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EQUATIONS FOR THE INDUCED VELOCITIES NEAR A LIFTING ROTOR

WITH NONUNIFORM AZIMUTHWISE VORTICITY DISTRIBUTION

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SUMMARY

Equations, which can be integrated on high-speed computing machines, are developed for all three components of induced velocity at an arbitrary point near the rotor and for an arbitrary harmonic variation of vorticity. Sample calculations for vorticity which varies as the sine of the azimuth angle indicate that the normal component of induced velocity is, in this case, uniform along either side of the lateral axis.

INTRODUCTION

The mutual interference between the rotors, wing, and tail of a helicopter or convertiplane may have a large effect upon the overall performance and stability of the aircraft. Since mutual interference may entail significant performance penalties, it is necessary to consider such effects in the initial stages of design. The accurate assessment of mutual interference, however, requires a knowledge of the flow field of a lifting rotor. The most complete study of the induced velocities near a lifting rotor is that of reference 1, which shows that the flow may be calculated with reasonable accuracy provided that a representative distribution of disk loading is assumed. The theory of reference 1, however, is based upon existing induced velocity calculations. Since these calculations consider only the case of uniform azimuthwise vorticity, the theory, as given in reference 1, can only be used to represent disk loadings for which the circulation distribution is circularly symmetrical. As a consequence, the flow field, according to reference 1, is symmetrical about the longitudinal plane of symmetry of the rotor. The measured flow field of reference 1, however, shows significant differences in the flow on opposite sides of the rotor. Thus, it is desirable to examine the possibility of extending the available theory to include the varying components of vorticity which are known to exist in the wake of the rotor.

The calculation of the flow field is greatly complicated by the inclusion of terms representing azimuthwise variations in circulation. The only investigation which has considered these effects is that described in reference 2, which examined the induced velocity distribution on the lateral axis of the rotor for $\sin \psi$ circulation. The actual calculation was based upon a crude approximation to the wake that is believed to correspond to this distribution of circulation; furthermore, it is not possible to extend the analysis of reference 2 to points other than those which lie on the lateral axis. Thus, it is necessary to consider the problem from a different viewpoint if completely general results are to be obtained.

The present paper develops equations for all three components of induced velocity at an arbitrary point near the rotor and for an arbitrary harmonic of the azimuthwise distribution of circulation. This derivation is accomplished without recourse to the approximations of reference 2. Unfortunately, the final integration (with respect to ψ) is not possible in closed form; however, numerical results may be obtained by the use of modern high-speed computing machines. As an example, the distribution of the normal component of induced velocity on the lateral axis is computed for $\sin \psi$ circulation. These calculated velocities are compared with those of reference 2.

SYMBOLS

$\bar{a}$ vector distance from point P in space to vortex element (fig. 1), ft

$A = x' \cos \psi + y' \sin \psi$

$B = z' \cos X - x' \sin X$

$C = (y' \cos \psi - x' \sin \psi) \cos X - z' \sin X \sin \psi$

$d\bar{s}$ vector length of vortex element, ft

$\left(\dfrac{d\Gamma}{dL}\right)_0$ constant portion of wake vorticity, ft/sec

$f(\psi)$ Fourier sine-cosine series $\Big($ normalized with respect to the constant term $\left(\dfrac{d\Gamma}{dL}\right)_0\Big)$ describing the azimuthwise variation of vorticity in the outer wake; the negative derivative of $f(\psi)$ describes the corresponding vorticity in the inner wake

$\bar{i}, \bar{j}, \bar{k}$ unit direction vectors along X-, Y-, and Z-axes, respectively

L   running coordinate along edge of wake, ft

P   arbitrary point in space

$\bar{q}$ vector induced velocity at P, ft/sec

R   rotor radius, ft

R_c distance between P and edge of rotor disk at ψ (fig. 1),

$$\sqrt{R^2 + x^2 + y^2 + z^2 - 2R(x \cos \psi + y \sin \psi)}, \text{ ft}$$

R_0 distance between P and center of rotor (fig. 1),

$$\sqrt{x^2 + y^2 + z^2}, \text{ ft}$$

r   vortex-element radius, measured parallel to tip-path plane from wake axis (fig. 1(b)), ft

$\bar{s}$ vector distance from origin to surface of cylindrical wake, ft

u,v,w   induced velocity components directed parallel to X-, Y-, and Z-axes, respectively, ft/sec

V   forward speed of rotor, ft/sec

w_0 normal component of induced velocity at center of rotor with uniform radial and azimuthwise circulation, positive upward, ft/sec

X,Y,Z   Cartesian coordinate system centered in rotor, X positive rearward, Y positive on advancing side of rotor, and Z positive upward

x,y,z   Cartesian coordinates of point P (fig. 1), ft

α angle of attack of rotor tip-path plane, radians

Γ blade circulation, ft^2/sec

λ rotor inflow ratio, $\dfrac{V \sin \alpha + w_0}{\Omega R}$

μ rotor tip-speed ratio, $\dfrac{V \cos \alpha}{\Omega R}$

X   wake skew angle, angle between axis of tip-path plane and axis of wake, $\tan^{-1} \frac{-\mu}{\lambda}$, radians

ψ azimuth angle, measured in direction of rotation from downwind position (X-axis), radians

Ω rotor rotational speed, radians/sec

Primes on symbols denote nondimensionalization with respect to R.

THEORY

Assumed Wake

The shape of the wake assumed as a basis for the present calculations is an extension of that used for previous rotary-wing induced-velocity calculations (refs. 2 and 3, for example). Its general characteristics are repeated herein for completeness.

The circulation on the blades is considered uniform along the radius. (This assumption is in no way a restriction, since the results obtained under it can be converted to correspond with any arbitrary radial circulation distribution by the methods of ref. 1.) The free vortices leave the blade tip and are carried off with the speed and direction of the mean flow at the rotor. These vortices thus lie on the surface of an elliptic cylinder (fig. 1(a)). The vortices are assumed to be so closely spaced that the cylinder may be considered to be a sheet of continuous vorticity. This assumption restricts the analysis to obtaining only the time-averaged value of the induced velocity (ref. 4). The axial component of vorticity is assumed to be negligible. This assumption is equivalent to stating that the rotor tip speed is infinite (ref. 5), so that the circulation is proportional merely to the local disk loading.

The essential addition in the present analysis is that the wake vorticity is allowed to vary in any arbitrary manner with the azimuth position. As a consequence of this azimuthwise vorticity variation, the blades must also shed vorticity at their trailing edges. Figure 2 illustrates this radial vorticity for a simple case in which the circulation increases by an arbitrary constant over one sector of the rotor disk. In practice, however, the circulation changes are continuous, so that the wake cylinder will be completely filled with radial vorticity of a strength proportional to the derivative of the circulation.

The effect of the constant part of the blade circulation itself on the induced velocities is zero on a time-averaged basis. This result is not obtained, however, for the harmonic components of the blade circulation. Nevertheless, this contribution is ignored in the present analysis since it may be obtained relatively easily, and since it is zero in the only numerical example considered herein.

The derivations which follow include all three components of the induced velocity. For convenience, the effects of the outer wake (the circumferential vorticity around the cylinder) and the inner wake (the radial vorticity within the cylinder) are considered separately.

