equations (33).

EFFECT OF VERY SMALL ANGLES OF ATTACK (FOR $\sigma=0$)

The effect of very small angles of attack on two-dimensional power law bodies at hypersonic speeds can also be found. This is done herein.

If a two-dimensional power law body is at angle of attack α, the equation of the upper surface becomes $\overline{r}_b=\eta_{b,0}C\overline{x}^m-\alpha\overline{x}$, or

$$r_b=x^m(\eta_{b,0}+\epsilon x^{1-m}) \tag{42}$$

where

$$\epsilon\equiv-\alpha/\delta$$

Assuming $\epsilon\ll1$, the resulting flow (in the upper half plane) can be found from the perturbation equations with

$$\mu=\beta/2 \tag{43}$$

Numerical solutions have been obtained and these are tabulated in table II. For $\mu=\beta=0$, equations (33) apply.

The lift per unit span can be found by noting that the perturbation solution is antisymmetric about the $\overline{x}$-axis. The local lift coefficient is then

$$\Delta c_{p,b}\equiv(c_{p,b})_{\text{bottom}}-(c_{p,b})_{\text{top}}$$

$$=-4\epsilon m^2\delta^2F_2(\eta_{b,0})x^{m-1}$$

The net lift per unit span $\hat{L}$ is (for $m\neq0$)

$$\hat{L}=q\overline{L}\int_0^1\Delta c_{p,b}\,dx$$

$$=-4\epsilon m\delta^2\overline{L}qF_2(\eta_{b,0}) \tag{44}$$

The lift coefficient, referenced to $\overline{L}$, is (recalling $\epsilon\equiv-\alpha/\delta$)

$$C_L\equiv\frac{\hat{L}}{\overline{L}q}=-4\epsilon m\delta^2F_2(\eta_{b,0})$$

$$=4m\delta F_2(\eta_{b,0})\alpha \tag{45}$$

The moment about the leading edge $\hat{M}$ is

$$\hat{M}=q\overline{L}^2\int_0^1 x(\Delta c_{p,b})\,dx$$

Integration yields

$$\frac{\hat{M}}{\overline{L}^2q}=4\frac{m^2}{1+m}\delta F_2(\eta_{b,0})\alpha \tag{46}$$

The center of pressure $\overline{x}_{c.p.}\equiv\hat{M}/\hat{L}$ is then given by

$$\frac{\overline{x}_{c.p.}}{\overline{L}}=\frac{m}{m+1} \tag{47}$$

The lift problem, for $\sigma=1$, is formulated in appendix G. It has also been treated in reference 11 using Newtonian theory.

EFFECT OF BLUNTING THE NOSE OF VERY SLENDER WEDGES AND CONES

The effect of blunting the nose of very slender wedges and cones is now considered. It is assumed that the wedge or cone is sufficiently slender so that the major contribution to drag is due to the blunt nose. The zero-order flow is then a constant energy ($\beta=1$) flow. The divergence of the body downstream of the nose induces a small perturbation in this zero-order flow.

Sketch (a) indicates the flow field considered, and sketch (b) indicates the corresponding physical flow. In the present example, the body shape is given by

$$\overline{r}_b=\delta_b\overline{x} \tag{48}$$

where $\overline{r}_b$ is measured from the surface of the zero-order body (sketch (b)) and δ_b is the semivertex angle of the wedge or cone. From equations (13b) and (48) it follows that

$$\mu=\frac{\sigma+1}{2} \tag{49}$$

for the present problem. The equations of motion have been integrated numerically for $\mu=(\sigma+1)/2$; $\beta=1$; $\sigma=0,1$; and $\gamma=1.15$, 1.4, 1.67. The results are given in table II. The shock shape, pressure distribution, and drag can be found from equations (30) to (32). The solution is valid provided that $(r_b/R_0)^{\sigma+1}\ll1$.

Define the following quantities:

$$\bar{\bar{R}} \equiv \left(\frac{2\delta_b^2}{C_{D_N}}\right)^{\frac{1}{\sigma+1}} \frac{\bar{R}}{\bar{r}_N} \tag{50a}$$

$$\bar{\bar{x}} \equiv \left(\frac{2\delta_b^2}{C_{D_N}}\right)^{\frac{1}{\sigma+1}} \frac{\delta_b \bar{x}}{\bar{r}_N} \tag{50b}$$

$$K \equiv \frac{1}{2} \left[\frac{2(\sigma+3)^2}{I}\right]^{\frac{1}{\sigma+3}} \tag{50c}$$

where K is a function of γ, σ. Equations (30) and (31) can then be written

$$\bar{\bar{R}} = K(\bar{\bar{x}})^{\frac{2}{\sigma+3}}\left\{1 + a_2\left[\frac{1}{K}(\bar{\bar{x}})^{\frac{\sigma+1}{\sigma+3}}\right]^{\sigma+1}\right\} \tag{51a}$$

$$\frac{c_{p,b}}{\delta_b^2} = 2F_0(0)\left[\frac{2K}{\sigma+3}(\bar{\bar{x}})^{-\frac{\sigma+1}{\sigma+3}}\right]^2$$
$$\left\{1 + \frac{F_2(0)}{F_0(0)}\left[\frac{1}{K}(\bar{\bar{x}})^{\frac{\sigma+1}{\sigma+3}}\right]^{\sigma+1}\right\} \tag{51b}$$

The quantities $\bar{\bar{R}}$ and $c_{p,b}/\delta_b^2$ are functions only of $\bar{\bar{x}}$ for a given σ, γ.

The problem of a blunted wedge or cone has also been treated by Chernyi in references 12 and 13. An account of his method is given in appendix II. The method is approximate, but becomes more exact as γ approaches 1. It has the advantage of not being restricted to small values of $\bar{\bar{x}}$ as is the perturbation analysis of the present report. Chernyi presents curves of shock shape and surface pressure distribution against $\bar{\bar{x}}$ for $\gamma=1.4$ and $\sigma=0,1$. Chernyi's equations are solved analytically for small $\bar{\bar{x}}$ in appendix II, and the resulting expressions for K, a_2, $F_0(0)$, and $F_2(0)$ are compared in table III with the numerical results of the present report. The agreement becomes poorer as γ departs from 1, particularly for $F_0(0)$ and a_2.

APPENDIX A

SYMBOLS

a_2 constant defining shock perturbation (eq. (9))

C constant (eqs. (6))

C_D drag coefficient (eq. (27))

C_{D_N} drag coefficient for impulsive drag addition at $\bar{x}=0$ (eq. (28))

c_p local pressure coefficient

D_N impulsive drag addition at $\bar{x}=0$

$D(\bar{x})$ forebody drag up to station $\bar{x}$

$F_0(\eta),F_2(\eta)$ pressure similarity variables (eq. (12c))

g a constant (eqs. (D6b), (D11b), (D16b)), or ratio of stagnation enthalpies (eq. (F3b))

g_w wall- to free-stream stagnation enthalpy ratio (eq. (39b))

h_{st} stagnation enthalpy

I function of σ,γ tabulated in ref. 1

J_1,J_2,J_3 quantities defining boundary-layer thickness (eq. (F9))

$\bar{L}$ streamwise length of body

M free-stream Mach number

m zero-order power law exponent (eq. (6))

N perturbation power law exponent defined by equation (9)

Pr Prandtl number

$\bar{p}$ pressure

q dynamic pressure, $\bar{\rho}_\infty \bar{u}_\infty^2/2$

$\bar{R}(\bar{x})$ lateral coordinate of shock

$\bar{r}$ lateral coordinate (normal to $\bar{x}$-axis)

$\bar{r}_b(\bar{x})$ lateral coordinate of body

$\bar{r}_N$ semithickness or radius (at $\bar{x}=0$) of blunt-nosed two-dimensional or axisymmetric body

$\bar{T}$ temperature

$\bar{u}$ velocity in $\bar{x}$-direction

$\bar{v}$ velocity in $\bar{z}$- or $\bar{r}$-directions

$\bar{w}$ velocity in θ-direction of cylindrical coordinate system

$\bar{x},\bar{y},\bar{z}$ Cartesian coordinates with $\bar{x}$ in stream direction and origin at nose of body

α angle of attack

β alternative zero-order shock shape parameter, $2\left(\dfrac{1}{m}-1\right)\Big/(\sigma+1)$

$\tilde{\beta}$ eq. (F6)

γ ratio of specific heats

δ characteristic slope, $\bar{R}_0(\bar{L})/\bar{L}$

$\bar{\delta}^*$ boundary-layer displacement thickness

ϵ small quantity, defined by eqs. (20b) and (29) for specified body perturbation

η lateral coordinate similarity variable, r/R_0 (eq. (10))

$\eta_{b,0}$ $\bar{r}_{b,0}/\bar{R}_0$, tabulated in ref. 1 as function of γ, σ, and β (denoted by η_b therein)

θ cylindrical coordinate (sketch (e), appendix G)

$\theta_0(\eta)$ zero-order stream function similarity variable (eq. (D1))

μ alternative perturbation power law exponent (eq. (17))

$\bar{\mu}_\infty$ free-stream viscosity

ξ x (eq. (10))

$\bar{\rho}$ density

σ 0,1 for zero-order flows that are two-dimensional or axisymmetric, respectively

$\varphi_0(\eta),\varphi_2(\eta)$ similarity variables for v (eq. (12b))

$\psi_0(\eta),\psi_2(\eta)$ similarity variables for ρ (eq. (12d))

$\Omega_2(\eta)$ similarity variables for w (eq. (G5))

Subscripts:

b quantity evaluated at perturbed body surface

$b,0$ quantity evaluated at zero-order body surface

s quantity evaluated at (or just downstream of) perturbed shock

0 zero-order solution

2 perturbation quantity

∞ undisturbed free-stream value

Superscripts:

$(\bar{})$ barred quantities are dimensional

() unbarred quantities are nondimensional (eqs. (1))

$()'$ primes indicate differentiation with respect to η

10

APPENDIX B

BOUNDARY CONDITIONS AT $\eta=1$ AND $\eta=\eta_{b,0}$ FOR PERTURBED FLOW

It is convenient to satisfy the perturbed flow boundary conditions at the zero-order shock location ($\eta=1$) and at the zero-order body location ($\eta=\eta_{b,0}$). The appropriate boundary conditions are found herein.

Boundary conditions at $\eta=1$:

(c)
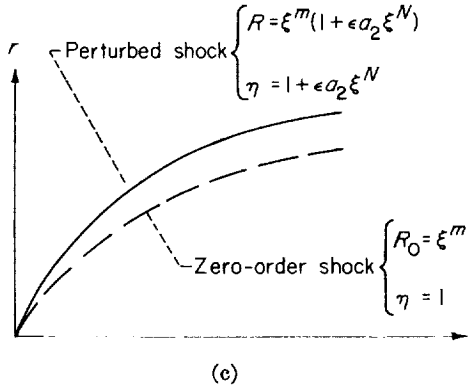
The zero-order shock location is at $\eta=1$, while the perturbed shock is at $\eta=1+\epsilon a_2\xi^N$, as indicated in sketch (c). The flow on the downstream side of the perturbed shock (denoted by subscript s) is given by (from eqs. (5b) to (5d) and (12a))

$$v_s=\frac{2}{\gamma+1}\, m\xi^{m-1}\left[1+\epsilon a_2\left(1+\frac{N}{m}\right)\xi^N\right] \quad \text{(B1a)}$$

$$p_s=\frac{2}{\gamma+1}\,(m\xi^{m-1})^2\left[1+2\epsilon a_2\left(1+\frac{N}{m}\right)\xi^N\right] \quad \text{(B1b)}$$

$$\rho_s=\frac{\gamma+1}{\gamma-1} \quad \text{(B1c)}$$

Let $Q=Q(\eta)$ be the variation, with η, of any flow quantity, at a fixed station ξ. Expanding in a Taylor series about $\eta=1$ then gives

$$Q(\eta)=Q(1)+Q'(1)(\eta-1)+\cdots \quad \text{(B2)}$$

If η is taken to be the perturbed shock location, $1+\epsilon a_2\xi^N$, equation (B2) becomes

$$Q_s=Q(1)+\epsilon a_2\xi^N\, Q'(1)+\cdots \quad \text{(B3)}$$

Consider Q to be the transverse velocity r. Equations (12b) and (B3) then yield

$$v_s=m\xi^{m-1}\{[(\varphi_0(1)+\epsilon\xi^N\varphi_2(1)]+\epsilon a_2\xi^N\varphi_0'(1)+\cdots\} \quad \text{(B4)}$$

Equating equations (B1a) and (B4) and collecting terms of order ϵ yield

$$\frac{\varphi_2(1)}{a_2}=\frac{2}{\gamma+1}\left(1+\frac{N}{m}\right)-\varphi_0'(1) \quad \text{(B5)}$$

Similarly, identifying Q with p and ρ yields, respectively,

$$\frac{F_2(1)}{a_2}=\frac{4}{\gamma+1}\left(1+\frac{N}{m}\right)-F_0'(1) \quad \text{(B6a)}$$

$$\frac{\psi_2(1)}{a_2}=-\psi_0'(1) \quad \text{(B6b)}$$

Equations (B5) and (B6) give the boundary conditions on φ_2, F_2, and ψ_2 at $\eta=1$ and appear in the body of the report as equations (18).

