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CONFIGURATION FACTORS FOR EXCHANGE OF RADIANT ENERGY BETWEEN AXISYMMETRICAL SECTIONS OF CYLINDERS, CONES, AND HEMISPHERES AND THEIR BASES

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CONFIGURATION FACTORS FOR EXCHANGE OF RADIANT ENERGY BETWEEN AXISYMMETRICAL SECTIONS OF CYLINDERS, CONES, AND HEMISPHERES AND THEIR BASES

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SUMMARY

Radiation-interchange configuration factors are derived for axisymmetrical sections of cylinders, cones, and hemispheres radiating internally to annular and circular sections of their bases and to other axisymmetrical sections. The general procedure of obtaining configuration factors is outlined and the results are presented in the form of equations, tables, and figures.

INTRODUCTION

The high temperatures which are encountered in components of aerospace vehicles have brought about a renewed interest in heat transfer by radiation. For certain components, the heat transferred by radiation can be shown to overshadow that transferred by conduction. In the present paper axisymmetric radiation for some common axisymmetric shapes is studied.

Radiative transfer of heat from one area to another depends, among other things, upon the fraction of the radiant energy emitted by one area which is intercepted by a second area. This fraction is identified by several names, such as the configuration factor, the interchange factor, the angle factor, or the geometric view factor, and is a function of the geometrical relation of the areas involved. In the present paper, the term configuration factor will be used to designate the fraction.

Configuration factors are available for radiation between various surfaces (see refs. 1, 2, and 3) but, for the most part, the areas which are involved are plane. The purpose of the present paper is to provide configuration factors for some of the more common nonplanar surfaces. Some of the configuration factors presented herein are obtained, in appendix A, by integrating the basic equation which defines the factor and the remainder are obtained, in appendix B, by the application of configuration-factor algebra. The techniques of configuration-factor algebra make it possible, in some situations, to obtain the desired configuration factor from available factors without the need for integration.

In addition to being listed in tables and given in the form of equations, the results are presented in carpet plots which permit an estimate of the magnitude of a given factor and show the effect of varying the proportions of the surfaces involved.

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SYMBOLS

A	area
a	radius of the base of a surface of revolution
C	circular area
F	configuration factor defined by equation (4)
H	height of a cone
j,k,m,n	integers
L_m^n	length between the mth and nth planes
M	nondimensional parameter, r_1/a
N	nondimensional parameter, L_o^1/a
q	energy per unit time
R_m^n	area of ring between the mth and nth planes
r	radius of circle
S	distance between centers of the areas exchanging radiant energy
T	absolute temperature
x,y,z	Cartesian coordinates

ρ,θ,z polar coordinates

ρ,θ,φ spherical coordinates

σ Stefan-Boltzmann constant

ψ half the apex angle of a cone

ψ_n angle between the normal to the area A_n and the line between centers of the area A_n and the area which intercepts radiation from A_n

ω solid angle

Subscripts:

0,1,2,3,j,k,n identification of an area, plane, or point

C_j circular area in the base of a body of revolution

$C_{j,k}$ an annular area in the base of a body of revolution $(C_{j,k} = C_j - C_k)$

dA_1,dA_2 from an area dA_1 to an area dA_2

j,k from an area j to an area k

Superscripts:

1,2,3,a,H,m,n identification of an area, plane, or point

ANALYSIS

Black-Body Radiation Between Two Isothermal Surfaces of Arbitrary Orientation

Consider the exchange of radiant energy between two isothermal black surfaces, A_1 and A_2, of arbitrary orientation separated by a nonabsorbing medium as shown in figure 1. The energy per unit time leaving the first elementary surface dA_1 in the direction of the second dA_2 is given by (see ref. 1)

$$dq_{dA_1} = \frac{\sigma}{\pi} T_1^4 \cos \psi_1 \, dA_1 \tag{1}$$

where

q energy, per unit time

σ Stefan-Boltzmann constant

T_1 absolute temperature of dA_1

ψ_1 angle between normal to dA_1 and line between centers of areas dA_1 and dA_2

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The portion of the energy per unit time leaving dA_1 which is intercepted by dA_2 depends upon the solid angle $d\omega$ subtended by dA_2 and can be expressed as

$$dq_{dA_1,dA_2} = \frac{\sigma}{\pi} T_1^4 \cos \psi_1 \, dA_1 \, d\omega \tag{2}$$

where

$$d\omega = \frac{\cos \psi_2}{S^2} dA_2$$

and S is the distance between the centers of the areas dA_1 and dA_2.

The energy per unit time which leaves the surface dA_1 and is intercepted by the surface dA_2 (eq. (2)) can therefore be expressed as

$$dq_{dA_1,dA_2} = \frac{\sigma}{\pi} T_1^4 \frac{\cos \psi_1 \cos \psi_2}{S^2} dA_1 \, dA_2 \tag{3}$$

By defining

$$F_{A_1,A_2} = \frac{1}{\pi A_1} \int_{A_1} \int_{A_2} \frac{\cos \psi_1 \cos \psi_2}{S^2} dA_2 \, dA_1 \tag{4}$$

equation (3) becomes

$$q_{A_1,A_2} = \sigma T_1^4 A_1 F_{A_1,A_2} \tag{5}$$

The term F_{A_1,A_2} depends only upon the geometric configuration of the two surfaces and is known as the configuration factor. The configuration factor is defined as the fraction of the total energy per unit time which leaves a surface and is intercepted by a second surface. The above procedure can be repeated to determine the energy per unit time which leaves the second surface and is intercepted by the first with the following results:

$$q_{A_2,A_1} = \sigma T_2^4 A_2 F_{A_2,A_1} \tag{6}$$

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where

$$F_{A_2,A_1} = \frac{1}{\pi A_2} \int_{A_2} \int_{A_1} \frac{\cos \psi_1 \cos \psi_2}{S^2} \, dA_1 \, dA_2 \tag{7}$$

It can be seen from equations (4) and (7) that

$$A_1 F_{A_1,A_2} = A_2 F_{A_2,A_1} \tag{8}$$

Equation (8) is known as the reciprocity theorem and, as is seen in appendix B, is very useful in the application of configuration-factor algebra. For brevity, whenever the areas involved are understood, equation (8) is written as

$$A_1 F_{1,2} = A_2 F_{2,1}$$

The net exchange of radiant energy between A_1 and A_2 of figure 1, obtained as the difference between equations (5) and (6) and simplified by the reciprocity theorem, is

$$q_{net} = \sigma A_1 F_{1,2}\left(T_1^4 - T_2^4\right) \tag{9}$$

Black-Body Radiation in Closed Systems

Consider a closed system composed of n isothermal black-body surfaces separated by a nonabsorbing medium. The net heat flow result as presented for two isothermal black-body surfaces can be extended to include the n isothermal black surfaces in the following manner.

According to the Stefan-Boltzmann law, the radiant energy leaving an isothermal surface j is

$$\sigma T_j^4 A_j \tag{10}$$

The radiant energy incident upon the area A_j from all other surfaces in the system is

$$\sum_{k=1}^{n} \sigma T_k^4 A_k F_{k,j} \tag{11}$$

so that the net exchange of energy becomes

$$q_{net} = \sigma T_j^4 A_j - \sum_{k=1}^{n} \sigma T_k^4 A_k F_{k,j} \tag{12}$$

Finally, equation (12) can be reduced by applying the reciprocity theorem to obtain the following equation:

$$q_{net} = \sigma A_j \left[T_j^4 - \sum_{k=1}^{n} T_k^4 F_{j,k} \right] \tag{13}$$

With a knowledge of the configuration factors $F_{j,k}$, equation (13) can be used to obtain heat flows or temperature distributions in a closed system. Reference 4 demonstrates the use of equation (13) when radiant heat transfer is accompanied by heat transfer by conduction.

Configuration Factors

In practice, configuration factors can be obtained by experimental, numerical, and analytical means. (See, for example, ref. 1.) In the present paper, some configuration factors are found directly from equation (4) and some indirectly from equation (4) through configuration-factor algebra.

The shapes considered are surfaces of revolution (cylinder, cone, and hemisphere) with the ends closed by plane surfaces. All areas considered are axisymmetrical and therefore the resulting configuration factors are applicable only to surfaces exhibiting axisymmetrical temperature distributions.

In the study of radiant heat transmission within a system composed of a body of revolution and a base plane six types of general configuration factors are required. These six types of configuration factors, which are derived in the present paper, are shown in figure 2 for the case where the surface of revolution is a cylinder. Similar configuration factors have been determined for the cone and hemisphere.

In all six configurations, the surface of revolution is divided into rings by one or more planes which are parallel to the base at heights of L_0^1, L_0^2, . . . L_0^n. The subscripts and superscripts indicate the planes between which the length is measured (zero being the base plane) so that the rings between these planes will be known as R_0^1, R_0^2, . . . R_0^n.

Areas in the base plane considered are either circular or annular and are designated by C_0, C_1, . . . C_n and by $C_{n-1,n}$, respectively. The circular area C_0 represents the full base of the cylinder so that $C_{n-1} > C_n$. The annular region $C_{n-1,n}$ represents the region between the circular areas C_{n-1} and C_n so that

$$C_{n-1,n} = C_{n-1} - C_n$$

Since the circular and annular areas are normally in the base plane, there is usually no need to specify the plane in which they lie. However, for a few cases it is necessary to specify the plane and this will be done by the use of superscripts. For example, $C_{n-1,n}^m = C_{n-1}^m - C_n^m$ so that the annular region is in the mth plane and is equal to the area contained between concentric circles in that plane. Whenever C terms appear without superscripts the area is understood to be in the base plane.

By using this method, R_0^1,C_1 would indicate a ring on the surface of revolution extending from the base plane to the first plane above the base exchanging radiant energy with a circular area C_1 in the plane of the base. In the same manner $R_1^2,C_{1,2}$ would indicate a ring lying between the first and second planes above the base exchanging radiant energy with an annular area, $C_1 - C_2$, in the base plane.

By using this symbolism, the six configurations presented in figure 2 are designated as R_0^1,C_1, R_1^2,C_1, $R_0^1,C_{1,2}$, $R_1^2,C_{1,2}$, R_2^3,R_0^1, and R_1^2,R_1^2.

The derivation of the configuration factors is presented in the appendixes. The configuration factors for the geometry of R_0^1,C_1 are obtained by integration of equation (4) for cylinders, cones, and hemispheres in appendix A. With the exception of R_2^3,R_0^1 for the hemisphere, the remaining configuration factors are obtained by using configuration-factor algebra and the equations derived for the geometry of R_0^1,C_1. The use of configuration-factor algebra is explained and demonstrated in appendix B where configuration factors are given for the geometries of R_1^2,C_1, $R_0^1,C_{1,2}$, $R_1^2,C_{1,2}$, R_2^3,R_0^1, and R_1^2,R_1^2. The geometry R_2^3,R_0^1 for the hemisphere is not amenable to the use of configuration-factor algebra. This situation results from the fact that configuration-factor algebra depends to a large extent on dealing with similar surfaces and the geometry of R_2^3,R_0^1 results in spherical segments which are not hemispheres. Therefore, the result for this case is obtained by integration in appendix A.

RESULTS AND DISCUSSION

Since all the configuration factors given in the present paper are obtained with one exception $\left(R_2^3R_0^1 \text{ for a hemisphere}\right)$ from three primary equations (A7), (A16), and (A26) involving the geometry of $R_0^1C_1$ and configuration-factor algebra, only evaluations of these three equations will be discussed in any detail in this section. The configuration factor derived in appendix A for $R_2^3R_0^1$ when the surface of revolution is a hemisphere is elementary and will not require discussion.

Table I is a summary which indicates, by number, the proper equation to use for the cases previously described. In addition to the specific surfaces of revolution treated in detail, fundamental equations are indicated for use with an arbitrary surface of revolution.

Cylinders

The configuration factor for the geometry R_0^1,C_1 when the surface of revolution is a cylinder is derived in appendix A and is given by equation (A7). Equation (A7) is given in a nondimensional form by equation (A8) which is

$$F_{R_0^1,C_1} = \frac{1}{4N}\left[\sqrt{N^4 + 2N^2(1 + M^2) + (1 - M^2)^2} - (1 - M^2) - N^2\right]$$

The nondimensional parameters are

$$M = r_1/a$$

$$N = L_0^1/a$$

where a is the radius of the base, r_1 is the radius of C_1, and L_0^1 is the height of R_0^1. (See fig. 3.)

The term M is a ratio of the radius of area C_1 to the radius of the base of the cylinder C_0, whereas the term N is a slenderness ratio.

Table II presents results from the nondimensional equation (A8) for a wide range of r_1/a and L_0^1/a. The data of table II are also given in the form of a carpet plot in figure 4.

