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THE USE OF HIGHER-ORDER DIFFERENCE METHODS IN BEAM

VIBRATION ANALYSIS

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SUMMARY

    Simple and higher-order difference methods for the solution for the natural
frequencies of vibration of a uniform beam are compared. The same basic higher-
order method is used throughout for the interior cells, but three different
methods of boundary-condition representation are given.

    Tables and graphs of the error in mode frequencies, as compared with a
continuous beam, are given for the various methods as a function of the number
of cells. It is concluded that higher-order methods improve accuracy for a
given number of cells, with essentially no change in the quantity of com-
puting equipment required.

I. INTRODUCTION

One of the most commonly encountered problems of structural dynamics is that of the lateral vibrations of beams. With the advent on a large scale of analog and digital computers of greatly improved speed and computational efficiency, it is worthwhile to take another look at the methods available for performing these computations. This report is principally concerned with improvements in analog computer methods of analysis of the beam-vibration problem. In particular, it is concerned with methods applicable to the electronic differential analyzer. Nevertheless, it will be seen that the matrix formulation of the problem and most of the results are directly applicable to digital computation as well.

Until quite recently, all attempts to use analog computers in the solution of partial differential equations by difference techniques were restricted to simple methods with accuracies of second order in the cell size at best. In 1956, Fisher[1] suggested the use of higher-order differences in the solution of these problems on the analog computer and indicated some of the advantages to be expected. However, he was not concerned in his examples with the beam equation and did not study the problem of boundary-condition representation using higher-order methods.

This report considers in detail the adequate representation of boundary conditions when higher-order methods are used, and presents the results of rather extensive computations which enable one to obtain a feel for advantages and disadvantages of each method. The analysis in the main body of the report is concerned with uniform beams. The mode-frequency errors for a tapered cantilever beam are considered in Appendix B and comparisons are made with the continuous beam.

No analysis of mode-shape errors has been included. However, spot checks have shown these errors to be comparable in magnitude with the frequency errors.

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II. SYMBOLS

A	mode-amplitude matrix
a_0, a_1, etc.	constant coefficients
B	$-[T]^{-1} [K]$
B_1, B_2, etc.	difference matrices
b	an element of matrix B
b_0, b_1, etc.	constant coefficients
C	capacitance, constraint matrix
C_1, C_2, etc.	constants
D	difference matrix
E	Young's modulus
F	constraint force
f	lateral force per unit length
G	constant
H	constant
h	cell size
I	moment of inertia of cross section, unit matrix
i	mode number
K	system matrix $[m]^{-1} [k]$
k	stiffness matrix
k_θ	bending stiffness of a cell
L	inductance, total length of beam

M   bending moment

m   mass, mass matrix

N   total number of cells

n   station designation, number of degrees of freedom

R   resistance

S $S = 0$ is the characteristic equation

T $-[K][B]^{-1}$. Also the time-dependent factor in the beam deflection

t   time

V   shear force

x   position along the beam

Y   deflection amplitude

y   lateral deflection

β eigenvalue for a continuous beam of unit length

Δ a small perturbation of the designated quantity

ϵ frequency error

ζ dimensionless lateral deflection

η the square of a mode frequency

θ beam slope

λ eigenvalue for beam of length N

ξ dimensionless position along beam

ρ mass per unit length

τ dimensionless time

ω eigenvalue for difference beam of unit length

$(\quad)_{ij}$ indices referring to row and column, respectively

$(\quad)_o$ reference value

$[\quad]^T$ transpose of the designated matrix

$(^-)$ computed in the modal $(\overline{y})$ coordinate system

$\widetilde{=}$ approximately equal to

$|\quad|$ amplitude of a sinusoidally varying quantity, determinant of a matrix

$(\quad)$ row matrix, or usual parenthesis notation

$[\quad]$ rectangular matrix, or usual brackets notation

$\{\quad\}$ column matrix, or usual braces notation

III. CONTINUOUS BEAMS

A. THE BEAM EQUATION

We will restrict ourselves to the study of the lateral vibrations of a thin beam where linearity is assumed. The partial differential equation describing the motion is

$$\frac{\partial^2}{\partial x^2}\left(EI\,\frac{\partial^2 y}{\partial x^2}\right) + \rho\,\frac{\partial^2 y}{\partial t^2} = f(x,t) \tag{1}$$

where

$$
\begin{aligned}
x &= \text{position along beam}\\
y &= \text{lateral deflection}\\
t &= \text{time}\\
EI &= \text{bending stiffness}\\
\rho &= \text{mass per unit length}\\
f &= \text{lateral force per unit length}
\end{aligned}
$$

It can be seen that Eq. (1) is of fourth order in x and second order in t. For our purposes it is often convenient to write four equations of first order in x.

$$\theta = \frac{\partial y}{\partial x} \tag{2}$$

$$M = EI\,\frac{\partial \theta}{\partial x} \tag{3}$$

$$V = \frac{\partial M}{\partial x} \tag{4}$$

$$\rho\,\frac{\partial^2 y}{\partial t^2} = -\frac{\partial V}{\partial x} + f(x,t) \tag{5}$$

where

$$
\begin{aligned}
\theta &= \text{slope of beam}\\
M &= \text{bending moments}\\
V &= \text{shear force}
\end{aligned}
$$

This set of partial differential equations is equivalent to Eq. (1). We will see that the analog computer approach to the problem is based upon this latter set of equations rather than the original fourth-order equation.

B. SOLUTIONS OF THE BEAM EQUATION

1. General Solution.—In general, the parameters ρ and EI are functions of x. Initially, however, we will assume that each is constant. Furthermore, we will assume that units are chosen such that the coefficients are unity. Our principal interest is in the solution of the homogeneous equation, so let us set $f(x,t) = 0$. Then Eq. (1) becomes

$$\frac{\partial^4 y}{\partial x^4} + \frac{\partial^2 y}{\partial t^2} = 0 \tag{6}$$

It may be seen that essentially the same equation can be obtained by transforming Eq. (1) to dimensionless form and again setting $f(x,t) = 0$. Let

$$\zeta = \frac{y}{L}$$

$$\xi = \frac{x}{L}$$

$$\tau = \frac{t}{L^2} \sqrt{\frac{EI}{\rho}}$$

where L is the length of the beam. Then

$$\frac{\partial^4 \zeta}{\partial \xi^4} + \frac{\partial^2 \zeta}{\partial \tau^2} = 0 \tag{7}$$

Equation (6) can be solved by the method of separation of variables.[2,3] The solution is of the form

$$y = Y(x)\, T(t) \tag{8}$$

$$|M| \cong -2C_1\beta\left(1 - \sqrt{\beta}\, x + \frac{\beta^2 x^4}{4!} - \frac{\beta^{\frac{5}{2}} x^5}{5!} + \ldots\right) \tag{22}$$

$$|V| \cong -2C_1\beta^{\frac{3}{2}}\left(-1 + \frac{\beta^{\frac{3}{2}} x^3}{3!} - \frac{\beta^2 x^4}{4!} + \frac{\beta^{\frac{7}{2}} x^7}{7!} - \frac{\beta^4 x^8}{8!} + \ldots\right) \tag{23}$$

3. <u>The</u> <u>Free-Free</u> <u>Beam</u>.—For the case of a free-free beam, the boundary conditions are

$$M(0) = M(1) = 0 \tag{24}$$

$$V(0) = M(1) = 0 \tag{25}$$

Substituting into Eq. (9), we obtain

$$-C_1 + C_3 = 0 \tag{26}$$

$$-C_2 + C_4 = 0 \tag{27}$$

$$\left(\cosh\sqrt{\beta} - \cos\sqrt{\beta}\right)C_1 + \left(\sinh\sqrt{\beta} - \sin\sqrt{\beta}\right)C_2 = 0 \tag{28}$$

$$\left(\sinh\sqrt{\beta} + \sin\sqrt{\beta}\right)C_1 + \left(\cosh\sqrt{\beta} - \cos\sqrt{\beta}\right)C_2 = 0 \tag{29}$$

giving, as before,

$$\frac{C_2}{C_1} = \frac{\cos\sqrt{\beta} - \cosh\sqrt{\beta}}{\sinh\sqrt{\beta} - \sin\sqrt{\beta}} \tag{30}$$

The eigenvalue equation is the same as for the built-in case, namely,

$$1 - \cos\sqrt{\beta}\,\cosh\sqrt{\beta} = 0 \tag{31}$$

The value $\beta = 0$ has physical significance in this case but since the mode involves no elastic deformation, it is omitted. The remaining eigenvalues are identical with those for the built-in case.

The amplitude of the motion is

$$Y(x) = 2C_1\left(1 + \frac{\beta^2 x^4}{4!} + \frac{\beta^4 x^8}{8!} + \ldots\right) + 2C_2\left(\sqrt{\beta}\, x + \frac{\beta^{\frac{5}{2}} x^5}{5!} + \frac{\beta^{\frac{9}{2}} x^9}{9!} + \ldots\right) \tag{32}$$

giving the approximate amplitudes

$$Y(x) \cong 2C_1\left(1 - \sqrt{\beta}\, x + \frac{\beta^2 x^4}{4!} - \frac{\beta^{\frac{5}{2}} x^5}{5!} + \frac{\beta^4 x^8}{8!} - \frac{\beta^{\frac{9}{2}} x^9}{9!} + \ldots\right) \tag{33}$$

$$|\theta| \cong 2C_1\sqrt{\beta}\left(-1 + \frac{\beta^{\frac{3}{2}} x^3}{3!} - \frac{\beta^2 x^4}{4!} + \frac{\beta^{\frac{7}{2}} x^7}{7!} - \frac{\beta^4 x^8}{8!} + \ldots\right) \tag{34}$$

$$|M| \cong 2C_1\beta\left(\frac{\beta x^2}{2!} - \frac{\beta^{\frac{3}{2}} x^3}{3!} + \frac{\beta^3 x^6}{6!} - \frac{\beta^{\frac{7}{2}} x^7}{7!} + \ldots\right) \tag{35}$$

$$|V| \cong 2C_1\beta^{\frac{3}{2}}\left(\sqrt{\beta}\, x - \frac{\beta x^2}{2!} + \frac{\beta^{\frac{5}{2}} x^5}{5!} - \frac{\beta^3 x^6}{6!} + \ldots\right) \tag{36}$$

If we compare the solution for the free-free beam with that for a beam built-in at both ends, we note some important similarities. We have seen that the nonzero eigenvalues are identical. Beyond that, the roles of the deflection y and the bending moment M are interchanged in the two cases as may be seen by comparing Eqs. (20) and (35) and also Eqs. (22) and (33). Similarly, the roles of the slope θ and the shear force V are interchanged. This similarity will hold even for the finite difference solution which will be obtained later. Therefore it will not be necessary to carry out separate calculations for built-in and free-free beams.

4. The Cantilever Beam.—The boundary conditions in this case are

$$y(0) = 0 \tag{37}$$
$$\theta(0) = 0 \tag{38}$$
$$M(1) = 0 \tag{39}$$
$$V(1) = 0 \tag{40}$$

Applying these boundary conditions to the general solution given by Eq. (9), we obtain

$$C_1 + C_3 = 0 \tag{41}$$

$$C_2 + C_4 = 0 \tag{42}$$

$$\left(\cos \sqrt{\beta} + \cosh \sqrt{\beta}\right)C_1 + \left(\sin \sqrt{\beta} + \sinh \sqrt{\beta}\right)C_2 = 0 \tag{43}$$

$$\left(\sinh \sqrt{\beta} - \sin \sqrt{\beta}\right)C_1 + \left(\cos \sqrt{\beta} + \cosh \sqrt{\beta}\right)C_2 = 0 \tag{44}$$

From Eqs. (43) and (44) we obtain the eigenvalue equation

$$1 + \cos \sqrt{\beta} \cosh \sqrt{\beta} = 0 \tag{45}$$

and also

$$\frac{C_2}{C_1} = - \frac{\cos \sqrt{\beta} + \cosh \sqrt{\beta}}{\sin \sqrt{\beta} + \sinh \sqrt{\beta}} \tag{46}$$

the ratio being approximately -1 for all modes higher than the first.

The first five eigenvalues are

$$\beta_1 = 3.516015$$
$$\beta_2 = 22.03449$$
$$\beta_3 = 61.69721$$
$$\beta_4 = 120.9019$$
$$\beta_5 = 199.8595$$

Note that, except for i = 1, the value of β_i in this case in approximately equal to β_{i-1} for the built-in or free-free case.

The amplitudes are given by the equations

$$Y(x) = -2C_1\left[\frac{\beta x^2}{2!} + \frac{\beta^3 x^6}{6!} + \frac{\beta^5 x^{10}}{10!} + \ldots + \frac{C_2}{C_1}\left(\frac{\beta^{\frac{3}{2}} x^3}{3!} + \frac{\beta^{\frac{7}{2}} x^7}{7!} + \frac{\beta^{\frac{11}{2}} x^{11}}{11!} + \ldots\right)\right] \tag{47}$$

$$|\theta| = -2C_1\sqrt{\beta}\left[\sqrt{\beta}\,x + \frac{\beta^{\frac{5}{2}}x^5}{5!} + \frac{\beta^{\frac{9}{2}}x^9}{9!} + \ldots + \frac{C_2}{C_1}\left(\frac{\beta x^2}{2!} + \frac{\beta^3 x^6}{6!} + \frac{\beta^5 x^{10}}{10!} + \ldots\right)\right]$$
(48)

$$|M| = -2C_1\beta\left[1 + \frac{\beta^2 x^4}{4!} + \frac{\beta^4 x^8}{8!} + \ldots + \frac{C_2}{C_1}\left(\sqrt{\beta}\,x + \frac{\beta^{\frac{5}{2}}x^5}{5!} + \frac{\beta^{\frac{9}{2}}x^9}{9!} + \ldots\right)\right]$$
(49)

$$|V| = -2C_1\beta^{\frac{3}{2}}\left[\frac{\beta^{\frac{3}{2}}x^3}{3!} + \frac{\beta^{\frac{7}{2}}x^7}{7!} + \ldots + \frac{C_2}{C_1}\left(1 + \frac{\beta x^2}{2!} + \frac{\beta^4 x^8}{8!} + \ldots\right)\right]$$
(50)

Each of these expressions contains the same powers of x as was obtained for the corresponding built-in case in Eqs. (20) to (23).

5. <u>The Pinned Beam</u>.—The boundary conditions for a pinned beam are

$$y(0) = y(1) = 0$$
(51)

$$M(0) = M(1) = 0$$
(52)

Applying these conditions to the solution given in Eq. (9), we obtain

$$C_1 = C_3 = C_4 = 0$$
(53)

and therefore

$$Y(x) = C_2 \sin\sqrt{\beta}\,x$$
(54)

where the eigenvalues are

$$\beta_i = i^2\pi^2 \qquad (i = 1, 2, 3, \ldots)$$
(55)

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The power series form of the solution is

$$Y(x) = C_2 \left(\sqrt{\beta} \, x - \frac{\beta^{\frac{3}{2}} x^3}{3!} + \frac{\beta^{\frac{5}{2}} x^5}{5!} - \frac{\beta^{\frac{7}{2}} x^7}{7!} + \ldots \right) \qquad (56)$$

Also

$$|\theta| = C_2 \sqrt{\beta} \left(1 - \frac{\beta x^2}{2!} + \frac{\beta^2 x^4}{4!} - \frac{\beta^3 x^6}{6!} + \ldots \right) \qquad (57)$$

$$|M| = C_2 \beta \left(- \sqrt{\beta} \, x + \frac{\beta^{\frac{3}{2}} x^3}{3!} - \frac{\beta^{\frac{5}{2}} x^3}{5!} + \ldots \right) \qquad (58)$$

$$|V| = C_2 \beta^{\frac{3}{2}} \left(-1 + \frac{\beta x^2}{2!} - \frac{\beta^2 x^4}{4!} + \ldots \right) \qquad (59)$$

The eigenvalues in this case are not close to those calculated in the previous cases. In fact, for the higher modes (large i), the eigenvalues tend toward positions midway between those for the built-in, free-free, or cantilever beams.

IV. DIFFERENCES OF ORDER h^2

Previously we have obtained solutions for the partial differential equations describing a continuous uniform beam. Unfortunately, the analog computer cannot solve partial differential equations directly but must solve the set of ordinary differential equations which are obtained by finite difference approximations to the spatial derivative. Thus we will obtain solution amplitudes at a discrete set of points rather than continuously as a function of x. At the given points, however, the finite difference solution should closely approximate the solution for the continuous beam.

A. FINITE DIFFERENCE EQUATIONS

An analog computer approach to the beam-vibration problem using finite difference equations of order h^2 has been given by Howe and Howe[4] and others.[5,6] The procedure in each case is based essentially upon Eq. (60).

$$\left.\frac{\partial y}{\partial x}\right|_{n+\frac{1}{2}} \cong \frac{-y_n + y_{n+1}}{h} - \frac{h^2}{24}\left.\frac{\partial^3 y}{\partial x^3}\right|_{n+\frac{1}{2}} \tag{60}$$

The interval between stations is designated by h. Subscripts refer to the station at which y (or one of its derivatives) is calculated. The last term is the first error term and is not included in the computer mechanization. It is given to provide an estimate of the accuracy of the approximation. Equation (60) can be derived by writing the Taylor expansions about station $n+\frac{1}{2}$ for y_n and y_{n+1} and solving for $\left.\frac{\partial y}{\partial x}\right|_{n+\frac{1}{2}}$ after eliminating $y_{n+\frac{1}{2}}$.

Using the approximation of Eq. (60) and again assuming that EI and ρ are unity, the beam equations (2), (3), (4), and (5) can be written in difference form:

$$\theta_{n+\frac{1}{2}} = \frac{-y_n + y_{n+1}}{h} \tag{61}$$

$$M_n = \frac{-\theta_{n-\frac{1}{2}} + \theta_{n+\frac{1}{2}}}{h} \tag{62}$$

$$V_{n+\frac{1}{2}} = \frac{-M_n + M_{n+1}}{h} \tag{63}$$

$$\ddot{y}_n = \frac{V_{n-\frac{1}{2}} - V_{n+\frac{1}{2}}}{h} \qquad (64)$$

Note that displacements and bending moments are calculated at integer stations whereas the slopes and shear forces are calculated at half-integer stations. If one calculates all quantities only at integer stations, the result is to double the effective cell length and thereby to quadruple the first error term. Therefore we will use "staggered" stations when finite difference approximations to first derivatives are used.

The analog computer circuit representing a typical internal cell is shown in Fig. 1. It requires five amplifiers per cell if one reverses signs at adjacent cells.

Another approach to the problem is to approximate the second spatial derivative rather than the first derivative. In this case the finite difference approximation is

$$\left.\frac{\partial^2 y}{\partial x^2}\right|_n \cong \frac{y_{n-1} - 2y_n + y_{n+1}}{h^2} - \frac{h^2}{12}\left.\frac{\partial^4 y}{\partial x^4}\right|_n \qquad (65)$$

and the corresponding beam equations solved on the computer are

$$M_n = \frac{y_{n-1} - 2y_n + y_{n+1}}{h^2} \qquad (66)$$

$$\ddot{y}_n = \frac{-M_{n-1} + 2M_n - M_{n+1}}{h^2} \qquad (67)$$

The analog computer circuit for this case requires only 3 amplifiers per cell and is shown in Fig. 2. This circuit is subject to exactly the same finite difference errors as the 5-amplifier circuit and, in fact, can be obtained as a direct reduction of that circuit.

B. PHYSICAL INTERPRETATION

There are two ways in which one can give physical meaning to finite difference approximations to the beam equations.

The first method is to consider a finite difference expression as an approximation to a spatial derivative in the continuous beam at a given point. For example, the right-hand side of Eq. (64) approximates $-\frac{\partial V}{\partial x}\big|_n$, which, in accordance with Eq. (5) for a continuous beam, is proportional to the acceleration $\ddot{y}_n$ of a local differential element.

The second method is to think of the set of finite difference equations as representing <u>exactly</u> a lumped physical system whose response approximates that of the continuous beam. The calculated values of accelerations, shear forces, etc., at the various stations have their real counterparts in the lumped physical system. This approach is helpful in gaining physical insight into the meaning of the equations, particularly for simple differences. Two lumped physical representations of this finite difference approximation to the beam equations are shown in Fig. 3.

In Fig. 3a the beam is approximated by a series of massless levers connected by pin joints. At each joint a spring produces a moment (corresponding to the bending moment) which is proportional to the difference in slope angle θ of the adjacent levers. The mass of the beam is lumped into point masses at the joints. The deflection y is measured at the joints while the slope θ is measured at the midpoint of each lever.

Figure 3b shows an electrical circuit analog of the lumped mechanical system of Fig. 3a. It also is described by finite difference Eqs. (61) to (64). The displacement velocities are represented by voltages and the shear forces and bending moments are represented by currents. The lumped bending stiffnesses are represented by inductors, while the lumped masses are represented by capacitors. The transformers, each of turns ratio 1 to h, perform the coordinate transformations relating slopes to deflections, and can be considered as the analog of the massless levers.