BOUNDARY CONDITIONS AT $\eta=\eta_{b,0}$

The boundary condition that the flow velocity be tangent to the body surface, $r_b=dr_b/d\xi$, is now considered. The cases $\beta<1$ and $\beta=1$ are treated separately.

Case $\beta<1$.—The zero-order body is located at $\eta=\eta_{b,0}$, while the perturbed body is located at $\eta_b=\eta_{b,0}+\epsilon\xi^N$ as indicated in sketch (d).

(d)
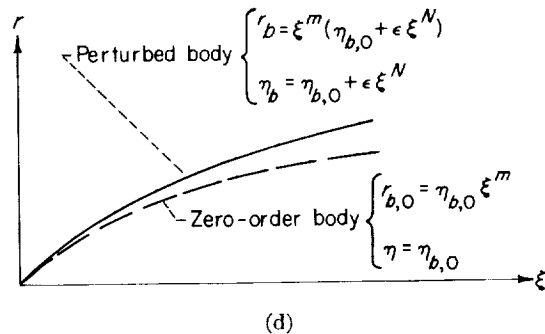
Expanding equation (12b) in a Taylor series about $\eta_{b,0}$, at constant ξ yields

$$v_b \equiv m\xi^{m-1}(\varphi_0 + \epsilon\xi^N\varphi_2)_{\eta=\eta_b} = m\xi^{m-1}\{[\varphi_0(\eta_{b,0})$$
$$+ \varphi_0'(\eta_{b,0})\epsilon\xi^N + \ldots] + \epsilon\xi^N\varphi_2(\eta_{b,0}) + \ldots\} \quad (B7)$$

But for $r_b = \xi^m(\eta_{b,0} + \epsilon\xi^N)$

$$\frac{dr_b}{d\xi} = m\xi^{m-1}\left[\eta_{b,0} + \epsilon\left(1 + \frac{N}{m}\right)\xi^N\right] \quad (B8)$$

Equating equations (B7) and (B8) and collecting terms of ϵ then give

$$\varphi_2(\eta_{b,0}) = 1 + \frac{N}{m} - \varphi_0'(\eta_{b,0}) \quad (B9)$$

For $\beta < 1$ (from ref. 1)

$$\varphi_0'(\eta_{b,0}) = 1 - \frac{\gamma-\beta}{\gamma}(\sigma+1) \quad (B10)$$

Substitution of equation (B10) into equation (B9) yields

$$\varphi_2(\eta_{b,0}) = (\sigma+1)(\gamma+\gamma\mu-\beta)/\gamma \quad (B11)$$

which appears as equation (19a) in the body of the report.

Case $\beta = 1$. For $\beta = 1$, the zero-order body shape is $r_{b,0} = 0$, and the perturbed body shape is taken to be of the form

$$\left.\begin{array}{l} r_b = \xi^m(\epsilon\xi^N)^{1/(\sigma+1)} \\ \eta_b = (\epsilon\xi^N)^{1/(\sigma+1)} \end{array}\right\} \quad (B12)$$

Expanding φ_0 in a Taylor series about $\eta_{b,0} = 0$

and noting $\epsilon\xi^N \equiv (\epsilon\xi^N)^{1/(\sigma+1)}\eta_b^\sigma$ permit v_b to be written

$$v_b \equiv m\xi^{m-1}(\varphi_0 + \epsilon\xi^N\varphi_2)_{\eta=\eta_b}$$
$$= m\xi^{m-1}\{[\varphi_0(0) + \varphi_0'(0)(\epsilon\xi^N)^{1/(\sigma+1)} + \ldots]$$
$$+ (\epsilon\xi^N)^{1/(\sigma+1)}[\eta_b^\sigma\varphi_2(\eta_b)]\} \quad (B13)$$

But

$$\frac{dr_b}{d\xi} = m\xi^{m-1}\left[\left(1 + \frac{1}{\sigma+1}\frac{N}{m}\right)(\epsilon\xi^N)^{1/(\sigma+1)}\right] \quad (B14)$$

Equating equations (B13) and (B14), noting $\varphi_0(0) = 0$, $\varphi_0'(0) = 1/\gamma$ (from ref. 1), and collecting terms of order $(\epsilon\xi^N)^{1/(\sigma+1)}$ then yield to this order

$$\eta_b^\sigma\varphi_2(\eta_b) = (\gamma+\gamma\mu-1)/\gamma \quad (B15)$$

From equation (D18) it is seen that equation (B15) can also be written

$$\lim_{\eta \to 0} (\eta^\sigma\varphi_2) = (\gamma+\gamma\mu-1)/\gamma \quad (B16)$$

which appears as equation (19b) in the body of the report.

It was necessary to take r_b of the form indicated by equations (B12) in order that equations (B13) and (B14) be consistent. Conversely, the required form for r_b could be deduced from equations (B13) and (B14). That is, let $r_b = \xi^m(\epsilon\xi^N)^k$ where k is an unknown constant. Using this expression in equations (B13) and (B14) and noting $\lim_{\eta \to 0} \eta^\sigma\varphi_2(\eta) =$ constant (from eq. (D18)) then show that these equations are consistent only when $k = 1/(\sigma+1)$

APPENDIX C

PRESSURE DISTRIBUTION IN VICINITY OF BODY SURFACE

The pressure distribution in the vicinity of the body surface is now considered.

The local pressure is given as a function of F_0 and F_2 by equation (12c). Expanding in a Taylor series about $\eta_{b,0}$ shows

$$F_0(\eta_b) + \epsilon \xi^N F_2(\eta_b) = [F_0(\eta_{b,0}) + F_0'(\eta_{b,0})(\eta_b - \eta_{b,0}) + \ldots] + \epsilon \xi^N [F_2(\eta_{b,0}) + \ldots] \quad \text{(C1)}$$

Noting $F_0'(\eta_{b,0}) = 0$, from reference 1, and collecting terms of order ϵ^0 and ϵ^1 yield

$$\left. \begin{array}{l} F_0(\eta_b) = F_0(\eta_{b,0}) \\ F_2(\eta_b) = F_2(\eta_{b,0}) \end{array} \right\} \quad \text{(C2)}$$

Thus, the pressure at η_b is the same as that at $\eta_{b,0}$ (for each value of ξ).

13

APPENDIX D

SPECIAL INTEGRALS AND ASYMPTOTIC SOLUTIONS

In the present section the perturbation equations of motion are written in a form wherein the dependent variables are $\varphi_2/(\eta-\varphi_0)$, F_2/F_0, and ψ_2/ψ_0. Some special integrals are then obtained. Finally, asymptotic solutions, valid near $\eta=\eta_{b,0}$, are found.

Following the procedure used to derive equations (55) in reference 1 allows equations (17) to be written, respectively,

$$\left(\frac{\varphi_2}{\eta-\varphi_0}\right)' - \left(\frac{\psi_2}{\psi_0}\right)' + \frac{\theta_0'}{\theta_0}\left(\frac{\varphi_2}{\eta-\varphi_0}\right) + \mu\frac{\theta_0'}{\theta_0}\left(\frac{\psi_2}{\psi_0}\right) = 0 \tag{D1a}$$

$$-(\sigma+1)\left(\frac{\theta_0}{\theta_0'}\right)^2\left(\frac{\varphi_2}{\eta-\varphi_0}\right)' + \frac{\eta^\sigma F_0}{\theta_0'}\left(\frac{F_2}{F_0}\right)'$$

$$+(\sigma+1)\frac{\theta_0}{\theta_0'}\left[\mu-\frac{\beta}{2}-\frac{1+2\sigma}{1+\sigma}+\frac{2\theta_0\theta_0''}{(\theta_0')^2}\right]\left(\frac{\varphi_2}{\eta-\varphi_0}\right)$$

$$+\eta^\sigma\frac{F_0'}{\theta_0'}\left(\frac{F_2}{F_0}-\frac{\psi_2}{\psi_0}\right) = 0 \tag{D1b}$$

$$\gamma\left(\frac{\varphi_2}{\eta-\varphi_0}\right)' - \left(\frac{F_2}{F_0}\right)' + (\gamma-\beta)\frac{\theta_0'}{\theta_0}\left(\frac{\varphi_2}{\eta-\varphi_0}\right) + \mu\frac{\theta_0'}{\theta_0}\left(\frac{F_2}{F_0}\right) = 0 \tag{D1c}$$

where θ_0 is a stream function similarity variable defined by (ref. 1)

$$\theta_0 \equiv exp\left[(\sigma+1)\int_\eta^1\frac{d\eta}{\varphi_0-\eta}\right]$$

(The zero-order quantities φ_0, F_0, and ψ_0 are expressed in terms of θ_0 in eqs. (40) of ref. 1.) Integration of equations (D1a) and (D1c) yields, respectively,

$$\frac{\psi_2}{\psi_0} = \frac{\varphi_2}{\eta-\varphi_0} + (1+\mu)\,\theta_0^\mu\int\frac{\varphi_2}{\eta-\varphi_0}\frac{d\theta_0}{\theta_0^{1+\mu}} + E_2\theta_0^\mu \tag{D2a}$$

and

$$\frac{F_2}{F_0} = \gamma\frac{\varphi_2}{\eta-\varphi_0} + (\gamma\mu+\gamma-\beta)\,\theta_0^\mu\int\frac{\varphi_2}{\eta-\varphi_0}\frac{d\theta_0}{\theta_0^{1+\mu}} + G_2\theta_0^\mu \tag{D2b}$$

where E_2 and G_2 are constants. The latter equations correspond to equations (56) in reference 1.

14

Elimination of the indefinite integral between equations (D2a) and (D2b) yields the following special integral (of the continuity and energy equations):

$$\beta\frac{\varphi_2}{\eta-\varphi_0} + (\gamma\mu+\gamma-\beta)\frac{\psi_2}{\psi_0} - (1+\mu)\frac{F_2}{F_0} = (\text{constant})\,\theta_0^\mu \tag{D3}$$

where the constant can be evaluated in terms of the boundary conditions at $\eta=1$. If $\mu=\beta$ (i.e., $N=2(1-m)$) and the subscript 2 is replaced by the subscript 1, the above equations become identical with the corresponding equations in reference 1.

Asymptotic solutions of equations (D1) valid near $\eta\approx\eta_{b,0}$ (i.e., $\theta_0\approx0$) will now be found using the procedure of appendix D in reference 1. Assume $\varphi_2/(\eta-\varphi_0)$ of the form

$$\frac{\varphi_2}{\eta-\varphi_0} = \theta_0^{\tilde{N}}[(\tilde{N}-\mu)L_2 + (\tilde{N}+P-\mu)M_2\theta_0^P + \dots] \tag{D4a}$$

where $\tilde{N}$, P, L_2, and M_2 are constants. The corresponding values of ψ_2 and F_2 are then (from eqs. (D2))

$$\frac{\psi_2}{\psi_0} = \theta_0^{\tilde{N}}[(\tilde{N}+1)L_2 + (\tilde{N}+P+1)M_2\theta_0^P + \dots] + E_2\theta_0^\mu \tag{D4b}$$

$$\frac{F_2}{F_0} = \theta_0^{\tilde{N}}\{(\gamma\tilde{N}+\gamma-\beta)L_2$$

$$+[\gamma(\tilde{N}+P+1)-\beta]M_2\theta_0^P + \dots\} + G_2\theta_0^\mu \tag{D4c}$$

Appropriate values of $\tilde{N}$ and P as well as the ratio M_2/L_2 can be found by substituting equations (D4) into the momentum equation (eq. (D1b)) and considering $\theta_0\approx0$. This will be done for $0<\beta<1$, $\beta=0$, and $\beta=1$, respectively. Since equations (D1) are equivalent to a third-order linear equation, three independent asymptotic solutions can be found.