Cones

The configuration factor for the geometry of R_0^1,C_1 when the surface of revolution is a cone is derived in appendix A and is given by equation (A16), which is a lengthy equation that results from the evaluation of a nonelementary integral. Because a large number of terms in the equation must be defined, it will not be repeated in the text and reference should be made to appendix A. Table III gives the results of the evaluation of a nondimensional form of equation (A16) for combinations of L_0^1/H and r_1/a between 0.1 and 1.0 and for cone half-angles of 5°, 10°, and 20°. The dimensions L_0^1 and r_1 are as shown in figure 5.

The data of table III are presented in the form of a carpet plot in figure 6. Figure 6 contains three parts, one for each of the half-angles considered. As expected, the evaluation of equation (A17), shows that the cone results approach the results obtained for the cylinder as the base angle approaches $\pi/2$.

Hemispheres

The configuration factor for the geometry of R_0^1, C_1 when the surface of revolution is a hemisphere is derived in appendix A and is given by equation (A26). Equation (A26) is given in nondimensional form by equation (A27), which is

$$F_{R_0^1, C_1} = \frac{1}{4N}\left[\sqrt{\left(1 - M^2\right)^2 + 4M^2N^2} - \left(1 - M^2\right)\right]$$

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where the nondimensional parameters are, as for the cylinder,

$$M = r_1/a$$

$$N = L_0^1/a$$

and the dimensions L_0^1, r_1, and a are as shown in figure 7. Table IV presents results of the evaluation of equation (A27) for combinations of r_1/a and L_0^1/a between 0.1 and 1.0.

The data of table IV are presented in the form of a carpet plot in figure 8. Figure 8 shows that $F_{R_0^1, C_1}$ is constant for all values of N when $M = 1$.

CONCLUDING REMARKS

Configuration factors are presented which can be used in heat-transfer studies involving nonplanar surfaces at high temperatures. These configuration factors have been derived for axisymmetrical sections of cylinders, cones, and hemispheres radiating internally to circular and annular regions of their bases or to other axisymmetric sections. Some of the factors were obtained by integrating fundamental equations expressed in terms of convenient coordinates. The remainder of the factors were obtained by utilizing configuration-factor algebra and the results of the integrations. The use of configuration-factor algebra is explained and demonstrated. The calculated radiation configuration factors are given in tables and plots.

Langley Research Center,
National Aeronautics and Space Administration,
Langley Air Force Base, Va., July 20, 1961.

APPENDIX A

DERIVATION OF CONFIGURATION FACTORS FOR CYLINDERS, CONES, AND HEMISPHERES BY INTEGRATION

The equation for the portion of the total radiation from an elemental area dA_1 which is incident upon an elemental area dA_2 is derived in the body of the paper as

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$$F_{A_1,A_2} = \frac{1}{\pi A_1} \int_{A_1} \int_{A_2} \frac{\cos \psi_1 \cos \psi_2}{S^2} \, dA_2 \, dA_1 \tag{A1}$$

where

S distance between dA_1 and dA_2

ψ_1, ψ_2 angles between the line joining the areas dA_1 and dA_2 and the normals to the respective areas.

The configuration factor will be derived for the most general case of a section of a body of revolution extending from the base plane to a given plane above it exchanging radiant energy with an axisymmetrical, circular region located in the plane of the base as shown in figures 3, 5, and 7 for cylinders, cones, and hemispheres, respectively. The parameters of equation (A1), ψ_1, ψ_2, and S, must be expressed in terms of the chosen coordinates so that the equation can be integrated.

In order to keep the solution as general as possible, the integration of equation (A1) will be carried out over surfaces designated as A_1 and A_2. The area A_1 will represent the area on the surface of revolution R_0^1 and the area A_2 will represent the circular area C_1 in the base. This will apply throughout the derivation of the configuration factor for the geometry of R_0^1,C_1 for cylinders, cones, and hemispheres. For the geometry of R_2^3,R_0^1 for a hemsiphere, the areas A_1 and A_2 represent the upper and lower rings, respectively. However, in all cases the limits will be written in general terms.

Configurations Factors for the Geometry R_0^1, C_1

Cylinder.- When the surface of revolution is a cylinder (see fig. 3), the parameters of equation (A1) in terms of the polar coordinates ρ, θ, and z become

$$\cos \psi_1 = \frac{a - \rho_2 \cos(\theta_1 - \theta_2)}{S} \tag{A2}$$

$$\cos \psi_2 = \frac{z}{S} \tag{A3}$$

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$$S^2 = z^2 + a^2 + \rho_2{}^2 - 2a\rho_2\cos(\theta_1 - \theta_2) \tag{A4}$$

$$dA_1 = a\ d\theta_1\ dz$$

$$dA_2 = \rho_2\ d\rho_2\ d\theta_2$$

Integration over θ_2, from 0 to 2π (after substituting eqs. (A2), (A3), and (A4) into eq. (A1)), gives the configuration factor from the area dA_1 to the differential ring in the base $2\pi\rho_2\ d\rho_2$ as

$$dA_1F_{dA_1,2\pi\rho_2 d\rho_2} = \frac{2z\left[a\left(z^2 + a^2 + \rho_2{}^2\right) - 2a\rho_2{}^2\right]\rho_2\ d\rho_2\ dA_1}{\left[\left(z^2 + a^2 + \rho_2{}^2\right)^2 - 4a^2\rho_2{}^2\right]^{3/2}} \tag{A5}$$

The configuration factor from the differential area dA_1 to the finite area A_2 can be obtained from equation (A5) by integrating over ρ_2, from 0 to r_1, which after rearranging gives

$$dA_1F_{dA_1,C_1} = \frac{z}{2a}\left\{\frac{z^2 + a^2 + r_1{}^2}{\left[z^4 + 2\left(a^2 + r_1{}^2\right)z^2 + \left(a^2 - r_1{}^2\right)^2\right]^{1/2}} - 1\right\} dA_1 \tag{A6}$$

The following equation, obtained by integrating equation (A6) over θ_1 from 0 to 2π and over z from 0 to L_0^1, gives the configuration factor from the area A_1 to the area A_2:

$$F_{R_0^1,C_1} = \frac{1}{4aL_0^1}\left[\sqrt{\left(L_0^1\right)^4 + 2\left(L_0^1\right)^2\left(a^2 + r_1^2\right) + \left(a^2 - r_1^2\right)^2} - \left(a^2 - r_1^2\right) - \left(L_0^1\right)^2\right] \tag{A7}$$

or in nondimensional form

$$F_{R_0^1,C_1} = \frac{1}{4N}\left[\sqrt{N^4 + 2N^2\left(1 + M^2\right) + \left(1 - M^2\right)^2} - \left(1 - M^2\right) - N^2\right] \tag{A8}$$

where

$$M = r_1/a$$

$$N = L_0^1/a$$

Results obtained by evaluating equation (A8) in the range $0.1 \leqq M \leqq 1.0$ and $0.2 \leqq N \leqq 200$ are given in table II and figure 4.

A special case presents itself when A_2 becomes the full area of the base of the cylinder ($M = 1$). Equation (A8) then reduces to

$$F_{R_0^1,C_0} = \frac{1}{4}\left(\sqrt{N^2 + 4} - N\right) \tag{A9}$$

The reciprocity theorem can be employed to determine the configuration factor from the base of the cylinder to the walls F_{C_0,R_0^1}.

<u>Cones</u>.- When the surface of revolution is a cone (fig. 5), the parameters of equation (A1) can be expressed as

$$\cos \psi_1 = \frac{\cos \psi}{S}\left[a - \rho_2 \cos\left(\theta_1 - \theta_2\right)\right] \tag{A10}$$

$$\cos \psi_2 = \frac{z}{S} \tag{A11}$$

$$S^2 = z^2 + \rho_1{}^2 + \rho_2{}^2 - 2\rho_1\rho_2 \cos(\theta_1 - \theta_2) \tag{A12}$$

$$dA_2 = \rho_2 \, d\rho_2 \, d\theta_2$$

where

$$\psi = \text{Half apex angle}$$

$$a = \rho_1 + z \tan \psi$$

Integration over θ_2 after substituting equations (A10), (A11), and (A12) into equation (A1) gives the configuration factor from the area dA_1 to the differential ring in the base as

$$dA_1 \, F_{dA_1, 2\pi\rho_2 d\rho_2} = \frac{2z \cos \psi \, dA_1 \left[a\left(z^2 + \rho_1{}^2 + \rho_2{}^2\right) - 2\rho_1\rho_2{}^2\right]\rho_2 \, d\rho_2}{\left[\left(z^2 + \rho_1{}^2 + \rho_2{}^2\right)^2 - 4\rho_1{}^2\rho_2{}^2\right]^{3/2}} \tag{A13}$$

The configuration factor from the differential area dA_1 to the area C_1 is obtained by integrating equation (A13) over ρ_2 from 0 to r_1. It is

$$dA_1 F_{dA_1, C_1} = 2z \cos \psi \, dA_1 \left\{ \frac{\rho_1{}^4 - a\rho_1{}^3 + \rho_1{}^2\left(2z^2 - r_1{}^2\right) + a\rho_1\left(r_1{}^2 - z^2\right) + z^2\left(z^2 + r_1{}^2\right)}{4\rho_1{}^2\left[\left(z^2 + \rho_1{}^2\right)^2 + 2r_1{}^2\left(z^2 - \rho_1{}^2\right) + r_1{}^4\right]^{1/2}} - \frac{\rho_1{}^4 - a\rho_1{}^3 + 2z^2\rho_1{}^2 - az^2\rho_1 + z^2}{4\rho_1 z^2\left(z^2 - \rho_1{}^2\right)} \right\} \tag{A14}$$

Since the area A_1 is a surface of revolution, where

$$dA_1 = \rho_1 \sec \psi \, d\theta_1 \, dz$$

the configuration factor from an area R_0^1 to an area C_1 after integrating over θ_1 and collecting terms becomes

$$R_0^1 F_{R_0^1, C_1} = \pi \left[\int_0^{L_0^1} \frac{\left(a_1 z^3 + b_1 z^2 + c_1 z + d_1\right)dz}{\left(a_2 z^4 + b_2 z^3 + c_2 z^2 + d_2 z + e_2\right)^{1/2}} - \int_0^{L_0^1} \frac{a_3 z^3 + b_3 z^2 + c_3 z + d_3}{a_4 z^2 + b_4 z + c_4} \, dz \right] \tag{A15}$$

where

$$a_1 = \sec^4\psi$$

$$b_1 = -3a \tan \psi \sec^2\psi$$

$$c_1 = \left(3a^2 - r_1{}^2\right)\tan^2\psi + a^2 - r_1{}^2$$

$$d_1 = -a \tan \psi\left(a^2 - r_1{}^2\right)$$

$$a_2 = \sec^4\psi$$

$$b_2 = -4a \tan \psi \sec^2\psi$$

$$c_2 = 2a^2\left(2 \tan^2\psi + \sec^2\psi\right) + 2r_1{}^2\left(1 - \tan^2\psi\right)$$

$$d_2 = -4a \tan \psi\left(a^2 - r_1{}^2\right)$$

$$e_2 = \left(a^2 - r_1{}^2\right)^2$$

$$a_3 = \sec^4\psi$$

$$b_3 = -3a \tan \psi \sec^2\psi$$

$$c_3 = a^2\left(3 \tan^2\psi + 1\right)$$

$$d_3 = -a^3\tan \psi$$

$$a_4 = \sec^2\psi$$

$$b_4 = -2a \tan \psi$$

$$c_4 = a^2$$

For brevity, equation (A15) will be written as

$$F_{R_0^1, C_1} = \frac{\cos \psi \cot \psi\left(I_1 + I_2\right)}{L_0^1\left(H + L_1^H\right)} \tag{A16}$$

The first integral of equation (A15) is not an elementary integral and it is necessary to introduce elliptic functions in order to evaluate it. Reference 4 presents a method allowing integrals containing the square root of a quartic in the denominator of the integrand to be put into Legendre's standard form of an elliptic integral.