C. BOUNDARY CONDITIONS

We have previously given the equations stating the boundary conditions on a continuous beam for the various cases under consideration. We must now write corresponding equations for the finite difference beam. First, however, we must decide whether to end the beam at a point where lateral deflection and bending moment are computed (the "integer" stations), or at a point where the slope and shear force are computed (the "half-integer" stations).

Experience has shown[7] that higher accuracy can be obtained for a given amount of computation (or degrees of freedom) if the end occurs at a half-

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integer station for built-in or free ends. For the pinned case it is best to end the beam at an integer station.

1. <u>Built-In</u> <u>End</u>.—Previously we have seen that the displacement y and slope Θ are zero at a built-in end of a beam. Suppose, for example, that the beam ends at station $n+\frac{1}{2}$. We must approximate the deflection at a half-integer station where it is not usually defined. Using the Taylor expansion approach and keeping only the first error term, we find that

$$y_{n+\frac{1}{2}} \cong \frac{1}{2}\left(y_n + y_{n+1}\right) - \frac{h^2}{8} \left.\frac{\partial^2 y}{\partial x^2}\right|_{n+\frac{1}{2}} \tag{68}$$

Therefore, since $y_{n+\frac{1}{2}}$ and $\Theta_{n+\frac{1}{2}}$ are zero, we see from Eqs. (61) and (68) that

$$y_n = y_{n+1} = 0 \tag{69}$$

implying that there is no lateral displacement at the integer station adjacent to a built-in end.

2. <u>Free</u> <u>End</u>.—At a free end we find that the bending moment M and shear force V are zero. Using reasoning similar to that of the previous case, we find that for a beam ending at station $n+\frac{1}{2}$

$$M_n = M_{n+1} = 0 \tag{70}$$

3. <u>Pinned</u> <u>End</u>.—For a pinned end at station n the boundary conditions are quite straightforward, namely,

$$y_n = 0 \tag{71}$$

$$M_n = 0 \tag{72}$$

In case the pinned end occurs at station $n+\frac{1}{2}$, the boundary conditions are

$$y_n = -y_{n+1} \tag{73}$$

$$M_n = -M_{n+1} \tag{74}$$

4. The Use of Symmetry.—For vibration problems of a uniform beam where the same boundary condition applies at each end, one can make use of symmetry to reduce the required amount of computation. All the natural modes can be classified according to whether the deflection curve has even or odd symmetry about the midpoint of the beam. Consider, for example, a beam that is built-in at both ends. If one arranges the modes in order of increasing frequency, then modes 1, 3, 5, etc., will have mode shapes with even symmetry about the midpoint, whereas modes 2, 4, 6, etc., will exhibit odd symmetry about the midpoint. (See Fig. 4.) In either event, however, one need analyze only one-half of the beam since the motion of the other half can be deduced from symmetry.

For the case of even symmetry of the deflection curve, the bending moment curve will also show even symmetry, whereas the slope and shear force curves will have odd symmetry. Conversely, for odd symmetry in y and M, there will be even symmetry in θ and V.

The above symmetry assumptions are exact and therefore the same frequencies and mode shapes result as for the case where the equations are written for the complete beam.

D. ANALOG COMPUTER CIRCUITS

1. Cantilever Beam.—The analog computer circuit for the cantilever beam is useful in illustrating built-in and free end conditions. Two circuits are shown in Fig. 5. The first circuit requires 3 amplifiers per cell, the second, 5 amplifiers per cell. These are nominal values, however, and amplifiers can be saved in representing the end cells because of the boundary conditions. Thus it can be seen that the actual numbers of amplifiers required to represent a four-cell cantilever beam are 9 and 15, respectively. Cells can be added by adding standard circuits such as those in Fig. 1 or Fig. 2, representing interior cells.

2. Built-In Beam.—The computer circuits for a six-cell beam that is built-in at both ends are shown in Fig. 6. Note that only one-half of the beam is actually represented on the computer; the motion of the other half is inferred from symmetry. Circuits are shown for even or odd symmetry about the

center of the span. In each case the center occurs at a half-integer station.
Circuits for the case where the center occurs at an integer station could
have been derived by a similar process. This situation would arise for a
built-in beam with an odd number of cells, assuming, of course, that the ends
occur at half-integer stations.

3. Pinned Beam.—The computer circuits representing a six-cell beam
with both ends pinned are shown in Fig. 7. The circuit for the case of odd
symmetry about the center, Fig. 7(b), is particularly simple because of the
additional symmetry about the quarter points along the span.

The circuits to be used in the higher-order methods that we consider are
quite similar in general form to those of Figs. 5(b), 6, and 7. However,
most amplifiers will have four rather than two inputs.

E. MATRIX FORMULATION OF THE DIFFERENCE EQUATIONS

In the analysis of specific cases of finite difference beams, it is con-
venient to write the equations in matrix form. It is particularly important
to use this formulation if digital computations are to be performed.

In this report the beam equations are, in general, written as four first-
order difference equations. Using matrix notation, they are as follows:

$$\{\Theta\} \ = \ [B_1] \ \{y\} \tag{75}$$

$$\{M\} \ = \ [B_2] \ \{\Theta\} \tag{76}$$

$$\{V\} \ = \ [B_3] \ \{M\} \tag{77}$$

$$\{\ddot{y}\} \ = \ [B_4] \ \{V\} \tag{78}$$

or

$$\{\ddot{y}\} \ + \ [K] \ \{y\} \ = \ 0 \tag{79}$$

where

$$[K] \ = \ -[B_4][B_3][B_2][B_1] \tag{80}$$

Although the B matrices are rectangular in general, the K matrix must be square. The natural frequencies are obtained from the determinantal equation

$$|K - \lambda^2 I| = 0 \tag{81}$$

where λ is the natural frequency in rad/sec and I is the unit matrix. For the case where the K matrix is n x n, there will be n values of λ^2 (roots) corresponding to the n natural frequencies of the system.

As an example, consider the case of a six-cell beam that is built-in at both ends. We will write the equations for modes with even symmetry at the center. As before, EI and ρ are assumed to be unity. The matrix equations are

$$\begin{Bmatrix} \theta_{\frac{3}{2}} \\ \theta_{\frac{5}{2}} \end{Bmatrix} = \frac{1}{h} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} y_2 \\ y_3 \end{Bmatrix} \tag{82}$$

$$\begin{Bmatrix} M_1 \\ M_2 \\ M_3 \end{Bmatrix} = \frac{1}{h} \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix} \begin{Bmatrix} \theta_{\frac{3}{2}} \\ \theta_{\frac{5}{2}} \end{Bmatrix} \tag{83}$$

$$\begin{Bmatrix} V_{\frac{3}{2}} \\ V_{\frac{5}{2}} \end{Bmatrix} = \frac{1}{h} \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} M_1 \\ M_2 \\ M_3 \end{Bmatrix} \tag{84}$$

$$\begin{Bmatrix} \ddot{y}_2 \\ \ddot{y}_3 \end{Bmatrix} = \frac{1}{h} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} V_{\frac{3}{2}} \\ V_{\frac{5}{2}} \end{Bmatrix} \tag{85}$$

Assuming a unit cell size (h = 1) and performing the matrix multiplications, we obtain

$$\begin{Bmatrix} \ddot{y}_2 \\ \ddot{y}_3 \end{Bmatrix} + \begin{bmatrix} 6 & -3 \\ -3 & 2 \end{bmatrix} \begin{Bmatrix} y_2 \\ y_3 \end{Bmatrix} = 0 \tag{86}$$

According to Eq. (81), the roots are obtained from

$$\begin{vmatrix} (6-\lambda^2) & -3 \\ -3 & (2-\lambda^2) \end{vmatrix} = 0 \tag{87}$$

giving the characteristic equation

$$\lambda^4 - 8\lambda^2 + 3 = 0 \tag{88}$$

The roots are

$$\lambda_{1,2}^2 = 0.394449, \ 7.60555$$

or

$$\lambda_{1,2} = 0.628052, \ 2.75732$$

These are the actual frequencies for the modes with even symmetry and
h = 1. For our purposes, however, it is convenient to assume that the total
length of the beam is unity. This implies that $h = 1/N$, where N is the number
of cells, and results in multiplying each frequency by N^2. Denoting this
normalized frequency by ω, we obtain

$$\omega_1 = 22.6099$$

$$\omega_2 = 99.2815$$

The same basic matrix formulation will be used in the computations through-out the report. The use of higher-order differences will add to the numerical complexity but will not change the form of the equations.

F. RESULTS

The results of computations of the normalized frequency parameter ω using differences of order h^2 are summarized in Tables II and III. The computations are for built-in (or free-free) and cantilever beams for various numbers of cells. The finite difference error is plotted against the cell number N in Figs. 13 and 14. These results are essentially the same as those given previously by Howe and Howe[4] and Michie,[7] except that the higher mode frequencies are also included.

G. ORTHOGONALITY

We have seen previously that the equations of motion can be put in the form

$$\{\ddot{y}\} + [K] \{y\} = 0 \tag{79}$$

We can also write

$$[m] \{\ddot{y}\} + [k] \{y\} = 0 \tag{89}$$

where m and k are the mass and stiffness matrices, respectively. Comparing Eqs. (79) and (89), we see that

$$[K] = [m]^{-1} [k] \tag{90}$$

Now, for the case of the first-order difference equations under consideration here, both the m and k matrices are symmetric. In fact, the m matrix is diagonal. It is always possible in these linear bilateral undamped systems (whose kinetic and potential energies can be written as positive-definite

quadratic forms) to find a coordinate transformation which will simultaneously diagonalize the m and k matrices. In this case the natural modes of the system are said to be orthogonal.

The orthogonality of the modes is not surprising since we have already seen (Fig. 3) that these equations represent exactly a lumped mechanical or electrical system composed of linear bilateral elements. However, when we discuss higher-order difference methods, we will find that the k matrix is not always symmetric, in which case the modes are not orthogonal.

V. DIFFERENCES OF ORDER h^4

We have given a brief review of the calculation of eigenvalues for a uniform beam, using simple finite difference approximations to the spatial derivatives. In this way we have obtained some physical feeling for the main features of this method and perhaps some insight into the more promising approaches to the use of higher-order differences. In the remainder of the report we will consider several higher-order methods and indicate some of the advantages as well as problems in their use.

A. CHOICE OF THE BASIC DIFFERENCE EQUATION

Even before one gets into the questions concerning the representation of boundary conditions, there are a number of possible choices one can make concerning the basic higher-order difference equation to be used. First is the question whether the approximation ought to be with respect to the first, second, or possibly the fourth spatial derivative. We can immediately reject the latter possibility on the grounds that it is quite sensitive to component errors in the analog computer circuit.[8,9] Also there is the disadvantage that other variables of interest such as θ, M, and V are not directly available even though three amplifiers are required per cell.

The choice between approximating two second-order equations or four first-order equations is more difficult. Let us consider each possibility in turn.

The most obvious approximation of order h^4 for the second derivative with respect to x is given by the central difference equation

$$\left.\frac{\partial^2 y}{\partial x^2}\right|_n \cong \frac{-y_{n-2} + 16y_{n-1} - 30y_n + 16y_{n+1} - y_{n+2}}{12h^2} + \frac{h^4}{90}\left.\frac{\partial^6 y}{\partial x^6}\right|_n \tag{91}$$

Using this approximation, the equations set up on the computer are of the form

$$M_n = \frac{1}{12h^2}\left(-y_{n-2} + 16y_{n-1} - 30y_n + 16y_{n+1} - y_{n+2}\right) \tag{92}$$

$$\ddot{y}_n = \frac{1}{12h^2}\left(M_{n-2} - 16M_{n-1} + 30M_n - 16M_{n+1} + M_{n+2}\right) \tag{93}$$

where unity values are assumed for EI and ρ. The analog computer circuit requires three amplifiers per cell and up to five inputs per amplifier. (See Fig. 8.)

Consider now some of the higher-order approximations to $\frac{\partial y}{\partial x}$. The usual central difference approximation of order h^4 is

$$\frac{\partial y}{\partial x}\bigg|_n \cong \frac{1}{12h}\left(y_{n-2} - 8y_{n-1} + 8y_{n+1} - y_{n+2}\right) + \frac{h^4}{30}\frac{\partial^5 y}{\partial x^5}\bigg|_n \qquad (94)$$

The resulting beam equations are

$$\Theta_n = \frac{1}{12h}\left(y_{n-2} - 8y_{n-1} + 8y_{n+1} - y_{n+2}\right) \qquad (95)$$

$$M_n = \frac{1}{12h}\left(\Theta_{n-2} - 8\Theta_{n-1} + 8\Theta_{n+1} - \Theta_{n+2}\right) \qquad (96)$$

$$V_n = \frac{1}{12h}\left(M_{n-2} - 8M_{n-1} + 8M_{n+1} - M_{n+2}\right) \qquad (97)$$

$$\ddot{y}_n = \frac{1}{12h}\left(-V_{n-2} + 8V_{n-1} - 8V_{n+1} + V_{n+2}\right) \qquad (98)$$

where EI and ρ are assumed to be of unit value. The analog computer circuit requires nine amplifiers per cell in this case because each of the four output variables must be generated with both signs. (Each cell would require 2 integrators, 3 summers, and 4 inverters.)

Another possibility is to try a higher-order approximation using staggered stations, i.e., calculate the derivatives at the midpoints between stations where the function is defined. A central difference approximation of this sort is the following:

$$\frac{\partial y}{\partial x}\bigg|_{n+\frac{1}{2}} \cong \frac{1}{24h}\left(y_{n-1} - 27y_n + 27y_{n+1} - y_{n+2}\right) + \frac{3h^4}{640}\frac{\partial^5 y}{\partial x^5}\bigg|_{n+\frac{1}{2}} \qquad (99)$$

This approximation is seen to have a smaller error term than that for Eq. (94). Furthermore, it will be seen that less analog computer equipment is required to implement it.

Using the approximation of Eq. (99), the beam equations are

$$\Theta_{n+\frac{1}{2}} = \frac{1}{24h}\left(y_{n-1} - 27y_n + 27y_{n+1} - y_{n+2}\right) \tag{100}$$

$$M_n = \frac{1}{24h}\left(\Theta_{n-\frac{3}{2}} - 27\Theta_{n-\frac{1}{2}} + 27\Theta_{n+\frac{1}{2}} - \Theta_{n+\frac{3}{2}}\right) \tag{101}$$

$$V_{n+\frac{1}{2}} = \frac{1}{24h}\left(M_{n-1} - 27M_n + 27M_{n+1} - M_{n+2}\right) \tag{102}$$

$$\ddot{y}_n = \frac{1}{24h}\left(-V_{n-\frac{3}{2}} + 27V_{n-\frac{1}{2}} - 27V_{n+\frac{1}{2}} + V_{n+\frac{3}{2}}\right) \tag{103}$$

where, as before, unit values are assumed for EI and ρ. Note that y and M are calculated at integer stations whereas Θ and V are calculated at half-integer stations.

The analog computer circuit representation of Eqs. (100) to (103) requires 5 amplifiers per cell with up to 4 inputs per amplifier. The circuit for a typical cell is shown in Fig. 9. The signs of amplifier outputs alternate on successive cells.

In spite of the fact that the circuit of Fig. 9 requires 5 amplifiers per cell compared to 3 amplifiers per cell for the circuit of Fig. 8, the computations in this report are based on the beam equations as given by Eqs. (100) to (103). The reasons for this choice are (1) the greater accuracy of the basic difference equation, (2) greater ease and flexibility in applying boundary conditions and forcing functions, and (3) less sensitivity to computer component errors.

B. BOUNDARY CONDITIONS AND RESULTS

Earlier we saw that the representation of boundary conditions was relatively straightforward for the case of difference equations of order h^2. When one uses higher-order differences, the problem of applying boundary conditions be-

comes more complex. Many approaches are possible and the accuracy of the re-

$$
\begin{Bmatrix} \theta_{\frac{3}{2}} \\ \theta_{\frac{5}{2}} \\ \theta_{\frac{7}{2}} \\ \cdot \\ \cdot \\ \cdot \end{Bmatrix} = \frac{1}{5400h} \begin{bmatrix} 4800 & 0 & 0 & 0 & \cdots \\ -6025 & 6066 & -225 & 0 & \\ 225 & -6075 & 6075 & -225 & \cdots \\ \cdot & & & \cdot & \\ \cdot & & & \cdot & \\ \cdot & & & \cdot & \end{bmatrix} \begin{Bmatrix} y_2 \\ y_3 \\ y_4 \\ \cdot \\ \cdot \\ \cdot \end{Bmatrix} \qquad (106)
$$

Thus we find that we can omit y_1 as a coordinate in our analysis because of the constraint expressed in Eq. (105). This was also true when simple approximations of order h^2 were used, but in that case y_1 was equal to zero.

In a similar manner, using the approximations of Table I and Eq. (99), we can write the remaining beam equations for the case of a built-in end.

$$
\begin{Bmatrix} M_1 \\ M_2 \\ M_3 \\ \cdot \\ \cdot \\ \cdot \end{Bmatrix} = \frac{1}{24h} \begin{bmatrix} 17 & 9 & -5 & 1 & 0 & \cdots \\ -27 & 27 & -1 & 0 & 0 & \cdots \\ 1 & -27 & 27 & -1 & 0 & \cdots \\ \cdot & & & & \cdot & \\ \cdot & & & & \cdot & \\ \cdot & & & & \cdot & \end{bmatrix} \begin{Bmatrix} \theta_{\frac{3}{2}} \\ \theta_{\frac{5}{2}} \\ \theta_{\frac{7}{2}} \\ \cdot \\ \cdot \end{Bmatrix} \qquad (107)
$$

$$
\begin{Bmatrix} V_{\frac{1}{2}} \\ V_{\frac{3}{2}} \\ V_{\frac{5}{2}} \\ \cdot \\ \cdot \\ \cdot \end{Bmatrix} = \frac{1}{24h} \begin{bmatrix} -93 & 229 & -225 & 111 & -22 & \cdots \\ -22 & 17 & 9 & -5 & 1 & \\ 1 & -27 & 27 & -1 & 0 & \cdots \\ \cdot & & & & \cdot & \\ \cdot & & & & \cdot & \\ \cdot & & & & \cdot & \end{bmatrix} \begin{Bmatrix} M_1 \\ M_2 \\ M_3 \\ \cdot \\ \cdot \\ \cdot \end{Bmatrix} \qquad (108)
$$

$$\begin{Bmatrix} \ddot{y}_2 \\ \ddot{y}_3 \\ \ddot{y}_4 \\ \cdot \\ \cdot \\ \cdot \end{Bmatrix} = \frac{1}{24h} \begin{bmatrix} -1 & 27 & -27 & 1 & 0 & \cdots \\ 0 & -1 & 27 & -27 & 1 & \cdots \\ 0 & 0 & -1 & 27 & -27 & \cdots \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \end{bmatrix} \begin{Bmatrix} V_{\frac{1}{2}} \\ V_{\frac{3}{2}} \\ V_{\frac{5}{2}} \\ \cdot \\ \cdot \\ \cdot \end{Bmatrix} \tag{109}$$

Let us apply Eqs. (106) to (109) for the specific case of a 6-cell beam that is built-in at both ends. Only the modes with even symmetry about the center will be considered. The equations of motion are

$$\begin{Bmatrix} \theta_{\frac{3}{2}} \\ \theta_{\frac{5}{2}} \end{Bmatrix} = \frac{1}{5400h} \begin{bmatrix} 4800 & 0 \\ -6025 & 5841 \end{bmatrix} \begin{Bmatrix} y_2 \\ y_3 \end{Bmatrix} \tag{110}$$

$$\begin{Bmatrix} M_1 \\ M_2 \\ M_3 \end{Bmatrix} = \frac{1}{24h} \begin{bmatrix} 17 & 8 \\ -27 & 27 \\ 1 & -26 \end{bmatrix} \begin{Bmatrix} \theta_{\frac{3}{2}} \\ \theta_{\frac{5}{2}} \end{Bmatrix} \tag{111}$$

$$\begin{Bmatrix} V_{\frac{1}{2}} \\ V_{\frac{3}{2}} \\ V_{\frac{5}{2}} \end{Bmatrix} = \frac{1}{24h} \begin{bmatrix} -93 & 207 & -114 \\ -22 & 18 & 4 \\ 1 & -27 & 26 \end{bmatrix} \begin{Bmatrix} M_1 \\ M_2 \\ M_3 \end{Bmatrix} \tag{112}$$

$$\begin{Bmatrix} \ddot{y}_2 \\ \ddot{y}_3 \end{Bmatrix} = \frac{1}{24h} \begin{bmatrix} -1 & 27 & -27 \\ 0 & -1 & 26 \end{bmatrix} \begin{Bmatrix} V_{\frac{1}{2}} \\ V_{\frac{3}{2}} \\ V_{\frac{5}{2}} \end{Bmatrix} \tag{113}$$

Letting h = 1 and applying Eqs. (80) and (81), we obtain

$$\begin{vmatrix} (5.22099 - \lambda^2) & -2.77553 \\ -4.29387 & (2.85815 - \lambda^2) \end{vmatrix} = 0 \tag{114}$$

giving the characteristic equation

$$\lambda^4 - 8.07914\lambda^2 + 3.00462 = 0 \tag{115}$$

and roots

$$\lambda_{1,2} = 0.625142, \ 2.7^{-}278$$

which can be normalized to

$$\omega_1 = 22.5051$$

$$\omega_2 = 99.8203$$

Comparing these results with those for the case where simple differences were used, one finds that the h^4 approximation gives somewhat better accuracy for the first mode and approximately the same results for the higher mode. This improvement due to using an h^4 approximation seems to be rather slight. However, if one looks at the accuracy of a given mode as the number of cells is increased, the h^4 approximation converges toward the correct result more rapidly, as it must, of course, in the limit. See Table II and Fig. 15 for a summary of the results.