CASE $0<\beta<1$

In the vicinity of $\theta_0\approx0$, equation (D1b) becomes

$$\frac{d(F_2/F_0)}{d\theta_0} + \frac{\beta\eta_{b,0}^{1-\sigma}}{2F_0(\eta_{b,0})}\left(\frac{F_2}{F_0}-\frac{\psi_2}{\psi_0}\right) \approx 0 \tag{D5}$$

Substitution of equations (D4) into equation (D5) yields the following three independent solutions:

(1) For $\tilde{N}=-\left(\dfrac{\gamma-\beta}{\gamma}\right)$, $P=1$, $E_2=G_2=0$, $L_2=-\gamma L_{2,1}$, $M_2=\gamma M_{2,1}$:

$$\left.\begin{aligned}
\frac{\varphi_{2,1}}{\eta-\varphi_0}&=\theta_0^{-\left(\frac{\gamma-\beta}{\gamma}\right)}\left[(\gamma+\gamma\mu-\beta)\,L_{2,1}\right.\\
&\left.\qquad+(\beta-\gamma\mu)\,M_{2,1}\theta_0+\ \ldots\ \right]\\
\frac{\psi_{2,1}}{\psi_0}&=\theta_0^{-\left(\frac{\gamma-\beta}{\gamma}\right)}\left[-\beta L_{2,1}+(\gamma+\beta)\,M_{2,1}\theta_0\right.\\
&\left.\qquad\qquad\qquad\qquad+\ \ldots\ \right]\\
\frac{F_{2,1}}{F_0}&=\theta_0^{-\left(\frac{\gamma-\beta}{\gamma}\right)}(0+\gamma^2 M_{2,1}\theta_0+\ \ldots)\\
\frac{M_{2,1}}{L_{2,1}}&=\frac{-\beta\eta_{b,0}^{1-\sigma}}{2\gamma F_0(\eta_{b,0})}\\
&\equiv f
\end{aligned}\right\}\quad\text{(D6a)}$$

(2) For $\tilde{N}=0$, $P=1$, $E_2=G_2=0$, $L_2=-L_{2,2}$:

$$\left.\begin{aligned}
\frac{\varphi_{2,2}}{\eta-\varphi_0}&=\mu L_{2,2}+(1-\mu)\,M_{2,2}\theta_0+\ \ldots\\
\frac{\psi_{2,2}}{\psi_0}&=-L_{2,2}+2M_{2,2}\theta_0+\ \ldots\\
\frac{F_{2,2}}{F_0}&=-(\gamma-\beta)\,L_{2,2}+(2\gamma-\beta)\,M_{2,2}\theta_0+\ \ldots\\
\frac{M_{2,2}}{L_{2,2}}&=\frac{(\gamma-\beta-1)\,\beta\eta_{b,0}^{1-\sigma}}{2(2\gamma-\beta)\,F_0(\eta_{b,0})}\\
&\equiv g
\end{aligned}\right\}$$

$$\text{(D6b)}$$

(3) For $\tilde{N}=\mu$, $P=1$, $G_2=0$, $L_2=0$:

$$\left.\begin{aligned}
\frac{\varphi_{2,3}}{\eta-\varphi_0}&=\theta_0^\mu(0+M_{2,3}\theta_0+\ \ldots)\\
\frac{\psi_{2,3}}{\psi_0}&=\theta_0^\mu[E_{2,3}+(2+\mu)M_{2,3}\theta_0+\ \ldots]\\
\frac{F_{2,3}}{F_0}&=\theta_0^\mu\{0+[\gamma(2+\mu)-\beta]M_{2,3}\theta_0+\ \ldots\}\\
\frac{M_{2,3}}{E_2}&=\frac{\beta\eta_{b,0}^{1-\sigma}}{2(1+\mu)[\gamma(\mu+2)-\beta]F_0(\eta_{b,0})}\\
&\equiv h
\end{aligned}\right\}\text{(D6c)}$$

Equations (D6) can be linearly added to find the behavior of $\varphi_2/(\eta-\varphi_0)$, ψ_2/ψ_0, and F_2/F_0 near $\theta_0\approx 0$. Replacing the constants $L_{2,1}$, $L_{2,2}$, and E_2 by A, $-B$, and D, respectively, then yields

$$\frac{\varphi_2}{\eta-\varphi_0}=A\theta_0^{-\left(\frac{\gamma-\beta}{\gamma}\right)}[(\gamma+\gamma\mu-\beta)+(\beta-\gamma\mu)f\theta_0]$$
$$+B[-\mu-(1-\mu)g\theta_0]+D\theta_0^\mu(h\theta_0)+\ \ldots\quad\text{(D7a)}$$

$$\frac{\psi_2}{\psi_0}=A\theta_0^{-\left(\frac{\gamma-\beta}{\gamma}\right)}[-\beta+(\gamma+\beta)f\theta_0]+B(1-2g\theta_0)$$
$$+D\theta_0^\mu[1+(2+\mu)h\theta_0]+\ \ldots\quad\text{(D7b)}$$

$$\frac{F_2}{F_0}=A\theta_0^{-\left(\frac{\gamma-\beta}{\gamma}\right)}(\gamma^2 f\theta_0)+B[(\gamma-\beta)-(2\gamma-\beta)g\theta_0]$$
$$+D\theta_0^\mu[\gamma(2+\mu)-\beta]h\theta_0+\ \ldots\quad\text{(D7c)}$$

The constant A can be found by satisfying equation (B11). This is done as follows. Recall from reference 1 that, near $\theta_0\approx 0$,

$$\left.\begin{aligned}
\theta_0\approx&\left\{\left(\frac{\gamma+1}{\gamma-1}\frac{\gamma-\beta}{\gamma}\right)\right.\\
&\left.\left[\frac{\gamma+1}{2}F_0(\eta_{b,0})\right]^{1/\gamma}(\eta^{\sigma+1}-\eta_{b,0}^{\sigma+1})\right\}^{\frac{\gamma}{\gamma-\beta}}\\
&\eta-\varphi_0=(\sigma+1)\frac{\theta_0}{\theta_0'}
\end{aligned}\right\}\ \text{(D8)}$$

Evaluating equation (D7a) at $\eta_{b,0}$ and applying equation (B11) then yield

$$A=\frac{\gamma+1}{\gamma-1}\frac{\sigma+1}{\gamma}\left[\frac{\gamma+1}{2}F_0(\eta_{b,0})\right]^{1/\gamma}\eta_{b,0}^\sigma\quad\text{(D9)}$$

for the present case.

CASE $\beta=0$

The momentum equation becomes, near $\theta_0\approx 0$,

$$-(\sigma+1)\left(\frac{\theta_0^2}{\theta_0'}\right)\frac{d\,[\varphi_2/'\eta-\varphi_0)]}{d\theta_0}+F_0\eta^\sigma\,\frac{d\,(F_2/F_0)}{d\theta}$$
$$+\frac{\theta_0}{\theta_0'}[(\sigma+1)\mu-(1+2\sigma)]\frac{\varphi_2}{\eta-\varphi_0}$$
$$+\sigma\frac{\theta_0}{\theta_0'}\left(\frac{\psi_2}{\psi_0}-\frac{F_2}{F_0}\right)\approx 0\quad\text{(D10)}$$

where

$$\theta_0'\approx(\sigma+1)\eta^\sigma\,\frac{\gamma+1}{\gamma-1}\left[\frac{\gamma+1}{2}F_0(\eta_{b,0})\right]^{1/\gamma}$$

The following three independent solutions result:

(1) For $\tilde{N}=-1$, $P=2$, $G_2=E_2=0$,
$$-(1+\mu)L_2=L_{2,1}:$$

$$\frac{\varphi_{2,1}}{\eta-\varphi_0}=\theta_0^{-1}[L_{2,1}+(1-\mu)M_{2,1}\theta_0^2+\ldots]$$

$$\frac{\psi_{2,1}}{\psi_0}=\theta_0^{-1}(0+2M_{2,1}\theta_0^2+\ldots)$$

$$\frac{F_{2,1}}{F_0}=\theta_0^{-1}(0+2\gamma M_{2,1}\theta_0^2+\ldots)$$

$$\frac{M_{2,1}}{L_{2,1}}=-\frac{(\gamma-1)[\sigma-(\sigma+1)\mu]}{4\gamma(\sigma+1)\eta_{b,0}^{2\sigma}\left[\frac{\gamma+1}{2}F_0(\eta_{b,0})\right]^{\frac{\gamma+1}{\gamma}}}$$

$$\equiv f$$

(D11a)

(2) For $\tilde{N}=0$, $P=2$, $E_2=G_2=0$:

$$\frac{\varphi_{2,2}}{\eta-\varphi_0}=-\mu L_{2,2}+(2-\mu)M_{2,2}\theta_0^2+\ldots$$

$$\frac{\psi_{2,2}}{\psi_0}=L_{2,2}+3M_{2,2}\theta_0^2+\ldots$$

$$\frac{F_{2,2}}{\psi_0}=\gamma L_{2,2}+3\gamma M_{2,2}\theta_0^2+\ldots$$

$$\frac{M_{2,2}}{L_{2,2}}=\frac{(\gamma-1)[(\sigma+1)\mu^2-(1+2\sigma)\mu+\sigma(\gamma-1)]}{12\gamma(\sigma+1)\eta_{b,0}^{2\sigma}\left[\frac{\gamma+1}{2}F_0(\eta_{b,0})\right]^{\frac{\gamma+1}{\gamma}}}$$

$$\equiv g$$

(D11b)

(3) For $\tilde{N}=\mu$, $P=2$, $G_2=L_2=0$:

$$\frac{\varphi_{2,3}}{\eta-\varphi_0}=\theta_0^\mu(0+2M_{2,3}\theta_0^2+\cdots)$$

$$\frac{\psi_{2,3}}{\psi_0}=\theta_0^\mu[E_2+(3+\mu)M_{2,3}\theta_0^2+\cdots]$$

$$\frac{F_{2,3}}{F_0}=\theta_0^\mu[0+\gamma(3+\mu)M_{2,3}\theta_0^2+\cdots]$$

$$\frac{M_{2,3}}{E_2}=\frac{-\sigma(\gamma-1)}{2\gamma(\sigma+1)(3+\mu)(2+\mu)\eta_{b,0}^{2\sigma}\left[\frac{\gamma+1}{2}F_0(\eta_b)\right]^{\frac{\gamma+1}{\gamma}}}$$

$$\equiv h$$

(D11c)

Replacing the constants $L_{2,1}$, $L_{2,2}$, and E_2 by A, B, and D, respectively, and linearly adding equations (D11) then yield

$$\frac{\varphi_2}{\eta-\varphi_0}=A\theta_0^{-1}[1+(1-\mu)f\theta_0^2]+B[-\mu+(2-\mu)g\theta_0^2]$$
$$+D\theta_0^\mu(2h\theta_0^2)+\cdots\quad\text{(D12a)}$$

$$\frac{\psi_2}{\psi_0}=A\theta_0^{-1}(2f\theta_0^2)+B(1+3g\theta_0^2)$$
$$+D\theta_0^\mu[1+(3+\mu)h\theta_0^2]+\ldots\quad\text{(D12b)}$$

$$\frac{F_2}{F_0}=A\theta_0^{-1}(2\gamma f\theta_0^2)+B(\gamma+3\gamma g\theta_0^2)$$
$$+D\theta_0^\mu[\gamma(3+\mu)h\theta_0^2]+\ldots\quad\text{(D12c)}$$

which defines the behavior of the dependent variables near $\theta_0\approx0$. From equations (B11), (D8), and (D12a), the value of A is

$$A=\frac{\gamma+1}{\gamma-1}(\sigma+1)(1+\mu)\left[\frac{\gamma+1}{2}F_0(\eta_{b,0})\right]^{1/\gamma}\eta_{b,0}^\sigma\quad\text{(D13)}$$

for the present case.