By making use of reference 5 to evaluate the elliptic integrals the first integral of equation (A15) becomes:

$$
\begin{aligned}
I_1 = \frac{(q - p)}{\sec^2\psi}\Bigg(\Bigg(&\left[\frac{W}{K_9(x + 1)^2} + \frac{2\left(VK_9 - WK_{10}\right)}{K_9^2(x + 1)} - \frac{2\left(VK_9 - WK_{10}\right)}{K_9^2 x}\right]\sqrt{G(x)}\,\Bigg|_{-\frac{p}{q}}^{\frac{L_0^1 - p}{q - L_0^1}} \\
&+ \frac{\left(K_1K_9 - WK_{12}\right)K_9 + K_{12}\left(VK_9 - WK_{10}\right)}{K_9^2\sqrt{K_6K_7}}\left[E\left(\varphi_1, k\right) + E\left(\varphi_2, k\right)\right] \\
&+ \frac{K_{12}\left(VK_9 - WK_{10}\right)}{K_9^2}\,\frac{K_5}{K_8}\sqrt{\frac{K_7}{K_6}}\left[\mathrm{dn}\varphi_1\mathrm{cs}\varphi_1 + E\left(\varphi_1, k\right) + \mathrm{dn}\varphi_2\mathrm{cs}\varphi_2 + E\left(\varphi_2, k\right)\right] \\
&+ \frac{K_9\left(UK_9 - WK_{11}\right) - 2K_{10}\left(VK_9 - WK_{10}\right)}{K_9^2}\left(\frac{\delta}{s\sqrt{K_6K_7}\,\sqrt{s^2 - k^2}}\right)\Bigg\{\left[F\left(\varphi_2, k\right)\right. \\
&\left. + F\left(\varphi_1, k\right)\right]\left[\frac{\sqrt{s^2 - k^2}}{s\delta} - Z(A, k)\right] + \sum_{m=1}^{\infty}\frac{\bar{q}^m\sin(2m\varpi)\sin\left(2m\bar{v}_1\right)}{m\sinh(2m\bar{p})} \\
&+ \sum_{m=1}^{\infty}\frac{\bar{q}^m\sin(2m\varpi)\sin\left(2m\bar{v}_2\right)}{m\sinh(2m\bar{p})} + \frac{1}{2}\log_e\left[\frac{\sin\left(\varpi + \bar{v}_2\right)\left(\sqrt{s^2 - k^2} - s\,\mathrm{dn}\varphi_2\right)}{\sin\left(\varpi - \bar{v}_2\right)\left(\sqrt{s^2 - k^2} + s\,\mathrm{dn}\varphi_2\right)}\right] \\
&+ \frac{1}{2}\log_e\left[\frac{\sin\left(\varpi + \bar{v}_1\right)\left(\sqrt{s^2 - k^2} - s\,\mathrm{dn}\varphi_1\right)}{\sin\left(\varpi - \bar{v}_1\right)\left(\sqrt{s^2 - k^2} + s\,\mathrm{dn}\varphi_1\right)}\right]\Bigg\}\Bigg)\Bigg)
\end{aligned}
\qquad \text{(A17)}
$$

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where p and q are roots of the equation

$$(\alpha + \bar{\alpha} - \beta - \bar{\beta})\eta^2 + 2(\beta\bar{\beta} - \alpha\bar{\alpha})\eta + \alpha\bar{\alpha}(\beta + \bar{\beta}) - \beta\bar{\beta}(\alpha + \bar{\alpha}) = 0$$

and where α and β are the nonconjugate roots of the quartic

$$z^4 + \frac{b_2}{a_2} z^3 + \frac{c_2}{a_2} z^2 + \frac{d_2}{a_2} z + \frac{e_2}{a_2} = 0$$

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and

$$A = \sin^{-1}\sqrt{\frac{K_6}{K_5 + K_6}}$$

$\mathrm{cs}\ \varphi_n = \cot \varphi_n$

$\mathrm{dn}\ \varphi_n$ Jacobi elliptic function, $\sqrt{1 - k^2\sin^2\varphi_n}$

$E(\varphi_n, k)$ incomplete elliptic integral of the second kind

$F(\varphi_n, k)$ incomplete elliptic integral of the first kind

$G(x) = (K_5 + K_6 x)(K_7 + K_8 x)$

k modulus, $\dfrac{K_6K_7 - K_5K_8}{K_6K_7}$

k' complementary modulus, $\sqrt{1 - k^2}$

K,E complete elliptic integrals of the first and second kinds, respectively

K' complete elliptic integral of the first kind with a modulus of k'

$K_1 = a_1q^3 + b_1q^2 + c_1q + d_1$

$K_2 = 3a_1pq^2 + b_1q^2 + 2b_1pq + 2c_1q + c_1p + 3d_1$

$K_3 = 3a_1p^2q + 2b_1pq + b_1p^2 + c_1q + 2c_1p + 3d_1$

$$K_4 = a_1p^3 + b_1p^2 + c_1p + d_1$$

$$K_5 = p^2 - 2p\mathrm{Re}(\alpha) + |\alpha|^2$$

$$K_6 = q^2 - 2q\mathrm{Re}(\alpha) + |\alpha|^2$$

$$K_7 = p^2 - 2p\mathrm{Re}(\beta) + |\beta|^2$$

$$K_8 = q^2 - 2q\mathrm{Re}(\beta) + |\beta|^2$$

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$$K_9 = -2(K_5 + K_6)(K_7 + K_8)$$

$$K_{10} = 3(K_5K_8 + K_6K_7 + 2K_6K_8)$$

$$K_{11} = -(K_5K_8 + K_6K_7 + 6K_6K_8)$$

$$K_{12} = 2K_6K_8$$

$$\bar{p} = \frac{K'}{2K}$$

$$\bar{q} = e^{-2\bar{p}}$$

$$s = \sqrt{\frac{K_5 + K_6}{K_6}}$$

$$U = K_2 - 3K_1$$

$$\bar{v}_n = \frac{\pi F(\varphi_n, k)}{2k}$$

$$V = K_3 - 2K_2 + 3K_1$$

$$W = K_4 - K_3 + K_2 - K_1$$

$Z(A,k)$ Jacobi Zeta function, $E(A,k) - \frac{E}{K} F(A,k)$

$$\delta = \sqrt{\frac{K_5}{K_6}}$$

$$\varphi_1 = \tan^{-1}\left(-\frac{p}{q}\sqrt{\frac{K_6}{K_5}}\right)$$

$$\varphi_2 = \tan^{-1}\left(\frac{L_0^1 - p}{q - L_0^1}\sqrt{\frac{K_6}{K_5}}\right)$$

$$\bar{\omega} = \frac{\pi F(A,k)}{2K}$$

The second integral of equation (A15) is an elementary integral which upon integration becomes

$$I_2 = L_0^1 \frac{\sec^2\psi}{2}\left(2H \sin^2\psi + L_0^1\right) \tag{A18}$$

Equation (A16) represents the fraction of the total radiant energy which leaves surface A_1, the frustum of a right cone, and is intercepted by surface A_2, a plane circular area in the base of the cone. If the intercepting area A_2 is the entire base of the cone, the results are greatly simplified. The procedure remains unchanged up to the integration of equation (A15) which now contains two elementary integrals whose integration yields

$$F_{R_0^1,C_0} = \frac{1}{2\left(H + L_1^H\right)}\left[\sqrt{\left(L_0^1\right)^2 \csc^2\psi + 4HL_1^H} + \csc\psi\left(2H \sin^2\psi - L_0^1\right)\right] \tag{A19}$$

The configuration factor for the complete cone exchanging radiant energy with the complete base becomes

$$F_{R_0^H,C_0} = \sin\psi \tag{A20}$$

The configuration factor from the base of the cone to the walls can be found from the reciprocity theorem with the aid of configuration-factor algebra. (See appendix B.)

Hemisphere.- When the surface of revolution is a hemisphere (fig. 7), the parameters of equation (A1) can be expressed as

$$\cos\psi_1 = \frac{2\left[a^2 - \rho_1\rho_2 \cos\left(\theta_1 - \theta_2\right)\right]}{aS} \tag{A21}$$

$$\cos \psi_2 = \frac{z}{S} \tag{A22}$$

$$S^2 = a^2 + \rho_2^2 - 2\rho_1\rho_2 \cos(\theta_1 - \theta_2) \tag{A23}$$

and

$$dA_1 = a \, d\theta_1 \, dz$$

$$dA_2 = \rho_2 \, d\rho_2 \, d\theta_2$$

Integration over θ_2 after substituting equations (A21), (A22), and (A23) into equation (A1) gives the configuration factor from the area dA_1 to the differential ring in the base as

$$dA_1F_{dA_1,2\pi\rho_2 d\rho_2} = \frac{2z \, dA_1\left[\left(\rho_1^2 + z^2\right)^2 + \left(z^2 - \rho_1^2\right)\rho_2^2\right]\rho_2 \, d\rho_2}{a\left[\rho_2^4 + 2\rho_2^2\left(z^2 - \rho_1^2\right) + \left(\rho_1^2 + z^2\right)^2\right]^{3/2}} \tag{A24}$$

where a is the radius of the hemisphere. Integration of equation (A24) over ρ_2 from 0 to r_1 gives, after some rearranging, the configuration factor from the differential area dA_1 to the area C_1 as

$$dA_1F_{dA_1,C_1} = \frac{zr_1^2 dA_1}{a\left[r_1^4 + 2r_1^2\left(2z^2 - \varepsilon^2\right) + a^4\right]^{1/2}} \tag{A25}$$

Integrating over θ_1 from 0 to 2π gives

$$dA_1F_{2\pi a dz,C_1} = \frac{2\pi r_1^2 z \, dz}{\left[\left(a^2 - r_1^2\right)^2 + 4r_1^2z^2\right]^{1/2}}$$

Integrating over z from 0 to L_0^1 gives the configuration factor from area R_0^1 to area C_1 as

$$F_{R_0^1, C_1} = \frac{1}{4aL_0^1}\left[\sqrt{4r_1^{\,2}\left(L_0^1\right)^2 + \left(a^2 - r_1^{\,2}\right)^2} - \left(a^2 - r_1^{\,2}\right)\right] \tag{A26}$$

where r_1 is the radius of the circular area in the equatorial plane, and L_0^1 is the vertical height of the area R_0^1.

Equation (A26) may be put in nondimensional form by letting

$$M = r_1/a$$

$$N = L_0^1/a$$

so that the configuration factor becomes

$$F_{R_0^1, C_1} = \frac{1}{4N}\left[\sqrt{\left(1 - M^2\right)^2 + 4M^2N^2} - \left(1 - M^2\right)\right] \tag{A27}$$

An interesting and useful result can be obtained from equation (A25). If A_2 is taken as the total area of the base (i.e., $r_1 = a$), equation (A25) reduces to

$$dA_1F_{dA_1, C_0} = \frac{dA_1}{2} \tag{A28}$$

By integrating over A_1 the following result is obtained:

$$F_{A_1, C_0} = \frac{1}{2} \tag{A29}$$

Now since A_1 has not been specified it follows that the configuration factor from any area on the surface of a hemisphere to the equatorial plane is one-half.

Configuration Factors for the Geometry R_2^3, R_0^1 for the Hemisphere

When the surfaces exchanging radiant energy are rings on the surface of a hemisphere (fig. 9), the parameters of equation (A1) become

$$\cos \psi_1 = \cos \psi_2$$

$$= \frac{a}{S}\left[1 - \sin \theta_1 \sin \theta_2 \cos\left(\varphi_1 - \varphi_2\right) - \cos \theta_1 \cos \theta_2\right] \tag{A30}$$

and

$$S^2 = 2a^2\left[1 - \sin \theta_1 \sin \theta_2 \cos\left(\varphi_1 - \varphi_2\right) - \cos \theta_1 \cos \theta_2\right] \tag{A31}$$

where φ and θ are as shown in figure 9.

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Substitution of equations (A30) and (A31) into equation (A1) yields

$$dA_1 F_{dA_1,dA_2} = \frac{dA_1\ dA_2}{4\pi a^2} \tag{A32}$$

Since A_2 is a surface of revolution

$$dA_2 = a^2 \sin \theta_2\ d\theta_2\ d\varphi_2$$

integration of equation (A31) is simplified and gives the configuration factor from the differential area dA_1 to the ring R_0^1 as

$$dA_1 F_{dA_1,R_0^1} = \frac{L_0^1}{2a}\ dA_1 \tag{A33}$$

Integration over A_1 where

$$dA_1 = a^2 \sin \theta_1\ d\theta_1\ d\varphi_1$$

gives the configuration factor from a ring to a ring as

$$R_2^3 F_{R_2^3,R_0^1} = \pi L_0^1 L_2^3 \tag{A34}$$

The areas can be expressed as

$$R_2^3 = 2\pi a L_2^3$$

and

$$R_0^1 = 2\pi a L_0^1$$

so that

$$\left.\begin{aligned} F_{R_2^3, R_0^1} &= \frac{1}{2}\,\frac{L_0^1}{a} \\ F_{R_0^1, R_2^3} &= \frac{1}{2}\,\frac{L_2^3}{a} \end{aligned}\right\} \tag{A35}$$

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where L_0^1 and L_2^3 are the vertical heights of the hemispherical segments.

APPENDIX B

CONFIGURATION-FACTOR ALGEBRA

It is possible to determine configuration factors for many cases from existing configuration factors by utilizing the technique of configuration-factor algebra (see the section entitled "Geometric Flux Algebra" in ref. 1) and the previously mentioned reciprocity theorem. The basic requirement involved in this technique is that the unknown configuration factors be of such a nature that they can be expressed as sums and differences of known configuration factors. The procedure is best explained by an example.