3. <u>Free Beam</u>.—Previously we have seen that the representation of a free end is obtained from that for a built-in end by interchanging the roles of y and M and also those of θ and V. Furthermore, we found for a continuous beam that the nonzero mode frequencies are identical for the cases where both ends are built-in or both ends are free. This is also true when finite difference methods are used. To illustrate this, let us calculate the mode frequencies (even symmetry only) of a 6-cell beam that is free at both ends.

The equations of motion are

$$
\begin{Bmatrix} \theta_{\frac{1}{2}} \\ \theta_{\frac{3}{2}} \\ \theta_{\frac{5}{2}} \end{Bmatrix} = \frac{1}{24h}
\begin{bmatrix} -93 & 207 & -114 \\ -22 & 18 & 4 \\ 1 & -27 & 26 \end{bmatrix}
\begin{Bmatrix} y_1 \\ y_2 \\ y_3 \end{Bmatrix}
\tag{116}
$$

$$\begin{Bmatrix} M_2 \\ M_3 \end{Bmatrix} = \frac{1}{24h} \begin{bmatrix} 1 & -27 & 27 \\ 0 & 1 & -26 \end{bmatrix} \begin{Bmatrix} \theta_{\frac{1}{2}} \\ \theta_{\frac{3}{2}} \\ \theta_{\frac{5}{2}} \end{Bmatrix} \tag{117}$$

$$\begin{Bmatrix} V_{\frac{3}{2}} \\ V_{\frac{5}{2}} \end{Bmatrix} = \frac{1}{5400h} \begin{bmatrix} 4800 & 0 \\ -6025 & 5841 \end{bmatrix} \begin{Bmatrix} M_2 \\ M_3 \end{Bmatrix} \tag{118}$$

$$\begin{Bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \\ \ddot{y}_3 \end{Bmatrix} = \frac{1}{24h} \begin{bmatrix} -17 & -8 \\ 27 & -27 \\ -1 & 26 \end{bmatrix} \begin{Bmatrix} V_{\frac{3}{2}} \\ V_{\frac{5}{2}} \end{Bmatrix} \tag{119}$$

Performing the matrix multiplications for the case where h = 1, we obtain from Eq. (81) that

$$\begin{vmatrix} (0.20619-\lambda^2) & -0.00031 & -0.20588 \\ -2.16868 & (5.46771-\lambda^2) & -3.29903 \\ 1.23959 & -3.64483 & (2.40524-\lambda^2) \end{vmatrix} = 0 \tag{120}$$

giving the characteristic equation

$$\lambda^2(\lambda^4 - 8.07914\lambda^2 + 3.00462) = 0 \tag{121}$$

Comparing Eqs. (115) and (121), we see that the nonzero roots are identical even though the K matrices are quite different.

All computations in this report applicable to either a built-in or free beam were actually set up considering the built-in case, using equations similar to (110) to (113).

4. <u>Cantilever Beam</u>.—One can use the methods developed for obtaining h^4 accuracy in the representation of a built-in end or a free end to write the equations of motion for a cantilever beam. The minimum number of cells required for h^4 accuracy is five. Other methods to be considered later will be

applicable with fewer cells.

As an example of a typical calculation, consider a six-cell cantilever beam. The equations of motion are

$$
\begin{Bmatrix} \theta_{\frac{3}{2}} \\ \theta_{\frac{5}{2}} \\ \theta_{\frac{7}{2}} \\ \theta_{\frac{9}{2}} \\ \theta_{\frac{11}{2}} \\ \theta_{\frac{13}{2}} \end{Bmatrix}
= \frac{1}{5400h}
\begin{bmatrix}
4800 & 0 & 0 & (& 0 \\
-6025 & 6066 & -225 & (& 0 \\
225 & -6075 & 6075 & -225 & 0 \\
0 & 225 & -6075 & 6075 & -225 \\
-225 & 1125 & -2025 & -3825 & 4950 \\
4950 & -24975 & 50625 & -51525 & 20925
\end{bmatrix}
\begin{Bmatrix} y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{Bmatrix}
\tag{122}
$$

$$
\begin{Bmatrix} M_1 \\ M_2 \\ M_3 \\ M_4 \\ M_5 \end{Bmatrix}
= \frac{1}{24h}
\begin{bmatrix}
17 & 9 & -5 & 1 & 0 & 0 \\
-27 & 27 & -1 & 0 & 0 & 0 \\
1 & -27 & 27 & -1 & 0 & 0 \\
0 & 1 & -27 & 27 & -1 & 0 \\
0 & 0 & 1 & -27 & 27 & -1
\end{bmatrix}
\begin{Bmatrix} \theta_{\frac{3}{2}} \\ \theta_{\frac{5}{2}} \\ \theta_{\frac{7}{2}} \\ \theta_{\frac{9}{2}} \\ \theta_{\frac{11}{2}} \\ \theta_{\frac{13}{2}} \end{Bmatrix}
\tag{123}
$$

$$
\begin{Bmatrix} V_{\frac{1}{2}} \\ V_{\frac{3}{2}} \\ V_{\frac{5}{2}} \\ V_{\frac{7}{2}} \\ V_{\frac{9}{2}} \\ V_{\frac{11}{2}} \end{Bmatrix}
= \frac{1}{5400h}
\begin{bmatrix}
-20925 & 51525 & -50625 & 24975 & -4950 \\
-4950 & 3825 & 2025 & -1125 & 225 \\
225 & -6075 & 6075 & -225 & 0 \\
0 & 225 & -6075 & 6075 & -225 \\
0 & 0 & 225 & -6066 & 6025 \\
0 & 0 & 0 & 0 & -4800
\end{bmatrix}
\begin{Bmatrix} M_1 \\ M_2 \\ M_3 \\ M_4 \\ M_5 \end{Bmatrix}
\tag{124}
$$

$$
\begin{Bmatrix} \ddot{y}_2 \\ \ddot{y}_3 \\ \ddot{y}_4 \\ \ddot{y}_5 \\ \ddot{y}_6 \end{Bmatrix} = \frac{1}{24h} \begin{bmatrix} -1 & 27 & -27 & 1 & 0 & 0 \\ 0 & -1 & 27 & -27 & 1 & 0 \\ 0 & 0 & -1 & 27 & -27 & 1 \\ 0 & 0 & 0 & -1 & 27 & -27 \\ 0 & 0 & 1 & -5 & 9 & 17 \end{bmatrix} \begin{Bmatrix} V_{\frac{1}{2}} \\ V_{\frac{3}{2}} \\ V_{\frac{5}{2}} \\ V_{\frac{7}{2}} \\ V_{\frac{9}{2}} \\ V_{\frac{11}{2}} \end{Bmatrix} \tag{125}
$$

The large numbers in Eqs. (122) and (124) result directly from using the one-sided approximations of order h^4 as given in part A of Table I.

Note the similarities in the matrices of Eqs. (122) and (124). Reading from left to right, beginning with the first row of (122), one sees the negative of the numbers obtained in reading from right to left beginning with the last row of (124). A similar relationship exists between Eqs. (123) and (125), except for sign. This property is common to all finite difference analyses of the cantilever beam and results from the complementary character of y and M and also of Θ and V.

Setting h = 1, one obtains the characteristic equations for the system:

$$
\lambda^{10} - 29.01745\lambda^8 + 221.62314\lambda^6 - 462.52608\lambda^4 + 148.71888\lambda^2 - 1.37737 = 0 \tag{126}
$$

giving the normalized frequencies

$$
\begin{aligned}
\omega_1 &= 3.51687 \\
\omega_2 &= 22.1688 \\
\omega_3 &= 59.3364 \\
\omega_4 &= 99.8114 \\
\omega_5 &= 153.687
\end{aligned}
$$

Note that all except the lowest mode frequency are roughly equal to corresponding frequencies for the 6-cell built-in case. An even closer agreement occurs in the percentage error for corresponding cases. In fact, for large N and comparing the higher modes, one finds essentially identical results for the built-in and cantilever beams. (See Tables II and III.)

C. ORTHOGONALITY

We have seen that the use of approximations of order h^2 leads to orthogonal modes. Also, one can obtain a lumped physical system composed of linear, bilateral elements that is represented exactly by the equations of motion. This is no longer true for the h^4 approximation using one-sided differences at the boundaries. If one writes the equations of motion, the m matrix is symmetric but the k matrix is not. [See Eq. (120), for example.] Thus the system is nonphysical, i.e., a passive system of linear bilateral elements cannot be obtained for which the equations are an exact mathematical representation. Also, the modes are not orthogonal, implying that one cannot obtain a coordinate transformation that diagonalizes the m and k matrices simultaneously.

To illustrate this point, consider again the 6-cell built-in beam for the case of even symmetry. For each mode, one can calculate an amplitude ratio that indicates the relative magnitude of the motions of y_2 and y_3.

$$\frac{A_3}{A_2} = \frac{5.22099 - \lambda^2}{2.77553} \tag{127}$$

The results for this case are given by the matrix

$$[A] = \begin{bmatrix} 1.00000 & 1.00000 \\ 1.74028 & -0.88897 \end{bmatrix} \tag{128}$$

where each column corresponds to a natural mode and the amplitude of the y_2 motion is arbitrarily set equal to unity.

The original mass matrix for this system is

$$[m] = \begin{bmatrix} \dfrac{85}{81} & \dfrac{-2}{225} \\[2ex] \dfrac{-2}{225} & \dfrac{626}{625} \end{bmatrix} \tag{129}$$

The deviations from a unit matrix in this case are brought about by the elimination of y_1 using the constraint equation

$$y_1 = \frac{2}{9} y_2 - \frac{1}{25} y_3 \tag{130}$$

The stiffness matrix for the original system is

$$[k] = [m][K] = \begin{bmatrix} 5.51698 & -2.93800 \\ -4.34715 & 2.88740 \end{bmatrix} \tag{131}$$

Consider now the coordinate transformation

$$\{y\} = [A]\{\bar{y}\} \tag{132}$$

The mass matrix in terms of the barred coordinate system is

$$[\bar{m}] = [A]^T[m][A] = \begin{bmatrix} 4.05185 & -0.50770 \\ -0.50770 & 1.85671 \end{bmatrix} \tag{133}$$

and the stiffness matrix is

$$[\bar{k}] = [A]^T[k][A] = \begin{bmatrix} 1.58347 & -3.90342 \\ -0.19842 & 14.27502 \end{bmatrix} \tag{134}$$

Thus we see that neither the mass nor the stiffness matrix is diagonalized by the transformation to generalized modal coordinates, and therefore the modes are not orthogonal. Furthermore, the $\bar{k}$ matrix is not symmetric.

We have defined orthogonality of modes in terms of a simultaneous diagonalization of the mass and stiffness matrices by a coordinate transformation. It should be noted, however, that a nonorthogonality of modes in this sense does not necessarily imply dependence or coupling of modes. In the above case, for example, the matrix $[\bar{m}]^{-1}[\bar{k}]$ turns out to be diagonal.

38

VI.   THE USE OF SYMMETRY IN REPRESENTING BOUNDARY CONDITIONS

The use of an h^4 approximation in a straightforward manner has been shown to require the use of one-sided differences near the ends, whereas central differences can be used in middle portion. Another approach to the problem of representing boundary conditions is to assume a virtual or image beam extending beyond the actual beam and to use values of y, θ, M, and V in this imaginary beam, as necessary, in the standard central difference equations. This approach will now be investigated.

A.   SYMMETRY ASSUMPTIONS

In establishing the basic approach to the problem, we will again use the central difference approximation

$$\frac{\partial y}{\partial x}\bigg|_{n+\frac{1}{2}} \cong \frac{1}{24h}\left(y_{n-1} - 27y_n + 27y_{n+1} - y_{n+2}\right) + \frac{3h^4}{640}\frac{\partial^5 y}{\partial x^5}\bigg|_{n+\frac{1}{2}} \qquad (99)$$

using staggered stations such that y and M are calculated at integer stations whereas θ and V are calculated at half-integer stations. The beam ends occur at half-integer stations.

Let us now consider appropriate symmetry assumptions for a cantilever beam, chosen to illustrate free and built-in end conditions. (See Fig. 10.) First we note that the assumption of even symmetry in y about the built-in end will give zero slope at that end, using Eq. (99). (The assumption of odd symmetry here would not give the proper result.) Successive differentiation of y with respect to x gives alternating odd and even functions, as shown by the dashed lines where they deviate from the solid lines. Thus the assumed beam shows a discontinuity in V (and the slope of M). However, one can see from the basic approximation given in Eq. (99) that the presence of discontinuities will amount to an increase in the magnitude of the higher derivatives at this point and thereby increase the error in the approximation.

To avoid these errors due to discontinuities, we will arbitrarily assume continuous curves at the boundaries even though these assumptions are not physically consistent. We are interested in obtaining the best possible estimates for the derivatives of y within the actual beam, and therefore we

make the most promising assumptions concerning the virtual beam.

The symmetry assumptions that we will use can be summarized as follows:

(1) If the derivative of the function approaches zero as the end of the actual beam is approached, assume even symmetry about the end.

(2) If the derivative of the function does not approach zero as the end is approached, assume odd symmetry about the end, possibly with an offset to avoid discontinuities.

The procedure by which these assumptions are incorporated into the equations of motion is quite straightforward except for the extrapolation of M at a built-in end or the similar extrapolation of y at a free end. Taking the case of the built-in end, a functional form must be assumed for M near the boundary. Referring to the analytical solution near a built-in end, as given by Eq. (22) or (49), we find that the magnitude of M varies according to a linear plus a fourth-order term in x plus other higher-order terms. So a logical choice of the functional form of M near the boundary might be

$$M = a_0 + a_1 x \pm a_4 x^4 \tag{135}$$

where the last term takes the plus or minus sign depending on whether x is positive or negative. This assumption results in the equations

$$M_0 = \frac{1}{232} [503 M_1 - 310 M_2 + 39 M_3] \tag{136}$$

and

$$M_{-1} = \frac{1}{232} [735 M_1 - 542 M_2 + 39 M_3] \tag{137}$$

where the built-in end occurs at station $\frac{1}{2}$.

Another possibility is to assume that M is an offset odd function having a linear plus cubic variation with x. Thus we might assume that

$$M = b_0 + b_1 x + b_3 x^3 \tag{138}$$

...is assumption avoids having discontinuous derivatives at x = 0 and gives numerically simpler equations. In this case

$$M_0 = \frac{1}{3}[7M_1 - 5M_2 + M_3] \tag{139}$$

$$M_{-1} = \frac{1}{3}[10M_1 - 8M_2 + M_3] \tag{140}$$

Computations were run for many cases, using either Eq. (135) or (138), and the results were compared. The differences were fairly small and, if anything, favored the linear plus cubic assumption. All results given in this report, assuming symmetry at the boundaries, are based upon a linear plus cubic variation of M near a built-in end, i.e., upon Eqs. (139) and (140).

The assumptions concerning the form of y near a built-in end are the same as we used previously for the h^4 case. Again we can eliminate y_1 by using the equation

$$y_1 = \frac{2}{9} y_2 - \frac{1}{25} y_3 \tag{130}$$

Of course a similar relation exists for the value of M at the station nearest a free end.

B. EQUATIONS FOR A BUILT-IN BEAM

1. General Case.—We have seen how we can use symmetry to extrapolate the variables of the problem beyond the actual limits of the beam. Using these extrapolated values and the basic central difference approximation of Eq. (99), we can write general equations for a built-in beam similar to those given by h^4 approximation in Eqs. (106) to (109). They are

$$
\begin{Bmatrix} \theta_{\frac{3}{2}} \\ \theta_{\frac{5}{2}} \\ \theta_{\frac{7}{2}} \\ \cdot \\ \cdot \\ \cdot \end{Bmatrix} = \frac{1}{5400h}
\begin{bmatrix}
4775 & 9 & 0 & 0 & \cdots \\
-6025 & 6066 & -225 & 0 & \\
225 & -6075 & 6075 & -225 & \cdots \\
\cdot & & & \cdot & \\
\cdot & & & \cdot & \\
\cdot & & & \cdot &
\end{bmatrix}
\begin{Bmatrix} y_2 \\ y_3 \\ y_4 \\ \cdot \\ \cdot \\ \cdot \end{Bmatrix} \tag{141}
$$

$$\begin{Bmatrix} M_1 \\ M_2 \\ M_3 \\ \cdot \\ \cdot \\ \cdot \end{Bmatrix} = \frac{1}{24h} \begin{bmatrix} 26 & -1 & 0 & 0 & \cdots \\ -27 & 27 & -1 & 0 & \\ 1 & -27 & 27 & -1 & \cdots \\ \cdot & & & \cdot & \\ \cdot & & & \cdot & \\ \cdot & & & \cdot & \end{bmatrix} \begin{Bmatrix} \theta_{\frac{3}{2}} \\ \theta_{\frac{5}{2}} \\ \theta_{\frac{7}{2}} \\ \cdot \\ \cdot \\ \cdot \end{Bmatrix} \tag{142}$$

$$\begin{Bmatrix} V_{\frac{1}{2}} \\ V_{\frac{3}{2}} \\ V_{\frac{5}{2}} \\ \cdot \\ \cdot \\ \cdot \end{Bmatrix} = \frac{1}{72h} \begin{bmatrix} -98 & 124 & -26 & 0 & \cdots \\ -74 & 76 & -2 & 0 & \\ 3 & -81 & 81 & -3 & \cdots \\ \cdot & & & \cdot & \\ \cdot & & & \cdot & \\ \cdot & & & \cdot & \end{bmatrix} \begin{Bmatrix} M_1 \\ M_2 \\ M_3 \\ \cdot \\ \cdot \\ \cdot \end{Bmatrix} \tag{143}$$

$$\begin{Bmatrix} \ddot{y}_2 \\ \ddot{y}_3 \\ \ddot{y}_4 \\ \cdot \\ \cdot \\ \cdot \end{Bmatrix} = \frac{1}{24h} \begin{bmatrix} -1 & 27 & -27 & 1 & 0 & \cdots \\ 0 & -1 & 27 & -27 & 1 & \\ 0 & 0 & -1 & 27 & -27 & \cdots \\ \cdot & & & & \cdot & \\ \cdot & & & & \cdot & \\ \cdot & & & & \cdot & \end{bmatrix} \begin{Bmatrix} V_{\frac{1}{2}} \\ V_{\frac{3}{2}} \\ V_{\frac{5}{2}} \\ \cdot \\ \cdot \\ \cdot \end{Bmatrix} \tag{144}$$

A comparison of the above equations with those for the h^4 approximation as given by Eqs. (106) to (109) shows considerable similarity. It should be noted, however, that the use of symmetry at the boundary never requires more than four inputs per amplifier in the analog computer circuit, whereas the h^4 approximation requires up to five inputs per amplifier.