Induced Velocities of Outer Wake

The induced velocities of the outer wake are found by integrating the Biot-Savart law over the entire wake. Thus,

$$d\bar{q} = \frac{1}{4\pi} \frac{d\Gamma}{dL} \frac{d\bar{s} \times \bar{a}}{|\bar{a}|^3} dL \tag{1}$$

From figure 1(a), the following quantities may be determined by inspection:

$$\bar{s} = \bar{i}(R \cos \psi + L \sin X) + \bar{j}(R \sin \psi) + \bar{k}(-L \cos X)$$

$$d\bar{s} = \left[\bar{i}(-R \sin \psi) + \bar{j}(R \cos \psi) + \bar{k}(0)\right] d\psi$$

$$\bar{a} = \bar{i}(R \cos \psi + L \sin X - x) + \bar{j}(R \sin \psi - y) + \bar{k}(-L \cos X - z)$$

Substituting these values into equation (1) and integrating yields

$$\bar{q} = \frac{R}{4\pi} \int_0^{2\pi} \int_0^{\infty} \frac{d\Gamma}{dL} \frac{\begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ -\sin \psi & \cos \psi & 0 \\ (R \cos \psi + L \sin X - x) & (R \sin \psi - y) & (-L \cos X - z) \end{vmatrix} dL\, d\psi}{\left[(R \cos \psi + L \sin X - x)^2 + (R \sin \psi - y)^2 + (-L \cos X - z)^2\right]^{3/2}} \tag{2}$$

Normal component.- The normal (or $\bar{k}$) component of induced velocity is, from equation (2),

$$w = \frac{R}{4\pi} \int_0^{2\pi} \frac{d\Gamma}{dL} \int_0^{\infty} \frac{x \cos \psi + y \sin \psi - R - L \sin \chi \cos \psi}{\left[R^2 + x^2 + y^2 + z^2 - 2R(x \cos \psi + y \sin \psi) + 2L(z \cos \chi - x \sin \chi + R \sin \chi \cos \psi) + L^2 \right]^{3/2}} \, dL \, d\psi \qquad (3)$$

The integration with respect to L may be accomplished with the aid of items 162 and 170 of reference 6.

$$w = \frac{R}{4\pi} \int_0^{2\pi} \frac{d\Gamma}{dL} \left. \frac{(x \cos \psi + y \sin \psi - R)(L + z \cos \chi - x \sin \chi + R \sin \chi \cos \psi) + \sin \chi \cos \psi \left[(z \cos \chi - x \sin \chi + R \sin \chi \cos \psi)L + R^2 + x^2 + y^2 + z^2 - 2R(x \cos \psi + y \sin \psi) \right]}{\left[R^2 + x^2 + y^2 + z^2 - 2R(x \cos \psi + y \sin \psi) - (z \cos \chi - x \sin \chi + R \sin \chi \cos \psi)^2 \right] \sqrt{R^2 + x^2 + y^2 + z^2 - 2R(x \cos \psi + y \sin \psi) + 2L(z \cos \chi - x \sin \chi + R \sin \chi \cos \psi) + L^2}} \right|_0^{\infty} d\psi \qquad (4)$$

Substituting limits and combining terms reduces equation (4) to

$$w = \frac{R}{4\pi} \int_0^{2\pi} \frac{d\Gamma}{dL} \frac{\left[(x \cos \psi + y \sin \psi - R) - \sin \chi \cos \psi \sqrt{R^2 + x^2 + y^2 + z^2 - 2R(x \cos \psi + y \sin \psi)} \right]}{\left[R^2 + x^2 + y^2 + z^2 - 2R(x \cos \psi + y \sin \psi) \right] \left[\sqrt{R^2 + x^2 + y^2 + z^2 - 2R(x \cos \psi + y \sin \psi)} - x \sin \chi + R \sin \chi \cos \psi \right]} \, d\psi \qquad (5)$$

Introducing nondimensional coordinates and noting that

$$\sqrt{R^2 + x^2 + y^2 + z^2 - 2R(x \cos \psi + y \sin \psi)} = R_c$$

results in

$$w = \frac{-1}{4\pi} \int_0^{2\pi} \frac{d\Gamma}{dL} \frac{1 - (x' \cos \psi + y' \sin \psi) + R_c' \sin \chi \cos \psi}{\left[R_c' + (\cos \psi - x') \sin \chi + z' \cos \chi \right] R_c'} \, d\psi \qquad (6)$$

At the center of the rotor, $x' = y' = z' = 0$, so that equation (6) reduces to

$$w = \frac{-1}{4\pi} \int_0^{2\pi} \frac{d\Gamma}{dL} \, d\psi \qquad (7)$$

If, now, $\frac{d\Gamma}{dL}$ is expressed as $\left(\frac{d\Gamma}{dL}\right)_0 f(\psi)$, where $\left(\frac{d\Gamma}{dL}\right)_0$ is the constant part of the vorticity, and where $f(\psi)$ is a Fourier series in ψ (normalized with respect to the constant term), it is evident that the only term which can produce a normal induced velocity at the center of the rotor is the constant term. Thus, the induced velocity at the center of the rotor is

$$w_0 = \frac{-1}{2}\left(\frac{d\Gamma}{dL}\right)_0 \tag{8}$$

so that finally

$$\frac{w}{w_0} = \frac{1}{2\pi}\int_0^{2\pi} f(\psi) \frac{\left[1 - (x'\cos\psi + y'\sin\psi) + R_c'\sin X \cos\psi\right]d\psi}{\left[R_c' + (\cos\psi - x')\sin X + z'\cos X\right]R_c'} \tag{9}$$

Longitudinal component.- The longitudinal (or $\bar{i}$) component of the induced velocity is, from equation (2),

$$u = \frac{-R}{4\pi}\int_0^{2\pi}\int_0^\infty \frac{d\Gamma}{dL} \frac{(z + L\cos X)\cos\psi \, dL \, d\psi}{\left[R^2 + x^2 + y^2 + z^2 - 2R(x\cos\psi + y\sin\psi) + 2L(z\cos X - x\sin X + R\sin X\cos\psi) + L^2\right]^{3/2}} \tag{10}$$

The integration follows in precisely the same manner as that for the normal component. The final expression is found to be

$$\frac{u}{w_0} = \frac{-1}{2\pi}\int_0^{2\pi} f(\psi) \frac{(z' + R_c'\cos X)\cos\psi \, d\psi}{\left[R_c' + (\cos\psi - x')\sin X + z'\cos X\right]R_c'} \tag{11}$$

Lateral component.- From equation (2), the lateral (or $\bar{j}$) component is

$$v = \frac{-R}{4\pi}\int_0^{2\pi}\int_0^\infty \frac{d\Gamma}{dL} \frac{(z + L\cos X)\sin\psi \, dL \, d\psi}{\left[R^2 + x^2 + y^2 + z^2 - 2R(x\cos\psi + y\sin\psi) + 2L(z\cos X - x\sin X + R\sin X\cos\psi) + L^2\right]^{3/2}} \tag{12}$$

It is apparent that equation (12) differs from equation (10) only by a factor, $\tan X$, which does not enter into the integration with

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respect to L. Thus, the final expression for the lateral component of induced velocity may be written immediately, on comparison with equation (11), as

$$\frac{v}{w_0} = \frac{-1}{2\pi} \int_0^{2\pi} f(\psi) \frac{(z' + R_c' \cos X)\sin \psi \, d\psi}{\left[R_c' + (\cos \psi - x')\sin X + z' \cos X\right]R_c'} \qquad (13)$$

Induced Velocities of Inner Wake

The induced velocities of the inner wake are found by integrating over the wake, where now, from figure 1(b),

$$\bar{s} = \bar{i}(r \cos \psi + L \sin X) + \bar{j}(r \sin \psi) + \bar{k}(-L \cos X)$$

$$d\bar{s} = \left[\bar{i}(\cos \psi) + \bar{j}(\sin \psi) + \bar{k}(0)\right] dr$$

$$\bar{a} = \bar{i}(r \cos \psi + L \sin X - x) + \bar{j}(r \sin \psi - y) + \bar{k}(-L \cos X - z)$$