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Example

Suppose that it is necessary to determine the configuration factor from a ring on the surface of a cylinder R_1^2 to an annular region $C_{1,2}$ of the base as shown in figure 10. It will be shown that it is possible to determine this configuration factor from the configuration factors for other geometries of figure 10 which are readily obtained from equation (A7).

Equation (A7) gives the configuration factor for a particular geometry. That is, the surface of the cylinder must extend from the intersection of the base plane to any height above this intersection. The intercepting area is also restricted in that it must be a circular area the center of which is on the axis of revolution. It is therefore necessary to express the desired configuration factor in terms of those which have been obtained. This can be done as follows. From figure 10, it can be seen that

$$R_1^2 F_{R_1^2, C_1} = R_0^2 F_{R_0^2, C_1} - R_0^1 F_{R_0^1, C_1} \tag{B1}$$

and

$$R_1^2 F_{R_1^2, C_2} = R_0^2 F_{R_0^2, C_2} - R_0^1 F_{R_0^1, C_2} \tag{B2}$$

Since $C_{1,2} = C_1 - C_2$, then

$$R_1^2 F_{R_1^2,C_{1,2}} = R_1^2 F_{R_1^2,C_1} - R_1^2 F_{R_1^2,C_2} \tag{B3}$$

The left-hand side of equation (B3) can be expressed as the difference of the right-hand sides of equations (B1) and (B2) as follows:

$$\begin{aligned} F_{R_1^2,C_{1,2}} = \frac{1}{R_1^2}\Bigg[& R_0^2\left(F_{R_0^2,C_1} - F_{R_0^2,C_2}\right) \\ & - R_0^1\left(F_{R_0^1,C_1} - F_{R_0^1,C_2}\right)\Bigg] \end{aligned} \tag{B4}$$

If equation (A7) is used to determine the configuration factors within the brackets, $F_{R_1^2,C_{1,2}}$ can be expressed in terms of the cylinder dimensions as

$$\begin{aligned} F_{R_1^2,C_{1,2}} = \frac{1}{4aL_1^2}\Bigg[& \sqrt{\left(L_0^2\right)^4 + 2\left(a^2 + r_1{}^2\right)\left(L_0^2\right)^2 + \left(a^2 - r_1{}^2\right)^2} \\ & - \sqrt{\left(L_0^2\right)^4 + 2\left(a^2 + r_2{}^2\right)\left(L_0^2\right)^2 + \left(a^2 - r_2{}^2\right)^2} \\ & + \sqrt{\left(L_0^1\right)^4 + 2\left(a^2 + r_2{}^2\right)\left(L_0^1\right)^2 + \left(a^2 - r_2{}^2\right)^2} \\ & - \sqrt{\left(L_0^1\right)^4 + 2\left(a^2 + r_1{}^2\right)\left(L_0^1\right)^2 + \left(a^2 - r_1{}^2\right)^2}\Bigg] \end{aligned} \tag{B5}$$

Although figure 10 and the preceding example involve a cylinder, the procedure applies equally as well when the surface of revolution is a cone or a hemisphere.

The above example is a simple application of configuration-factor algebra presented in order to introduce the basic ideas which will now be used to obtain the configuration factors for the geometries of

R_1^2,C_1, $R_0^1,C_{1,2}$, $R_1^2,C_{1,2}$, and R_1^2,R_1^2 for cylinders, cones, and hemispheres as well as R_2^3,R_0^1 for cylinders and cones. (See fig. 2 for examples of these geometries.)

Configuration Factors for the Geometry R_1^2,C_1

The configuration factor for the geometry of R_1^2,C_1 can be obtained from the equation derived for the geometry of R_0^1,C_1 through configuration-factor algebra and is given in general terms by

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$$F_{R_1^2,C_1} = \frac{1}{R_1^2}\left(R_0^2 F_{R_0^2,C_1} - R_0^1 F_{R_0^1,C_1}\right) \tag{B6}$$

where R_0^1, R_0^2, and R_1^2 are ring areas.

Cylinder.- For a cylindrical surface of revolution, equation (A7) can be used to determine the terms on the right-hand side of equation (B6) so that the configuration factor in terms of the dimensions of the cylinder becomes

$$F_{R_1^2,C_1} = \frac{1}{4aL_1^2}\left[\sqrt{\left(L_0^2\right)^4 + 2\left(L_0^2\right)^2\left(a^2 + r_1^2\right) + \left(a^2 - r_1^2\right)^2}\right.$$
$$\left. - \sqrt{\left(L_0^1\right)^4 + 2\left(L_0^1\right)^2\left(a^2 + r_1^2\right) + \left(a^2 - r_1^2\right)^2} + \left(L_0^1\right)^2 - \left(L_0^2\right)^2\right] \tag{B7}$$

Cone.- For a conical surface of revolution, equation (A16) can be used to determine the terms on the right-hand side of equation (B6) so that the configuration factor in terms of the dimensions of the cone becomes

$$F_{R_1^2,C_1} = \frac{\cos\psi\,\cot\psi}{L_1^2\left(L_1^H + L_2^H\right)}\left[\left(I_1 + I_2\right)\Big|_{L=L_0^2} - \left(I_1 + I_2\right)\Big|_{L=L_0^1}\right]\Bigg|_{\rho=r_1} \tag{B8}$$

where the integrals I_1 and I_2 are given by equations (A17) and (A18), respectively.

The first two integrals of equation (B8) are to be evaluated for a frustum of height L_0^2 and the second two are to be evaluated for a frustum of height L_0^1 while all four integrals are to be evaluated for an intercepting area of radius r_1.

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Hemisphere.- For a hemispherical surface of revolution, equation (A26) can be used to determine the terms on the right-hand side of equation (B6) so that the configuration factor in terms of the dimensions of the hemisphere becomes

$$F_{R_1^2, C_1} = \frac{1}{4aL_1^2}\left[\sqrt{4r_1^2\left(L_0^2\right)^2 + \left(a^2 - r_1^2\right)^2} - \sqrt{4r_1^2\left(L_0^1\right)^2 + \left(a^2 - r_1^2\right)^2}\right] \tag{B9}$$

Configuration Factors for the Geometry $R_0^1, C_{1,2}$

The configuration factor for the geometry of $R_0^1, C_{1,2}$ can be obtained from the equation derived for the geometry of R_0^1, C_1 through configuration-factor algebra and is given in general terms by

$$F_{R_0^1, C_{1,2}} = F_{R_0^1, C_1} - F_{R_0^1, C_2} \tag{B10}$$

Cylinder.- For a cylindrical surface of revolution, equation (A7) can be used to determine the terms on the right-hand side of equation (B10) so that the configuration factor in terms of the dimensions of the cylinder becomes

$$\begin{aligned} F_{R_0^1, C_{1,2}} = \frac{1}{4aL_0^1}\Bigg[&\sqrt{\left(L_0^1\right)^4 + 2\left(L_0^1\right)^2\left(a^2 + r_1^2\right) + \left(a^2 - r_1^2\right)^2} \\ &- \sqrt{\left(L_0^1\right)^4 + 2\left(L_0^1\right)^2\left(a^2 + r_2^2\right) + \left(a^2 - r_2^2\right)^2} + r_1^2 - r_2^2\Bigg] \end{aligned} \tag{B11}$$

Cone.- For a conical surface of revolution, equation (A16) can be used to determine the terms on the right-hand side of equation (B10) so that the configuration factor in terms of the dimensions of the cone becomes

$$F_{R_0^1,C_{1,2}} = \frac{\cos\psi \cot\psi}{2L_0^1\left(H + L_1^H\right)}\left[\left(I_1 + I_2\right)\Big|_{\rho=r_1} - \left(I_1 + I_2\right)\Big|_{\rho=r_2}\right]\Bigg|_{L=L_0^1} \tag{B12}$$

Note that two integrals are evaluated for $\rho = r_1$, two for $\rho = r_2$, and all four for a height L_0^1.

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Hemisphere.- For a hemispherical surface of revolution, equation (A26) can be used to determine the terms on the right-hand side of equation (B10) so that the configuration factor in terms of the dimensions of the hemisphere becomes

$$F_{R_0^1,C_{1,2}} = \frac{1}{4aL_0^1}\left[\sqrt{4r_1^2\left(L_0^1\right)^2 + \left(a^2 - r_1^2\right)^2} - \sqrt{4r_2^2\left(L_0^1\right)^2 + \left(a^2 - r_2^2\right)^2} + r_1^2 - r_2^2\right] \tag{B13}$$

Configuration Factors for the Geometry of $R_1^2,C_{1,2}$

The configuration factor for the geometry of $R_1^2,C_{1,2}$ can be obtained from the equation derived for the geometry of R_0^1,C_1 through configuration-factor algebra. This has been performed in the section entitled "Example" and is given by equation (B4).

Cylinder.- Equation (B5) gives the configuration factor for the geometry of $R_1^2,C_{1,2}$ for a cylindrical surface of revolution.

Cone.- For a conical surface of revolution, equation (A16) can be used to determine the terms on the right-hand side of equation (B4) so that the configuration factor in terms of the dimension of the cone becomes

$$F_{R_1^2,C_{1,2}} = \frac{\cos\psi \cot\psi}{L_1^2\left(L_1^H + L_2^H\right)}\left\{\left[\left(I_1 + I_2\right)\Big|_{\rho=r_1} - \left(I_1 + I_2\right)\Big|_{\rho=r_2}\right]\Bigg|_{L=L_0^2} - \left[\left(I_1 + I_2\right)\Big|_{\rho=r_1} - \left(I_1 + I_2\right)\Big|_{\rho=r_2}\right]\Bigg|_{L=L_0^1}\right\} \qquad \text{(B14)}$$

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Again it is to be noted that the sum of the integrals must be evaluated for the correct combinations of ρ and L.

Hemisphere.- For a hemispherical surface of revolution, equation (A26) can be used to determine the terms on the right-hand side of equation (B4) so that the configuration factor in terms of the dimensions of the hemisphere becomes

$$F_{R_1^2,C_{1,2}} = \frac{1}{4aL_1^2}\left[\sqrt{4r_1^{\,2}\left(L_0^2\right)^2 + \left(a^2 - r_1^{\,2}\right)^2} - \sqrt{4r_2^{\,2}\left(L_0^2\right)^2 + \left(a^2 - r_2^{\,2}\right)^2} - \sqrt{4r_1^{\,2}\left(L_0^1\right)^2 + \left(a^2 - r_1^{\,2}\right)^2} + \sqrt{4r_2^{\,2}\left(L_0^1\right)^2 + \left(a^2 - r_2^{\,2}\right)^2}\right] \qquad \text{(B15)}$$

Configuration Factors for the Geometry of R_2^3,R_0^1

The configuration factor for the geometry of R_2^3,R_0^1 can be obtained from the equation derived for the geometry of R_0^1,C_1 through configuration-factor algebra and is given in general terms by

$$F_{R_2^3,R_0^1} = \frac{1}{R_2^3}\left[\left(R_1^3F_{R_1^3,C_0^1} - R_1^2F_{R_1^2,C_0^1}\right) - \left(R_0^3F_{R_0^3,C_0} - R_0^2F_{R_0^2,C_0}\right)\right] \qquad \text{(B16)}$$

Cylinder.- For a cylindrical surface of revolution, equation (A9) (a special case of (A7) when the intercepting area is the full base of the cylinder) can be used to determine the terms on the right-hand side of equation (B16) so that the configuration factor in terms of the dimensions of the cylinder becomes

$$F_{R_2^3,R_0^1} = \frac{1}{4aL_2^3}\left[2L_0^1L_2^3 + L_1^3\sqrt{\left(L_1^3\right)^2 + 4a^2} - L_1^2\sqrt{\left(L_1^2\right)^2 + 4a^2}\right.$$
$$\left. - L_0^3\sqrt{\left(L_0^3\right)^2 + 4a^2} + L_0^2\sqrt{\left(L_0^2\right)^2 + 4a^2}\right] \qquad \text{(B17)}$$

Cone.- For a conical surface of revolution, equation (A19) (a special case of equation (A16) when the intercepting area is the full area of the base of the cone) can be used to determine the terms on the right-hand side of equation (B16) so that the configuration factor in terms of the dimensions of the cone becomes

$$F_{R_2^3,R_0^1} = \frac{1}{2L_2^3\left(L_2^H + L_3^H\right)}\left[L_1^3\sqrt{\left(L_1^3\right)^2\csc^2\psi + 4L_1^HL_3^H} - L_1^2\sqrt{\left(L_1^2\right)^2\csc^2\psi + 4L_1^HL_2^H}\right.$$
$$- L_0^3\sqrt{\left(L_0^3\right)^2\csc^2\psi + 4HL_3^H} + L_0^2\sqrt{\left(L_0^2\right)^2\csc^2\psi + 4HL_2^H}$$
$$\left. + 2L_0^1L_2^3\cos\psi\cot\psi\right] \qquad \text{(B18)}$$

Hemisphere.- For a hemispherical surface of revolution, equation (A26) cannot be applied to equation (B16) to determine the configuration factor for the geometry of R_2^3,R_0^1. This is because the first two terms on the right of equation (B16) cannot be obtained from equation (A26), which is for hemispheres, since the surface of revolution above plane 1 is a spherical segment. Therefore, the equation for the configuration factor between rings on the surface of a hemisphere is derived in appendix A (eq. (A35)) by integrating equation (4) over R_0^1 and R_2^3.