2. The 6-Cell Case.—As an example of a mode calculation for a specific case, consider again the modes with even symmetry about the center for a 6-cell beam built-in at both ends. The equations of motion (using symmetry assumptions at the boundaries) are

$$\left\{\begin{array}{c} \theta_{\frac{3}{2}} \\ \theta_{\frac{5}{2}} \end{array}\right\} = \frac{1}{5400h} \begin{bmatrix} 4775 & 9 \\ -6025 & 5841 \end{bmatrix} \left\{\begin{array}{c} r_2 \\ r_3 \end{array}\right\} \tag{145}$$

$$\left\{\begin{array}{c} M_1 \\ M_2 \\ M_3 \end{array}\right\} = \frac{1}{24h} \begin{bmatrix} 26 & -1 \\ -27 & 27 \\ 1 & -26 \end{bmatrix} \left\{\begin{array}{c} \theta_{\frac{3}{2}} \\ \theta_{\frac{5}{2}} \end{array}\right\} \tag{146}$$

$$\left\{\begin{array}{c} V_{\frac{1}{2}} \\ V_{\frac{3}{2}} \\ V_{\frac{5}{2}} \end{array}\right\} = \frac{1}{72h} \begin{bmatrix} -98 & 124 & -26 \\ -74 & 76 & -2 \\ 3 & -81 & 78 \end{bmatrix} \left\{\begin{array}{c} \mu_1 \\ \mu_2 \\ \mu_3 \end{array}\right\} \tag{147}$$

$$\left\{\begin{array}{c} \ddot{y}_2 \\ \ddot{y}_3 \end{array}\right\} = \frac{1}{24h} \begin{bmatrix} -1 & 27 & -27 \\ 0 & -1 & 26 \end{bmatrix} \left\{\begin{array}{c} V_{\frac{1}{2}} \\ V_{\frac{3}{2}} \\ V_{\frac{5}{2}} \end{array}\right\} \tag{148}$$

Setting h = 1 and performing the matrix multiplications, one obtains

$$\begin{vmatrix} (8.04779-\lambda^2) & -4.39000 \\ -4.39275 & (2.91454-\lambda^2) \end{vmatrix} = 0 \tag{149}$$

giving the characteristic equation

$$\lambda^4 - 10.96233\lambda^2 + 4.17146 = 0 \tag{150}$$

The roots are

$$\lambda_{1,2} = 0.62828, 3.25078$$

and the normalized roots are

$$\omega_1 = 22.6182$$

$$\omega_2 = 117.028$$

Recalling that the corresponding normalized frequencies for the continuous beam are

$$\beta_1 = 22.3733$$

$$\beta_2 = 120.903$$

we see that the above results give a somewhat larger error for the first mode than does the h^4 approximation, but the error in the higher mode is much smaller.

An alternate method can be used, particularly with the use of a digital computer, to set up the equations of motion and to solve them. This method, which indicates more clearly the symmetry assumptions in its formulation, is given in Appendix A.

3. Results.—The results of calculations for built-in and cantilever beams, using symmetry assumptions at the boundaries, are summarized in Tables II and III and in Figs. 17 and 18.

Comparing these results with those for the h^4 approximation, we note that for a practical range of N (say 6 to 16) the use of symmetry gives lower accuracy on the low frequency modes but better accuracy on the higher modes. For example, consider the case of an 8-cell cantilever beam having a total of seven modes. Using symmetry at the boundaries, the first four modes have a frequency error well within 1% and even the sixth mode has an error of only 10%, approximately. By contrast, the use of h^4 approximations at the boundaries gives better accuracy for the first three modes but the fourth mode has an error of nearly 5% and the sixth mode has an error of approximately 20%.

C. ORTHOGONALITY

It turns out when boundary conditions are imposed using symmetry that the resulting modes of vibration are nearly, but not exactly, orthogonal. This means that this approach also results in a nonphysical system in the sense that a lumped model cannot be built of linear bilateral elements.

To compare the orthogonality properties with those for the h^4 approximation, let us consider again the example of the 5-cell, built-in beam. The mass matrix is again

$$[m] = \begin{bmatrix} \dfrac{85}{81} & \dfrac{-2}{225} \\[2mm] \dfrac{-2}{225} & \dfrac{626}{625} \end{bmatrix} \tag{129}$$

The stiffness matrix is

$$[k] = [m][K] = \begin{bmatrix} 8.48426 & -4.63270 \\[2mm] -4.47131 & 2.95823 \end{bmatrix} \tag{151}$$

The amplitude ratios are calculated from

$$\frac{A_2}{A_1} = \frac{8.04779 - \lambda^2}{4.39000} \tag{152}$$

yielding the modal matrix

$$[A] = \begin{bmatrix} 1.00000 & 1.00000 \\[2mm] 1.74329 & -0.57309 \end{bmatrix} \tag{153}$$

In the modal coordinate system the mass matrix is

$$[\bar{m}] \; = \; [A]^T[m][A] \; = \; \begin{bmatrix} 4.06232 & 0.03676 \\ 0.03676 & 1.38957 \end{bmatrix} \tag{154}$$

The stiffness matrix is

$$[\bar{k}] \; = \; [A]^T[k][A] \; = \; \begin{bmatrix} 1.60356 & 0.38348 \\ 0.01451 & 14.68446 \end{bmatrix} \tag{155}$$

Thus neither matrix is diagonalized by the transformation, but the result is considerably closer than was the case for the h^4 approximation, as can be seen from Eqs. (133) and (134).

VII. THE PASSIVE CIRCUIT METHOD

We have seen in the case of the simple approximation of order h^2 that an electrical circuit exists which the difference equations describe exactly. However, when one goes to higher-order differences and uses the methods we have described for establishing the boundary conditions, one finds that the resulting equations are nonphysical, i.e., no passive linear circuit analogy exists.

It can be shown that the nonphysical character of the higher-order methods discussed thus far is due to the boundary-condition representation rather than the basic difference approximation. In this section a passive circuit analogy will be obtained for the basic higher-order approximations of Eqs. (100) to (103). By imposing boundary conditions on this circuit and writing the corresponding equations, one can obtain a set of mutually orthogonal modes.

D
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6
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A. PASSIVE CIRCUIT ANALOGY

We have been using the following finite difference approximations to the beam equations.

$$\theta_{n+\frac{1}{2}} = \frac{1}{24h}\left(y_{n-1} - 27y_n + 27y_{n+1} - y_{n+2}\right) \tag{100}$$

$$M_n = \frac{1}{24h}\left(\theta_{n-\frac{3}{2}} - 27\theta_{n-\frac{1}{2}} + 27\theta_{n+\frac{1}{2}} - \theta_{n+\frac{3}{2}}\right) \tag{101}$$

$$V_{n+\frac{1}{2}} = \frac{1}{24h}\left(M_{n-1} - 27M_n + 27M_{n+1} - M_{n+2}\right) \tag{102}$$

$$\ddot{y}_n = \frac{1}{24h}\left(-V_{n-\frac{3}{2}} + 27V_{n-\frac{1}{2}} - 27V_{n+\frac{1}{2}} + V_{n+\frac{3}{2}}\right) \tag{103}$$

Setting h = 1, and representing generalized velocities by voltages and generalized forces by currents, one can show that the passive circuit of Fig. 11 is described by these equations. Note that three transformers, one inductor, and one capacitor are required per cell.

The rather large transformer requirement of Fig. 11 compared with the h^2 approximation of Fig. 3(b) makes it impractical to use this passive circuit

directly. Instead we can write the circuit equations and solve them using a differential analyzer or a digital computer. The equipment requirements in this case are almost the same as for the other higher-order methods but are actually slightly less due to the details of boundary representation.

B. THE BUILT-IN BEAM

1. Symmetry Assumptions.—To obtain a proper representation of a built-in end, it is convenient to make certain symmetry assumptions consistent with the given boundary conditions. We will assume that the deflection curve has even symmetry about the end and zero slope at the boundary. Thus y and M are even functions of x, while Θ and V are odd functions of x, as shown by the dashed lines of Fig. 10.

The circuit that has been used to represent a built-in end is shown in Fig. 12. The beam end is at the center of the diagram (station $\frac{1}{2}$) with the virtual beam on the left and the actual beam on the right. The autotransformer at the bottom of the figure represents the constraint relating y_1 and y_2.

2. Constraint Equation.—It can be shown for a uniform beam that

$$y_{n+\frac{1}{2}} \cong \frac{1}{16}\left(-y_{n-1} + 9y_n + 9y_{n+1} - y_{n+2}\right) + \frac{h^6}{512} \left.\frac{\partial^6 y}{\partial x^6}\right|_{n+\frac{1}{2}} \tag{156}$$

So if we assume that y has even symmetry about the built-in end at station $\frac{1}{2}$ and also set $y_{\frac{1}{2}}$ equal to zero, we obtain the approximate equality

$$y_1 \cong \frac{1}{9} y_2 \tag{157}$$

This is used as the constraint equation in the passive circuit approach.

It can be seen from Fig. 12 that the autotransformer representing the constraint equation causes a certain relationship to exist between voltages and also between currents. In mechanical terms, a constraint equation relating coordinates also implies forces of constraint to be exerted on the system in such a way that no work is done. These forces of constraint have been ignored in the previous two approaches to the problem and this resulted in a lack of symmetry in the stiffness matrices.

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To calculate the effect of these forces of constraint, consider a beam described by the matrix equation

$$[m]\{\ddot{y}\} = [D]\{V\} \tag{158}$$

Now suppose a constraint is imposed as described by

$$y_1 = (C)\{y\} \tag{159}$$

where the summation is over all y except y_1. (The parentheses denote a row matrix.) Let us include the constraint forces in the analysis and also separate the y_1 equation from the others. Then we obtain

$$m_1\ddot{y}_1 = (D_1)\{V\} + F_1 \tag{160}$$

$$[m]\{\ddot{y}\} = [D]\{V\} + \{F\} \tag{161}$$

where the F are constraint forces acting on the beam. Note that Eq. (161) does not contain $\ddot{y}_1$ and the D matrix does not contain a D_1 row.

Now multiply (160) by y_1 and premultiply (161) by $\{y\}^T$. Adding, we obtain

$$y_1 m_1 \ddot{y}_1 + \{y\}^T[m]\{\ddot{y}\} = y_1(D_1)\{V\} + \{y\}^T[D]\{V\} + F_1 y_1 + \{y\}^T\{F\} \tag{162}$$

Since the constraint does no work, we can write

$$F_1 y_1 + \{y\}^T\{F\} = 0 \tag{163}$$

and the last two terms of (162) drop out. Finally, substituting for y_1 from Eq. (159), dividing out $\{y\}^T$, and rearranging, we obtain

$$m_1(C)\{\ddot{y}\}(C)^T + [m]\{\ddot{y}\} = (D_1)\{V\}(C)^T + [D]\{V\} \tag{164}$$

This gives the equations of motion with y_1 omitted and constraint forces included. For example, in the present case with a simple constraint given by Eq. (157), the effective mass at y_2 is $82/81$ times its value before the constraint was applied. More complicated constraints will result in inertial coupling, but the effective inertia matrix will always be symmetric.

3. General Case.—The complete equations of motion for a built-in beam can be written now using the passive circuit method. They are

$$
\begin{Bmatrix} \theta_{\frac{3}{2}} \\ \theta_{\frac{5}{2}} \\ \theta_{\frac{7}{2}} \\ \cdot \\ \cdot \end{Bmatrix} = \frac{1}{216h}
\begin{bmatrix}
217 & -9 & 0 & 0 & \cdots \\
-242 & 243 & -9 & 0 & \\
9 & -243 & 243 & -9 & \cdots \\
\cdot & & & \cdot & \\
\cdot & & & \cdot & \\
\cdot & & & \cdot &
\end{bmatrix}
\begin{Bmatrix} y_2 \\ y_3 \\ y_4 \\ \cdot \\ \cdot \\ \cdot \end{Bmatrix} \tag{165}
$$

$$
\begin{Bmatrix} M_1 \\ M_2 \\ M_3 \\ \cdot \\ \cdot \\ \cdot \end{Bmatrix} = \frac{1}{24h}
\begin{bmatrix}
26 & -1 & 0 & 0 & \cdots \\
-27 & 27 & -1 & 0 & \\
1 & -27 & 27 & -1 & \cdots \\
\cdot & & & \cdot & \\
\cdot & & & \cdot & \\
\cdot & & & \cdot &
\end{bmatrix}
\begin{Bmatrix} \theta_{\frac{3}{2}} \\ \theta_{\frac{5}{2}} \\ \theta_{\frac{7}{2}} \\ \cdot \\ \cdot \\ \cdot \end{Bmatrix} \tag{166}
$$

$$
\begin{Bmatrix} V_{\frac{3}{2}} \\ V_{\frac{5}{2}} \\ V_{\frac{7}{2}} \\ \cdot \\ \cdot \\ \cdot \end{Bmatrix} = \frac{1}{24h}
\begin{bmatrix}
-26 & 27 & -1 & 0 & \cdots \\
1 & -27 & 27 & -1 & \\
0 & 1 & -27 & 27 & \cdots \\
\cdot & & & \cdot & \\
\cdot & & & \cdot & \\
\cdot & & & \cdot &
\end{bmatrix}
\begin{Bmatrix} M_1 \\ M_2 \\ M_3 \\ \cdot \\ \cdot \\ \cdot \end{Bmatrix} \tag{167}
$$

$$\begin{Bmatrix} \ddot{y}_2 \\ \ddot{y}_3 \\ \ddot{y}_4 \\ \cdot \\ \cdot \\ \cdot \end{Bmatrix} = \frac{1}{1968h} \begin{bmatrix} 1953 & -2178 & 8_\cdot & 0 & \cdots \\ -82 & 2214 & -2214 & 82 & \\ 0 & -82 & 2214 & -2214 & \cdots \\ \cdot & & & \cdot & \\ \cdot & & & \cdot & \\ \cdot & & & \cdot & \end{bmatrix} \begin{Bmatrix} V_{\frac{3}{2}} \\ V_{\frac{5}{2}} \\ V_{\frac{7}{2}} \\ \cdot \\ \cdot \\ \cdot \end{Bmatrix} \tag{168}$$

The amount of analog computer equipment needed for the passive circuit method is slightly less than for either of the other higher-order methods. There are two reasons for this. First, the value of V at a built-in end (or of Θ at a free end) need not be calculated. Second, the signs are optimum in that no extra amplifiers are required for inversion at any point.

The equations of motion for stations near a free end can be obtained most easily from the above equations by exchanging the roles of y and M and also Θ and V. Of course they could also be obtained directly from the passive circuit, remembering the constraint equation relating the values of M at the last two stations.

4. The 6-Cell Case.—Again let us consider the even symmetry modes of a 6-cell beam built-in at both ends. Using the passive circuit approach, the equations of motion are

$$\begin{Bmatrix} \Theta_{\frac{3}{2}} \\ \Theta_{\frac{5}{2}} \end{Bmatrix} = \frac{1}{216h} \begin{bmatrix} 217 & -9 \\ -242 & 234 \end{bmatrix} \begin{Bmatrix} y_2 \\ y_3 \end{Bmatrix} \tag{169}$$

$$\begin{Bmatrix} M_1 \\ M_2 \\ M_3 \end{Bmatrix} = \frac{1}{24h} \begin{bmatrix} 26 & -1 \\ -27 & 27 \\ 1 & -26 \end{bmatrix} \begin{Bmatrix} \Theta_{\frac{3}{2}} \\ \Theta_{\frac{5}{2}} \end{Bmatrix} \tag{170}$$

$$\begin{Bmatrix} V_{\frac{3}{2}} \\ V_{\frac{5}{2}} \end{Bmatrix} = \frac{1}{24h} \begin{bmatrix} -26 & 27 & -1 \\ 1 & -27 & 26 \end{bmatrix} \begin{Bmatrix} M_1 \\ M_2 \\ M_3 \end{Bmatrix} \tag{171}$$

$$\begin{Bmatrix} \ddot{y}_2 \\ \ddot{y}_3 \end{Bmatrix} = \frac{1}{1968h} \begin{bmatrix} 1953 & -2178 \\ -82 & 2132 \end{bmatrix} \begin{Bmatrix} V_{\frac{3}{2}} \\ V_{\frac{5}{2}} \end{Bmatrix} \qquad (172)$$

Setting h = 1, we can solve for the eigenvalues from

$$\begin{vmatrix} (8.47527-\lambda^2) & -4.54772 \\ -4.60386 & (2.99140-\lambda^2) \end{vmatrix} = 0 \qquad (173)$$

or

$$\lambda^4 - 11.46667\lambda^2 + 4.41583 = 0 \qquad (174)$$

The normalized roots are

$$\omega_1 = 22.7395$$

$$\omega_2 = 119.765$$

In this case the first mode frequency is 1.6% high and the second mode frequency is 0.94% low.

C. RESULTS

The results of mode frequency calculations for a uniform beam using the passive circuit method are summarized in Tables II and III and in Figs. 19 and 20.

Comparing these results with those obtained previously, using other methods, we note that the passive circuit method gives very good over-all accuracy for cases where the number of cells is 9 or more. On the other hand, the symmetry method appears to give generally better results for smaller values of N. Of course, the h^4 method is still best for accuracies of the order of 0.25% or better.

One way of comparing the various methods is to obtain an approximate number
of cells required per wavelength to obtain a given accuracy. For 1% error,
the ordinary h^2 approximation requires 12 cells per wavelength. In contrast,
the h^4 approximation requires 6-1/2 cells per wavelength, the symmetry meth-
od requires 5 cells per wavelength, and the passive circuit method requires
4-1/2 cells per wavelength.

D. ORTHOGONALITY

In contrast with the higher-order methods considered previously, the pas-
sive circuit method produces orthogonal modes. The orthogonality property is
a result of the symmetry of the original m and k matrices.

We can illustrate this point by referring again to the 6-cell case. From
Eq. (173) we see that the amplitude ratios for the two modes can be calculated
using

$$\frac{A_2}{A_1} = \frac{8.47527 - \lambda^2}{4.54772} \tag{175}$$

The modal matrix is

$$[A] = \begin{bmatrix} 1.00000 & 1.00000 \\ 1.77590 & -0.57005 \end{bmatrix} \tag{176}$$

The original mass matrix is

$$[m] = \begin{bmatrix} \dfrac{82}{81} & 0 \\ 0 & 1 \end{bmatrix} \tag{177}$$

The corresponding stiffness matrix is

$$[k] = \begin{bmatrix} 8.57990 & -4.60386 \\ -4.60386 & 2.99140 \end{bmatrix} \tag{178}$$

After the transformation

$$\{y\} \; = \; [A]\{\overline{y}\} \tag{132}$$

the resulting mass and stiffness matrices are

$$[\overline{m}] \; = \; [A]^T[m][A] \; = \; \begin{bmatrix} 4.16616 & 0 \\ 0 & 1.33730 \end{bmatrix} \tag{179}$$

$$[\overline{K}] \; = \; [A]^T[k][A] \; = \; \begin{bmatrix} 1.66223 & 0 \\ 0 & 14.80080 \end{bmatrix} \tag{180}$$

The diagonal nature of these matrices implies orthogonality of modes, and also that the motion of each of the modal coordinates is independent of the others.

It should be noted in passing that the matrix

$$[\lambda^2] \; = \; [A]^{-1}[K][A] \tag{181}$$

is diagonal for all three methods and the numbers along the main diagonal are the squares of the mode frequencies.[10] This does not imply orthogonality, however, as we have defined it.

VIII. SOURCES OF ERROR

In this section we will consider the principal sources of error in the methods that have been suggested for the solution of beam vibration problems. These errors are (1) finite difference errors due to the basic difference approximation; (2) finite difference errors in the representation of boundary conditions, and (3) component errors. The error curves that have been presented so far represent a combination of error sources (1) and (2) above for various specific cases. Now we will consider them with somewhat more generality.

A. FINITE DIFFERENCE ERRORS

1. The h^2 Approximation. We have seen previously for a pinned beam that, when proper symmetry assumptions are made, there is no additional error due to the method of representing boundary conditions and so the entire computed error is due to the basic finite difference approximation. So consider now a uniform beam pinned at each end at an integer station.

Recall again the basic h^2 approximation

$$\left.\frac{\partial y}{\partial x}\right|_{n+\frac{1}{2}} \cong \frac{-y_n + y_{n+1}}{h} - \frac{h^2}{24}\left.\frac{\partial^3 y}{\partial x^3}\right|_{n+\frac{1}{2}} \qquad (60)$$

Repeated application of Eq. (60) yields

$$\left.\frac{\partial^2 y}{\partial x^2}\right|_{n} \cong \frac{y_{n-1} - 2y_n + y_{n+1}}{h^2} - \frac{h^2}{12}\left.\frac{\partial^4 y}{\partial x^4}\right|_{n} \qquad (65)$$

Similarly, one can obtain

$$\left.\frac{\partial^4 y}{\partial x^4}\right|_{n} \cong \frac{y_{n-2} - 4y_{n-1} + 6y_n - 4y_{n+1} + y_{n+2}}{h^4} - \frac{h^2}{6}\left.\frac{\partial^6 y}{\partial x^6}\right|_{n} \qquad (182)$$

which is the result of applying Eq. (60) four times in sequence.