Substituting these expressions into equation (1) and integrating yields

$$\bar{q} = \frac{-1}{4\pi} \int_0^{2\pi} \int_0^R \int_0^\infty \frac{-d}{d\psi}\left(\frac{d\Gamma}{dL}\right) \frac{\begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \cos \psi & \sin \psi & 0 \\ (r \cos \psi + L \sin X - x) & (r \sin \psi - y) & (-L \cos X - z) \end{vmatrix} dL \, dr \, d\psi}{\left[(r \cos \psi + L \sin X - x)^2 + (r \sin \psi - y)^2 + (-L \cos X - z)^2\right]^{3/2}} \qquad (14)$$

Normal component.- From equation (14), the normal (or $\bar{k}$ component of induced velocity is

$$w = \frac{1}{4\pi} \int_0^{2\pi} \int_0^R \int_0^\infty \frac{\frac{d}{d\psi}\left(\frac{d\Gamma}{dL}\right)(x \sin \psi - y \cos \psi - L \sin X \sin \psi)dL \, dr \, d\psi}{\left[r^2 + x^2 + y^2 + z^2 - 2r(x \cos \psi + y \sin \psi) + 2L(z \cos X - x \sin X + r \sin X \cos \psi) + L^2\right]^{3/2}} \qquad (15)$$

The integration of equation (15) with respect to L is similar to the same integration for the outer wake (eq. 4) and may be accomplished with the aid of items 162 and 170 of reference 6; thus

$$w = \frac{1}{4\pi} \int_0^{2\pi} \int_0^R \frac{d}{d\psi}\left(\frac{d\Gamma}{dL}\right)\left\{\frac{(x\sin\psi - y\cos\psi)(L + z\cos\chi - x\sin\chi + r\sin\chi\cos\psi) + \sin\chi\sin\psi(z\cos\chi - x\sin\chi + r\sin\chi\cos\psi)}{[r^2 + x^2 + y^2 + z^2 - 2r(x\cos\psi + y\sin\psi) - (z\cos\chi - x\sin\chi + r\sin\chi\cos\psi)^2]\sqrt{r^2 + x^2 + y^2 + z^2 - 2r(x\cos\psi + y\sin\psi) + 2L(z\cos\chi - x\sin\chi + r\sin\chi\cos\psi) + L^2}}\right\}\Bigg|_0^\infty dr\,d\psi$$

(16)

Substituting limits reduces equation (16) to

$$w = \frac{1}{4\pi} \int_0^{2\pi} \int_0^R \frac{d}{d\psi}\left(\frac{d\Gamma}{dL}\right)\left\{\frac{x\sin\psi - y\cos\psi - \sin\chi\sin\psi\sqrt{r^2 + x^2 + y^2 + z^2 - 2r(x\cos\psi + y\sin\psi)}}{\left[\sqrt{r^2 + x^2 + y^2 + z^2 - 2r(x\cos\psi + y\sin\psi) + z\cos\chi - x\sin\chi + r\sin\chi\cos\psi}\right]\sqrt{r^2 + x^2 + y^2 + z^2 - 2r(x\cos\psi + y\sin\psi)}}\right\}dr\,d\psi$$

(17)

The integration with respect to r is somewhat involved. The general forms are integrated in the appendix. The integration of equation (17) is carried out with the aid of these general forms, equations (A5) and (A14). Note the identity,

$$-2(\sin\chi\cos\psi - 1)(x\cos\psi + y\sin\chi)(z\cos\chi - x\sin\chi) - (x^2 + y^2 + z^2)(\sin^2\chi\cos^2\psi - 1) - \left[+(x\cos\psi + y\sin\psi) + (z\cos\chi - x\sin\chi)\right]^2 = \left[y\cos\psi - x\sin\psi)\cos\chi - z\sin\chi\sin\psi\right]^2$$

(18)

Thus,

$$w = \frac{1}{4\pi} \int_0^{2\pi} \frac{d}{d\psi}\left(\frac{d\Gamma}{dL}\right)\left\{\frac{2\cos\chi}{1 - \sin^2\chi\cos^2\psi}\tan^{-1}\left[\frac{(1 - \sin\chi\cos\psi)\left(\sqrt{r^2 + x^2 + y^2 + z^2 - 2r(x\cos\psi + y\sin\psi)} - z\right) + (x\cos\psi + y\sin\psi - z) + (x\cos\psi + y\sin\psi) + (z\cos\chi - x\sin\chi) + r\sin\chi\cos\psi}{(y\cos\psi - x\sin\psi)\cos\chi - z\sin\chi\sin\psi}\right]\right.$$

$$+ \frac{\sin^2\chi\sin\psi\cos\psi}{1 - \sin^2\chi\cos^2\psi}\log_e\sqrt{r^2 + x^2 + y^2 + z^2 - 2r(x\cos\psi + y\sin\psi) + (z\cos\chi - x\sin\chi + y\sin\psi) + (z\cos\chi - x\sin\chi) + r\sin\chi\cos\psi}$$

$$\left.+ \frac{\sin\chi\sin\psi}{1 - \sin^2\chi\cos^2\psi}\log_e 2\left|r - (x\cos\psi + y\sin\psi) - \sqrt{r^2 + x^2 + y^2 + z^2 - 2r(x\cos\psi + y\sin\psi)}\right|\right\}\Bigg|_0^R d\psi$$

(19)

Upon substitution of limits, this equation becomes

$$w = \frac{1}{4\pi} \int_0^{2\pi} \frac{d}{d\psi}\left(\frac{df}{d\psi}\right)\left\{\frac{2\cos\chi}{1-\sin^2\chi\,\cos^2\psi}\,\tan^{-1}\left[\frac{(1-\sin\chi\cos\psi)\left(\sqrt{R^2+x^2+y^2+z^2-2R(x\cos\psi+y\sin\psi)}-R\right)+(x\cos\psi+y\sin\psi)+(z\cos\chi-x\sin\chi)}{(y\cos\psi-x\sin\psi)\cos\chi-z\sin\chi\sin\psi}\right]\right.$$

$$-\frac{2\cos\chi}{1-\sin^2\chi\,\cos^2\psi}\,\tan^{-1}\left[\frac{(1-\sin\chi\cos\psi)\sqrt{x^2+y^2+z^2}+(x\cos\psi+y\sin\psi)+(z\cos\chi-x\sin\chi)}{(y\cos\psi-x\sin\psi)\cos\chi-z\sin\chi\sin\psi}\right]$$

$$+\frac{\sin^2\chi\,\sin\psi\,\cos\psi}{1-\sin^2\chi\,\cos^2\psi}\,\log_e\left|\frac{\sqrt{R^2+x^2+y^2+z^2-2R(x\cos\psi+y\sin\psi)}+(z\cos\chi-x\sin\chi)+R\sin\chi\cos\psi}{\sqrt{x^2+y^2+z^2}+(z\cos\chi-x\sin\chi)}\right|$$

$$\left.+\frac{\sin\chi\,\sin\psi}{1-\sin^2\chi\,\cos^2\psi}\,\log_e\left|\frac{(x\cos\psi+y\sin\psi)+\sqrt{R^2+x^2+y^2+z^2-2R(x\cos\psi+y\sin\psi)}-R}{\sqrt{x^2+y^2+z^2}+(x\cos\psi+y\sin\psi)}\right|\right\}d\psi$$

or, in nondimensional form

$$\frac{w}{w_0}=\frac{-1}{2\pi}\int_0^{2\pi}\frac{df(\psi)}{d\psi}\left\{\left[\frac{2\cos\chi}{1-\sin^2\chi\,\cos^2\psi}\left(\tan^{-1}\left|\frac{(1-\sin\chi\cos\psi)(R_c'-1)-A-B}{C}\right|-\tan^{-1}\left|\frac{(1-\sin\chi\cos\psi)R_0'+A+B}{C}\right|\right)\right.\right.$$

$$\left.\left.+\frac{\sin^2\chi\,\sin\psi\,\cos\psi}{1-\sin^2\chi\,\cos^2\psi}\log_e\left|\frac{R_c'+B+\sin\chi\cos\psi}{R_0'+B}\right|+\frac{\sin\chi\,\sin\psi}{1-\sin^2\chi\,\cos^2\psi}\log_e\left|\frac{R_c'+A-1}{R_0'+A}\right|\right]\right\}d\psi \tag{21}$$