Configuration Factors for the Geometry R_1^2, R_1^2

Radiation in a closed system composed of several isothermal surfaces is further complicated by surfaces of positive curvature. A surface of positive curvature A_j intercepts a portion of its own radiated energy and a configuration factor of the form $F_{j,j}$ must be determined. In order to determine $F_{j,j}$, use is made of the fact that if a surface A_j is surrounded by (n - 1) other surfaces, forming a closed system, the sum of the configuration factors between A_j and all surfaces is one, or

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$$\sum_{k=1}^{n} F_{j,k} = 1 \tag{B19}$$

If the surface A_j can see itself, equation (B19) can be written as

$$F_{j,j} = 1 - \sum_{k=1}^{j-1} F_{j,k} - \sum_{k=j+1}^{n} F_{j,k} \tag{B20}$$

For a surface of revolution, equation (B20) can be replaced by

$$F_{R_1^2,R_1^2} = 1 - F_{R_1^2,C_0^1} - F_{R_1^2,C_0^2} \tag{B21}$$

where R_1^2 is the ring formed by the intersections of planes 1 and 2, which are perpendicular to the axis of revolution. Areas C_0^1 and C_0^2 are circular areas in planes 1 and 2 bounded by the surface of revolution. The last two terms in equation (B21) represent the fractions of radiation from R_1^2 which fall on surfaces below and above R_1^2, respectively.

Cylinder.- If the surface of revolution is a cylinder

$$C_0 = C_0^1 = C_0^2$$

and equation (B21) reduces to

$$F_{R_1^2,R_1^2} = 1 - 2F_{R_1^2,C_0^1} \tag{B22}$$

The configuration factor $F_{R_1^2,C_0^1}$ from the walls of a cylinder to the base can be obtained from equation (A9), and equation (B22) may then be written in terms of the dimensions of the cylinder as

$$F_{R_1^2,R_1^2} = 1 + \frac{L_1^2}{2a} - \sqrt{1 + \left(\frac{L_1^2}{2a}\right)^2} \tag{B23}$$

It can be seen from equation (B23) that $F_{R_1^2,R_1^2}$ for a cylinder is dependent upon the height of the ring but not upon the position of the ring above the base of the cylinder.

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Cone.- For a conical surface of revolution

$$F_{R_1^2,C_0^2} = F_{R_1^2,R_2^H} \tag{B24}$$

where R_2^H is all the surface area above R_1^2.

By using equation (B24), equation (B21) becomes

$$F_{R_1^2,R_1^2} = 1 - F_{R_1^2,C_0^1} - F_{R_1^2,R_2^H} \tag{B25}$$

The portion of the energy leaving R_1^2 which is intercepted by the circular area C_0^1 can be determined from equation (A19) as

$$F_{R_1^2,C_0^1} = \frac{1}{2\left(L_1^H + L_2^H\right)}\left[\sqrt{\left(L_1^2\right)^2 \csc^2\psi + 4L_1^H L_2^H} + \csc\psi\left(2L_1^H \sin^2\psi - L_1^2\right)\right] \tag{B26}$$

Configuration-factor algebra for the exchange of radiant energy between R_1^2 and R_2^H gives

$$R_1^2 F_{R_1^2,R_2^H} = R_2^H F_{R_2^H,R_1^2} = R_2^H F_{R_2^H,C_0^2} - \left(R_1^H F_{R_1^H,C_0^1} - R_1^2 F_{R_1^2,C_0^1}\right) \tag{B27}$$

By using equations (A19) and (A20), equation (B27) yields

$$F_{R_1^2,R_2^H} = \frac{1}{2\left(L_1^H + L_2^H\right)}\left[\sqrt{\left(L_1^2\right)^2 \csc^2\psi + 4L_1^H L_2^H} - \csc\psi\left(2L_2^H \sin^2\psi + L_1^2\right)\right] \tag{B28}$$

Substitution of equations (B26) and (B28) into equation (B25) gives the configuration factor for a ring of a cone radiating to itself as

$$F_{R_1^2,R_1^2} = 1 - \frac{1}{L_1^H + L_2^H}\left[\sqrt{\left(L_1^2\right)^2 \csc^2\psi + 4L_1^H L_2^H} - L_1^2 \cos\psi \cot\psi\right] \tag{B29}$$

It can be seen from equation (B29) that the portion of its own radiated energy that a ring on the surface of a cone receives is dependent upon the relative position of the ring as well as on the height of the ring.

<u>Hemisphere</u>.- For a hemispherical surface of revolution, since the total height equals a, equation (B21) can be written

$$F_{R_1^2,R_1^2} = 1 - F_{R_1^2,C_0^1} - F_{R_1^2,R_2^a} \tag{B30}$$

where

$$F_{R_1^2,C_0^1} = \frac{R_0^2 F_{R_0^2,C_0} - R_0^1 F_{R_0^1,C_0} + R_1^2 F_{R_1^2,R_0^1}}{R_1^2} \tag{B31}$$

By using equations (A29) and (A35), equation (B30) becomes

$$F_{R_1^2,R_1^2} = \frac{L_1^2}{2a} \tag{B32}$$

Here again, as for a cylinder, $F_{R_1^2,R_1^2}$ is dependent upon the height of the ring L_1^2 and not upon its position above the base plane.

TABLE II.- CONFIGURATION FACTORS FOR GEOMETRY R_0^1,C_1 FOR CYLINDERS $\left(F_{R_0^1,C_1}\right)$

$\frac{L_0^1}{a}$	Configuration factors for r_1/a of -									
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.2	1.00×10^{-3}	4.0×10^{-3}	1.06×10^{-2}	1.80×10^{-2}	3.10×10^{-2}	5.15×10^{-2}	8.40×10^{-2}	1.41×10^{-1}	2.46×10^{-1}	4.53×10^{-1}
.4	1.76	7.0	1.68	3.13	5.25	8.30	1.27×10^{-1}	1.91	2.82	4.10
1.0	2.50	1.01×10^{-2}	2.30	4.16	6.64	9.80	1.37	1.85	2.42	3.09
2.0	2.00	8.0×10^{-3}	1.80	3.22	5.05	7.29	9.98×10^{-2}	1.31	1.67	2.11
4.0	1.18	4.70	1.06	1.89	2.94	4.24	5.78	7.55×10^{-2}	9.56×10^{-2}	1.18
10.0	5.0×10^{-4}	1.98	4.45×10^{-3}	7.92×10^{-3}	1.24	1.78	2.43	3.17	4.01	4.95×10^{-2}
20.0	2.5	1.0	2.25	3.99	6.24×10^{-3}	8.98×10^{-3}	1.23	1.60	2.02	2.49
40.0	1.25	5.0×10^{-4}	1.13	2.0	3.12	4.50	6.12×10^{-3}	8.0×10^{-3}	1.01	1.25
100.0	5.00×10^{-5}	2.0	4.50×10^{-4}	8.0×10^{-4}	1.25	1.80	2.45	3.20	4.04×10^{-3}	5.0×10^{-3}
200.0	2.50	1.0	2.25	4.0	6.25×10^{-4}	8.90×10^{-4}	1.23	1.60	2.02	2.50

TABLE III.- CONFIGURATION FACTORS FOR GEOMETRY R_0^1, C_1 FOR CONES $\left(F_{R_0^1, C_1}\right)$

$\frac{L_0^1}{H}$	Configuration factors for r_1/a of -									
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
					$\psi = 5^\circ$					
0.1	2.84×10^{-3}	1.40×10^{-2}	2.59×10^{-2}	4.64×10^{-2}	7.36×10^{-2}	1.08×10^{-1}	1.51×10^{-1}	2.00×10^{-1}	2.60×10^{-1}	3.27×10^{-1}
.2	2.16	8.63×10^{-3}	1.94	3.45	5.39	7.80×10^{-2}	1.07	1.40	1.77	2.19
.3	1.64	6.56	1.48	2.61	4.09	5.90	8.07×10^{-2}	1.05	1.33	1.65
.4	1.34	5.36	1.21	2.13	3.33	4.81	6.58	8.57×10^{-2}	1.09	1.34
.5	1.15	4.61	1.04	1.84	2.87	4.16	5.66	7.38	9.35×10^{-2}	1.15
.6	1.06	4.14	9.31×10^{-3}	1.65	2.57	3.72	5.08	6.62	8.38	1.03
.7	9.58×10^{-4}	3.83	8.61	1.52	2.38	3.43	4.70	6.12	7.75	9.56×10^{-2}
.8	9.09	3.63	8.17	1.44	2.26	3.26	4.46	5.81	7.36	9.07
.9	8.82	3.52	7.92	1.40	2.19	3.16	4.32	5.63	7.14	8.80
1.0	8.72	3.49	7.84	1.39	2.8	3.14	4.27	5.58	7.06	8.72
					$\psi = 10^\circ$					
0.1	2.59×10^{-3}	1.06×10^{-2}	2.47×10^{-2}	4.60×10^{-2}	7.61×10^{-2}	1.18×10^{-1}	1.73×10^{-1}	2.47×10^{-1}	3.42×10^{-1}	4.62×10^{-1}
.2	3.21	1.29	2.94	5.30	8.38	1.23	1.70	2.26	2.92	3.68
.3	2.90	1.16	2.62	4.69	7.35	1.06	1.45	1.91	2.43	3.01
.4	2.52	1.01	2.28	4.06	6.35	9.16×10^{-2}	1.25	1.63	2.07	2.56
.5	2.23	8.95×10^{-3}	2.02	3.59	5.61	8.09	1.10	1.44	1.82	2.25
.6	2.03	8.13	1.83	3.26	5.09	7.34	9.99×10^{-2}	1.31	1.65	2.04
.7	1.89	7.56	1.70	3.03	4.74	6.83	9.29	1.21	1.54	1.90
.8	1.79	7.20	1.62	2.89	4.51	6.50	8.84	1.16	1.46	1.81
.9	1.74	6.99	1.57	2.80	4.38	6.31	8.59	1.12	1.42	1.75
1.0	1.73	6.95	1.56	2.78	4.34	6.25	8.51	1.11	1.41	1.74
					$\psi = 20^\circ$					
0.1	1.55×10^{-3}	6.40×10^{-3}	1.52×10^{-2}	2.93×10^{-2}	5.12×10^{-2}	8.54×10^{-2}	1.41×10^{-1}	2.32×10^{-1}	3.82×10^{-1}	6.08×10^{-1}
.2	3.07	1.25×10^{-2}	2.92	5.47	9.10	1.41×10^{-1}	2.09	2.98	4.11	5.50
.3	3.91	1.58	3.62	6.58	1.05×10^{-1}	1.57	2.21	2.99	3.91	5.0
.4	4.12	1.66	3.75	6.73	1.06	1.55	2.14	2.83	3.64	4.55
.5	4.03	1.61	3.64	6.50	1.02	1.47	2.02	2.65	3.37	4.19
.6	3.85	1.54	3.47	6.17	9.66×10^{-2}	1.39	1.90	2.49	3.15	3.90
.7	3.67	1.47	3.30	5.88	9.19	1.32	1.80	2.36	2.98	3.69
.8	3.53	1.41	3.18	5.66	8.84	1.27	1.73	2.26	2.87	3.54
.9	3.45	1.38	3.10	5.52	8.62	1.24	1.69	2.21	2.79	3.45
1.0	3.42	1.37	3.08	5.47	8.55	1.23	1.68	2.19	2.77	3.42