We will again consider the equation for the continuous beam in simplified form.

$$\frac{\partial^4 y}{\partial x^4} + \frac{\partial^2 y}{\partial t^2} = 0 \tag{6}$$

As we have seen previously, the solution for the continuous pinned beam is of the form

$$y = \sum_i Y_i \sin\sqrt{\beta_i}\, x \,(G_i \cos \beta_i t + H_i \sin \beta_i t) \tag{183}$$

where

$$\beta_i = i^2 \pi^2 \quad (i = 1, 2, 3, \dots) \tag{55}$$

For the case of the ith mode, we can substitute Eqs. (182) and (183) into (6), obtaining

$$\frac{1}{h^4}\left(y_{n-2} - 4y_{n-1} + 6y_n - 4y_{n+1} + y_{n+2}\right) + \frac{h^2}{6}\beta_i^3 y_n - \beta_i^2 y_n = 0$$

or

$$\frac{1}{h^4}\left(y_{n-2} - 4y_{n-1} + 6y_n - 4y_{n+1} + y_{n+2}\right) = \beta_i^2 y_n\left(1 - \frac{h^2}{6}\beta_i\right) \tag{184}$$

Now the left-hand side of Eq. (184) is of the same form as the approximation to $\dfrac{\partial^4 y}{\partial x^4}\bigg|_n$ used in the finite difference solution. Therefore, the corresponding finite difference equation is

$$\frac{1}{h^4}\left(y_{n-2} - 4y_{n-1} + 6y_n - 4y_{n+1} + y_{n+2}\right) = \omega_i^2 y_n \tag{185}$$

where ω_i is the finite difference eigenvalue corresponding to β_i. So the right sides of Eqs. (184) and (185) can be equated if sinusoidal mode shapes are assumed in each case. Finally, noting that

$$h = \frac{1}{N} \qquad (186)$$

we obtain

$$\omega_i^2 = (i\pi)^4 \left[1 - \frac{1}{6}\left(\frac{i\pi}{N}\right)^2 + \ldots \right]$$

or

$$\omega_i = (i\pi)^2 \left[1 - \frac{1}{12}\left(\frac{i\pi}{N}\right)^2 + \ldots \right] \qquad (187)$$

where i is the mode number.

2. The h^4 Approximation.—A similar approach can be used to compute the frequency error for a pinned beam using higher-order differences. The basic approximation is

$$\left.\frac{\partial y}{\partial x}\right|_{n+\frac{1}{2}} \cong \frac{1}{24h}\left(y_{n-1} - 27y_n + 27y_{n+1} - y_{n+2} \right) + \frac{3h^4}{640}\left.\frac{\partial^5 y}{\partial x^5}\right|_{n+\frac{1}{2}} \qquad (99)$$

Using successive repetitions of Eq. (99), we obtain

$$\left.\frac{\partial^2 y}{\partial x^2}\right|_n \cong \frac{1}{(24h)^2}\left(y_{n-3} - 54y_{n-2} + 783y_{n-1} - 1460y_n + 783y_{n+1} - 54y_{n+2} + y_{n+3} \right) + \frac{3h^4}{320}\left.\frac{\partial^6 y}{\partial x^6}\right|_n$$
$$(188)$$

and

$$\left.\frac{\partial^4 y}{\partial x^4}\right|_n \cong \frac{1}{(24h)^4}\left(y_{n-6} + \ldots + y_{n+6} \right) + \frac{3h^4}{160}\left.\frac{\partial^8 y}{\partial x^8}\right|_n \qquad (189)$$

Proceeding in a manner similar to that for the h^2 approximation, we obtain

$$\omega_i^2 = \beta_i^2 - \frac{3h^4}{160} \beta_i^4$$

or

$$\omega_i = (i\pi)^2 \left[1 - \frac{3}{320}\left(\frac{i\pi}{N}\right)^4 + \dots \right] \tag{190}$$

Thus we see that the basic finite difference approximation tends to give frequencies that are too low. The first error term in Eq. (190) gives an accurate estimate of the error only for small i/N ratios and more terms are needed for most practical cases. The actual error curves are plotted in Fig. 21 for the h^4 approximation.

3. Maximum Errors.—From the tabulated results of the calculations of mode frequencies, it is seen that, as the i/N ratio increases, the error also increases. In the limit as i/N approaches unity, the error is independent of the boundary conditions and depends only on the basic approximation. This limiting value of the error could be obtained by evaluating the complete series indicated in Eq. (187) or (190) for $i/N = 1$. A simpler way to obtain this result is to note that as i/N approaches unity the motion at adjacent stations is equal in amplitude but opposite in sign. Knowing this, one can calculate the frequency directly.

For example, using the h^2 approximation and assuming $h = 1$, we find that

$$\ddot{y}_n + 16y_n = 0 \tag{191}$$

yielding the frequency

$$\lambda = 4$$

The corresponding frequency for the continuous beam is the same as for a pinned beam of unit length vibrating in its fundamental mode, namely, π^2. So the maximum error for the h^2 approximation is

$$\epsilon_{max} = \frac{\pi^2 - 4}{\pi^2} = 59.472\%$$

(192)

Performing a similar calculation for the h^4 case, we obtain

$$\ddot{y}_n + \left(\frac{7}{3}\right)^4 y_n = 0$$

(193)

or

$$\lambda = \frac{49}{9}$$

giving a maximum error

$$\epsilon_{max} = \frac{9\pi^2 - 49}{9\pi^2} = 44.836\%$$

(194)

B. BOUNDARY-CONDITION ERRORS

We have seen that an analysis of uniform pinned beams with various numbers of cells gives directly the finite difference error due to the basic central difference approximation. Furthermore, we note that the eigenvalues of the pinned case alternate with the eigenvalues of either the cantilever or built-in case. Consequently, if the method of applying boundary conditions introduced no additional error, one would expect the error curves for the cantilever or built-in beam to run between and roughly parallel to the error curves for the pinned beam. Instead, one finds that all the boundary-condition methods considered here result in raising the natural frequencies. This causes a general improvement in accuracy for i/N values greater than about 0.3 for the symmetry and passive circuit approximations. For the h^4 method the accuracy is improved for all i/N.

For the lower modes and large N, the error for the three built-in or cantilever cases mentioned above exceeds the error for the corresponding pinned case because the boundary-condition errors are of lower order in h than the basic central difference errors, and ultimately must predominate. For example, in the equations for a built-in beam using the simple "h^2" approximation, we find the expression

$$\theta_{\frac{3}{2}} = \frac{y_2}{h} \qquad\qquad (195)$$

[See Eq. (82).] The error in this case is actually of order h, as can be ascertained by the Taylor expansion method.

Similarly, the passive circuit and symmetry methods of approximating a built-in end use the expression

$$M_1 = \frac{1}{24h}\left(26\theta_{\frac{3}{2}} - \theta_{\frac{5}{2}}\right) \qquad\qquad (196)$$

which has a first error term of order h. On the other hand, each equation used in the h^4 approximation is truly of order h^4, and therefore the boundary-condition errors do not tend to dominate for large N.

A study of the error curves presented in Figs. 16 and 21 leads one to the conclusion that the first error term alone does not give a reliable estimate of the actual errors encountered. The i/N ratio must be the order of 0.1 or less before the first error term clearly predominates and by this time roundoff or component errors mask the desired truncation error.

To illustrate this point, it is of interest to calculate the rate at which the actual truncation errors decrease with increasing N as N approaches 16, i.e., at the right-hand edge of the error diagram. Rough calculations indicate that the h^2 and symmetry methods have an error that is decreasing as $h^{1.9}$, while the error for the passive circuit method is decreasing as $h^{3.0}$ and that for the h^4 method is decreasing as $h^{4.7}$ for increasing N.

The conclusion from the preceding analysis is that the method of representing boundary conditions should be chosen on the basis of the requirements of the problem and actual error curves, rather than assuming that an h^4 approximation is necessarily more accurate than an h^3 or lower approximation over the practical range of N. Our results show that so-called higher-

order methods are generally superior to simpler approximation methods. However, the choice between the various methods of representing boundary conditions should be made on the basis of the problem at hand.

C. COMPONENT ERRORS

In this section we will consider the sensitivity of the calculated roots or frequencies to small shifts or inaccuracies of the component values. This will be accomplished by perturbing one of the matrix elements and calculating the resulting frequency shift.

Suppose we write the characteristic equation in the form

$$S(\eta, b_{ij}) = 0 \qquad (197)$$

where the eigenvalue, $\eta = \lambda^2$, is the square of the frequency (rad/sec) of one of the modes and b_{ij} is the matrix element that is to be perturbed. For small deviations about the reference values η_o, b_{ijo} we can expand in the series

$$S(\eta, b_{ij}) = S(\eta_o, b_{ijo}) + \frac{\partial S}{\partial \eta}\bigg|_o \Delta\eta + \frac{\partial S}{\partial t_{ij}}\bigg|_o \Delta b_{ij} + \cdots \qquad (198)$$

where

$$\eta = \eta_o + \Delta\eta$$

$$b_{ij} = b_{ijo} + \Delta b_{ij}$$

From Eqs. (197) and (198) we obtain an expression for the shift in the root.

$$\Delta\eta = -\frac{\dfrac{\partial S}{\partial b_{ij}}\bigg|_o}{\dfrac{\partial S}{\partial \eta}\bigg|_o} \Delta b_{ij} \qquad (199)$$

To illustrate this method, let us make an error-sensitivity computation for a 6-cell beam that is built-in at both ends using symmetry assumptions at

the ends and assuming even symmetry about the center. We have seen that the equations of motion can be written in the form

$$\{\ddot{y}\} + [K]\{y\} = 0 \tag{79}$$

where

$$[K] = - [B_4][B_3][B_2][B_1] \tag{80}$$

For the case we are considering, the characteristic equation is of the form

$$\begin{vmatrix} (K_{11}-\eta) & K_{12} \\ K_{21} & (K_{11}-\eta) \end{vmatrix} = 0$$

or

$$S = \eta^2 - (K_{11} + K_{22})\eta + (K_{11}K_{22} - K_{12}K_{21}) = 0 \tag{200}$$

Let us suppose that an element in the B_1 matrix is to be varied. We can write

$$[K] = - [T][B_1] \tag{201}$$

where

$$[T] = [B_4][B_3][B_2] \tag{202}$$

Consider now the case at hand where

$$[B_1] = \frac{1}{5400h} \begin{bmatrix} 4775 & 9 \\ -6025 & 5341 \end{bmatrix}$$

and evaluate the sensitivity to a perturbation of b_{21}. Evaluation of $\Delta\eta$ according to Eq. (199), assuming $h = 1$, gives

$$\Delta\eta_1 = 0.157010 \; \Delta b_{21}$$

$$\Delta\eta_2 = 4.22168 \; \Delta b_{21}$$

for the first and second modes, respectively. In this case the actual roots are

$$\eta_1 = 0.394741$$

$$\eta_2 = 10.5676$$

So the fractional changes in the roots are

$$\frac{\Delta\eta_1}{\eta_1} = -0.443791 \; \frac{\Delta b_{21}}{b_{21}}$$

$$\frac{\Delta\eta_2}{\eta_2} = -0.445731 \; \frac{\Delta b_{21}}{b_{21}}$$

This implies that a 1% increase in the magnitude of b_{21} will cause a 0.222% decrease in the first mode frequency and a 0.223% decrease in the second mode frequency.

It has been pointed out by Clymer,[8] Fisher,[9] and others that, in general, the repeated use of finite difference approximations to the first spatial derivative will result in less component error sensitivity than using approximations to the second or fourth derivatives. To check this point, let us calculate the error sensitivity for the same problem using approximations to the second derivative. Splitting the matrix multiplication differently, we

will let

$$[K] = -[T][B] \qquad (203)$$

where

$$[T] = [B_4][B_3] \qquad (204)$$

$$[B] = [B_2][B_1] \qquad (205)$$

In this case, for h = 1, we find that

$$[B] = \begin{bmatrix} 58.1271 & -2.5037 \\ -130.2083 & 70.3125 \\ 72.0812 & -67.8088 \end{bmatrix} \times 10^{-5}$$

and

$$[T] = \begin{bmatrix} -1981 & 4115 & -2134 \\ 152 & -2182 & 2030 \end{bmatrix}$$

Again we will calculate the error sensitivity for b_{21}. Performing the numerical computations as indicated in the equation, we obtain

$$\Delta\eta_1 = -77.6587 \, \Delta b_{21}$$

$$\Delta\eta_2 = -4037.34 \, \Delta b_{21}$$

corresponding to

$$\frac{\Delta\eta_1}{\eta_1} = 0.256163 \frac{\Delta b_{21}}{b_{21}}$$

$$\frac{\Delta \eta_2}{\eta_2} = 0.497460 \frac{\Delta b_{21}}{b_{21}}$$

So in this case a 1% increase in the magnitude of b_{21} will cause a 0.128% increase in the first mode frequency and a 0.249% increase in the second mode frequency.

Comparing these results with those for the approximation to the first derivative, we do not have a very significant difference in sensitivity. In fact, neither case would be particularly troublesome from the standpoint of component error sensitivity. Further calculations of error sensitivities for other matrix elements show comparable results.

This same case, using approximations to the first derivative, was checked for error sensitivity on an actual analog computer with results in accordance with the above calculations. Percentage frequency shifts were generally less than half the percentage component change.

As the number of cells N is increased, the differences in error sensitivity between the first derivative and second derivative approximations should become more apparent. In any case the sensitivity to component errors will increase with N and large sensitivities should be noticed first in the second derivative approximation.

As Fisher has pointed out, certain elements in the matrix will show considerably more error sensitivity than others. In general the large terms are more sensitive to error. For our basic higher-order approximation

$$\frac{\partial y}{\partial x}\bigg|_{n+\frac{1}{2}} \cong \frac{1}{24h}\left(y_{n-1} - 27y_n + 27y_{n+1} - y_{n+2} \right) \tag{99}$$

we would expect a larger sensitivity to errors in the coefficients ±27 than in the coefficients ±1 or even the coefficient 24. The coefficients ±1 act as correction or trimming terms and do not need to be held to the same percentage accuracy.

The above discussion of error sensitivity applies to digital computers when considering roundoff errors. The magnitude of the roundoff error is, of

course, dependent upon the number of digits used in the computations and the cell number N. It is also dependent upon the differencing scheme used and the details of the digital program. For the methods used with IBM 704 computer in calculating frequencies for the various cases given in this report, roundoff errors of the order of 0.01% were first noticed with i/N ratios of approximately 0.1 or less on cantilever beam computations. Because the use of symmetry effectively halved the number of degrees of freedom, the calculations for the built-in beam did not suffer appreciably from roundoff errors.

For reasons of roundoff error and the general complexity of computation, N = 16 seems to be a rough upper limit on the number of cells to be used in analog or digital computations. This detracts somewhat from the utility of the h^4 method of representing boundary conditions because it is at large N that this method is particularly advantageous from the standpoint of finite difference or truncation errors. On the other hand, it accentuates the advantage of higher-order difference methods in general as compared to the simpler h^2 methods, particularly in the computation of the higher frequency modes.

IX. CONCLUSIONS

The principal conclusions gained from the investigations reported here are as follows.

1. The use of higher-order differences in the solution of beam-vibration problems on the analog computer improves accuracy with essentially the same amount of equipment as would be required for simpler difference methods. In the digital formulation of the corresponding eigenvalue problem, a similar improvement in accuracy is obtained with essentially the same computer effort, as measured by matrix sizes and numbers of operations. The improved accuracy is obtained by using more inputs per amplifier in the analog circuit and more nonzero elements per matrix in the digital computations.

2. The principal difficulty in the formulation of beam-vibration problems using higher-order differences lies in the proper representation of boundary conditions. This difficulty arises from the need for obtaining input data from more stations for each differencing operation to obtain greater accuracy than in the simpler approximations. As the boundary is approached, some of these stations needed for the basic higher-order central difference approximation lie beyond the end of the beam, and so other approximations or assumptions must be used. Three methods are presented for the representation of built-in or free ends, namely, (1) the use of one-sided differences of order h^4, (2) the use of symmetry assumptions at the boundary, and (3) the passive circuit method.

   Each method has its areas of special utility, as seen from the results of calculations for uniform beams. The h^4 method is best suited to relatively small i/N ratios, corresponding to accuracies of 0.25% or better. The symmetry method is most advantageous in obtaining moderately good accuracies (the order of 1%) for all but the highest modes with a minimum of equipment, particularly for a cell number $N < 9$. The passive circuit method has quite good accuracy in general, and in particular for the higher modes (large i/N ratios) where $N > 9$.

3. The discussion of errors centered on (1) truncation errors from the basic central difference approximation and also from the boundary-condition representation and (2) component and roundoff errors. Error considerations as well as the ease of representing boundaries were important in choosing to approximate the first spatial derivative rather than the second or higher derivatives and also in the use of staggered stations in the calculation of successive derivatives.

The methods proposed in this report are relatively insensitive to component or roundoff errors for $N < 16$. Larger values of N are not recommended because of the general computational complexity and also the increasing size of component or roundoff errors.

4. The h^4 and symmetry methods are nonphysical in the sense that no system composed of linear bilateral elements can be found that is represented exactly by the equations of motion. This is a result of the fact that the stiffness matrices are not symmetric in these cases. On the other hand, the h^2 and the passive circuit methods result in physically realizable systems. The latter two methods have the further advantage that the resulting modes are orthogonal, i.e., a transformation to modal coordinates simultaneously diagonalizes the mass and stiffness matrices.

University of Michigan,
   Ann Arbor, Mich., October 1961.

APPENDIX A

ALTERNATE MATRIX FORMULATION

The matrix formulation of the difference equations that we have used thus far involves the solution of Eq. (81) for its eigenvalues, the matrix K being obtained after three matrix multiplications. Any assumptions with regard to boundary conditions or constraints are put into the individual matrices before the matrix multiplications are performed. An alternate scheme, particularly adapted to digital computers, enables one to obtain the same K matrix without losing sight of the subsidiary assumptions.

As an example of the alternate method, consider again the problem of the 6-cell, built-in beam using symmetry assumptions at the ends. Equations (145) to (148) could be obtained from the following eight equations:

$$
\left\{ \begin{array}{c} \theta_{\frac{3}{2}} \\ \\ \theta_{\frac{5}{2}} \end{array} \right\}
= \frac{1}{24h}
\begin{bmatrix} 1 & -27 & 27 & -1 & 0 \\ 0 & 1 & -27 & 27 & -1 \end{bmatrix}
\left\{ \begin{array}{c} y_0 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \end{array} \right\}
\tag{A1}
$$

$$
\left\{ \begin{array}{c} y_0 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \end{array} \right\}
= \frac{1}{225}
\begin{bmatrix} 50 & -9 \\ 50 & -9 \\ 225 & 0 \\ 0 & 225 \\ 0 & 225 \end{bmatrix}
\left\{ \begin{array}{c} y_2 \\ y_3 \end{array} \right\}
\tag{A2}
$$

$$
\left\{ \begin{array}{c} M_1 \\ M_2 \\ M_3 \end{array} \right\}
= \frac{1}{24h}
\begin{bmatrix} 1 & -27 & 27 & -1 & 0 & 0 \\ 0 & 1 & -27 & 27 & -1 & 0 \\ 0 & 0 & 1 & -27 & 27 & -1 \end{bmatrix}
\left\{ \begin{array}{c} \theta_{-\frac{1}{2}} \\ \theta_{\frac{1}{2}} \\ \theta_{\frac{3}{2}} \\ \theta_{\frac{5}{2}} \\ \theta_{\frac{7}{2}} \\ \theta_{\frac{9}{2}} \end{array} \right\}
\tag{A3}
$$

$$
\left\{ \begin{array}{c} \theta_{-\frac{1}{2}} \\ \theta_{\frac{1}{2}} \\ \theta_{\frac{3}{2}} \\ \theta_{\frac{5}{2}} \\ \theta_{\frac{7}{2}} \\ \theta_{\frac{9}{2}} \end{array} \right\}
=
\begin{bmatrix} -1 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & -1 \end{bmatrix}
\left\{ \begin{array}{c} \theta_{\frac{3}{2}} \\ \theta_{\frac{5}{2}} \end{array} \right\}
\tag{A4}
$$

$$\begin{Bmatrix} V_{\frac{1}{2}} \\ V_{\frac{3}{2}} \\ V_{\frac{5}{2}} \end{Bmatrix} = \frac{1}{24h} \begin{bmatrix} 1 & -27 & 27 & -1 & 0 & 0 \\ 0 & 1 & -27 & 27 & -1 & 0 \\ 0 & 0 & 1 & -27 & 27 & -1 \end{bmatrix} \begin{Bmatrix} M_{-1} \\ M_0 \\ M_1 \\ M_2 \\ M_3 \\ M_4 \end{Bmatrix} \tag{A5}$$

$$\begin{Bmatrix} M_{-1} \\ M_0 \\ M_1 \\ M_2 \\ M_3 \\ M_4 \end{Bmatrix} = \frac{1}{3} \begin{bmatrix} 10 & -8 & 1 \\ 7 & -5 & 1 \\ 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 3 \end{bmatrix} \begin{Bmatrix} M_1 \\ M_2 \\ M_3 \end{Bmatrix} \tag{A6}$$

$$\begin{Bmatrix} \ddot{y}_2 \\ \ddot{y}_3 \end{Bmatrix} = \frac{1}{24h} \begin{bmatrix} -1 & 27 & -27 & 1 & 0 \\ 0 & -1 & 27 & -27 & 1 \end{bmatrix} \begin{Bmatrix} V_{\frac{1}{2}} \\ V_{\frac{3}{2}} \\ V_{\frac{5}{2}} \\ V_{\frac{7}{2}} \\ V_{\frac{9}{2}} \end{Bmatrix} \tag{A7}$$

$$\begin{Bmatrix} V_{\frac{1}{2}} \\ V_{\frac{3}{2}} \\ V_{\frac{5}{2}} \\ V_{\frac{7}{2}} \\ V_{\frac{9}{2}} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{Bmatrix} V_{\frac{1}{2}} \\ V_{\frac{3}{2}} \\ V_{\frac{5}{2}} \end{Bmatrix} \tag{A8}$$

These equations explicitly separate the basic difference equations and the boundary conditions.