In general, it is convenient to combine the two inverse tangent terms of equation (21) by means of item 648 of reference 6, to yield

$$\frac{w}{w_0}=\frac{-1}{2\pi}\int_0^{2\pi}\frac{df(\psi)}{d\psi}\left\{\left[\frac{2\cos\chi}{1-\sin^2\chi\,\cos^2\psi}\,\tan^{-1}\left|\frac{C(R_c'-R_0'-1)(1-\sin\chi\cos\psi)}{R_0'(R_c'-1)(1-\sin\chi\cos\psi)^2+(A+B)(R_c'+R_0'-1)(1-\sin\chi\cos\psi)+(A+B)^2+C^2}\right|\right.\right.$$

$$\left.\left.+\frac{\sin^2\chi\,\sin\psi\,\cos\psi}{1-\sin^2\chi\,\cos^2\psi}\log_e\left|\frac{R_c'+B+\sin\chi\cos\psi}{R_0'+B}\right|+\frac{\sin\chi\,\sin\psi}{1-\sin^2\chi\,\cos^2\psi}\log_e\left|\frac{R_c'+A-1}{R_0'+A}\right|\right]\right\}d\psi \tag{22}$$

(20)

In certain cases, the argument of the inverse tangent term becomes indeterminate. The limiting value, however, may be found by the repeated application of L'Hopital's rule. Limiting values for a few of the more important cases are given in the following table:

Location of P	Azimuth angle, ψ, radians	Arc tangent, radians
y = z = 0	0	$\pm \dfrac{\pi}{2}$
	π	0
x = z = 0	$\dfrac{\pi}{2}$	$\chi - \dfrac{\pi}{2}$
	$\dfrac{3\pi}{2}$	0

Longitudinal component.- The longitudinal (or $\bar{\text{i}}$) component of induced velocity is, from equation (14),

$$u = \frac{-1}{4\pi} \int_0^{2\pi} \int_0^R \int_0^\infty \frac{\frac{d}{d\psi}\left(\frac{d\Gamma}{dL}\right)(z + L\cos\chi)\sin\psi\, dL\, dr\, d\psi}{\left[r^2 + x^2 + y^2 + z^2 - 2r(x\cos\psi + y\sin\psi) + 2L(z\cos\chi - x\sin\chi + r\sin\chi\cos\psi) + L^2\right]^{3/2}} \tag{23}$$

Equation (23) may be integrated with respect to L to yield

$$i = \frac{-1}{4\pi} \int_0^{2\pi} \int_0^R \frac{\frac{d}{d\psi}\left(\frac{d\Gamma}{dL}\right)\left[\sin\psi\right]\left[z + \cos\chi\sqrt{r^2 + x^2 + y^2 + z^2 - 2r(x\cos\psi + y\sin\psi)}\right]dr\, d\psi}{\left[\sqrt{r^2 + x^2 + y^2 + z^2 - 2r(x\cos\psi + y\sin\psi)} + z\cos\chi - x\sin\chi + r\sin\chi\cos\psi\right]\sqrt{r^2 + x^2 + y^2 + z^2 - 2r(x\cos\psi + y\sin\psi)}} \tag{24}$$

Integrating with respect to r, by means of equations (A5) and (A14) of the appendix, results in

$$u = \frac{-1}{4\pi} \int_0^{2\pi} \frac{d}{d\psi}\left(\frac{d\Gamma}{dL}\right) \left\{ \frac{2 \sin X \sin^2\psi}{1 - \sin^2 X \cos^2\psi} \tan^{-1}\left[\frac{(1 - \sin X \cos\psi)\left(\sqrt{r^2 + x^2 + y^2 + z^2 - 2r(x\cos\psi + y\sin\psi)} - r\right) + (x\cos\psi + y\sin\psi) + (z\cos X - x\sin X)}{(y\cos\psi - x\sin\psi)\cos X - z\sin X \sin\psi}\right] \right.$$

$$- \frac{\sin X \cos X \sin\psi \cos\psi}{1 - \sin^2 X \cos^2\psi} \log_e \left|\sqrt{r^2 + x^2 + y^2 + z^2 - 2r(x\cos\psi + y\sin\psi)} + (z\cos X - x\sin X) + r\sin X\cos\psi\right|$$

$$\left. - \frac{\cos X \sin\psi}{1 - \sin^2 X \cos^2\psi} \log_e 2\left|r - (x\cos\psi + y\sin\psi) - \sqrt{r^2 + x^2 + y^2 + z^2 - 2r(x\cos\psi + y\sin\psi)}\right| \underbrace{\qquad}\right|_0^R \; d\psi \qquad (25)$$

After limits are substituted and quantities are nondimensionalized, equation (25) becomes

$$\frac{u}{w_0} = \frac{1}{2\pi} \int_0^{2\pi} \frac{df(\psi)}{d\psi} \left(\frac{2 \sin X \sin^2\psi}{1 - \sin^2 X \cos^2\psi} \left\{ \tan^{-1}\left[\frac{(1 - \sin X \cos\psi)(R_c' - 1) + A + B}{C}\right] - \tan^{-1}\left[\frac{(1 - \sin X \cos\psi)R_0' + A + B}{C}\right] \right\} \right.$$

$$\left. - \frac{\sin X \cos X \sin\psi \cos\psi}{1 - \sin^2 X \cos^2\psi} \log_e \left|\frac{R_c' + B + \sin X \cos\psi}{R_0' + B}\right| - \frac{\cos X \sin\psi}{1 - \sin^2 X \cos^2\psi} \log_e \left|\frac{R_c' + A - 1}{R_0' + A}\right| \right) d\psi \qquad (26)$$

or, when the inverse tangent terms are combined as before,

$$\frac{u}{w_0} = \frac{1}{2\pi} \int_0^{2\pi} \frac{df(\psi)}{d\psi} \left\{ \frac{2 \sin X \sin^2\psi}{1 - \sin^2 X \cos^2\psi} \tan^{-1}\left[\frac{C(R_c' - R_0' - 1)(1 - \sin X \cos\psi)}{R_0'(R_c' - 1)(1 - \sin X \cos\psi)^2 + (A + B)(R_c' + R_0' - 1)(1 - \sin X \cos\psi) + (A + B)^2 + C^2}\right] \right.$$

$$\left. - \frac{\sin X \cos X \sin\psi \cos\psi}{1 - \sin^2 X \cos^2\psi} \log_e \left|\frac{R_c' + B + \sin X \cos\psi}{R_0' + B}\right| - \frac{\cos X \sin\psi}{1 - \sin^2 X \cos^2\psi} \log_e \left|\frac{R_c' + A - 1}{R_0' + A}\right| \underbrace{\qquad}\right\} d\psi \qquad (27)$$

Lateral component.- From equation (14), the lateral (or J) component is

$$v = \frac{1}{4\pi}\int_0^{2\pi}\int_0^R\int_0^\infty \frac{\frac{d}{d\psi}\left(\frac{d\Gamma}{dL}\right)(\cos\psi)(z + L\cos\chi)dL\,dr\,d\psi}{\left[r^2 + x^2 + y^2 + z^2 - 2r(x\cos\chi - x\sin\chi + r\sin\chi\cos\psi) + 2L(z\cos\chi - x\sin\chi + r\sin\chi\cos\psi) + L^2\right]^{3/2}} \tag{28}$$