TABLE IV.- CONFIGURATION FACTORS FOR GEOMETRY R_0^1, C_1 FOR HEMISPHERES $\left(F_{R_0^1, C_1}\right)$

$\frac{L_0^1}{a}$	Configuration factors for r_1/a of -									
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.1	5.05×10^{-4}	2.08×10^{-3}	4.94×10^{-3}	9.50×10^{-3}	1.66×10^{-2}	2.80×10^{-2}	4.72×10^{-2}	8.50×10^{-2}	1.79×10^{-1}	5.00×10^{-1}
.2	1.01×10^{-3}	4.15	9.85	1.89×10^{-2}	3.28	5.45	8.95	1.52×10^{-1}	2.71	5.00
.3	1.51	6.20	1.47×10^{-2}	2.80	4.80	7.86	1.26×10^{-1}	2.00	3.17	5.00
.4	2.02	8.27	1.95	3.67	6.25	1.0×10^{-1}	1.55	2.34	3.47	5.00
.5	2.52	1.03×10^{-2}	2.40	4.52	7.56	1.19	1.78	2.58	3.64	5.00
.6	3.02	1.23	2.87	5.32	8.76	1.35	1.97	2.76	3.76	5.00
.7	3.51	1.43	3.30	6.06	9.85	1.49	2.13	2.91	3.85	5.00
.8	4.00	1.63	3.71	6.75	1.08×10^{-1}	1.60	2.25	3.03	3.95	5.00
.9	4.50	1.81	4.11	7.40	1.17	1.71	2.36	3.12	4.00	5.00
1.0	5.00	2.00	4.50	8.00	1.25	1.80	2.45	3.20	4.05	5.00

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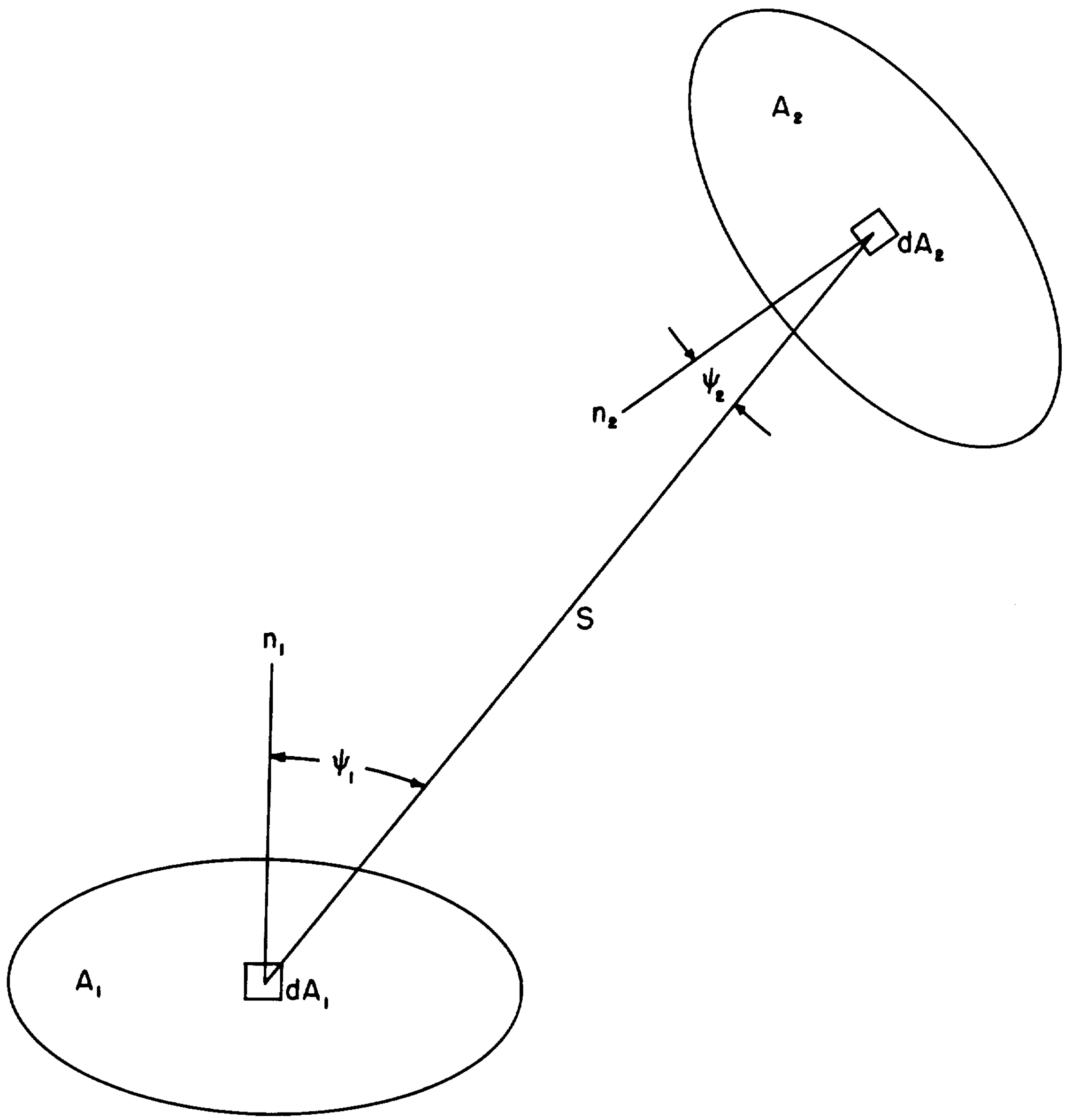

Figure 1.- Relative positions of isothermal black surfaces.

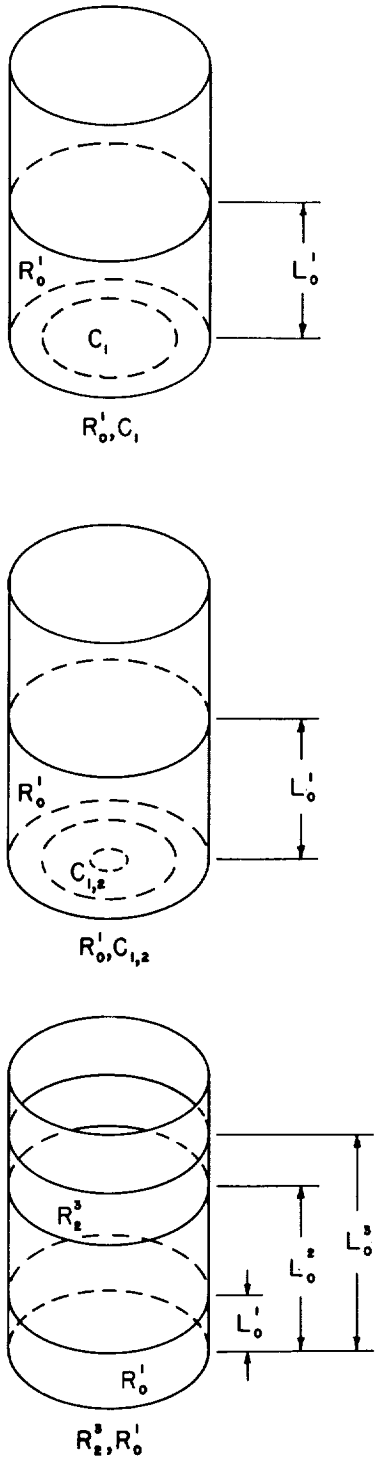

Figure 2.- Representations of configurations investigated.

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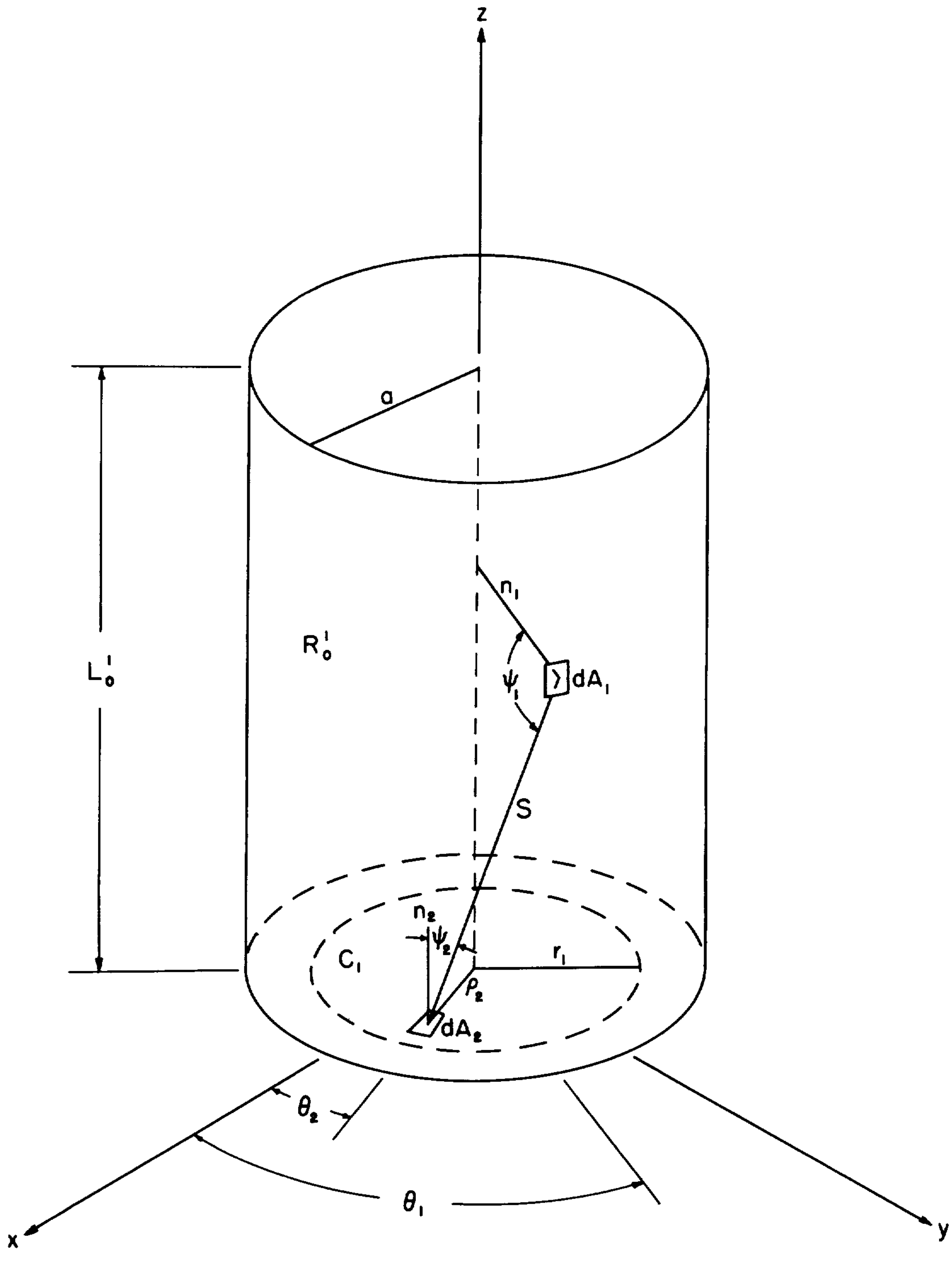

Figure 3.- Geometry of cylinder $\left(R_0^1, C_1\right)$.