APPENDIX B

TAPERED CANTILEVER BEAM

The analysis that has been made in the main body of the report has been concerned with the lateral vibrations of uniform beams. In this appendix we will indicate the sort of results to be expected when higher-order methods are applied to nonuniform beams.

ANALYTICAL SOLUTION

The case of a beam of unit width and with uniform mass per unit volume and a linearly decreasing depth has been analyzed by Siddall and Isakson.[11] Theoretical eigenvalues were obtained for the case where the depth at the free end is one-fifth that at the built-in end. (See Fig. 22.) For a beam of unit length, the linear density is

$$\rho = 2(1 - 0.8x) \qquad (B1)$$

and the bending stiffness is

$$EI = \frac{2}{3}(1 - 0.8x)^3 \qquad (B2)$$

The cantilever beam is built-in at $x = 0$.

Theoretical frequencies (rad/sec) for this case are

$$\beta_1 = 2.47829$$

$$\beta_2 = 9.08902$$

$$\beta_3 = 21.2953$$

FINITE DIFFERENCE FORMULATION

The same tapered beam was studied using the finite difference approach. In a manner similar to that for the uniform beam, the natural frequencies were

calculated using various boundary-condition assumptions.

For a nonuniform beam, new assumptions must be made concerning the method of lumping the mass and stiffness. Again we assume uniform cell size (h=1) and end the beam at half-integer stations. The total mass of each cell is lumped at its center. Similarly, the total effective bending stiffness of each cell (i.e., between consecutive half-stations) is lumped at its center.

Specifically, the mass lumped at the nth station is (see Fig. 22)

$$m_n = \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} \rho(x)\,dx \tag{B3}$$

Using higher-order differences, we obtain the acceleration at the nth station from

$$\ddot{y}_n = \frac{1}{24 m_n h}(-V_{n-\frac{3}{2}} + 27 V_{n-\frac{1}{2}} - 27 V_{n+\frac{1}{2}} + V_{n+\frac{3}{2}}) \tag{B4}$$

The lumped bending stiffness is

$$\frac{1}{(EI)_n} = \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} \frac{dx}{EI(x)} \tag{B5}$$

resulting in the bending moment equation

$$M_n = \frac{(EI)n}{24h}(\theta_{n-\frac{3}{2}} - 27\theta_{n-\frac{1}{2}} + 27\theta_{n+\frac{1}{2}} - \theta_{n+\frac{3}{2}}) \tag{B6}$$

The slope and shear force equations remain unaltered from the case of a uniform beam.

In general, the boundary conditions are imposed in the same manner as for a uniform beam. An exception occurs in the passive circuit case where the effective mass m_2 near the built-in end includes the reflected value (as seen from station 2) of the actual mass m_1. A similar reflection of bending stiffness occurs near a free end.

Also, it should be noted that the constraint equation

$$y_1 = \frac{2}{9}y_2 - \frac{1}{25}y_3 \tag{130}$$

is of order h^4 for the case of a nonuniform beam instead of order h^6 as was the case for the uniform beam.

RESULTS

Error curves for the tapered beam are shown in Figs. 23-26. Comparing these results with those for the uniform beam, one finds the various methods retain many of their characteristics in spite of a general reduction of accuracy. The h^2 method shows good accuracy for the fundamental mode while the higher-order methods (the symmetry and passive circuit methods, in particular) have better accuracy on the higher modes. The h^4 method suffers the largest loss of accuracy due the nonuniform mass and stiffness distributions.

The general lowering of mode frequencies compared to the uniform beam is probably due primarily to the mass lumping method which results in a slight positive shift in the center of mass location compared to the continuous beam. This is because the mass is lumped at the center of the cell whereas the actual center of mass for the cell is located slightly inboard of the center, i.e., toward the built-in end. Similarly, the compliance (inverse stiffness) for a cell is lumped at the center and is thus somewhat inboard of its compliance centroid. This also results in a lowering of the natural frequencies.

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TABLE I

ONE-SIDED APPROXIMATIONS OF ORDER h^4

A. General Case

$$\frac{\partial y}{\partial x}\bigg|_{\frac{1}{2}} \cong \frac{1}{24h}[-93y_1 + 229y_2 - 225y_3 + 111y_4 - 22y_5] + \frac{563}{640}h^4 \frac{\partial^5 y}{\partial x^5}\bigg|_{\frac{1}{2}}$$

$$\frac{\partial y}{\partial x}\bigg|_{\frac{3}{2}} \cong \frac{1}{24h}[-22y_1 + 17y_2 + 9y_3 - 5y_4 + y_5] - \frac{71}{1920}h^4 \frac{\partial^5 y}{\partial x^5}\bigg|_{\frac{3}{2}}$$

$$\frac{\partial \theta}{\partial x}\bigg|_1 \cong \frac{1}{24h}[-22\theta_{\frac{1}{2}} + 17\theta_{\frac{3}{2}} + 9\theta_{\frac{5}{2}} - 5\theta_{\frac{7}{2}} + \theta_{\frac{9}{2}}] - \frac{71}{1920}h^4 \frac{\partial^5 \theta}{\partial x^5}\bigg|_1$$

B. For $y_{\frac{1}{2}} = 0$

$$\frac{\partial y}{\partial x}\bigg|_{\frac{1}{2}} \cong \frac{1}{840h}[3675y_1 - 1225y_2 + 441y_3 - 75y_4] + \frac{7}{128}h^4 \frac{\partial^5 y}{\partial x^5}\bigg|_{\frac{1}{2}}$$

$$\frac{\partial y}{\partial x}\bigg|_{\frac{3}{2}} \cong \frac{1}{840h}[-1085y_1 + 1015y_2 - 63y_3 + 5y_4] + \frac{1}{1920}h^4 \frac{\partial^5 y}{\partial x^5}\bigg|_{\frac{3}{2}}$$

C. For $y_{\frac{1}{2}} = 0$, $\theta_{\frac{1}{2}} = 0$

$$y_1 \cong \frac{2}{9}y_2 - \frac{1}{25}y_3 + \frac{1}{48}h^4 \frac{\partial^4 y}{\partial x^4}\bigg|_{\frac{1}{2}} + \frac{3}{160}h^5 \frac{\partial^5 y}{\partial x^5}\bigg|_{\frac{1}{2}} + \frac{29}{2880}h^6 \frac{\partial^6 y}{\partial x^6}\bigg|_{\frac{1}{2}}$$

$$\frac{\partial y}{\partial x}\bigg|_{\frac{3}{2}} \cong \frac{1}{225h}[-225y_1 + 250y_2 - 9y_3] + \frac{1}{240}h^4 \frac{\partial^5 y}{\partial x^5}\bigg|_{\frac{3}{2}}$$

TABLE I (Continued)

D.   For $\left. \dfrac{\partial \theta}{\partial x} \right|_{\frac{1}{2}} = 0$

$$\left. \frac{\partial \theta}{\partial x} \right|_1 \cong \frac{1}{72h} \left[-91\theta_{\frac{1}{2}} + 99\theta_{\frac{3}{2}} - 9\theta_{\frac{5}{2}} + \theta_{\frac{7}{2}} \right] - \frac{7}{1920} h^4 \left. \frac{\partial^5 \theta}{\partial x^5} \right|_1$$

D
9
6
4

TABLE II

MODE FREQUENCIES OF A BUILT-IN OR FREE UNIFORM BEAM*

N	Mode	Continuous	Approximation Method			
			h^2	h^4	Symmetry	Passive Circuit
3	1	22.373	- 1.466% 22.045		3.831% 23.230	13.565% 25.408
4	1	22.373	1.135% 22.627		1.850% 22.787	4.863% 23.461
4	2	61.673	-17.961% 50.596		- 1.516% 60.738	1.527% 62.615
5	1	22.373	1.256% 22.654	0.796% 22.551	1.345% 22.674	2.700% 22.977
5	2	61.673	- 9.357% 55.902	-11.052% 54.857	0.028% 61.690	3.259% 63.683
5	3	120.90	-27.833% 87.246	-24.607% 91.155	- 8.313% 110.85	- 6.907% 112.55
6	1	22.373	1.059% 22.610	0.590% 22.505	1.095% 22.618	1.636% 22.739
6	2	61.673	- 5.609% 58.214	- 3.755% 59.357	0.328% 61.875	2.679% 63.325
6	3	120.90	-17.883% 99.281	-17.436% 99.820	- 3.201% 117.03	- 0.935% 119.77
6	4	199.86	-34.099% 131.71	-23.131% 153.63	-14.000% 171.88	-13.274% 173.33

*The percentages refer to the error in frequency as compared to the continuous beam.

TABLE II (Continued)

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N	Mode	Continuous	Approximation Method			
			h^2	h^4	Symmetry	Passive Circuit
7	1	22.373	0.858% 22.565	0.340% 22.449	0.912% 22.577	1.064% 22.611
7	2	61.673	- 3.707% 59.387	- 0.751% 61.210	0.460% 61.957	2.006% 62.910
7	3	120.90	-12.333% 105.99	- 9.992% 108.82	- 1.472% 119.12	0.711% 121.76
7	4	199.86	-24.307% 151.28	-19.494% 160.90	- 6.915% 186.04	- 5.424% 189.02
7	5	298.56	-38.354% 184.05	-23.617% 228.05	-18.412% 243.59	-18.000% 244.82
8	1	22.373	0.697% 22.529	0.192% 22.416	0.764% 22.544	0.729% 22.536
8	2	61.673	- 2.627% 60.053	0.091% 61.729	0.522% 61.995	1.493% 62.594
8	3	120.90	- 8.983% 110.04	- 4.806% 115.09	- 0.695% 120.06	1.092% 122.22
8	4	199.86	-18.063% 163.76	-14.140% 171.60	- 3.823% 192.22	- 2.086% 195.69
8	5	298.56	-29.163% 211.49	-20.117% 238.50	-10.480% 267.27	- 9.489% 270.23
8	6	416.99	-41.411% 244.31	-24.933% 313.02	-21.823% 325.99	-21.569% 327.05

TABLE II (Continued)

N	Mode	Continuous	Approximation Method			
			h²	h⁴	Symmetry	Passive Circuit
9	1	22.373	0.572% 22.501	0.10⁻% 22.39⁻	0.653% 22.519	0.523% 22.490
9	2	61.673	- 1.959% 60.465	0.24⁻% 61.82⁻	0.537% 62.004	1.129% 62.369
9	3	120.90	- 6.832% 112.64	- 2.026% 118.45	- 0.281% 120.56	1.092% 122.22
9	4	199.86	-13.905% 172.07	- 9.031% 181.81	- 2.272% 195.32	- 0.620% 198.62
9	5	298.56	-22.773% 230.57	-16.342% 249.77	- 6.391% 279.48	- 5.078% 283.40
9	6	416.99	-32.907% 279.77	-20.665% 330.82	-13.669% 359.99	-12.996% 362.80
9	7	555.17	-43.707% 312.52	-26.471% 408.21	-24.504% 419.13	-24.337% 420.06
10	1	22.373	0.474% 22.479	0.063% 22.387	0.550% 22.496	0.384% 22.459
10	2	61.673	- 1.518% 60.737	0.217% 61.807	0.525% 61.997	0.866% 62.207
10	3	120.90	- 5.368% 114.41	- 0.753% 119.99	- 0.033% 120.86	0.976% 122.08
10	4	199.86	-11.018% 177.84	- 5.244% 189.38	- 1.401% 197.06	0.030% 199.92
10	5	298.56	-18.214% 244.18	-12.165% 262.24	- 4.123% 286.25	- 2.726% 290.42
10	6	416.99	-26.629% 305.95	-17.506% 343.99	- 8.947% 379.68	- 7.959% 383.80
10	7	555.17	-35.865% 356.06	-21.478% 435.93	-16.449% 463.85	-15.972% 466.50
10	8	713.08	-45.493% 388.68	-27.959% 513.71	-26.649% 523.05	-26.534% 523.87

TABLE II (Continued)

N	Mode	Continuous	Approximation Method			
			h²	h⁴	Symmetry	Passive Circuit
11	1	22.373	0.402% 22.463	0.035% 22.381	0.478% 22.480	0.295% 22.439
11	2	61.673	- 1.213% 60.925	0.162% 61.773	0.499% 61.981	0.678% 62.091
11	3	120.90	- 4.334% 115.66	- 0.232% 120.62	0.116% 121.04	0.844% 121.92
11	4	199.86	- 8.936% 182.00	- 2.867% 194.13	- 0.866% 198.13	0.314% 200.49
11	5	298.56	-14.871% 254.16	- 8.377% 273.55	- 2.767% 290.30	- 1.440% 294.26
11	6	416.99	-21.919% 325.59	-14.185% 357.84	- 6.098% 391.56	- 4.959% 396.31
11	7	555.17	-29.812% 389.66	-18.328% 453.42	-11.366% 492.07	-10.620% 496.21
11	8	713.08	-38.248% 440.34	-22.501% 552.63	-18.844% 478.71	-18.499% 581.17
11	9	890.73	-46.918% 472.82	-29.311% 629.65	-28.398% 637.78	-28.317% 638.50

TABLE II (Continued)

N	Mode	Continuous	Approximation Method			
			h²	h⁴	Symmetry	Passive Circuit
12	1	22.373	0.340% 22.449	0.022% 22.373	0.416% 22.466	0.228% 22.424
12	2	61.673	- 0.992% 61.061	0.114% 61.743	0.467% 61.961	0.538% 62.005
12	3	120.90	- 3.573% 116.58	- 0.025% 120.87	0.199% 121.14	0.720% 121.77
12	4	199.86	- 7.395% 185.08	- 1.516% 196.83	- 0.520% 198.82	0.430% 200.72
12	5	298.56	-12.363% 261.65	- 5.446% 282.30	- 1.906% 292.87	- 0.717% 296.42
12	6	416.99	-18.322% 340.59	-10.856% 371.72	- 4.295% 399.08	- 3.132% 403.93
12	7	555.17	-25.097% 415.84	-15.494% 469.15	- 8.086% 510.28	- 7.171% 515.36
12	8	713.08	-32.468% 481.56	-19.124% 576.71	-13.590% 616.17	-13.021% 620.23
12	9	890.73	-40.206% 532.60	-23.619% 680.35	-20.910% 704.48	-20.653% 706.77
12	10	1088.1	-48.084% 564.93	-30.508% 756.16	-29.850% 763.32	-29.786% 763.98

TABLE II (Continued)

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N	Mode	Continuous	Approximation Method			
			h²	h⁴	Symmetry	Passive Circuit
13	1	22.373	0.295% 22.439	0.013% 22.376	0.362% 22.454	0.179% 22.413
13	2	61.673	- 0.827% 61.163	0.078% 61.721	0.431% 61.939	0.435% 61.941
13	3	120.90	- 2.994% 117.28	0.041% 120.95	0.248% 121.20	0.604% 121.63
13	4	199.86	- 6.219% 187.43	- 0.791% 198.28	- 0.285% 199.29	0.460% 200.78
13	5	298.56	-10.430% 267.42	- 3.423% 288.34	- 1.330% 294.59	- 0.301% 297.66
13	6	416.99	-15.526% 352.25	- 7.897% 384.06	- 3.103% 404.05	- 1.990% 408.69
13	7	555.17	-21.379% 436.48	-12.656% 484.91	- 5.904% 522.39	- 4.914% 527.89
13	8	713.08	-27.830% 514.63	-16.455% 595.74	-10.014% 641.67	- 9.278% 646.92
13	9	890.73	-34.710% 581.56	-19.988% 712.69	-15.612% 751.67	-15.170% 755.61
13	10	1088.1	-41.841% 632.83	-24.740% 818.86	-22.700% 841.13	-22.498% 843.26
13	11	1305.3	-49.054% 665.02	-31.564% 893.32	-31.073% 899.71	-31.027% 900.30

TABLE II (Continued)

N	Mode	Continuous	Approximation Method			
			h²	h⁴	Symmetry	Passive Circuit
14	1	22.373	0.255% 22.430	0.C09% 22.375	0.322% 22.445	0.147% 22.406
14	2	61.673	- 0.700% 61.241	0.C52% 61.705	0.397% 61.918	0.355% 61.892
14	3	120.90	- 2.548% 117.82	0.C58% 120.97	0.281% 121.24	0.521% 121.53
14	4	199.86	- 5.309% 189.25	- 0.110% 199.04	- 0.130% 199.60	0.440% 200.76
14	5	298.56	- 8.919% 271.93	- 2.124% 292.22	- 0.931% 295.79	- 0.060% 298.38
14	6	416.99	-13.314% 361.47	- 5.537% 393.90	- 2.281% 407.48	- 1.266% 411.71
14	7	555.17	-18.407% 452.98	- 9.936% 500.01	- 4.408% 530.70	- 3.412% 536.23
14	8	713.08	-24.074% 541.41	-13.955% 613.57	- 7.531% 659.38	- 6.696% 665.33
14	9	890.73	-30.191% 621.81	-17.294% 736.69	-11.844% 785.23	-11.249% 790.53
14	10	1088.1	-36.623% 689.63	-20.908% 860.61	-17.434% 898.40	-17.085% 902.19
14	11	1305.3	-43.231% 741.04	-25.837% 968.06	-24.263% 988.64	-24.107% 990.63
14	12	1542.1	-49.867% 773.10	-32.182% 1041.8	-32.112% 1046.9	-32.073% 1047.5

TABLE II (Continued)

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N	Mode	Continuous	Approximation Method			
			h^2	h^4	Symmetry	Passive Circuit
15	1	22.373	0.223% 22.423	0.000% 22.373	0.282% 22.436	0.116% 22.399
15	2	61.673	- 0.602% 61.302	0.034% 61.694	0.365% 61.898	0.293% 61.854
15	3	120.90	- 2.200% 118.24	0.058% 120.97	0.289% 121.25	0.438% 121.43
15	4	199.86	- 4.578% 190.71	- 0.215% 199.43	- 0.020% 199.82	0.425% 200.71
15	5	298.56	- 7.714% 275.53	- 1.320% 294.62	- 0.646% 296.63	0.077% 298.79
15	6	416.99	-11.540% 368.87	- 3.801% 401.14	- 1.698% 409.91	- 0.794% 413.68
15	7	555.17	-16.001% 466.34	- 7.536% 513.33	- 3.347% 536.59	- 2.392% 541.89
15	8	713.08	-21.005% 563.30	-11.526% 630.89	- 5.765% 671.97	- 4.891% 678.20
15	9	890.73	-26.455% 655.09	-14.961% 757.47	- 9.128% 809.42	- 8.429% 815.65
15	10	1088.1	-32.249% 737.24	-18.114% 891.04	-13.556% 940.62	-13.069% 945.89
15	11	1305.3	-38.275% 805.74	-21.857% 1020.0	-19.084% 1056.2	-18.800% 1059.9
15	12	1542.1	-44.413% 857.23	-26.859% 1127.9	-25.621% 1147.0	-25.498% 1148.9
15	13	1798.7	-50.564% 889.17	-33.296% 1199.8	-33.007% 1205.0	-32.979% 1205.5

TABLE II (Concluded)

N	Mode	Continuous	Approximation Method			
			h^2	h^4	Symmetry	Passive Circuit
16	1	22.373	0.197% 22.417	0.000% 22.373	0.250% 22.429	0.098% 22.395
16	2	61.673	- 0.524% 61.350	0.023% 61.687	0.334% 61.879	0.245% 61.824
16	3	120.90	- 1.911% 118.59	0.041% 120.95	0.289% 121.25	0.380% 121.36
16	4	199.86	- 3.993% 191.88	- 0.110% 199.64	0.050% 199.96	0.390% 200.64
16	5	298.56	- 6.736% 278.45	- 0.827% 296.09	- 0.439% 297.25	0.151% 299.01
16	6	416.99	-10.096% 374.89	- 2.583% 406.20	- 1.273% 411.68	- 0.484% 414.97
16	7	555.17	-14.030% 477.28	- 5.575% 524.22	- 2.576% 540.87	- 1.690% 545.79
16	8	713.08	-18.471% 581.37	- 9.254% 647.09	- 4.483% 681.11	- 3.612% 687.32
16	9	890.73	-23.343% 682.81	-12.761% 777.06	- 7.141% 827.12	- 6.382% 833.88
16	10	1088.1	-28.562% 777.32	-15.817% 915.96	-10.671% 971.99	-10.082% 978.39
16	11	1305.3	-34.046% 860.86	-18.946% 1058.0	-15.146% 1107.6	-14.748% 1112.8
16	12	1542.1	-39.700% 929.88	-22.800% 1190.5	-20.556% 1225.1	-20.329% 1228.6
16	13	1798.7	-45.438% 981.40	-27.809% 1298.5	-26.825% 1316.2	-26.725% 1318.0
16	14	2075.1	-51.173% 1013.2	-34.018% 1369.2	-33.786% 1374.0	-33.762% 1374.5