<div align="center">CASE $\beta=1$</div>

For this case, $\eta_{b,0}=0$ and (for $\theta_0\approx0$)

$$\frac{\theta_0}{\theta_0'\eta}\approx\frac{(\gamma-1)}{\gamma(\sigma+1)}$$

$$\frac{\theta_0''\theta_0}{(\theta_0')^2}\approx\frac{\gamma\sigma+1}{\gamma(\sigma+1)}$$

$$\eta^{1-\sigma}\theta_0\approx K_0\eta^{\frac{2\gamma+\sigma-1}{\gamma-1}}\approx K_0^{\frac{(\gamma-1)(\sigma-1)}{\gamma(\sigma+1)}}\theta_0^{\frac{2\gamma+\sigma-1}{\gamma(\sigma+1)}}$$

$$K_0\equiv\left(\frac{\gamma+1}{\gamma}\right)^{\frac{\gamma}{\gamma-1}}\left[\frac{\gamma+1}{2}F_0(0)\right]^{\frac{1}{\gamma-1}}$$

(D14)

The momentum equation can then be written

$$-\left(\frac{\gamma-1}{\gamma}\right)\theta_0\eta^{1-\sigma}\frac{d[\varphi_2/(\eta-\varphi_0)]}{d\theta_0}+F_0\frac{d(F_2/F_0)}{d\theta_0}$$

$$+(\gamma-1)\frac{[2\gamma(\sigma+1)\mu-3\gamma-\gamma\sigma+4]\eta^{1-\sigma}}{2\gamma^2(\sigma+1)}\frac{\varphi_2}{\eta-\varphi_0}$$

$$+\frac{3\gamma+\gamma\sigma-2}{2\gamma^2(\sigma+1)}\eta^{1-\sigma}\left(\frac{F_2}{F_0}-\frac{\psi_2}{\psi_0}\right)\approx0\quad\text{(D15)}$$

The resulting three independent asymptotic solutions are

(1) For $\tilde{N}=-\left(\dfrac{\gamma-1}{\gamma}\right)$, $P=\dfrac{2\gamma+\sigma-1}{\gamma(\sigma+1)}$, $G_2=E_2=0$, $L_2=-\gamma L_{2,1}$, $M_2=-\gamma M_{2,1}$:

$$\frac{\varphi_{2,1}}{\eta-\varphi_0}=\theta_0^{-\frac{\gamma-1}{\gamma}}\{[(\gamma-1)+\gamma\mu]L_{2,1}$$
$$+[(\gamma-1)+\gamma(\mu-P)]M_{2,1}\theta_0^P+ \ldots \}$$

$$\frac{\psi_{2,1}}{\psi_0}=\theta_0^{-\frac{\gamma-1}{\gamma}}[-L_{2,1}-(\gamma P+1)M_{2,1}\theta_0^P+ \ldots]$$

$$\frac{F_{2,1}}{F_0}=\theta_0^{-\frac{\gamma-1}{\gamma}}(0-\gamma^2 P M_{2,1}\theta_C^P+ \ldots)$$

$$\frac{M_{2,1}}{L_{2,1}}=\frac{[(\gamma-1)+\gamma\mu]K_0^{\frac{(\gamma-1)(\sigma-1)}{\gamma(\sigma+1)}}}{\gamma P[\gamma P-(\gamma-1)]F_0(0)}\left\{\left(\frac{\gamma-1}{\gamma}\right)^2\right.$$
$$+\frac{1}{2\gamma^2(\sigma+1)}\left[(\gamma-1)[2\gamma(\sigma+1)\mu\right.$$
$$\left.\left.-3\gamma-\gamma\sigma+4]+\frac{3\gamma+\gamma\sigma-2}{(\gamma-1)+\gamma\mu}\right]\right\}$$
$$\equiv f$$

(D16a)

(2) For $\tilde{N}=0$, $P=\dfrac{2\gamma+\sigma-1}{\gamma(\sigma+1)}$, $E_2=G_2=0$:

$$\frac{\varphi_{2,2}}{\eta-\varphi_0}=-\mu L_{2,2}+(P-\mu)M_{2,2}\theta_0^P+ \ldots$$

$$\frac{\psi_{2,2}}{\psi_0}=L_{2,2}+(P+1)M_{2,2}\theta_0^P+ \ldots$$

$$\frac{F_{2,2}}{F_0}=(\gamma-1)L_{2,2}+[\gamma(P+1)-1]M_{2,2}\theta_0^P+ \ldots$$

$$\frac{M_{2,2}}{L_{2,2}}=$$
$$\frac{\{\mu(\gamma-1)[2\gamma(\sigma+1)\mu-3\gamma-\sigma\gamma+4]\}K_0^{\frac{(\gamma-1)(\sigma-1)}{\gamma(\sigma+1)}}}{2\gamma^2(\sigma+1)P[\gamma(P+1)-1]F_0(0)}$$
$$+\frac{[(2-\gamma)(3\gamma+\gamma\sigma-2)]K_0^{\frac{(\gamma-1)(\sigma-)}{\gamma(\sigma+1)}}}{2\gamma^2(\sigma+1)P[\gamma(P+1)-1]F_0(0)}$$
$$\equiv g$$

(D16b)

(3) For $\tilde{N}=\mu$, $P=\dfrac{2\gamma+\sigma-1}{\gamma(\sigma+1)}$, $G_2=L_2=0$:

$$\frac{\varphi_{2,3}}{\eta-\varphi_0}=\theta_0^\mu(0+P M_{2,3}\theta_0^P+ \ldots)$$

$$\frac{\psi_{2,3}}{\psi_0}=\theta_0^\mu[E_2+(\mu+P+1)M_{2,3}\theta_0^P+ \ldots]$$

$$\frac{F_{2,3}}{F_0}=\theta_0^\mu\{0+[\gamma(\mu+P+1)-1]M_{2,3}\theta_0^P+ \ldots\}$$

$$\frac{M_{2,3}}{E_2}=\frac{(3\gamma+\gamma\sigma-2)K_0^{\frac{(\gamma-1)(\sigma-1)}{\gamma(\sigma+1)}}}{2\gamma^2(\sigma+1)(P+\mu)[\gamma(\mu+P+1)-1]F_0(0)}$$
$$\equiv h$$

(D16c)

Replacing $L_{2,1}$, $L_{2,2}$, and E_2 by A, B, and D, respectively, gives

$$\frac{\varphi_2}{\eta-\varphi_0}=A\theta_0^{-\left(\frac{\gamma-1}{\gamma}\right)}\{(\gamma-1+\gamma\mu)+[\gamma-1$$
$$+\gamma(\mu-P)]f\theta_0^P\}+B[-\mu+(P-\mu)g\theta_0^P]$$
$$+D\theta_0^\mu(Ph\theta_0^P)+ \ldots \quad \text{(D17a)}$$

$$\frac{\psi_2}{\psi_0}=A\theta_0^{-\left(\frac{\gamma-1}{\gamma}\right)}[-1-(\gamma P+1)f\theta_0^P]+B[1+(P+1)g\theta_0^P]$$
$$+D\theta_0^\mu[1+(\mu+P+1)h\theta_0^P]+ \ldots \quad \text{(D17b)}$$

$$\frac{F_2}{F_0}=A\theta_0^{-\left(\frac{\gamma-1}{\gamma}\right)}(-\gamma^2 P f\theta_0^P)+B\{(\gamma-1)+[\gamma(P+1)$$
$$-1]g\theta_0^P\}+D\theta_0^\mu[\gamma(\mu+P+1)-1]h\theta_0^P+ \ldots \quad \text{(D17c)}$$

The constant A is found as follows. Considering only the leading terms, equation (D17a) can be expressed as

$$\eta^\sigma\varphi_2=\frac{\gamma-1}{\gamma}\frac{\gamma+\gamma\mu-1}{K_0^{(\gamma-1)/\gamma}}A+0(\eta^{1+\sigma}) \quad \text{(D18)}$$

Noting equation (B16), it is seen that

$$A=\frac{1}{\gamma}\frac{\gamma+1}{\gamma-1}\left[\frac{\gamma+1}{2}F_0(0)\right]^{1/\gamma} \quad \text{(D19)}$$

APPENDIX E

DERIVATION OF EXPRESSIONS FOR DRAG

The drag on the portion of the body upstream of any section $\bar{x}$, designated $D(\bar{x})$, can be found either by integrating the pressure distribution along the body or by computing the energy of the transverse flow at section $\bar{x}$. Both methods are used herein to obtain expressions for $D(\bar{x})$.

INTEGRATION OF PRESSURES ALONG BODY SURFACE

The cases $\beta < 1$ and $\beta = 1$ are treated separately. **Case $\beta < 1$.** The forebody drag is given by

$$\frac{D(\bar{x})}{q} = 2\pi^\sigma \int_0^{\bar{r}_b(\bar{x})} c_{p,b}(\bar{r}_b)^\sigma \, d\bar{r}_b \qquad (E1)$$

assuming, for $\sigma = 0$, that the body is symmetrical about the $\bar{x}$-axis. But, from equations (13a), (22a), and (23),

$$\bar{r}_b = \delta \bar{L} x^m (\eta_{b,0} + \epsilon x^N)$$

$$d\bar{r}_b = \delta \bar{L} m x^{m-1} \left[\eta_{b,0} + \epsilon \left(1 + \frac{N}{m} \right) x^N \right] dx$$

$$c_{p,b} = 2\delta^2 m^2 x^{2(m-1)} [F_0(\eta_{b,0}) + \epsilon F_2(\eta_{b,0}) x^N]$$

Substituting into equation (E1) gives

$$\frac{D(\bar{x})}{2\pi^\sigma m^3 q \delta^{\sigma+3} (\bar{L}\eta_{b,0})^{\sigma+1}} = 2F_0 (\eta_{b,0}) \int_0^x \left\{ x^{m(\sigma+3)-3} \right.$$

$$\left. + \epsilon \left[\frac{F_2(\eta_{b,0})}{F_0(\eta_{b,0})} + \frac{\sigma+1+\dfrac{N}{m}}{\eta_{b,0}} \right] x^{m(\sigma+3)-3+N} \right\} dx \qquad (E2)$$

Assuming

$$m > 2/(\sigma+3) \quad \text{(i.e., } \beta < 1) \qquad (E3a)$$

$$N > 2 - m(\sigma+3) \qquad (E3b)$$

integration of equation (E2) yields

$$\frac{D(\bar{x})}{2\pi^\sigma m^3 q \delta^{\sigma+3}(\bar{L}\eta_{b,0})^{\sigma+1}}$$

$$= \frac{2F_0(\eta_{b,0})}{m(\sigma+3)-2} \left\{ x^{m(\sigma+3)-2} + \frac{\epsilon[m(\sigma+3)-2]}{m(\sigma+3)-2+N} \right.$$

$$\times \left[\frac{F_2(\eta_{b,0})}{F_0(\eta_{b,0})} + \frac{\sigma+1+\dfrac{N}{m}}{\eta_{b,0}} \right] x^{m(\sigma+3)-2+N} \right\} \qquad (E4)$$

Case $\beta = 1$. For $\beta = 1$, the zero-order flow field corresponds to the flow over a flat plate ($\sigma = 0$) or circular cylinder ($\sigma = 1$) with semithickness or radius, respectively, equal to $\bar{r}_N$. Designate the known zero-order drag of the nose ($x = 0$) by D_N as in reference 1. Then

$$D(\bar{x}) = D_N + 2\pi^\sigma q \int_0^{\bar{r}_b(\bar{x})} c_{p,b}(\bar{r}_b)^\sigma d\bar{r}_b \qquad (E5)$$

where $\bar{r}_b$ is measured from the surface of the zero-order body. From equations (13b) and (22a),

$$\bar{r}_b = \delta \bar{L} \epsilon^{\frac{1}{\sigma+1}} x^{m+\frac{N}{\sigma+1}} \qquad (E6a)$$

$$d\bar{r}_b = \delta \bar{L} \epsilon^{\frac{1}{\sigma+1}} m(1+\mu) x^{m+\frac{N}{\sigma+1}-1} dx \qquad (E6b)$$

$$c_{p,b} = 2\delta^2 m^2 x^{2(m-1)} F_0(0) + 0(\epsilon) \qquad (E6c)$$

Substituting into equation (E5) and integrating yield, for $N > 0$,

$$D(\bar{x}) = D_N + \frac{16\pi^\sigma q(1+\mu)\delta^2 F_0(0)[\bar{R}_0(\bar{L})]^{\sigma+1}\epsilon}{(\sigma+3)^2(\sigma+1)\mu} x^N \qquad (E7)$$

wherein $m = 2/(\sigma+3)$.

DRAG FROM CONSIDERATION OF ENERGY OF TRANSVERSE FLOW

The forebody drag $D(\bar{x})$ can also be found by computing the energy of the transverse flow field as discussed in references 1 and 2.