Because equation (28) differs from equation (23) only by a factor $-\cot\psi$, which does not enter into the integrations, the expressions for the lateral component can be written immediately from equation (26) and (27) as

$$\frac{v}{w_0} = \frac{-1}{2\pi}\int_0^{2\pi}\frac{df(\psi)}{d\psi}\left(\frac{2\sin\chi\sin\psi\cos\psi}{1-\sin^2\chi\cos^2\psi}\left\{\tan^{-1}\left[\frac{(1-\sin\chi\cos\psi)(R_c'-1)+A+B}{C}\right] - \tan^{-1}\left[\frac{(1-\sin\chi\cos\psi)R_0'+A+B}{C}\right]\right\}\right.$$

$$\left. - \frac{\sin\chi\cos\chi\cos^2\psi}{1-\sin^2\chi\cos^2\psi}\log_e\left|\frac{R_c'+B+\sin\chi\cos\psi}{R_0'+B}\right| - \frac{\cos\chi\cos\psi}{1-\sin^2\chi\cos^2\psi}\log_e\left|\frac{R_c'+A-1}{R_0'+A}\right|\right)d\psi \tag{29}$$

or

$$\frac{v}{w_0} = \frac{-1}{2\pi}\int_0^{2\pi}\frac{df(\psi)}{d\psi}\left(\frac{2\sin\chi\sin\psi\cos\psi}{1-\sin^2\chi\cos^2\psi}\tan^{-1}\left[\frac{C(R_c'-R_0'-1)(1-\sin\chi\cos\psi)}{R_0'(R_c'-1)(1-\sin\chi\cos\psi)^2+(A+B)(R_c'+R_0'-1)(1-\sin\chi\cos\psi)+(A+B)^2+C^2}\right]\right.$$

$$\left. - \frac{\sin\chi\cos\chi\cos^2\psi}{1-\sin^2\chi\cos^2\psi}\log_e\left|\frac{R_c'+B+\sin\chi\cos\psi}{R_0'+B}\right| - \frac{\cos\chi\cos\psi}{1-\sin^2\chi\cos^2\psi}\log_e\left|\frac{R_c'+A-1}{R_0'+A}\right|\right)d\psi \tag{30}$$

DISCUSSION

Evaluation of Integrals With Respect to ψ

It will be observed that the equations for the induced velocities (eqs. (9), (11), (13), (22), (27), and (30)) have been left in the form of integrals with respect to ψ. It appears that these integrals cannot be evaluated in closed form; however, usable results can be obtained by numerical integration. This procedure is only practical when handled on automatic digital computing machines. No attempt

is made in this paper to provide general flow charts for the various harmonics of vorticity such as has been done previously for the case of constant vorticity (for example, ref. 3). A few calculations for the normal component of induced velocity on the lateral axis of the rotor have been made for a unit $\sin \psi$ vorticity. These cases are discussed in the following sections.

Distribution of Induced Velocity on Lateral Axis

The calculated normal induced velocities are presented in figure 3 for the case of the lateral axis and a unit $\sin \psi$ vorticity. The skew angle is $\tan^{-1} 4$. The numerical results indicate that the induced velocities are uniform on either side of the axis and discontinuous at the center of the rotor. This distribution corresponds to that of the assumed $\sin \psi$ vorticity distribution, which is also uniform on either side of the axis but discontinuous at the center of the rotor. This close correlation between vorticity (or disk load) distribution and the induced velocities on the lateral axis was also noted in the experimental measurements of reference 1.

The contribution of the inner and outer wakes individually is also shown in figure 3. The contribution of each part of the wake is a linear function of radial position – the outer wake portion increasing, and the inner wake portion decreasing with radius. Except for positions close to the edge of the rotor disk ($y' > 0.85$), the inner wake produces the largest contribution to the total induced velocity for $\sin \psi$ vorticity.

Reference 2 treats the induced velocities due to $\sin \psi$ vorticity by means of a cruder approximation to the rotor wake. It is interesting to note that the actual induced velocity distribution (ref. 5) on the lateral axis, as obtained under the assumptions of reference 2, should be essentially the same, except in magnitude, as that obtained in the present analysis. The simple analysis of reference 2 cannot, however, be extended to locations other than the lateral axis.

Effect of Skew Angle

Since the absolute value of the induced velocity for $\sin \psi$ vorticity is constant on the lateral axis, the variation with skew angle can be illustrated very compactly as in figure 4. Here the uniform value of induced velocity is shown as a function of $\tan X$ for a range of skew angles encompassing the major portion of the helicopter flight range. It is evident that the effect of $\sin \psi$ vorticity increases somewhat with skew angle. The corresponding induced velocity from reference 2 for unit $\sin \psi$ vorticity is $\frac{w}{w_0} = 2.0$, irrespective of skew angle. Thus, even

though a strict interpretation of the procedures of reference 2 leads to the correct shape of the velocity distribution, it also leads to a considerable overestimate of the magnitude of the induced velocities.

CONCLUDING REMARKS

Equations, which can be integrated by use of high-speed computing machines, have been developed for all three components of induced velocity at an arbitrary point near the rotor and for an arbitrary harmonic of the vorticity distribution. Results of this investigation of the induced velocities near a lifting rotor with nonuniform azimuthwise vorticity distribution show that for a vorticity distribution which varies as the sine of the azimuth angle, the normal component of induced velocity along either side of the lateral axis is uniform and antisymmetric about the center of the rotor.

Langley Research Center,
National Aeronautics and Space Administration,
Langley Field, Va., March 10, 1960.

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APPENDIX

EVALUATION OF CERTAIN INTEGRALS OCCURRING IN THE ANALYSIS

The analysis of the inner wake involves integrals of the form,

$$\int \frac{dx}{X + (cx + d)\sqrt{X}} \qquad \text{and} \qquad \int \frac{dx}{\sqrt{X} + cx + d}$$

where

$$X = a + bx + x^2$$

and where x is an arbitrary variable and a, b, c, and d are arbitrary constants.

Both forms may be evaluated by use of the substitution (item 235, ref. 6),

$$u = \sqrt{X} - x \qquad (A1)$$

The first form is the simpler of the two and will be evaluated first. The indicated substitution (eq. (A1)) reduces this form to

$$\int \frac{dx}{X + (cx + d)\sqrt{X}} = \int \frac{2\, du}{(c - 1)u^2 + (b - 2d)u + (db - a - ac)} \qquad (A2)$$

The right-hand side of equation (A2) may be integrated by item 67 of reference 6 to yield

$$\int \frac{dx}{X + (cx + d)\sqrt{X}} = \frac{4}{\sqrt{q}} \tan^{-1}\left[\frac{2(c - 1)u + b - 2d}{\sqrt{q}}\right] \qquad (A3)$$

where

$$q = 4db(c - 1) + 4a(1 - c^2) - (b - 2d)^2 \qquad (A4)$$

The final expression is obtained by returning equation (A3) to the original variable x which results in

$$\int \frac{dx}{X + (cx + d)\sqrt{X}} = \frac{4}{\sqrt{q}} \tan^{-1}\left[\frac{2(c - 1)\left(\sqrt{X} - x\right) + b - 2d}{\sqrt{q}}\right] \qquad (A5)$$

where q is as given previously in equation (A4).