TABLE III

MODE FREQUENCIES OF A UNIFORM CANTILEVER BEAM

N	Mode	Continuous	Approximation Method			
			h^2	h^4	Symmetry	Passive Circuit
2	1	3.5160	13.766% 4.0000		3.851% 3.6514	4.209% 3.6640
3	1	3.5160	6.030% 3.728		1.812% 3.5797	1.257% 3.5602
3	2	22.034	- 1.389% 21.728		3.980% 22.911	10.674% 24.386
4	1	3.5160	3.356% 3.634		1.135% 3.5559	0.549% 3.5353
4	2	22.034	1.035% 22.262		1.938% 22.461	5.328% 23.208
4	3	61.697	-17.941% 50.628		- 1.520% 60.759	1.483% 62.612
5	1	3.5160	2.142% 3.5913	0.060% 3.5181	0.774% 3.5432	0.284% 3.5260
5	2	22.034	1.153% 22.288	0.880% 22.228	1.402% 22.343	2.927% 22.679
5	3	61.697	- 9.333% 55.939	-11.665% 54.500	0.023% 61.711	3.232% 63.691
5	4	120.90	-27.841% 87.242	-23.956% 91.937	- 8.313% 110.85	- 6.898% 112.56

TABLE III (Continued)

N	Mode	Continuous	Approximation Method			
			h²	h⁴	Symmetry	Passive Circuit
6	1	3.5160	1.4818% 3.5681	0.028% 3.5170	0.555% 3.5355	0.168% 3.5219
6	2	22.034	0.976% 22.249	0.613% 22.169	1.135% 22.284	1.770% 22.424
6	3	61.697	- 5.589% 58.249	- 3.827% 59.336	0.324% 61.897	2.663% 63.340
6	4	120.90	-17.885% 99.277	-17.143% 99.811	- 3.201% 117.03	- 0.935% 119.77
6	5	199.86	-34.099% 131.71	-23.101% 153.69	-14.000% 171.88	-13.274% 173.33
7	1	3.5160	1.086% 3.5542	0.017% 3.5166	0.418% 3.5307	0.102% 3.5196
7	2	22.034	0.794% 22.209	0.545% 22.110	0.944% 22.242	1.148% 22.287
7	3	61.697	- 3.691% 59.420	- 0.772% 61.221	0.459% 61.980	1.997% 62.929
7	4	120.90	-12.333% 105.99	- 9.992% 108.82	- 1.472% 119.12	0.711% 121.76
7	5	199.86	-24.307% 151.28	-19.479% 160.93	- 6.915% 186.04	- 5.424% 189.02
7	6	298.56	-38.354% 184.05	-23.520% 228.04	-18.412% 243.59	-18.000% 244.82

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TABLE III (Continued)

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N	Mode	Continuous	Approximation Method			
			h^2	h^4	Symmetry	Passive Circuit
8	1	3.5160	0.828% 3.5451	0.009% 3.5163	0.319% 3.5272	0.063% 3.5182
8	2	22.034	0.644% 22.176	0.191% 22.076	0.790% 22.208	0.785% 22.207
8	3	61.697	- 2.614% 60.084	0.086% 61.750	0.520% 62.018	1.488% 62.615
8	4	120.90	- 8.991% 110.03	- 4.797% 115.10	- 0.695% 120.06	1.092% 122.22
8	5	199.86	-18.063% 163.76	-14.140% 171.60	- 3.823% 192.22	- 2.086% 195.69
8	6	298.56	-29.163% 211.49	-20.117% 238.50	-10.480% 267.27	- 9.489% 270.23
8	7	416.99	-41.411% 244.31	-24.931% 313.03	-21.823% 325.99	-21.569% 327.05
9	1	3.5160	0.648% 3.5388	0.006% 3.5162	0.236% 3.5243	0.043% 3.5175
9	2	22.034	0.531% 22.151	0.104% 22.057	0.667% 22.181	0.558% 22.157
9	3	61.697	- 1.948 60.495	0.242% 61.846	0.534% 62.028	1.130% 62.394
9	4	120.90	- 6.832% 112.64	- 2.026% 118.45	- 0.281% 120.56	1.092% 122.22
9	5	199.86	-13.905% 172.07	- 9.031% 181.81	- 2.272% 195.32	- 0.620% 198.62
9	6	298.56	-22.773% 230.57	-16.342% 249.77	- 6.391% 279.48	- 5.074% 283.41
9	7	416.99	-32.907% 279.77	-20.665% 330.82	-13.669% 359.99	-12.996% 362.80
9	8	555.17	-43.707% 312.52	-26.471% 408.21	-24.504% 419.13	-24.338% 420.05

TABLE III (Continued)

N	Mode	Continuous	Approximation Method			
			h^2	h^4	Symmetry	Passive Circuit
10	1	3.5160	0.509% 3.5339	0.0C3% 3.5161	0.148% 3.5212	0.028% 3.5170
10	2	22.034	0.440% 22.131	0.0€8% 22.049	0.567% 22.159	0.409% 22.124
10	3	61.697	- 1.509% 60.766	0.217% 61.8²1	0.525% 62.021	0.864% 62.230
10	4	120.90	- 5.368% 114.41	- 0.753% 119.99	- 0.033% 120.86	0.976% 122.08
10	5	199.86	-11.018% 177.84	- 5.244% 189.3€	- 1.401% 197.06	0.030% 199.92
10	6	298.56	-18.214% 244.18	-12.165% 262.24	- 4.123% 286.25	- 2.726% 290.42
10	7	416.99	-26.627% 305.96	-17.5C6% 343.99	- 8.947% 379.68	- 7.959% 383.80
10	8	555.17	-35.865% 356.06	-21.473% 435.93	-16.447% 463.86	-15.972% 466.50
10	9	713.08	-45.491% 388.69	-27.953% 513.71	-26.649% 523.05	-26.534% 523.87

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TABLE III (Concluded)

N	Mode	Continuous	Approximation Method			
			h^2	h^4	Symmetry	Passive Circuit
11	1	3.5160	0.395% 3.5299	0.003% 3.5161	0.054% 3.5179	0.023% 3.5168
11	2	22.034	0.368% 22.115	0.050% 22.045	0.486% 22.141	0.309% 22.102
11	3	61.697	- 1.204% 60.954	0.162% 61.797	0.499% 62.005	0.676% 62.114
11	4	120.90	- 4.334% 115.66	- 0.232% 120.62	0.116% 121.04	0.835% 121.91
11	5	199.86	- 8.936% 182.00	- 2.867% 194.13	- 0.866% 198.13	0.315% 200.49
11	6	298.56	-14.871% 254.16	- 8.377% 273.55	- 2.767% 290.30	- 1.444% 294.25
11	7	416.99	-21.919% 325.59	-14.185% 357.84	- 6.096% 391.57	- 4.959% 396.31
11	8	555.17	-29.811% 389.67	-18.328% 453.42	-11.366% 492.07	-10.618% 496.22
11	9	713.08	-38.250% 440.33	-22.501% 552.63	-18.844% 578.71	-18.497% 581.18
11	10	890.73	-46.918% 472.82	-29.311% 629.65	-28.398% 637.78	-28.316% 638.51

TABLE IV

PINNED BEAM, BASIC h^4 APPROXIMATION MODE FREQUENCIES

Mode	Continuous	Number of Cells N														
		2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	9.8696	-4.871%	-1.054%	-0.343%	-0.143%	-0.069%	-0.037%	-0.022%	-0.014%	-0.009%	-0.006%	-0.004%	-0.003%	-0.002%	-0.002%	-0.001%
		9.3889	9.7656	9.8357	9.8555	9.8628	9.8659	9.8674	9.8682	9.8687	9.8690	9.8692	9.8693	9.8694	9.8694	9.8695
2	39.478		-13.440%	-4.869%	-2.115%	-1.054%	-0.578%	-0.342%	-0.215%	-0.142%	-0.096%	-0.068%	-0.051%	-0.035%	-0.030%	-0.020%
			34.172	37.556	38.643	39.062	39.250	39.343	39.393	39.422	39.440	39.451	39.458	39.464	39.466	39.470
3	88.826			-19.759%	-9.363%	-4.870%	-2.747%	-1.656%	-1.053%	-0.700%	-0.483%	-0.343%	-0.251%	-0.187%	-0.143%	-0.110%
				71.275	80.509	84.500	86.386	87.355	87.891	88.204	88.397	88.521	88.603	88.660	88.699	88.728
4	157.91				-24.147%	-13.438%	-7.884%	-4.870%	-3.147%	-2.115%	-1.469%	-1.051%	-0.773%	-0.576%	-0.443%	-0.342%
					119.78	136.69	145.46	150.22	152.94	154.57	155.59	156.25	156.69	157.00	157.21	157.37
5	246.74					-27.296%	-16.892%	-10.789%	-7.133%	-4.872%	-3.425%	-2.472%	-1.824%	-1.374%	-1.054%	-0.819%
						179.39	205.06	220.12	229.14	234.72	238.29	240.64	242.24	243.35	244.14	244.72
6	355.31						-29.642%	-19.760%	-13.442%	-9.364%	-6.679%	-4.872%	-3.625%	-2.750%	-2.119%	-1.658%
							249.99	285.10	307.55	322.04	331.58	338.00	342.43	345.54	347.78	359.42
7	483.61							-31.443%	-22.146%	-15.810%	-11.478%	-8.482%	-6.375%	-4.870%	-3.776%	-2.969%
								331.55	376.51	407.15	428.10	442.59	452.78	460.06	465.35	469.25
8	631.65								-32.869%	-24.149%	-17.905%	-13.441%	-10.226%	-7.884%	-6.158%	-4.870%
									424.03	479.11	518.55	546.75	567.06	581.85	592.75	600.89
9	799.44									-34.028%	-25.847%	-19.759%	-15.244%	-11.885%	-9.364%	-7.456%
										527.41	592.81	641.48	677.57	704.43	724.58	739.83
10	986.96										-34.981%	-27.297%	-21.398%	-16.890%	-13.441%	-10.790%
											641.71	717.55	775.77	820.26	854.30	880.47
11	1194.2											-35.781%	-28.548%	-22.852%	-18.389%	-14.897%
												766.90	853.28	921.31	974.59	1016.3
12	1421.2												-36.462%	-29.637%	-24.149%	-19.758%
													902.98	999.97	1078.0	1140.4
13	1660.0													-37.0%	-30.600%	-25.312%
														1050.0	1157.6	1245.8
14	1934.4														-37.562%	-31.441%
															1207.8	1326.2
15	2220.7															-38.011%
																1376.6

Fig. I. The analog computer circuit for a typical beam cell using approximations of order h^2 to four first-order differential equations.
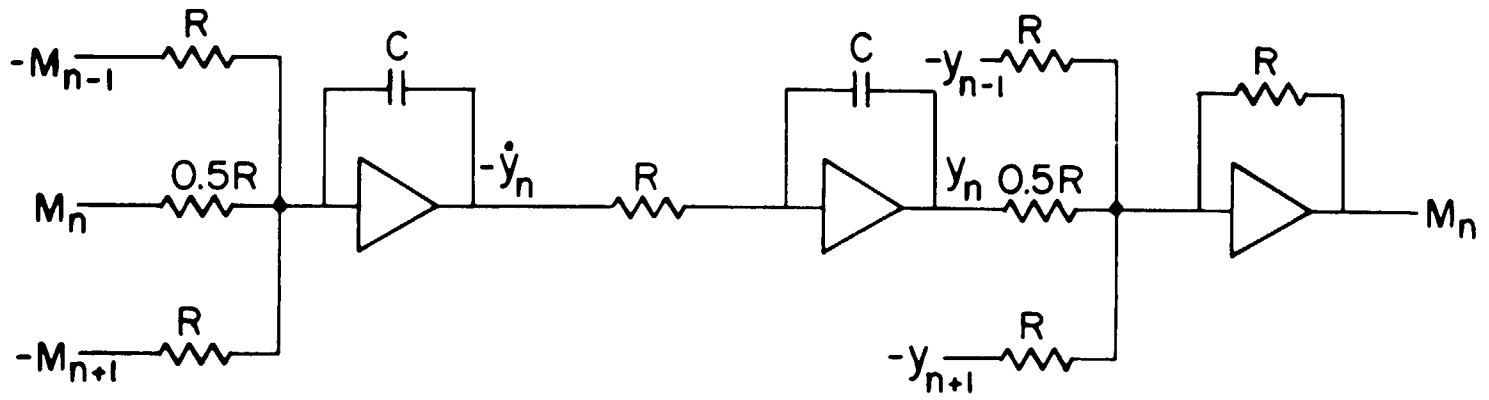
Fig. 2. The analog computer circuit for a typical beam cell using approximations of order h^2 to two second-order differential equations.

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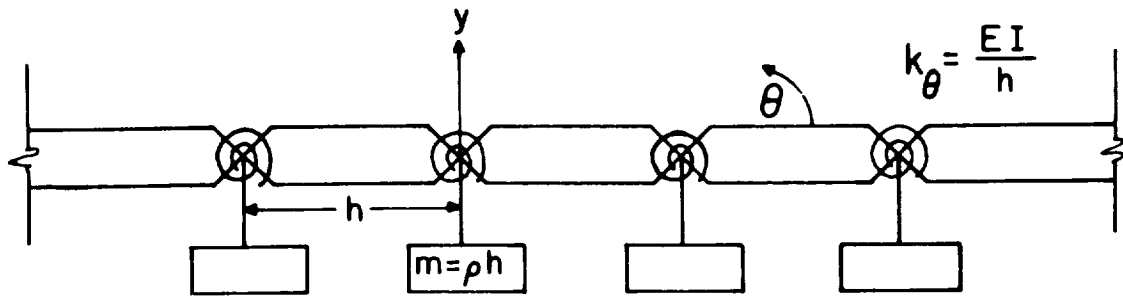
a.   Mechanical   representation   of   the   simple   finite
difference   approximation   to   a   uniform   beam.
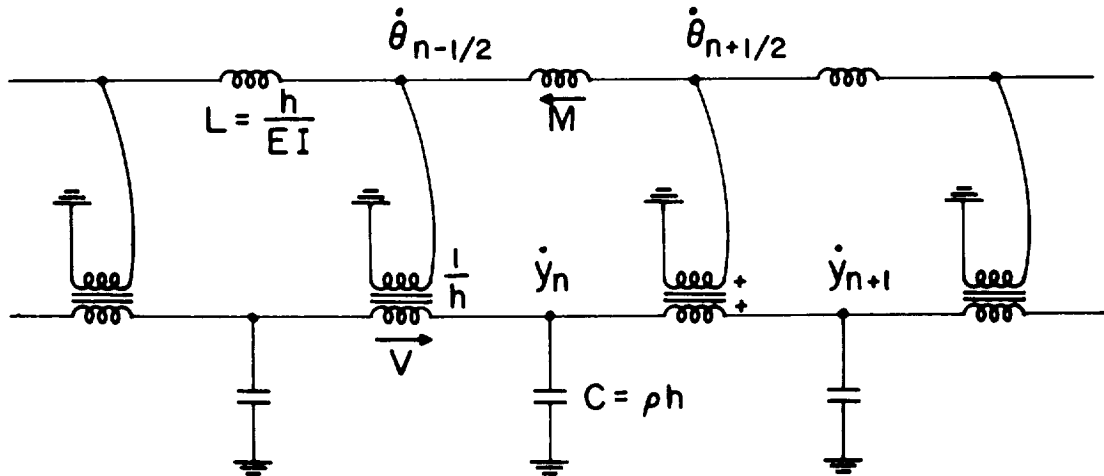
b.   Electrical   representation   of   the   simple   finite
difference   approximation   to   a   uniform   beam.

Fig. 3.
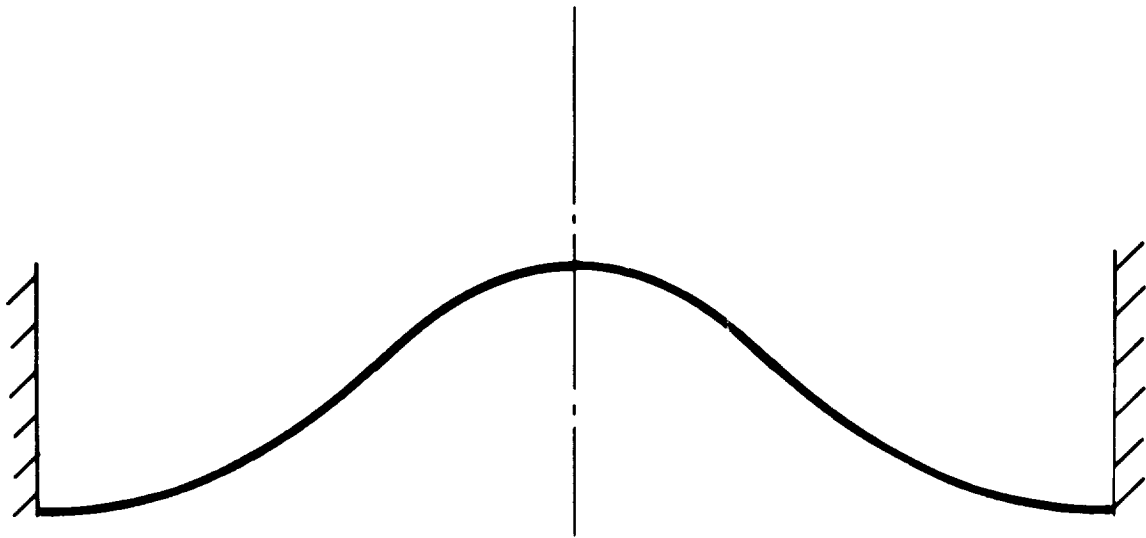
a. Even symmetry about center (symmetric mode).
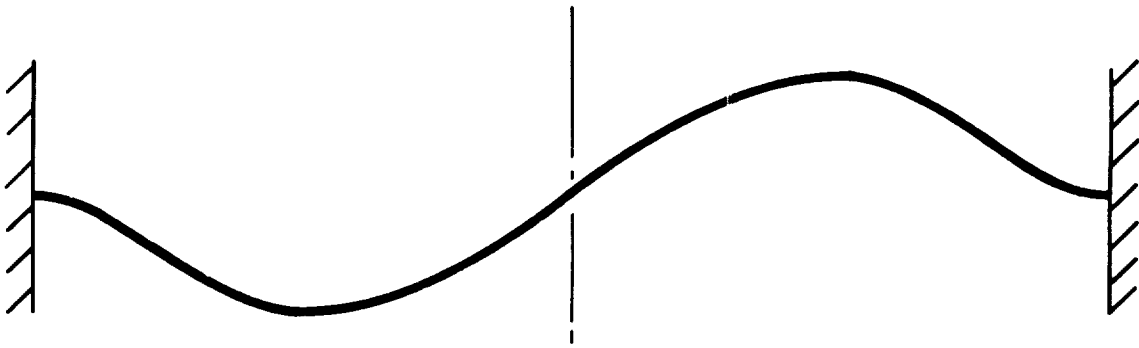
b. Odd symmetry about center (antisymmetric mode).