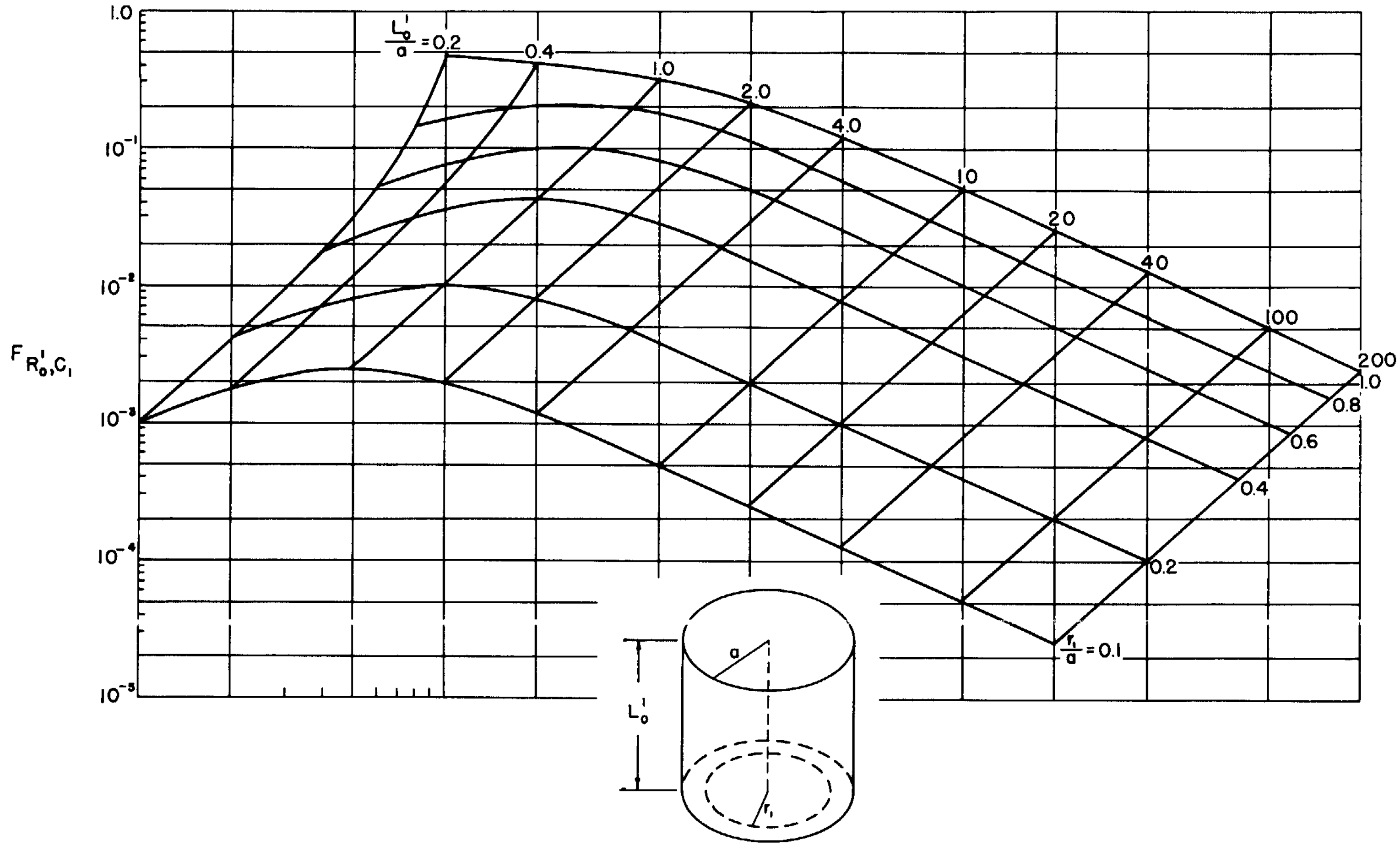

Figure 4.- Configuration factors for geometry R_0^1, C_1 for a cylinder.

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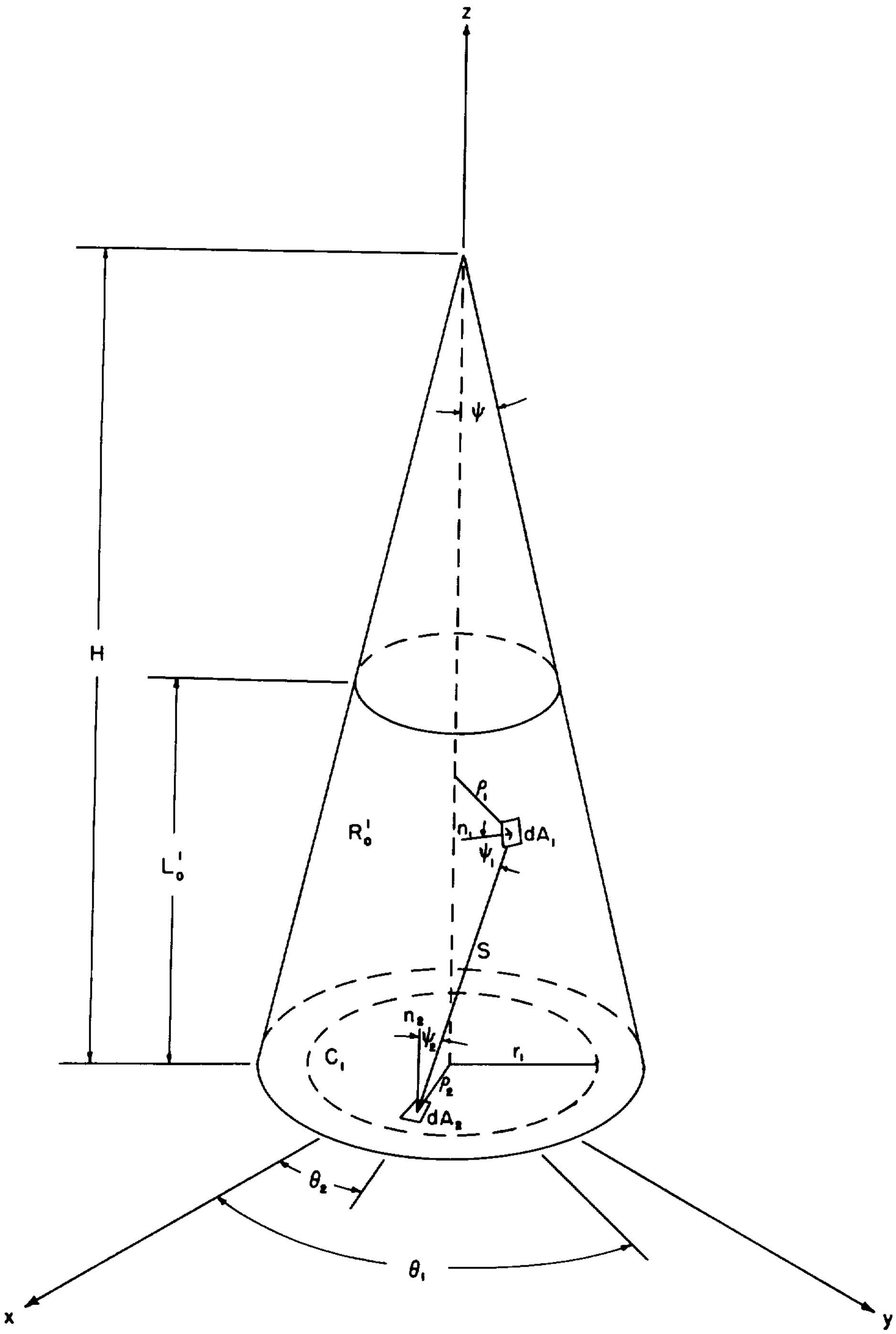

Figure 5.- Geometry of cone $\left(R_0^1, C_1\right)$.

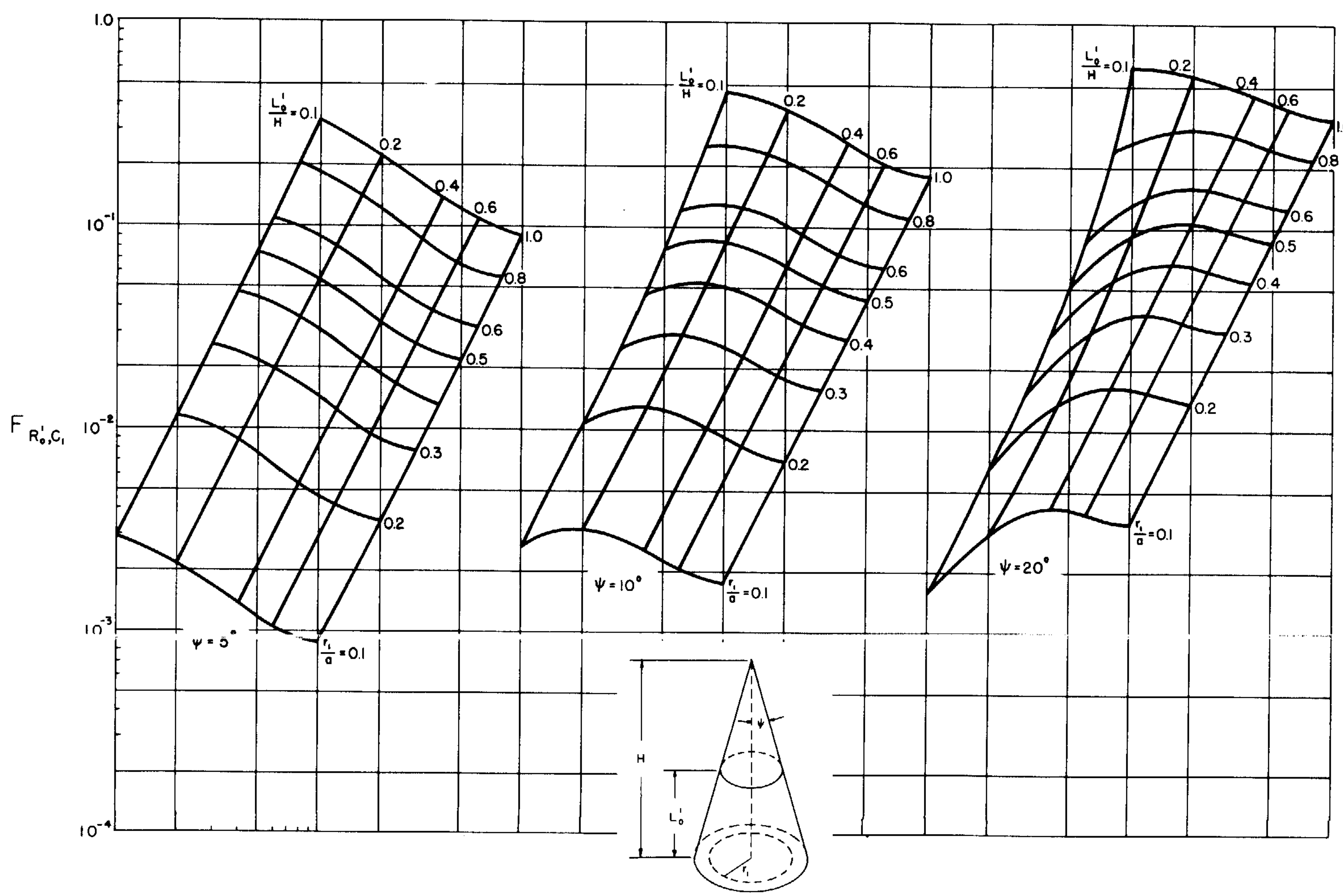

Figure 6.- Configuration factors for geometry R_0^1, C_1 for a cone.

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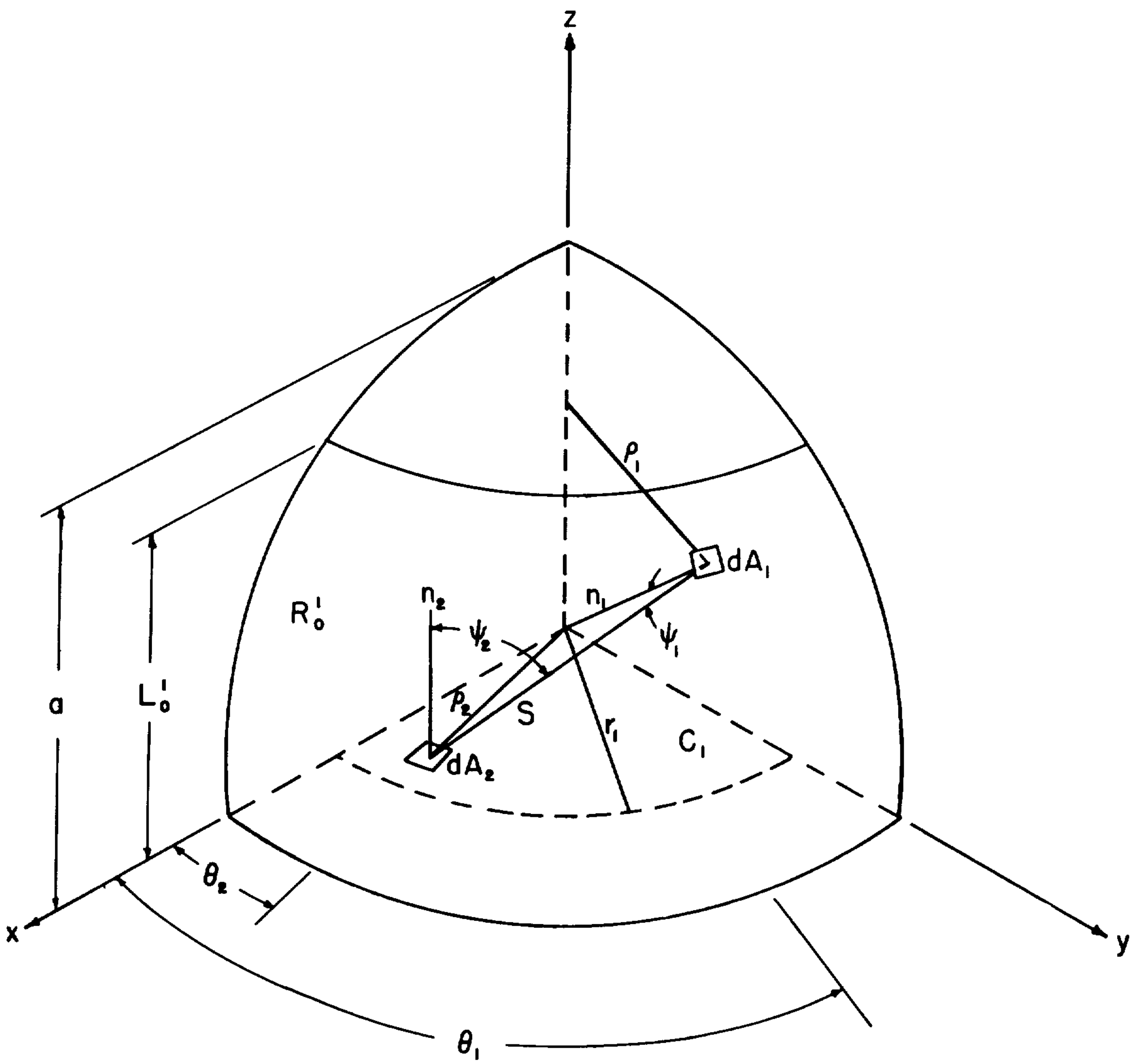

Figure 7.- Geometry of hemisphere $\left(R_0^1, C_1\right)$.