Let $\bar{E}$ be the energy (computed as a perturbation from the undisturbed free-stream value) per unit mass at any point. Then

$$\bar{\rho}\bar{E} = \bar{\rho}\left[C_r(\bar{T}-\bar{T}_\infty)+\frac{\bar{v}^2}{2}\right]$$

$$= \bar{\rho}\left[\frac{1}{\gamma-1}\left(\frac{\bar{p}}{\bar{\rho}}-\frac{\bar{p}_\infty}{\bar{\rho}_\infty}\right)+\frac{\bar{v}^2}{2}\right]$$

$$= \gamma M^2 \delta^2 \bar{p}_\infty \left(\frac{p}{\gamma-1}+\frac{1}{2}\rho v^2\right) \qquad (E8)$$

From references 1 and 2

$$D(\bar{x}) = 2\pi^\sigma \int_{\bar{r}_b(\bar{x})}^{\bar{R}(\bar{x})} \overline{\rho E} \bar{r}^\sigma \, d\bar{r} \qquad (E9)$$

where the integration is conducted at station $\bar{x}$. But

$$\left.\begin{array}{l}
\bar{r} = \delta \bar{L} x^m \eta \\[4pt]
d\bar{r} = \delta \bar{L} x^m \, d\eta \\[4pt]
\eta_s = 1 + \epsilon a_2 x^N \\[4pt]
\eta_b = \eta_{b,0} + \epsilon x^N \qquad \beta<1 \\[4pt]
 = (\epsilon x^N)^{1/(\sigma+1)} \qquad \beta=1
\end{array}\right\} \qquad (E10)$$

Substituting equations (12) and (E10) into equation (E9) and neglecting higher order terms yield

$$\frac{D(\bar{x})}{4q\pi^\sigma \delta^2 m^2 (\delta L)^{\sigma+1} x^{(\sigma+3)m-2}} = \int_{\eta_b}^{1+\epsilon a_2 x^N} \left\{\left(\frac{F_0}{\gamma-1}+\frac{\psi_0\varphi_0^2}{2}\right)\right.$$

$$\left. + \epsilon x^N\left[\frac{F_2}{\gamma-1}+\frac{\psi_0\varphi_0^2}{2}\left(\frac{\psi_2}{\psi_0}+\frac{2\varphi_2}{\varphi_0}\right)\right]\right\} \eta^\sigma \, d\eta$$

$$= \int_{\eta_{b,0}}^{1}\left(\frac{F_0}{\gamma-1}+\frac{\psi_0\varphi_0^2}{2}\right)\eta^\sigma \, d\eta + \epsilon x^N \int_{\eta_{b,0}}^{1}\left[\frac{F_2}{\gamma-1}\right.$$

$$\left. +\frac{\psi_0\varphi_0^2}{2}\left(\frac{\psi_2}{\psi_0}+\frac{2\varphi_2}{\varphi_0}\right)\right]\eta^\sigma \, d\eta + \int_{1}^{1+\epsilon a_2 x^N}\left(\frac{F_0}{\gamma-1}+\frac{\psi_0\varphi_0^2}{2}\right)\eta^\sigma \, d\eta$$

$$-\int_{\eta_{b,0}}^{\eta_b}\left(\frac{F_0}{\gamma-1}+\frac{\psi_0\varphi_0^2}{2}\right)\eta^\sigma \, d\eta \qquad (E11)$$

But, from equations (24) and (29) of reference 1

$$\left.\begin{array}{l}
\int_{\eta_{b,0}}^{1}\left(\dfrac{F_0}{\gamma-1}+\dfrac{\psi_0\varphi_0^2}{2}\right)\eta^\sigma \, d\eta \\[14pt]
\quad = \dfrac{m}{m(\sigma+3)-2}\eta_{b,0}^{\sigma+1}F_0(\eta_{b,0}) \qquad \beta<1 \\[14pt]
\quad = I \qquad\qquad\qquad\qquad\qquad\quad \beta=1
\end{array}\right\} \qquad (E12)$$

where I is tabulated as a function of γ and σ in reference 1. From the mean value theorem and equations (15),

$$\int_{1}^{1+\epsilon a_2 x^N}\left(\frac{F_0}{\gamma-1}+\frac{\psi_0\varphi_0^2}{2}\right)\eta^\sigma \, d\eta = \epsilon a_2 x^N\left(\frac{F_0}{\gamma-1}+\frac{\psi_0\varphi_0^2}{2}\right)_{\eta=1}$$

$$= \frac{4\epsilon a_2 x^N}{\gamma^2-1} \qquad (E13)$$

Similarly,

$$\left.\begin{array}{l}
\int_{\eta_{b,0}}^{\eta_b}\dfrac{F_0}{\gamma-1}\eta^\sigma \, d\eta = \dfrac{\eta_{b,0}^\sigma F_0(\eta_{b,0})}{\gamma-1}\epsilon x^N \qquad \beta<1 \\[14pt]
\quad = \dfrac{F_0(0)}{(\sigma+1)(\gamma-1)}\epsilon x^N \qquad \beta=1
\end{array}\right\} \qquad (E14)$$

Using the asymptotic forms noted in equations (43) of reference 1 it can be shown

$$\left.\begin{array}{l}
\dfrac{1}{2}\int_{\eta_{b,0}}^{\eta_b}\psi_0\varphi_0^2\eta^\sigma \, d\eta = \dfrac{1}{2}\dfrac{\gamma+1}{\gamma-1}\eta_{b,0}^{\sigma+2} \\[14pt]
\quad \left[\dfrac{\gamma+1}{2}F_0(\eta_{b,0})\right]^{1/\gamma}\epsilon x^N \qquad \text{for } \beta=0 \\[14pt]
\quad = \text{higher order} \qquad\qquad\qquad \text{for } \beta>0
\end{array}\right\}$$

$$(E15)$$

Define

$$I_2 \equiv \int_{\eta_{b,0}}^{1}\left[\frac{F_2}{\gamma-1}+\frac{\psi_0\varphi_0^2}{2}\left(\frac{\psi_2}{\psi_0}+\frac{2\varphi_2}{\varphi_0}\right)\right]\eta^\sigma \, d\eta \qquad (E16)$$

Substituting equations (E12) to (E16) into equation (E11) permits the latter to be written for $\beta=0$, $0<\beta<1$, and $\beta=1$, respectively,

$$\frac{D(\bar{x})}{4q\pi^\sigma \delta^2 m^2 (\delta L)^{\sigma+1} x^{(\sigma+3)m-2}}$$

$$= \frac{\eta_{b,0}^{\sigma+1}F_0(\eta_{b,0})}{\sigma+1}+\epsilon x^N\left\{I_2+\frac{4a_2}{\gamma^2-1}-\frac{\eta_{b,0}^\sigma F_0(\eta_{b,0})}{\gamma-1}\right.$$

$$\left. -\frac{1}{2}\frac{\gamma+1}{\gamma-1}\eta_{b,0}^{\sigma+2}\left[\frac{\gamma+1}{2}F_0(\eta_{b,0})\right]^{1/\gamma}\right\} \qquad \text{for } \beta=0$$

$$(E17a)$$

$$=\frac{m\,\eta_{b,0}^{\sigma+1}F_0(\eta_{b,0})}{m\,(\sigma+3)-2}+\epsilon x^N\left[I_2+\frac{4a_2}{\gamma^2-1}-\frac{\eta_{b,0}^{\sigma}F_0(\eta_{b,0})}{\gamma-1}\right]$$

$$\text{for }0<\beta<1 \quad \text{(E17b)}$$

$$=I+\epsilon x^N\left[I_2+\frac{4a_2}{\gamma^2-1}-\frac{F_0(0)}{(\sigma+1)(\gamma-1)}\right]$$

$$\text{for }\beta=1 \quad \text{(E17c)}$$

All the quantities on the right side of equations (17) are known from the zero-order solution except a_2 and I_2, which are found from the perturbation solution.

From equations (E17) it is seen that, when $m=2/(\sigma+3)$ (i.e., $\beta=1$), the zero-order drag rises discontinuously at $x=0$ and is constant for $x>0$. This flow was referred to as a "constant energy" flow in reference 1. Similarly, when $N=2-m$ $(\sigma+3)$, the portion of the drag associated with the perturbation solution rises discontinuously at $x=0$ and is constant for $x>0$. Equations (E17) are more general than equations (E4) and (E7), since the range of N is not restricted in the former equations.

APPENDIX F

BOUNDARY-LAYER GROWTH ON SLENDER POWER LAW BODIES AT HYPERSONIC SPEEDS

The boundary-layer displacement thickness on slender power law bodies at hypersonic speeds was found in reference 2 using the local similarity concept of reference 5. It was assumed in reference 2 that $\sigma = 1$, Prandtl number $= 1$, the body is insulated, and viscosity is proportional to temperature. A similar derivation for displacement thickness is presented herein except that the $\sigma = 0$ case is included and the assumption of an insulated body is replaced by the less restrictive assumption that the body surface temperature is the same at all stations along the body. All physical quantities are dimensional in the present section, and the superscript bar is omitted for convenience. In the present section the symbol μ represents viscosity.

Following reference 5, the independent variables s and y (where s is distance along the body and y is distance normal to the body) are replaced by

$$\tilde{\eta} = \frac{\rho_e u_e}{(2\tilde{s})^{1/2}} \int_0^y r_{b,0}^\sigma \frac{\rho}{\rho_e} \, dy \qquad \text{(F1a)}$$

$$\tilde{s} = \int_0^s \mu_e \rho_e u_e r_{b,0}^{2\sigma} \, ds \qquad \text{(F1b)}$$

The subscript e represents local free-stream conditions just outside the boundary layer. A stream function exists such that

$$\left. \begin{array}{l} \dfrac{\partial \psi}{\partial y} = \rho u r_{b,0}^\sigma \\[2mm] \dfrac{\partial \psi}{\partial r} = -\rho v r_{b,0}^\sigma \end{array} \right\} \qquad \text{(F2)}$$

Assume

$$\psi = (2\tilde{s})^{1/2} f(\tilde{\eta}) \qquad \text{(F3a)}$$

so that $f' = u/u_e$. Similarly, assume

$$\frac{h_{st}}{(h_{st})_e} = g(\tilde{n}) \qquad \text{(F3b)}$$

where h_{st} is the local stagnation enthalpy. If it is further assumed that

$$Pr = 1 \qquad \text{(F4a)}$$

$$\frac{\rho \mu}{\rho_e \mu_e} = 1 \qquad \text{(F4b)}$$

the boundary-layer momentum and energy equations become, respectively,

$$\left. \begin{array}{l} f''' + ff'' + \tilde{\beta}[g - (f')^2] = 0 \\[2mm] g'' + fg' = 0 \end{array} \right\} \qquad \text{(F5)}$$

with boundary conditions

$$f(0) = f'(0) = 0; f'(\tilde{\eta}) \to 1 \text{ as } \tilde{\eta} \to \infty$$

and

$$g(0) = g_w \text{ or } g'(0) = 0; g(\tilde{\eta}) \to 1 \text{ as } \tilde{\eta} \to \infty$$

Here

$$\tilde{\beta} \equiv \frac{2\tilde{s}}{M_e} \frac{dM_e}{d\tilde{s}} \qquad \text{(F6)}$$

Equations (F5) were solved numerically in references 6 and 7 for various values of $g(0)$ and $\tilde{\beta}$. [Note: $S \equiv g - 1$ therein.] The displacement thickness for these solutions is (ref. 5)

$$\delta^* = \frac{(2\tilde{s})^{1/2}}{r_{b,0}^\sigma \rho_e u_e} \int_0^\infty \left(\frac{\rho_e}{\rho} - \frac{u}{u_e} \right) d\tilde{n} \qquad \text{(F7)}$$

but

$$\frac{\rho_e}{\rho} = \left(1 + \frac{\gamma - 1}{2} M_e^2 \right) g - \frac{\gamma - 1}{2} M_e^2 (f')^2 \qquad \text{(F8)}$$

Substituting equation (F8) into equation (F7) permits the latter to be written

$$\delta^* = \frac{(2\tilde{s})^{1/2}}{r_{b,0}^\sigma \rho_e u_e} \left[\left(1 + \frac{\gamma - 1}{2} M_e^2 \right) (J_1 + J_3) + \frac{\gamma - 1}{2} M_e^2 J_2 \right] \qquad \text{(F9)}$$

21

where

$$J_1 \equiv \int_0^\infty (1-f')\, d\tilde{\eta}$$

$$J_2 \equiv \int_0^\infty f'(1-f')\, d\tilde{\eta}$$

$$J_3 \equiv \int_0^\infty (g-1)\, d\tilde{\eta}$$

For a given $g(0)$ and $\bar{\beta}$ the integrals J_2 and $(J_1 + J_3)$ can be evaluated from the numerical results in tables II of references 6 and 7. In particular,

$$\left.\begin{array}{l} J_2 = \left[\dfrac{l}{f''(0)}\right]_{\text{ref. 7}} \\[2ex] \dfrac{J_1 + J_3}{J_2} = [H_{inc}]_{\text{ref. 7}} \end{array}\right\} \quad (F10)$$

These results are summarized in table I of the present report. Additional values for the insulated wall case $(g(\tilde{\eta})=1)$ are also included in table I, as obtained from table I of reference 8.

The expression for δ^* and $\bar{\beta}$ can be simplified for the case of hypersonic flow over slender power law bodies. For such bodies

$$\left.\begin{array}{l} u_e = u_\infty \\[2ex] r_{b,0} = \eta_{b,0}\delta L\left(\dfrac{x}{L}\right)^m \\[2ex] \dfrac{p_e}{p_\infty} = \gamma M_\infty^2 \dfrac{F_0(\eta_{b,0})}{\eta_{b,0}^2}\left(\dfrac{dr_{b,0}}{dx}\right)^2 \end{array}\right\} \quad (F11)$$

Consistent with equation (F4b), a constant ω can be defined such that $\mu/T = \omega\mu_\infty/T_\infty$ or

$$\frac{\mu_e}{T_e} = \omega\,\frac{\mu_\infty}{T_\infty} \qquad (F12)$$

(If μ_e is not proportional to T_e, a mean value of ω is used.) Substituting equations (F11) and (F12) into equation (F1b) and assuming $s=x$

permit equation (F1b) to be integrated. Substitution into equation (F7) then yields, for $(\gamma-1)M_\infty^2 \gg 1$ and $(\gamma-1)M_e^2 \gg 1$,

$$\frac{\delta^*}{L} = \frac{\gamma-1}{\sqrt{2\gamma}}\sqrt{\frac{\omega}{R_{e,L}}}\,\frac{M_\omega}{\delta\sqrt{F_0(\eta_{b,0})}}\,\frac{J_2\left(1+\dfrac{J_1+J_3}{J_2}\right)}{m\sqrt{2m(\sigma+1)-1}}\left(\frac{x}{L}\right)^{\frac{3}{2}-m} \qquad (F13)$$

where $R_{e,L} \equiv \rho_\infty u_\infty L/\mu_\infty$. Similarly, since

$$M_e \equiv \frac{u_e}{a_e} = \frac{u_\infty}{a_e} \sim \frac{1}{a_e}$$

$$\tilde{s} \sim x^{2m(\sigma+1)-1}$$

$$\frac{da_e}{a_e} = \frac{\gamma-1}{2\gamma}\,\frac{d(p_e/p_\infty)}{(p_e/p_\infty)} \quad \begin{array}{l}\text{(isentropic flow external} \\ \text{to boundary layer)}\end{array}$$

equation (F6) becomes

$$\bar{\beta} = \frac{\gamma-1}{\gamma}\,\frac{2(1-m)}{2m(\sigma+1)-1} \qquad (F14)$$

Equations (F13) and (F14), together with table I, define the boundary-layer development on a slender power law body at hypersonic speeds. There appear to be several misprints in the final expression for δ^*/L in reference 2. (The integrals J_1 and J_2 are missing and m is in the numerator instead of the denominator therein.)