Evaluation of the second form is a more lengthy task but proceeds in the same manner. Making the same substitution (eq. (A1)) as before yields

$$\int \frac{dx}{\sqrt{X} + cx + d} = \frac{2(bu - a - u^2)du}{(b - 2u)\left[u^2(c - 1) + u(b - 2d) + (db - a - ac)\right]} \qquad (A6)$$

Now separate the right-hand side of equation (A6) into partial fractions to obtain

$$\int \frac{dx}{\sqrt{X} + cx + d} = \frac{1}{1 + c} \int \frac{2\,du}{b - 2u} + \frac{2c}{1 + c} \int \frac{u\,du}{\left[u^2(c - 1) + u(b - 2d) + (db - a - ac)\right]}$$

$$- \frac{2d}{1 + c} \int \frac{du}{\left[u^2(c - 1) + u(b - 2d) + (db - a - ac)\right]} \qquad (A7)$$

The first term may be integrated by inspection; the second term, by item 72 of reference 6 to give

$$\int \frac{dx}{\sqrt{X} + cx + d} = \frac{-1}{1 + c} \log_e(b - 2u) + \frac{c}{c^2 - 1} \log_e\left[u^2(c - 1) + u(b - 2d) + (db - a - ac)\right]$$

$$- \frac{bc - 2d}{c^2 - 1} \int \frac{du}{\left[u^2(c - 1) + u(b - 2d) + (db - a - ac)\right]} \qquad (A8)$$

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The remaining integral may be evaluated by means of item 67 of reference 6 to yield

$$\int \frac{dx}{\sqrt{X} + cx + d} = \frac{-1}{1 + c} \log_e(b - 2u)$$

$$+ \frac{c}{c^2 - 1} \log_e\left[u^2(c - 1) + u(b - 2d) + (db - a - ac)\right]$$

$$- \frac{bc - 2d}{c^2 - 1} \frac{2}{\sqrt{q}} \tan^{-1}\left[\frac{2(c - 1)u + b - 2d}{\sqrt{q}}\right] \tag{A9}$$

where q is defined in equation (A4).

Returning equation (A9) to the original variable x results in

$$\int \frac{dx}{\sqrt{X} + cx + d} = \frac{-(c - 1)}{c^2 - 1} \log_e\left(b + 2x - 2\sqrt{X}\right)$$

$$+ \frac{c}{c^2 - 1} \log_e\left[\left(\sqrt{X} - x\right)^2(c - 1) + \left(\sqrt{X} - x\right)(b - 2d) + (db - a - ac)\right]$$

$$- \frac{bc - 2d}{c^2 - 1} \frac{2}{\sqrt{q}} \tan^{-1}\left[\frac{2(c - 1)\left(\sqrt{X} - x\right) + b - 2d}{\sqrt{q}}\right] \tag{A10}$$

Note that

$$\left(\sqrt{X} - x\right)^2(c - 1) + \left(\sqrt{X} - x\right)(b - 2d) + (db - a - ac) = \left(b + 2x - 2\sqrt{X}\right)\left(\sqrt{X} + cx + d\right)$$

so that the logarithmic terms of equation (A10) may be combined to yield

$$\int \frac{dx}{\sqrt{X} + cx + d} = \frac{c}{c^2 - 1} \log_e\left(\sqrt{X} + cx + d\right) + \frac{1}{c^2 - 1} \log_e\left(b + 2x - 2\sqrt{X}\right)$$

$$- \frac{bc - 2d}{c^2 - 1} \frac{2}{\sqrt{q}} \tan^{-1}\left[\frac{2(c - 1)\left(\sqrt{X} - x\right) + b - 2d}{\sqrt{q}}\right] \tag{A11}$$

The presence of the logarithmic terms in equation (A11) occasions imaginary values for the integral if either $\left(\sqrt{X} + cx + d\right)$ or $\left(b + 2x - 2\sqrt{X}\right)$ is negative. Provided that these terms are monotonically negative, this result may be avoided as follows:

Rewrite equation (A7) as

$$\int \frac{dx}{\sqrt{X} + cx + d} = \frac{1}{1 + c} \int \frac{2\ du}{2u - b} - \frac{2c}{1 + c} \int \frac{u\ du}{\left[u^2(1 - c) + u(2d - b) - (db - a - ac)\right]}$$

$$- \frac{2d}{1 + c} \int \frac{du}{\left[u^2(c - 1) + u(b - 2d) + (db - a - ac)\right]} \tag{A12}$$

Integration, as in equation (A8), of the first two terms results in

$$\int \frac{dx}{\sqrt{X} + cx + d} = \frac{-1}{1 + c} \log_e(2u - b) + \frac{c}{c^2 - 1} \log_e\left[u^2(1 - c) + u(2d - b) - (db - a - ac)\right]$$

$$- \frac{bc - 2d}{c^2 - 1} \int \frac{du}{\left[u^2(c - 1) + u(b - 2d) + (db - a - ac)\right]} \tag{A13}$$

Comparison of equations (A8) and (A13) indicates that the only difference is a reversal in sign of the arguments of the logarithmic terms. Therefore, as long as these arguments are monotonically either positive or negative, the final expression may be rewritten as

$$\int \frac{dx}{\sqrt{X} + cx + d} = \frac{c}{c^2 - 1} \log_e\left|\sqrt{X} + cx + d\right| + \frac{1}{c^2 - 1} \log_e\left|b + 2x - 2\sqrt{X}\right|$$

$$- \frac{bc - 2d}{c^2 - 1} \frac{2}{\sqrt{q}} \tan^{-1}\left[\frac{2(c - 1)\left(\sqrt{X} - x\right) + b - 2d}{\sqrt{q}}\right] \tag{A14}$$

where, again,

$$q = 4db(c - 1) + 4a\left(1 - c^2\right) - (b - 2d)^2$$

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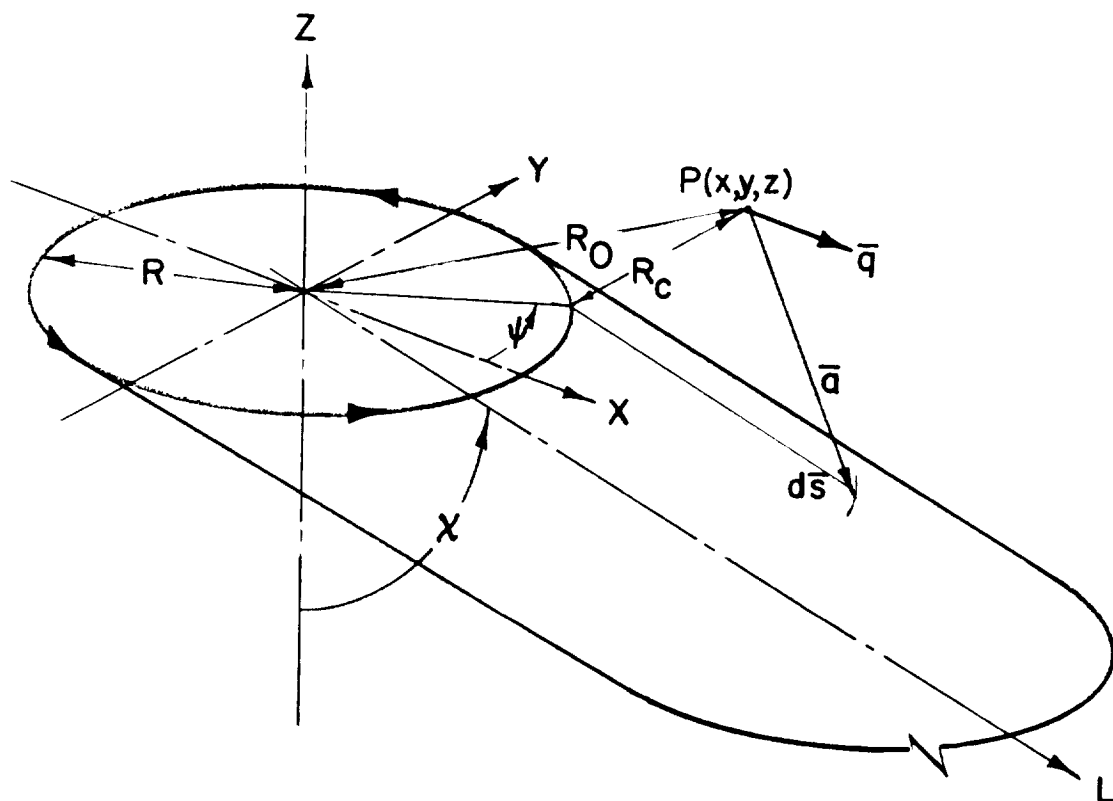
(a) Outer wake.