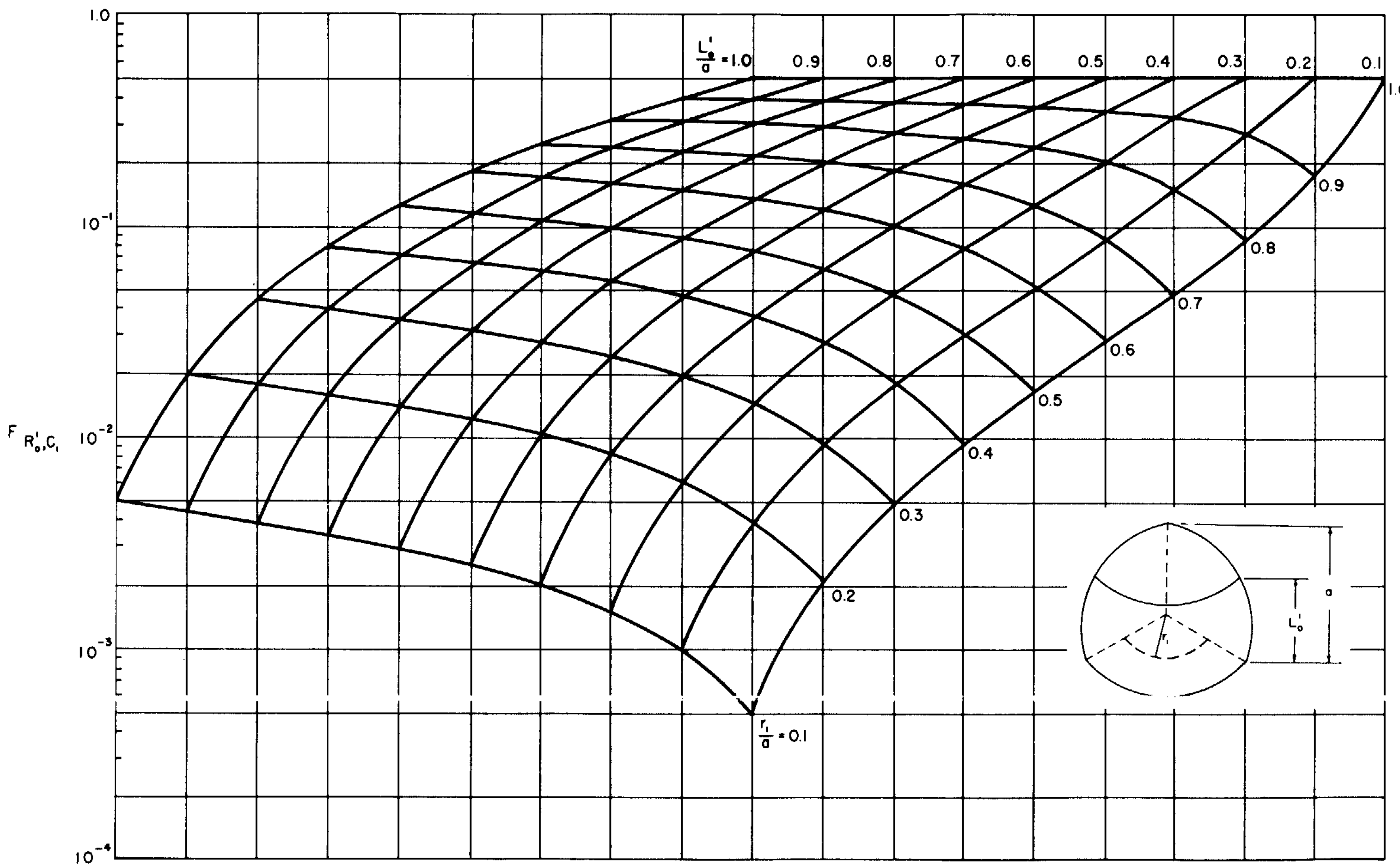

Figure 8.- Configuration factors for geometry R_0^1,C_1 for a hemisphere.

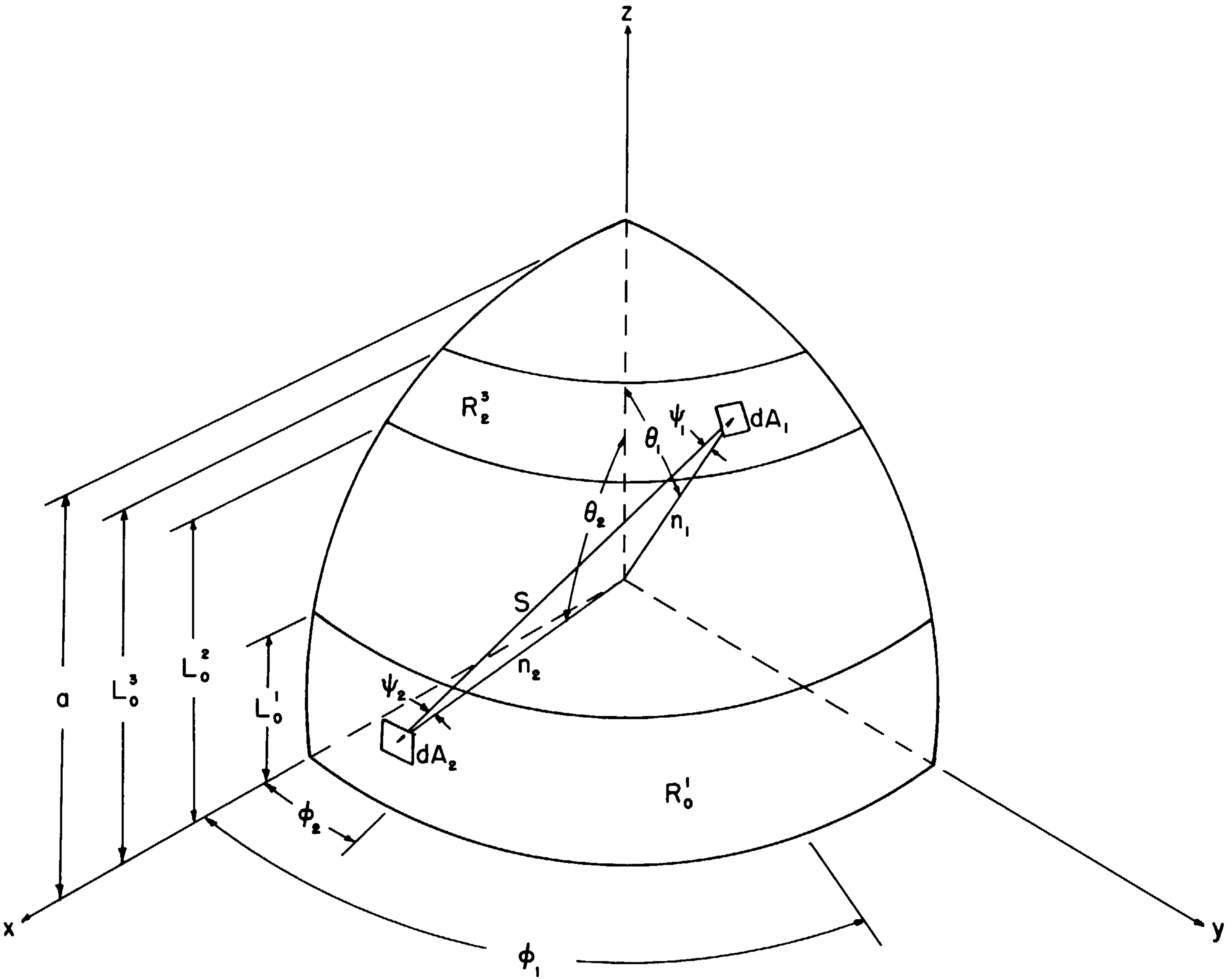

Figure 9.- Geometry for radiation between rings of a hemisphere.

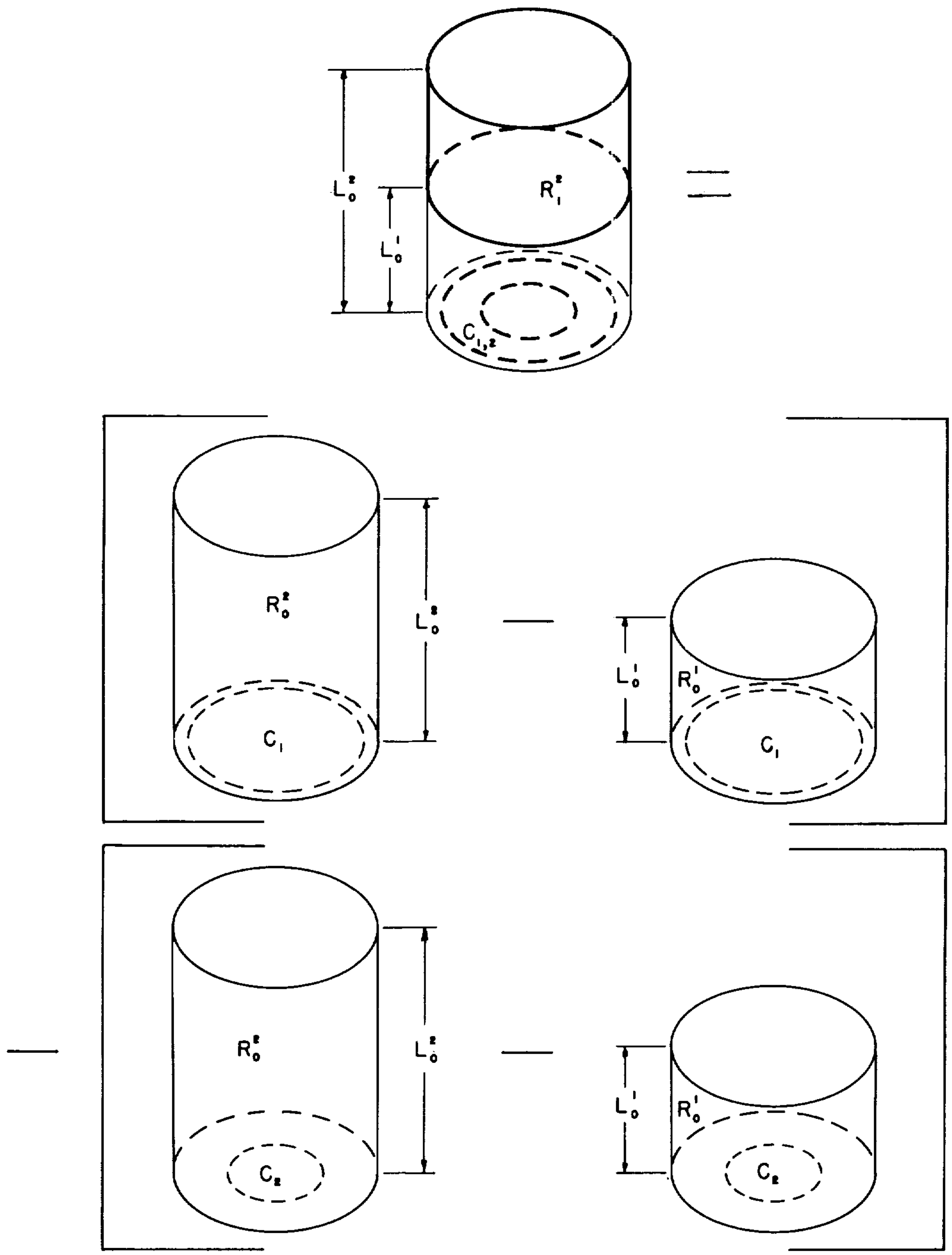

Figure 10.- Geometric representation of configuration-factor algebra.

$$R_1^2 F_{R_1^2, C_{1,2}} = \left[R_0^2 F_{R_0^2, C_1} - R_0^1 F_{R_0^1, C_1}\right] - \left[R_0^2 F_{R_0^2, C_2} - R_0^1 F_{R_0^1, C_2}\right].$$

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