The validity of equations (F13) may be questioned for $\beta=1$, $\sigma=1$, since $r_{b,0}=0$ for this case and the right sides of equations (F1) and (F2) are identically zero. If $r_{b,0}$ is replaced by r_N in equations (F1) and (F2) and if the assumptions incorporated in equations (F11) and (F12) are again applied to permit integration of equation (F16), it is found that s has a logarithmic singularity at $x=0$. It thus appears that the hypersonic slender body approximations must be abandoned in order to obtain a boundary-layer solution for the $\beta=1, \sigma=1$ case. Such a development is beyond the scope of the present report.

APPENDIX G

AXIALLY SYMMETRIC BODIES AT VERY SMALL ANGLES OF ATTACK

In the present section equations are developed which permit finding the effect of a power law lateral perturbation of the centerline of a slender power law body in a hypersonic stream. The flow about axially symmetric power law bodies at very small angles of attack is probably the most important application of these equations. The case $\beta = \sigma = 1$ is excluded from consideration.

First, the equations of motion will be put in cylindrical coordinate form. Let (x, r, θ) be a

(c)
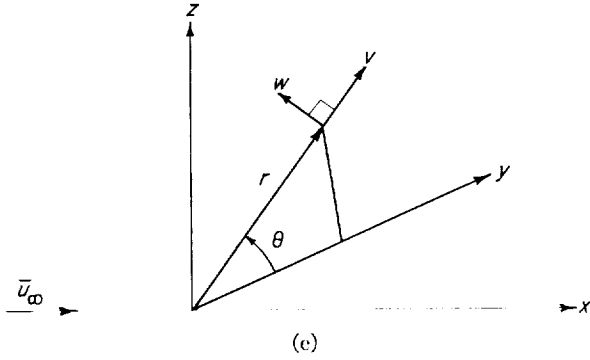
cylindrical coordinate system such that x is in the free-stream direction (see sketch (c)). Let r, w be the local velocities in the r, θ directions, respectively. Dependent and independent variables are assumed to be nondimensional (e.g., eqs. (1) and (7) of the present report and eqs. (8a) (8b) of ref. 4). The hypersonic slender body equations of motion (e.g., eqs. (9) and (10) of ref. 4) become, in cylindrical coordinates,

Continuity:

$$\frac{\partial \rho}{\partial x} + \frac{\partial \rho r}{\partial r} + \frac{\sigma \rho r}{r} + \frac{1}{r}\frac{\partial \rho w}{\partial \theta} = 0 \qquad \text{(G1a)}$$

r-Momentum:

$$\frac{\partial r}{\partial x} + r\frac{\partial r}{\partial r} + \frac{w}{r}\frac{\partial r}{\partial \theta} - \frac{w^2}{r} + \frac{1}{\rho}\frac{\partial p}{\partial r} = 0 \qquad \text{(G1b)}$$

θ-Momentum:

$$\frac{\partial w}{\partial x} + r\frac{\partial w}{\partial r} + \frac{w}{r}\frac{\partial w}{\partial \theta} + \frac{rw}{r} + \frac{1}{\rho r}\frac{\partial p}{\partial \theta} = 0 \qquad \text{(G1c)}$$

Energy:

$$\left(\frac{\partial}{\partial x} + r\frac{\partial}{\partial r} + \frac{w}{r}\frac{\partial}{\partial \theta}\right)\left(\frac{p}{\rho^\gamma}\right) = 0 \qquad \text{(G1d)}$$

Equations (G1) reduce to equations (3) for the case of two-dimensional or axisymmetric flow.

Assume a zero-order body and shock of the form $r_{b,0} = \eta_{b,0}x^m$ and $R_0 = x^m$, respectively. If the centerline of the body is displaced locally by a small amount

$$z = \epsilon x^{N+m} \qquad \text{(G2)}$$

in the x, z plane, the new body and shock locations are given by

$$r_b = x^m(\eta_{b,0} + \epsilon x^N \sin \theta) \qquad \text{(G3a)}$$

$$R = x^m(1 + \epsilon a_2 x^N \sin \theta) \qquad \text{(G3b)}$$

where a_2 is a constant. For the case of a body at positive angles of attack α,

$$\epsilon = -\alpha/\delta \qquad \text{(G4a)}$$

$$\left.\begin{array}{l} N = 1 - m \\ \mu = \beta/2 \end{array}\right\} \qquad \text{(G4b)}$$

From equation (G4a) it is seen that α must be small compared with δ.

Equations (G1) can be reduced to ordinary differential equations as follows. Introduce new independent variables ξ, η as defined by equations (10). The dependent variables may be expressed as

$$\left.\begin{array}{l} v = m\xi^{m-1}(\varphi_0 + \epsilon \xi^N \sin \theta \ \varphi_2) \\ w = m\,\epsilon \xi^{N+m-1}\cos \theta \ \Omega_2 \\ p = (m\xi^{m-1})^2(F_0 + \epsilon \xi^N \sin \theta \ F_2) \\ \rho = \psi_0 + \epsilon \xi^N \sin \theta \ \psi_2 \end{array}\right\} \qquad \text{(G5)}$$

where φ, F, ψ, and Ω are functions of η. Equations (G5) are consistent with the boundary conditions at the shock and at the body surface. Substitution of equations (G5) into equations (G1) gives zero-order equations identical with equations (14) and perturbation equations of the form

23

Continuity:

$$\frac{\varphi_2'}{\eta-\varphi_0}-\frac{\psi_2'}{\psi_0}+\left(\frac{\psi_0'}{\psi_0}+\frac{\sigma}{\eta}\right)\frac{\varphi_2}{\eta-\varphi_0}$$

$$+\left[\frac{\frac{\sigma}{\eta}\varphi_0+\varphi_0'+(\sigma+1)\mu}{\eta-\varphi_0}\right]\frac{\psi_2}{\psi_0}-\frac{\Omega_2}{\eta(\eta-\varphi_0)}=0 \quad \text{(G6a)}$$

r-Momentum:

$$\frac{\varphi_2'}{\eta-\varphi_0}-\frac{1}{(\eta-\varphi_0)^2}\frac{F_0}{\psi_0}\frac{F_2'}{F_0}$$

$$+\left[\frac{(\sigma+1)\left(\frac{\beta}{2}-\mu\right)-\varphi_0'}{\eta-\varphi_0}\right]\frac{\varphi_2}{\eta-\varphi_0}$$

$$+\left[\frac{F_0'}{(\eta-\varphi_0)^2\psi_0}\right]\frac{\psi_2}{\psi_0}=0 \quad \text{(G6b)}$$

θ-Momentum:

$$\Omega_2'+\left[\frac{(\sigma+1)\left(\frac{\beta}{2}-\mu\right)-\frac{\varphi_0}{\eta}}{\eta-\varphi_0}\right]\Omega_2-\frac{F_0}{\eta\psi_0(\eta-\varphi_0)}\frac{F_2}{F_0}=0$$

$$\text{(G6c)}$$

Energy:

$$\frac{F_2'}{F_0}-\gamma\frac{\psi_2'}{\psi_0}+\left(\gamma\frac{\psi_0'}{\psi_0}-\frac{F_0'}{F_0}\right)\frac{\varphi_2}{\eta-\varphi_0}+\gamma\left[\frac{\psi_0'}{\psi_0}+\frac{(\sigma+1)\mu}{\eta-\varphi_0}\right]\frac{\psi_2}{\psi_0}$$

$$-\left[\gamma\frac{\psi_0'}{\psi_0}+\frac{(\sigma+1)(\mu-\beta)}{\eta-\varphi_0}\right]\frac{F_2}{F_0}=0 \quad \text{(G6d)}$$

The boundary conditions at the shock can be shown to be satisfied if

$$\frac{\varphi_2(1)}{a_2}=\frac{2}{\gamma+1}[1+(\sigma+1)\mu]-\varphi_0'(1) \quad \text{(G7a)}$$

$$\frac{F_2(1)}{a_2}=\frac{4}{\gamma+1}[1+(\sigma+1)\mu]-F_0'(1) \quad \text{(G7b)}$$

$$\frac{\psi_2(1)}{a_2}=-\psi_0'(1) \quad \text{(G7c)}$$

$$\frac{\Omega_2(1)}{a_2}=\frac{-2}{\gamma+1} \quad \text{(G7d)}$$

The tangent flow boundary condition at the body surface is satisfied if

$$\varphi_2(\eta_{b,0})=(\sigma+1)(\gamma+\gamma\mu-\beta)/\gamma \qquad \beta<1 \quad \text{(G8a)}$$

$$\varphi_2(0)=(\gamma+\gamma\mu-1)/\gamma \qquad \sigma=0, \beta=1 \quad \text{(G8b)}$$

It can also be shown that $F_2(\eta_b)=F_2(\eta_{b,0})$.

Asymptotic expressions for the dependent variables, valid near $\eta_{b,0}$, are required if equations (G6) are to be numerically integrated. The derivation of these asymptotic expressions is now outlined. The cases $0<\beta<1$ and $\beta=0$ are considered. (The case $\sigma=0$, $\beta=1$ can be found from the equations in the body of the report, and the case $\sigma=1$, $\beta=1$ is not being considered.) First, the equations of motion are written in the form (treating $\varphi_2/(\eta-\varphi_0)$, F_2/F_0, ψ_2/ψ_0, and Ω_2/η as the dependent variables and θ_0 as the independent variable):

Continuity:

$$\frac{d[\varphi_2/(\eta-\varphi_0)]}{d\theta_0}-\frac{d(\psi_2/\psi_0)}{d\theta_0}$$

$$+\frac{1}{\theta_0}\left[\frac{\varphi_2}{\eta-\varphi_0}+\mu\frac{\psi_2}{\psi_0}-\frac{\Omega_2}{(\sigma+1)\eta}\right]=0 \quad \text{(G9a)}$$

r-Momentum:

$$\left.\begin{array}{c}\dfrac{d[\varphi_2/(\eta-\varphi_0)]}{d\theta_0}+\dfrac{k_1}{\theta_0}\dfrac{\varphi_2}{\eta-\varphi_0}-\dfrac{k_2}{\theta_0^{2-(\beta/\gamma)}}\dfrac{d(F_2/F_0)}{d\theta_0}\\[1em] +\dfrac{k_3}{\theta_0^{2-(\beta/\gamma)}}\left(\dfrac{\psi_2}{\psi_0}-\dfrac{F_2}{F_0}\right)=0 \text{ for } 0<\beta<1 \\[1.5em] \dfrac{d[\varphi_2/(\eta-\varphi_0)]}{d\theta_0}+\dfrac{k_1}{\theta_0}\dfrac{\varphi_2}{\eta-\varphi_0}-\dfrac{k_2}{\theta_0^2}\dfrac{d(F_2/F_0)}{d\theta_0}\\[1em] +\dfrac{k_3}{\theta}\left(\dfrac{\psi_2}{\psi_0}-\dfrac{F_2}{F_0}\right)=0 \text{ for } \beta=0\end{array}\right\} \quad \text{(G9b)}$$

θ-Momentum:

$$\frac{d(\Omega_2/\eta)}{d\theta_0}+\frac{k_5}{\theta_0}\frac{\Omega_2}{\eta}-\frac{k_4}{\theta_0^{1+(\beta/\gamma)}}\frac{F_2}{F_0}=0 \quad \text{(G9c)}$$