Figure 1.- Rotor wake system.
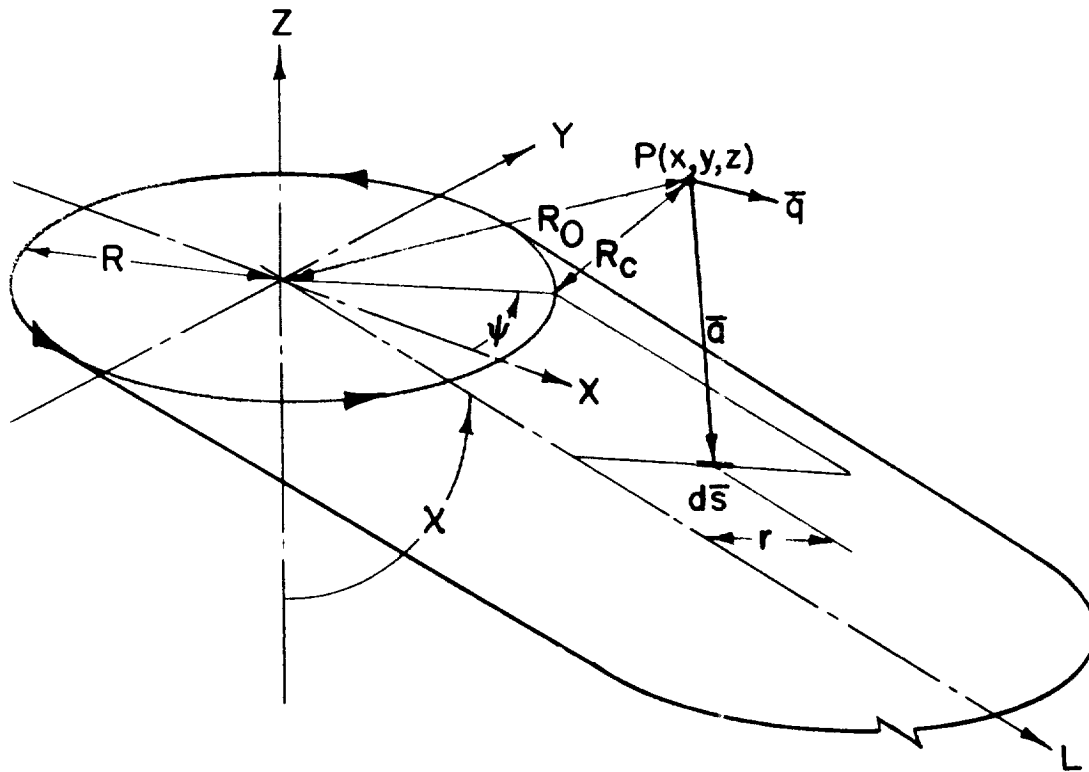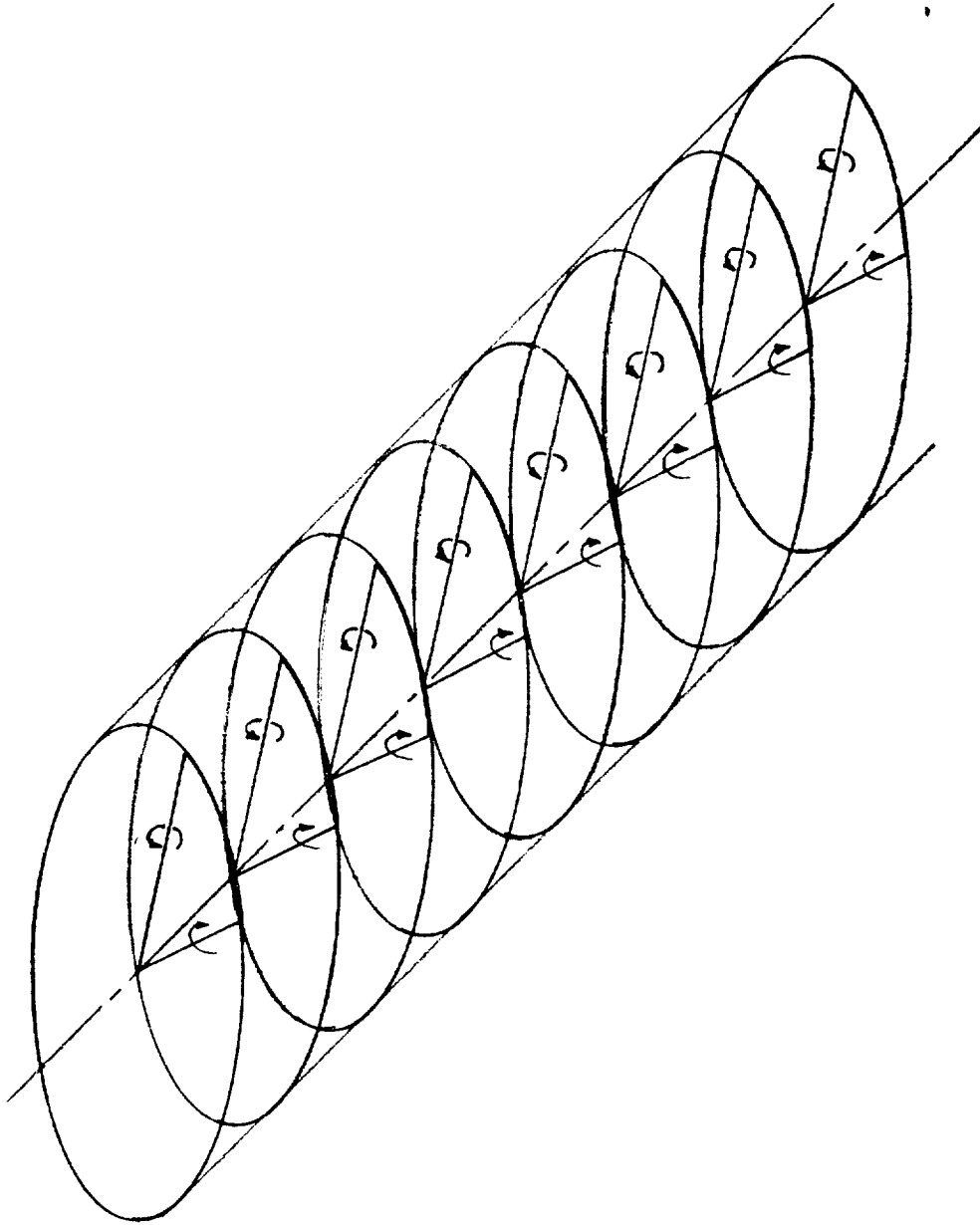
(b) Inner wake.

Figure 1.- Concluded.