Energy:

$$\frac{d(F_2/F_0)}{d\theta_0}-\gamma\frac{d(\psi_2/\psi_0)}{d\theta_0}+\frac{1}{\theta_0}\left(\beta\frac{\varphi_2}{\eta-\varphi_0}+\gamma\mu\frac{\psi_2}{\psi_0}-\mu\frac{F_2}{F_0}\right)=0$$

$$\text{(G9d)}$$

where, for $\theta_0\approx 0$ and $0\leq\beta<1$, the quantities k_1, k_2, ..., k_5 are constants given by

$$k_1=\frac{\beta}{2}-\mu+\frac{1-2\varphi_0'}{\sigma+1}=\frac{\beta}{2}-\mu+2\left(\frac{\gamma-\beta}{\gamma}\right)-\frac{1}{\sigma+1} \quad \text{(G10a)}$$

$$k_2=\left(\frac{1}{\sigma+1}\frac{\theta_0'}{\theta_0}\right)^2\frac{F_0}{\psi_0}\theta_0^{2-(\beta/\gamma)}=\frac{2\eta_{b,0}^{2\sigma}}{\gamma-1}\left[\frac{\gamma+1}{2}F_0(\eta_{b,0})\right]^{\frac{\gamma+1}{\gamma}} \quad \text{(G10b)}$$

$$k_3 = \frac{k_2}{F_0}\frac{dF_0}{d\theta_0} = \frac{\beta k_2 \eta_{b,0}^{1-\sigma}}{2F_0(\eta_{b,0})} \text{ for } 0<\beta<1$$

$$k_3 = \left(\frac{1}{\sigma+1}\frac{\theta_0'}{\theta_0}\right)^2 \frac{\theta_0}{\psi_0}\frac{dF_0}{d\theta_0} = \frac{-\sigma}{\sigma+1} \text{ for } \beta=0$$

(G10c)

$$k_4 = \frac{1}{\sigma+1}\frac{F_0}{\eta^2}\frac{\theta_0^{\beta/\gamma}}{\psi_0} = \frac{(\gamma-1)F_0(\eta_{b,0})}{(\sigma+1)(\gamma+1)\eta_{b,0}^2}\left[\frac{2}{(\gamma+1)F_0(\eta_{b,0})}\right]^{1/\gamma}$$

(G10d)

$$k_5 = \frac{\beta}{2}-\mu+\frac{1-2(\varphi_0/\eta)}{\sigma+1} = \frac{\beta}{2}-\mu-\frac{1}{\sigma+1}$$ (G10e)

CASE $0<\beta<1$

Assume the dependent variables to be of the form

$$\frac{F_2}{F_0} = \theta_0^{\tilde{N}+1}[F_{20}+\theta_0^{1-(\beta/\gamma)}F_{21}+\ldots]$$

$$\frac{\varphi_2}{\eta-\varphi_0} = \theta_0^{\tilde{N}}[\varphi_{20}+\theta_0^{1-(\beta/\gamma)}\varphi_{21}+\ldots]$$

$$\frac{\Omega_2}{\eta} = \theta_0^{\tilde{N}}[\Omega_{20}+\theta_0^{1-(\beta/\gamma)}\Omega_{21}+\ldots]$$

$$\frac{\psi_2}{\psi_0} = \theta_0^{\tilde{N}}[\psi_{20}+\theta_0^{1-(\beta/\gamma)}\psi_{21}+\ldots]$$

(G11)

where F_{20}, F_{21}, φ_{20}, φ_{21}, and so forth, are constants. Substituting equations (G11) into equations (G9) and collecting the coefficients of the lowest order terms, and the second lowest order terms, yield, respectively,

$$(\tilde{N}+1)\varphi_{20}-(\tilde{N}-\mu)\psi_{20}-\frac{\Omega_{20}}{\sigma+1}=0$$

$$-k_2(\tilde{N}+1)F_{20}+k_3\psi_{20}=0$$

$$(\tilde{N}+k_5)\Omega_{20}=0$$

$$\gamma(\mu-\tilde{N})\psi_{20}+\beta\varphi_{20}=0$$

(G12a)

$$\left(\tilde{N}+2-\frac{\beta}{\gamma}\right)\varphi_{21}-\left(\tilde{N}+1-\frac{\beta}{\gamma}-\mu\right)\psi_{21}-\frac{\Omega_{21}}{\sigma+1}=0$$

$$(\tilde{N}+k_1)\varphi_{20}-k_2\left(\tilde{N}+2-\frac{\beta}{\gamma}\right)F_{21}+k_3\psi_{21}=0$$

$$\left(\tilde{N}+1-\frac{\beta}{\gamma}+k_5\right)\Omega_{21}-k_4F_{20}=0$$

$$\gamma\left(\mu+\frac{\beta}{\gamma}-\tilde{N}-1\right)\psi_{21}+\beta\varphi_{21}=0$$

(G12b)

From equations (G12a), a nontrivial solution results if

$$\tilde{N} = -k_5, \mu, -\left(\frac{\gamma-\beta}{\gamma}\right), -1$$ (G13)

These define four independent asymptotic solutions. The constant coefficients in equations (G11) referenced to F_{20} may be expressed as

$$\frac{\psi_{20}}{F_{20}} = \frac{k_2(\tilde{N}+1)}{k_3}$$

$$\frac{\varphi_{20}}{F_{20}} = \frac{-\gamma(\mu-\tilde{N})}{\beta}\frac{\psi_{20}}{F_{20}}$$

$$\frac{\Omega_{20}}{F_{20}} = (\sigma+1)\left[(\tilde{N}+1)\frac{\varphi_{20}}{F_{20}}-(\tilde{N}-\mu)\frac{\psi_{20}}{F_{20}}\right]$$

$$\frac{\Omega_{21}}{F_{20}} = \frac{k_4}{\tilde{N}+1-\frac{\beta}{\gamma}+k_5}$$

$$\frac{\varphi_{21}}{F_{20}} = \frac{1}{(\sigma+1)\left(\tilde{N}+2-\frac{2\beta}{\gamma}\right)}\frac{\Omega_{21}}{F_{20}}$$

$$\frac{\psi_{21}}{F_{20}} = \frac{\beta}{\gamma\left(\tilde{N}+1-\mu-\frac{\beta}{\gamma}\right)}\frac{\varphi_{21}}{F_{20}}$$

$$\frac{F_{21}}{F_{20}} = \frac{1}{k_2\left(\tilde{N}+2-\frac{\beta}{\gamma}\right)}\left[k_3\frac{\psi_{21}}{F_{20}}+(\tilde{N}+k_1)\frac{\varphi_{20}}{F_{20}}\right]$$

(G14)

CASE $\beta=0$

Assume the dependent variables to be of the form

$$\frac{F_2}{F_0} = \theta_0^{\tilde{N}+2}(F_{20}+\theta_0^2F_{21}+\ldots)$$

$$\frac{\varphi_2}{\eta-\varphi_0} = \theta_0^{\tilde{N}}(\varphi_{20}+\theta_0^2\varphi_{21}+\ldots)$$

$$\frac{\Omega_2}{\eta} = \theta_0^{\tilde{N}}(\Omega_{20}+\theta_0^2\Omega_{21}+\ldots)$$

$$\frac{\psi_2}{\psi_0} = \theta_0^{\tilde{N}}(\psi_{20}+\theta_0^2\psi_{21}+\ldots)$$

(G15)

Substituting into equations (G9) and collecting the coefficients of the lowest order terms, and the

second lowest order terms, yield, respectively,

$$
\left.\begin{array}{c}
(\tilde{N}+1)\varphi_{20}-(\tilde{N}-\mu)\psi_{20}-\dfrac{\Omega_{20}}{\sigma+1}=0 \\[2mm]
(\tilde{N}+k_1)\varphi_{20}-k_2(\tilde{N}+2)F_{20}+k_3\psi_{20}=0 \\[2mm]
(\tilde{N}+k_5)\Omega_{20}=0 \\[2mm]
(\mu-\tilde{N})\psi_{20}=0
\end{array}\right\} \quad (G16a)
$$

$$
\left.\begin{array}{c}
(\tilde{N}+3)\varphi_{21}-(\tilde{N}+2-\mu)\psi_{21}-\dfrac{\Omega_{21}}{\sigma+1}=0 \\[2mm]
(\tilde{N}+2+k_1)\varphi_{21}-k_2(\tilde{N}+4)F_{21}+k_3\psi_{21}-k_3F_{20}=0 \\[2mm]
(\tilde{N}+2+k_5)\Omega_{21}-k_4F_{20}=0 \\[2mm]
(\tilde{N}+2-\mu)(F_{20}-\gamma\psi_{21})=0
\end{array}\right\}
$$

$$(G16b)$$

From equations (G16a), a nontrivial solution exists if

$$\tilde{N}=-k_5,\mu,-1,-2 \qquad (G17)$$

These define four independent asymptotic solutions. The coefficients in equations (G15), refer-

enced to F_{20}, are

$$
\left.\begin{array}{ll}
\dfrac{\varphi_{20}}{F_{20}}=\dfrac{k_2(\tilde{N}+2)}{\tilde{N}+k_1} & \text{for } \tilde{N}=-k_5,\ -1 \\[3mm]
\phantom{\dfrac{\varphi_{20}}{F_{20}}}=0 & \text{for } \tilde{N}=\mu,\ -2 \\[3mm]
\dfrac{\psi_{20}}{F_{20}}=\dfrac{k_2(\tilde{N}+2)}{k_3} & \text{for } \tilde{N}=\mu \\[3mm]
\phantom{\dfrac{\psi_{20}}{F_{20}}}=0 & \text{for } \tilde{N}=-k_5,\ -1,\ -2 \\[3mm]
\dfrac{\Omega_{20}}{F_{20}}=\dfrac{(\sigma+1)(\tilde{N}+1)(\tilde{N}+2)k_2}{\tilde{N}+k_1} & \text{for } \tilde{N}=-k_5 \\[3mm]
\phantom{\dfrac{\Omega_{20}}{F_{20}}}=0 & \text{for } \tilde{N}=\mu,\ -1,\ -2 \\[3mm]
\dfrac{\Omega_{21}}{F_{20}}=\dfrac{k_4}{\tilde{N}+2+k_5} \\[3mm]
\dfrac{\psi_{21}}{F_{20}}=\dfrac{1}{\gamma} \\[3mm]
\dfrac{\varphi_{21}}{F_{20}}=\dfrac{1}{\tilde{N}+3}\left[(\tilde{N}+2-\mu)\dfrac{\psi_{21}}{F_{20}}+\dfrac{1}{\sigma+1}\dfrac{\Omega_{21}}{F_{20}}\right] \\[3mm]
\dfrac{F_{21}}{F_{20}}=\dfrac{1}{k_2(\tilde{N}+4)}\left[(\tilde{N}+2+k_1)\dfrac{\varphi_{21}}{F_{20}}+k_3\dfrac{\psi_{21}}{F_{20}}-k_3\right]
\end{array}\right\}
$$

$$(G18)$$

APPENDIX H

APPROXIMATE SOLUTION FOR HYPERSONIC FLOW OVER SLENDER BLUNTED WEDGES AND CONES

Chernyi, in references 12 and 13, has developed an approximate solution for hypersonic flow over slender blunted wedges and cones. For the case of very slender wedges and cones, his approximate equations can be integrated in closed form and the results can be compared with those from the zero-order and perturbation equations of the present report. This is done herein.

First the derivation of Chernyi's equations will be outlined. The forebody drag up to station $\bar{x}$ is equal to the energy of the transverse flow at that station. Thus

$$D_N + 2\pi^\sigma \int_0^{\bar{x}} \bar{p}_b \bar{r}_b^\sigma \frac{d\bar{r}_b}{d\bar{x}} \, d\bar{x} = 2\pi^\sigma \int_{\bar{r}_b(\bar{x})}^{\bar{R}(\bar{x})} \left(\frac{\bar{p}}{\gamma-1} + \frac{\bar{\rho}\bar{r}^2}{2} \right) \bar{r}^\sigma d\bar{r} \quad \text{(H1)}$$

where D_N is the impulsive drag at $\bar{x} = 0$. For the present problem

$$\bar{r}_b = \delta_b \bar{x} \quad \text{(H2)}$$

Introduce the following nondimensional quantities:

$$\left.\begin{array}{ll} \bar{\bar{x}} = \left(\dfrac{2\delta_b^2}{C_{D_N}}\right)^{\frac{1}{\sigma+1}} \dfrac{\delta_b \bar{x}}{\bar{r}_N} & \bar{\bar{r}} = \left(\dfrac{2\delta_b^2}{C_{D_N}}\right)^{\frac{1}{\sigma+1}} \dfrac{\bar{r}}{\bar{r}_N} \\[4mm] \bar{\bar{p}} = \dfrac{\bar{p}}{\gamma M_\infty^2 \delta_b^2 \bar{p}_\infty} & \bar{\bar{\rho}} = \dfrac{\bar{\rho}}{\bar{\rho}_\infty} \qquad \bar{\bar{v}} = \dfrac{\bar{v}}{\delta_b \bar{u}_\infty} \end{array}\right\} \quad \text{(H3)}$$

where

$$C_{D_N} = D_N \Big/ \left(\frac{1}{2} \bar{\rho}_\infty \bar{u}_\infty^2\right) (2^{1-\sigma} \pi^\sigma \bar{r}_N^{\sigma+1})$$

Substituting equations (H2) and (H3) into equations (H1) yields

$$\frac{1}{2^\sigma} + \int_0^{\bar{\bar{x}}} \bar{\bar{p}}_b \bar{\bar{x}}^\sigma d\bar{\bar{x}} = \frac{1}{\gamma-1} \int_{\bar{\bar{r}}_b}^{\bar{\bar{R}}} \bar{\bar{p}} \bar{\bar{r}}^\sigma d\bar{\bar{r}} + \frac{1}{2} \int_{\bar{\bar{r}}_b}^{\bar{\bar{R}}} \bar{\bar{\rho}} \bar{\bar{v}}^2 \bar{\bar{r}}^\sigma d\bar{\bar{r}} \quad \text{(H4)}$$

Assuming $\bar{\bar{p}}$ and $\bar{\bar{v}}$ are constant in the integrals on the right side of equation (H4) and taking $\bar{\bar{p}} = \bar{\bar{p}}_b$ yield $\left[\text{noting } \bar{\bar{r}}_b = \bar{\bar{x}}, \text{ and, from continuity, } \int_{\bar{\bar{r}}_b}^{\bar{\bar{R}}} \bar{\bar{\rho}} \bar{\bar{r}}^\sigma d\bar{\bar{r}} = \bar{\bar{R}}^{\sigma+1}/(\sigma+1)\right]$

$$\frac{1}{2^\sigma} + \int_0^{\bar{\bar{x}}} \bar{\bar{p}}_b \bar{\bar{x}}^\sigma d\bar{\bar{x}} = \frac{\bar{\bar{p}}_b}{\gamma-1} \left(\frac{\bar{\bar{R}}^{\sigma+1} - \bar{\bar{x}}^{\sigma+1}}{\sigma+1} \right) + \frac{\bar{\bar{v}}^2}{2} \left(\frac{\bar{\bar{R}}^{\sigma+1}}{\sigma+1} \right) \quad \text{(H5)}$$

Chernyi further assumes that the entire mass of the disturbed flow is concentrated in the shock wave and moves with it. This assumption is nearly valid for strong shocks provided γ is near 1. Thus, Chernyi takes

$$\bar{\bar{v}} = d\bar{\bar{R}}/d\bar{\bar{x}}$$

$$= \dot{\bar{\bar{R}}} \quad \text{(H6a)}$$

and (from conservation of momentum of the mass concentrated in the shock wave)

$$\bar{\bar{p}}_b = \frac{1}{(2\bar{\bar{R}})^\sigma} \frac{d}{d\bar{\bar{x}}} (\bar{\bar{R}}^{\sigma+1} \dot{\bar{\bar{R}}}) \quad \text{(H6b)}$$

Substituting equations (H6) into equations (H5) yields

$$1 + \int_0^{\bar{\bar{x}}} \left(\frac{\bar{\bar{x}}}{\bar{\bar{R}}}\right)^\sigma \frac{d}{d\bar{\bar{x}}} (\bar{\bar{R}}^{\sigma+1}\dot{\bar{\bar{R}}}) d\bar{\bar{x}}$$

$$= \frac{\bar{\bar{R}}^{\sigma+1} - \bar{\bar{x}}^{\sigma+1}}{(\gamma-1)(2\bar{\bar{R}})^\sigma} \frac{d}{d\bar{\bar{x}}} (\bar{\bar{R}}^{\sigma+1}\dot{\bar{\bar{R}}}) + \frac{\bar{\bar{R}}^{\sigma+1}(\dot{\bar{\bar{R}}})^2}{2} \quad \text{(H7)}$$

Chernyi numerically integrated equation (H7) for $\gamma = 1.4$, $\sigma = 0, 1$; the results are plotted in references 12 and 13.

Equation (H7) can be solved by series expansion, for small $\bar{\bar{x}}$, and the results compared with those from the more exact zero-order and perturbation solutions of the present report. Thus assume $\bar{\bar{R}}$ of the form (for small $\bar{\bar{x}}$)

$$\bar{\bar{R}} = K \bar{\bar{x}}^{\frac{2}{\sigma+3}} \left[1 + a_2 \left(\frac{1}{K} \bar{\bar{x}}^{\frac{\sigma+1}{\sigma+3}}\right)^{\sigma+1} + \cdots \right] \quad \text{(H8)}$$

where K and a_2 are unknown constants. Substituting equation (A8) into equation (H7) and collecting terms of order $\bar{\bar{x}}^0$ and $\bar{\bar{x}}^{(\sigma+1)2/(\sigma+3)}$ give

$$\left.\begin{array}{l} K = \left(\dfrac{9\,(\gamma-1)}{2\gamma}\right)^{1/3} \\[4mm] a_2 = \dfrac{3\gamma-2}{2\,(2\gamma+1)} \end{array}\right\} \sigma=0 \quad \text{(H9a)}$$

$$\left.\begin{array}{l} K = \left(\dfrac{8\,(\gamma-1)}{\gamma}\right)^{1/4} \\[4mm] a_2 = \dfrac{2\gamma-1}{8\,(\gamma+1)} \end{array}\right\} \sigma=1 \quad \text{(H9b)}$$

The pressure coefficient on the body is given by

$$\frac{c_{p,b}}{\delta_b^2}=2\overline{\overline{p}}_b=\frac{2^{1-\sigma}}{\overline{\overline{R}}^\sigma}\frac{d}{d\overline{\overline{x}}}\left(\overline{\overline{R}}^{\sigma+1}\overline{\overline{\dot{R}}}\right) \qquad (H10)$$

Substituting equations (H8) into equation (H10) and comparing with equation (51b) show

$$\left.\begin{array}{ll}F_0(0)=1/2 & \sigma=0,1 \\[2mm] F_2(0)=\dfrac{5}{4}\dfrac{3\gamma-2}{2\gamma+1} & \sigma=0 \\[2mm] \quad=\dfrac{7}{8}\dfrac{2\gamma-1}{\gamma+1} & \sigma=1\end{array}\right\} \qquad (H11)$$

These expressions for K, a_2, $F_0(0)$, and $F_2(0)$ may be compared with equation (50c) and the numerical results of table II. Such a comparison is given in table III. Chernyi's approximate solution becomes less accurate as γ departs from 1.

Although the method of references 12 and 13 is inexact, particularly for γ not near 1, it has the advantage of not being restricted to small values of $\overline{\overline{x}}$ (as is the perturbation solution of the present report).

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TABLE I. BOUNDARY-LAYER PARAMETERS

g_w	$\tilde{\beta}$	J_2	$\dfrac{J_1+J_3}{J_2}$
0	-0.14	0.495	0.134
	0	.468	0
	.50	.4235	-.257
	2.00	.383	-.538
0.2	-0.14	0.504	0.692
	0	.468	.518
	.50	.409	.199
	1.50	.366	.083
0.6	-0.20	0.554	2.034
	0	.468	1.555
	.50	.380	1.185
	2.00	.294	.759
1	0	0.470	2.59
	.05	.452	2.53
	.10	.435	2.48
	.20	.408	2.41
	.40	.370	2.33
	.60	.336	2.27
	.80	.312	2.24
2	-0.10	0.543	5.67
	0	.468	5.18
	.30	.333	5.49
	.50	.274	6.01
	1.00	.1765	7.85

Results for $g_w \neq 1$ are from refs. 6 and 7 wherein $g_w=S_w+1$, $\tilde{\beta}\equiv\beta$, $J_2\equiv l/f_w''$, $(J_1+J_3)/J_2\equiv H_{inc}$. Results for $g_w=1$ are from ref. 8 wherein $\tilde{\beta}=\beta$, $J_2\equiv\theta$, $(J_1+J_3)/J_2\equiv H$.

TABLE II.—RESULTS OF NUMERICAL INTEGRATION OF PERTURBATION EQUATIONS

γ	β	$\sigma=0$				$\sigma=1$			
		Case	μ	a_2	$F_2(\eta_{b,0})$	Case	μ	a_2	$F_2(\eta_{b,0})$
1. 15	0	(a)	$-\frac{1}{2}$	1. 16	0. 798	(a)	$-\frac{1}{4}$	1. 05	0. 863
		(b)	0	1. 07	2. 00	------	0	1. 04	1. 96
	$\frac{1}{3}$	(a)	$-\frac{1}{4}$	1. 29	1. 18	(a)	0	1. 06	1. 64
		------	0	1. 12	1. 71				
		(b)	$\frac{1}{6}$	1. 04	2. 09				
	$\frac{1}{2}$	(a)	$-\frac{1}{8}$	1. 34	1. 39	------	0	1. 08	1. 49
		------	0	1. 17	1. 58	(a)	$\frac{1}{8}$	1. 02	1. 99
		(b)	$\frac{1}{4}$	. 974	2. 03				
	$\frac{3}{4}$	------	0	1. 40	1. 53	------	0	1. 18	1. 31
		(a)	$\frac{1}{16}$	1. 15	1. 45	(a)	$\frac{5}{16}$	. 734	1. 96
		(b)	$\frac{3}{8}$	. 693	1. 56				
	1	(a)	$\frac{1}{4}$	0. 325	0. 506	(c)	1	0. 0518	0. 503
		(b c)	$\frac{1}{2}$	. 188	. 483				
1. 4	0	(a)	$-\frac{1}{2}$	1. 44	0. 869	(a)	$-\frac{1}{4}$	1. 13	0. 843
		(b)	0	1. 20	2. 00	------	0	1. 09	1. 91
	$\frac{1}{3}$	(a)	$-\frac{1}{4}$	1. 77	1. 39	(a)	0	1. 14	1. 59
		------	0	1. 32	1. 76				
		(b)	$\frac{1}{6}$	1. 14	2. 03				
	$\frac{1}{2}$	(a)	$-\frac{1}{8}$	1. 86	1. 64	------	0	1. 19	1. 45
		------	0	1. 44	1. 67	(a)	$\frac{1}{8}$	1. 06	1. 87
		(b)	$\frac{1}{4}$	1. 02	1. 90				
	$\frac{3}{4}$	------	0	1. 95	1. 77	------	0	1. 38	1. 29
		(a)	$\frac{1}{16}$	1. 44	1. 53	(a)	$\frac{5}{16}$	. 655	1. 65
		(b)	$\frac{3}{8}$	. 660	1. 36				
	1	(a)	$\frac{1}{4}$	0. 443	0. 611	(c)	1	0. 0389	0. 643
		(b c)	$\frac{1}{2}$	. 217	. 555				
1. 67	0	(a)	$-\frac{1}{4}$	1. 74	0. 934	(a)	$-\frac{1}{4}$	1. 20	0. 823
		(b)	0	1. 33	2. 00	------	0	1. 15	1. 86
	$\frac{1}{3}$	(a)	$-\frac{1}{4}$	2. 28	1. 55	(a)	0	1. 22	1. 55
		------	0	1. 52	1. 78				
		(b)	$\frac{1}{6}$	1. 25	1. 99				
	$\frac{1}{2}$	(a)	$-\frac{1}{8}$	2. 40	1. 82	------	0	1. 29	1. 40
		------	0	1. 71	1. 73	(a)	$\frac{1}{8}$	1. 11	1. 80
		(b)	$\frac{1}{4}$	1. 10	1. 84				
	$\frac{3}{4}$	------	0	2. 48	1. 91	------	0	1. 53	1. 24
		(a)	$\frac{1}{16}$	1. 74	1. 59	(a)	$\frac{5}{16}$	. 645	1. 53
		(b)	$\frac{3}{8}$	. 701	1. 31				
	1	(a)	$\frac{1}{4}$	0. 546	0. 670	(c)	1	0. 0294	0. 828
		(b c)	$\frac{1}{2}$	. 244	. 605				

a Effect of boundary-layer development.
b Effect of small angles of attack.
c Effect of wedge and cone nose blunting.

TABLE III.—HYPERSONIC FLOW OVER VERY SLENDER BLUNT-NOSED WEDGES AND CONES

$(\beta=1, \mu=(\sigma+1)/2)$

σ	γ	Present numerical solution				Chernyi approximation (eqs. (II9) and (II11))			
		K	a_2	$F_0(0)$	$F_2(0)$	K	a_2	$F_0(0)$	$F_2(0)$
0	1.15	0.938	0.188	0.415	0.483	0.837	0.220	0.500	0.549
	1.40	1.23	.217	.325	.555	1.09	.289	.500	.724
	1.67	1.49	.244	.264	.605	1.22	.347	.500	.866
1	1.15	1.04	0.0518	0.411	0.503	1.01	0.0756	0.500	0.529
	1.40	1.34	.0389	.311	.643	1.23	.0938	.500	.656
	1.67	1.54	.0294	.241	.828	1.34	.110	.500	.764