Figure 2.- Schematic view of wake showing radial vorticity arising from a jump in circulation over one sector of the rotor disk.
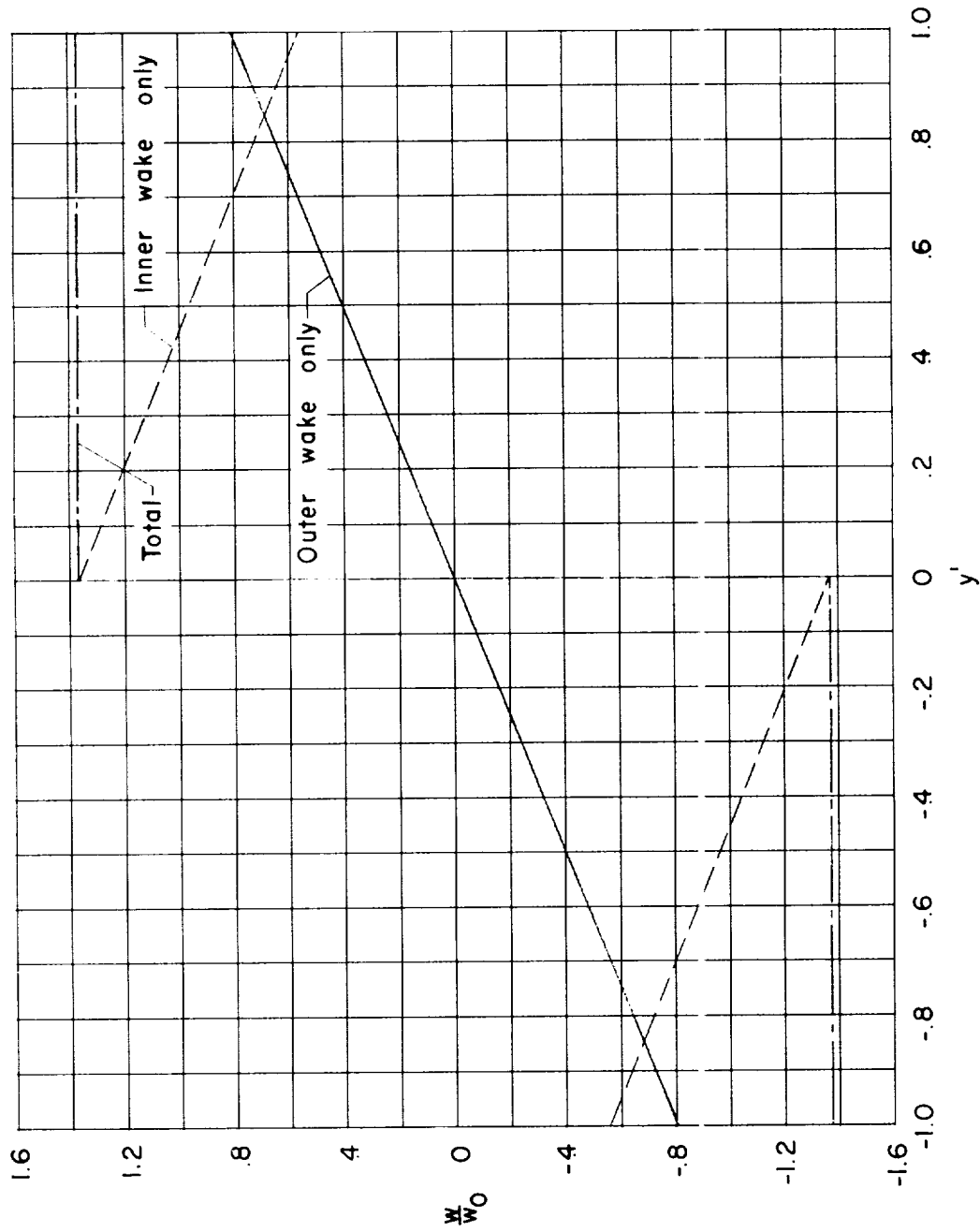
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Figure 3.- Normal component of induced velocity on lateral axis for unit $\sin \psi$ vorticity. $\chi = \tan^{-1} 4$.
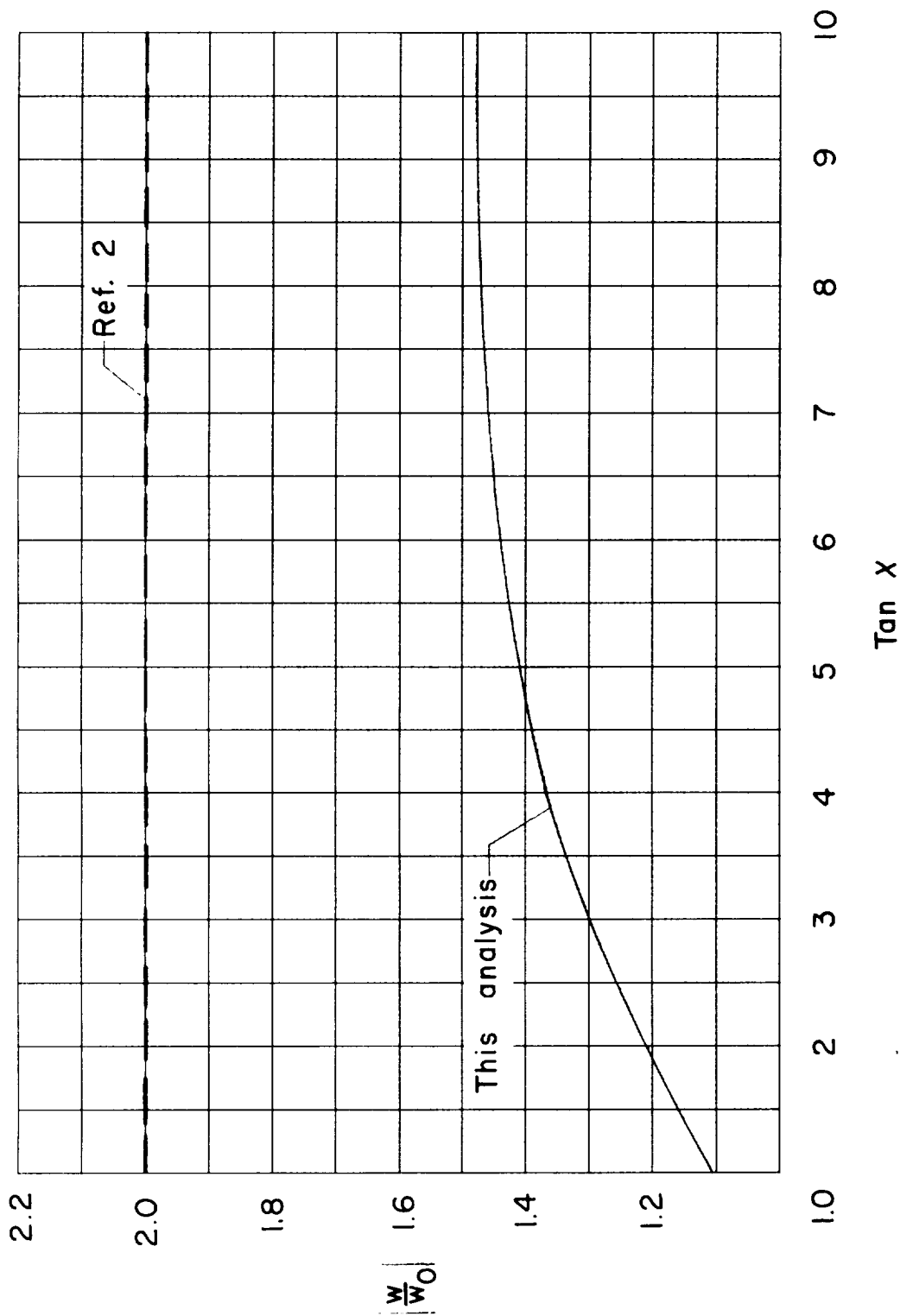
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Figure 4.- Absolute value of normal component of induced velocity on either side of lateral (Y) axis for unit $\sin \psi$ vorticity.