Fig. 4.
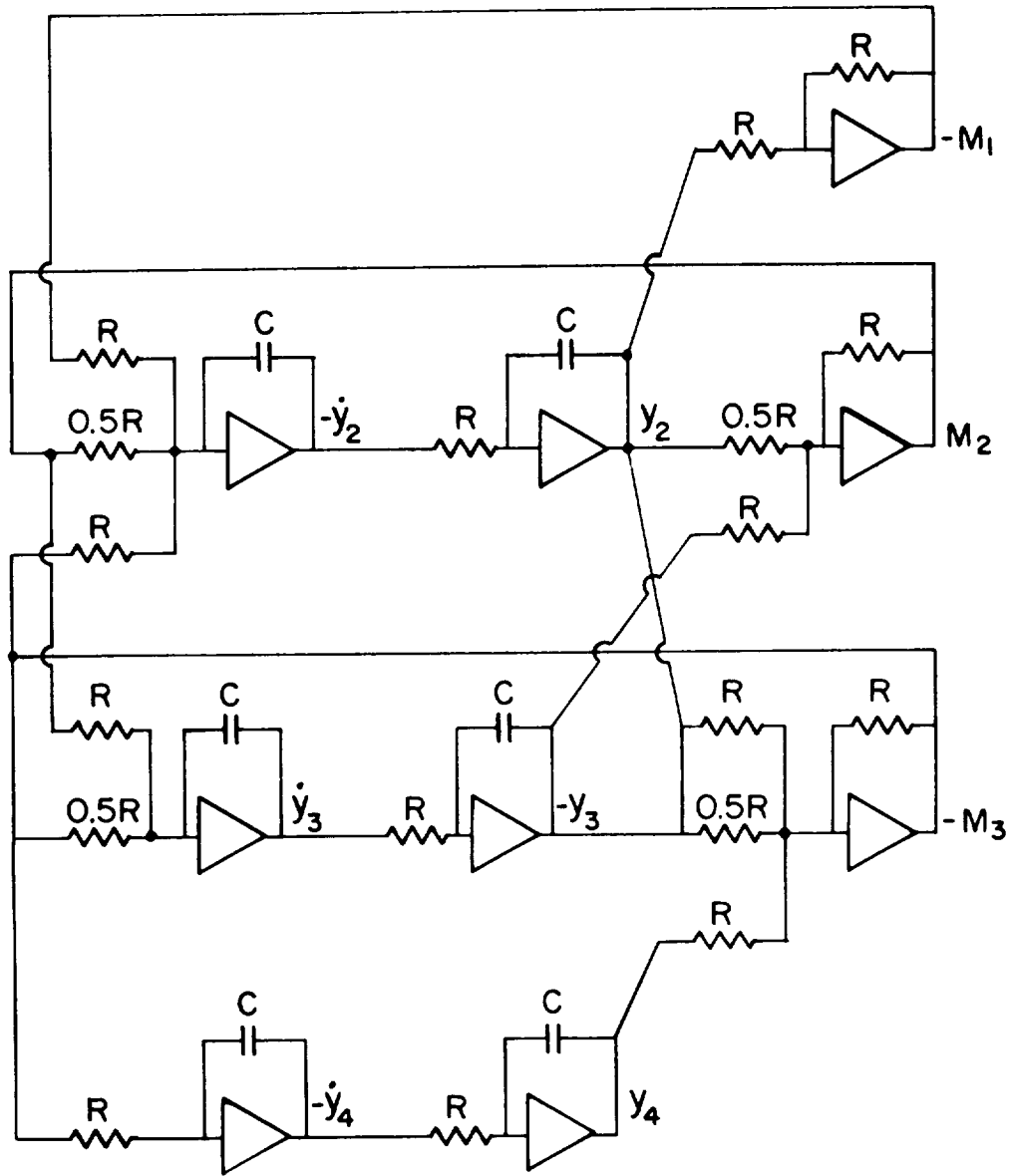
a.   Analog   computer   circuit   for   a   four-cell
cantilever   beam   using   differences   of   order h^2
and   second-order   equations.

Fig. 5.
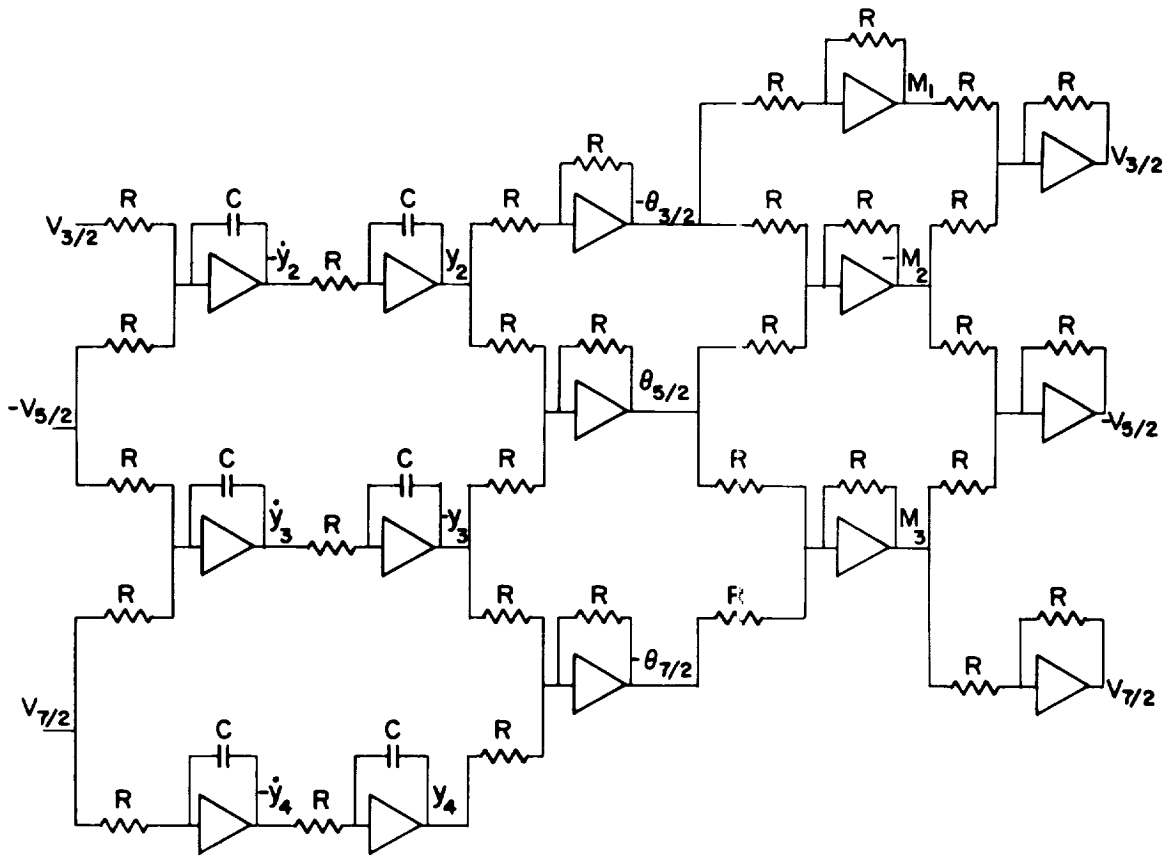
b. Analog computer circuit for a four-cell cantilever beam using differences of order h^2 and first-order equations. The built-in end occurs at station 1/2 and the free end at station 9/2.

Fig. 5.

a. Even symmetry
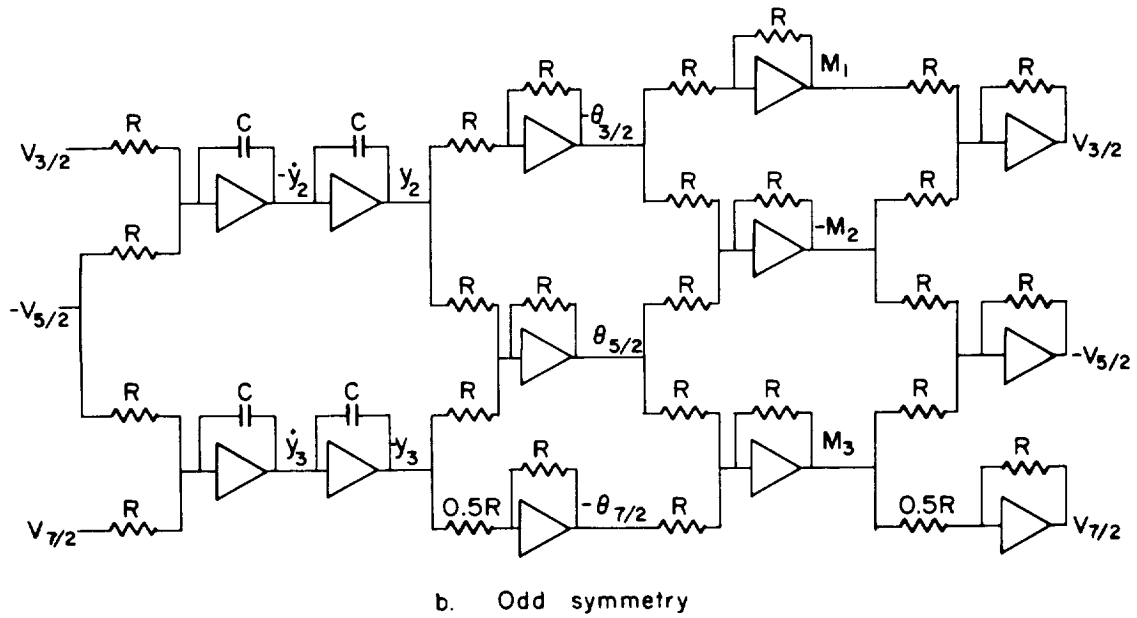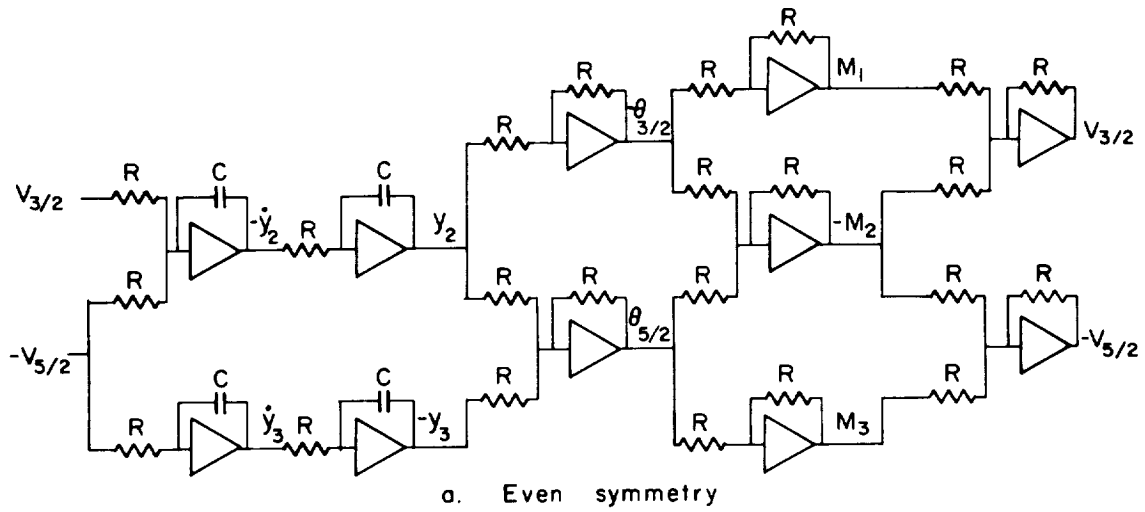
b. Odd symmetry

Fig. 6.   Analog computer circuit for a six-cell beam built-in at both ends.   Differences of order h^2 are used.   Only the left half of the beam is represented.
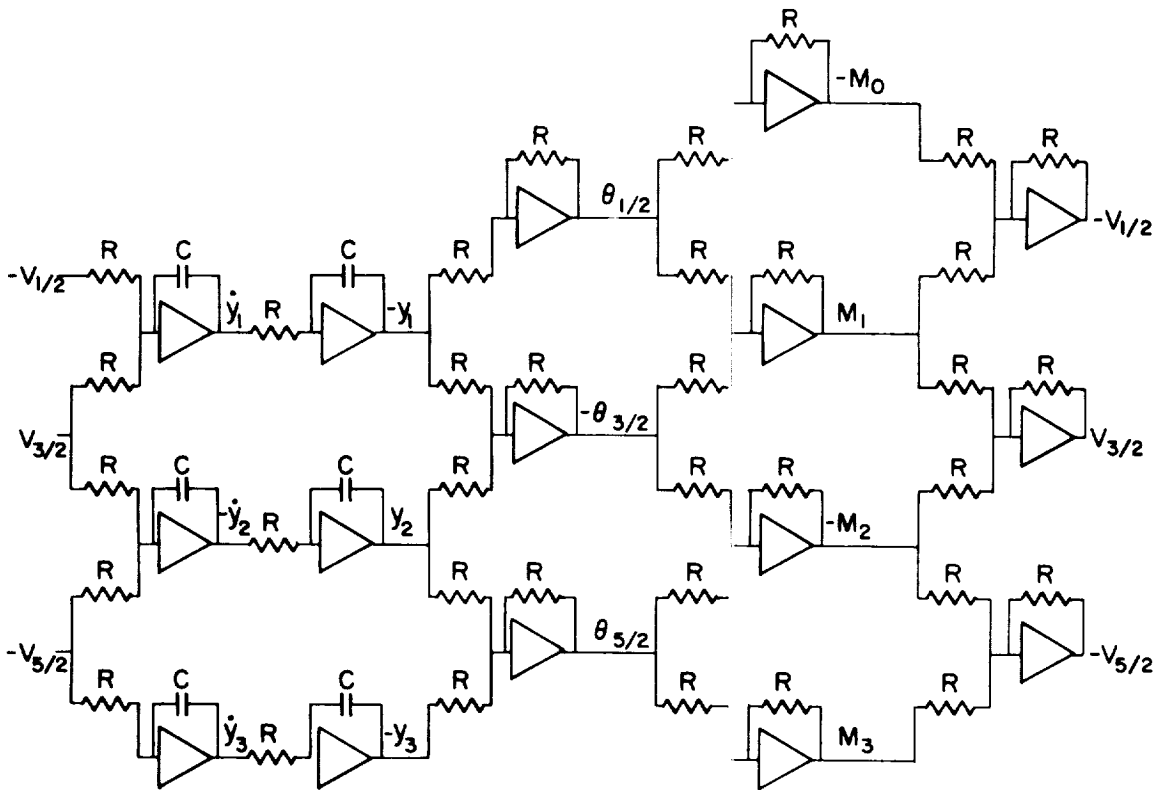
a. Even symmetry
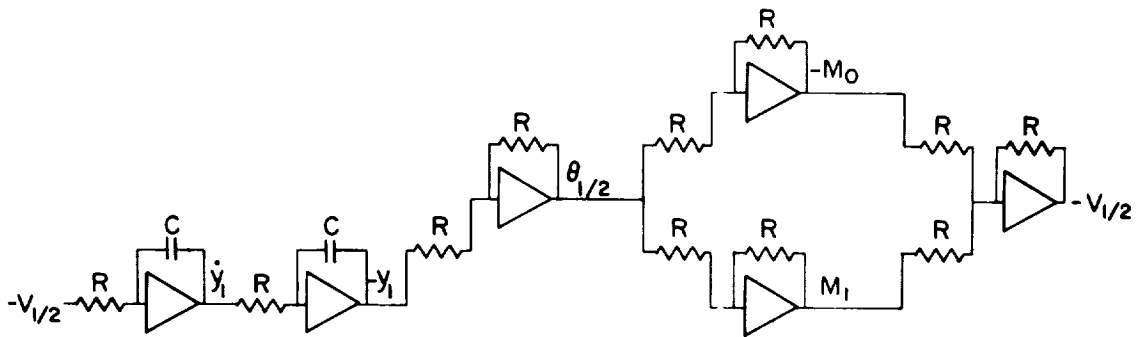
b. Odd symmetry

Fig. 7. Analog computer circuit for a six-cell beam that is pinned at both ends, using differences of order h^2. In (a.) the left half is represented, while in (b.) only the left quarter is represented, due to symmetry.
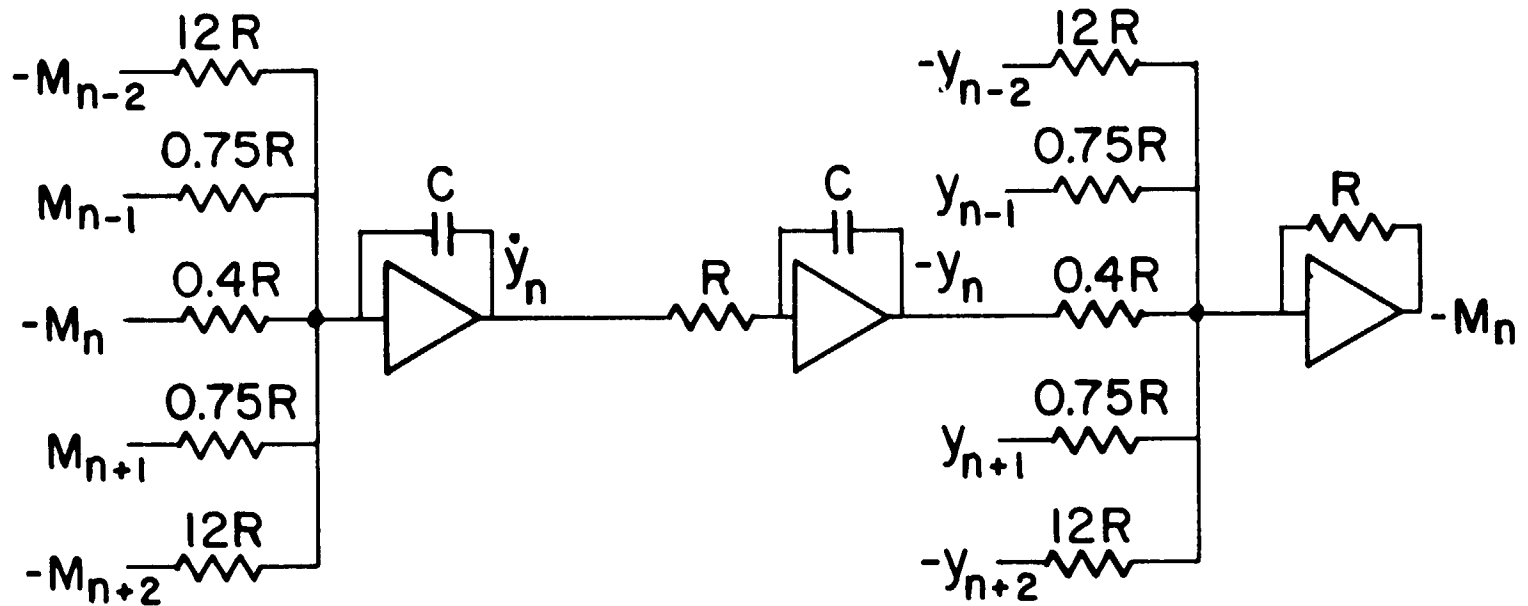
Fig. 8. Analog computer circuit for a typical cell using the central difference approximation of order h^4 to the second derivative in x.
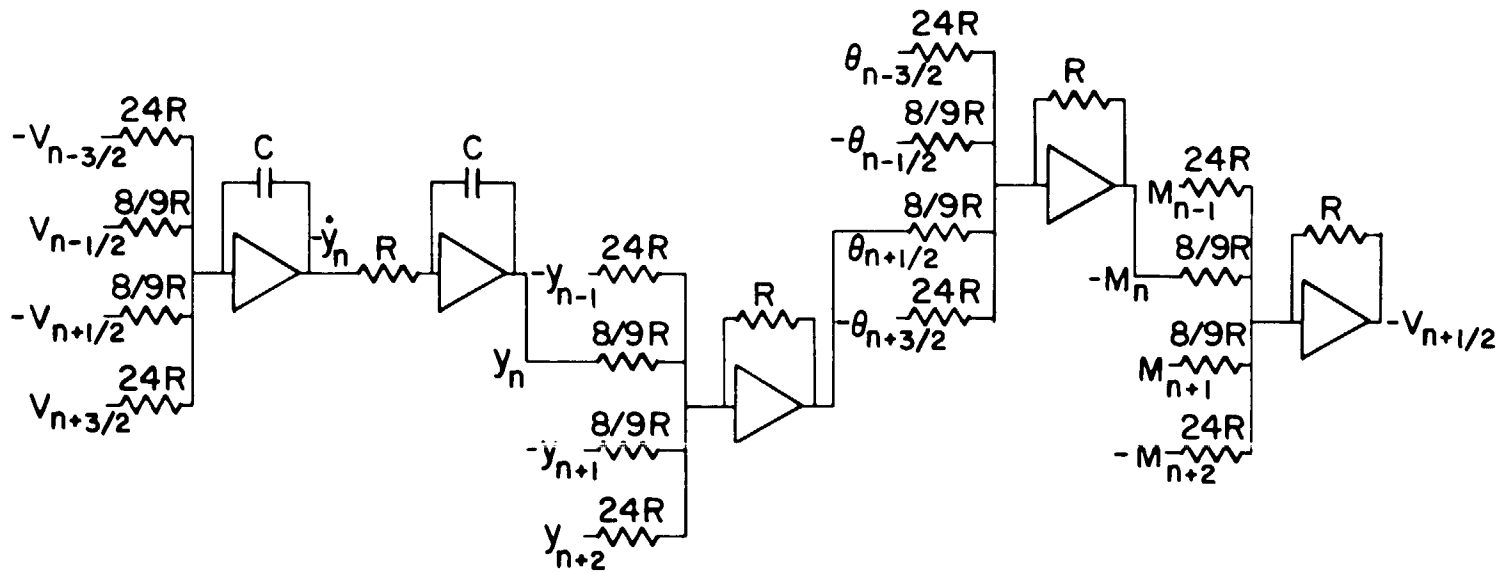
Fig. 9. Analog computer circuit for a typical cell using staggered stations and a central difference approximation of order h^4.
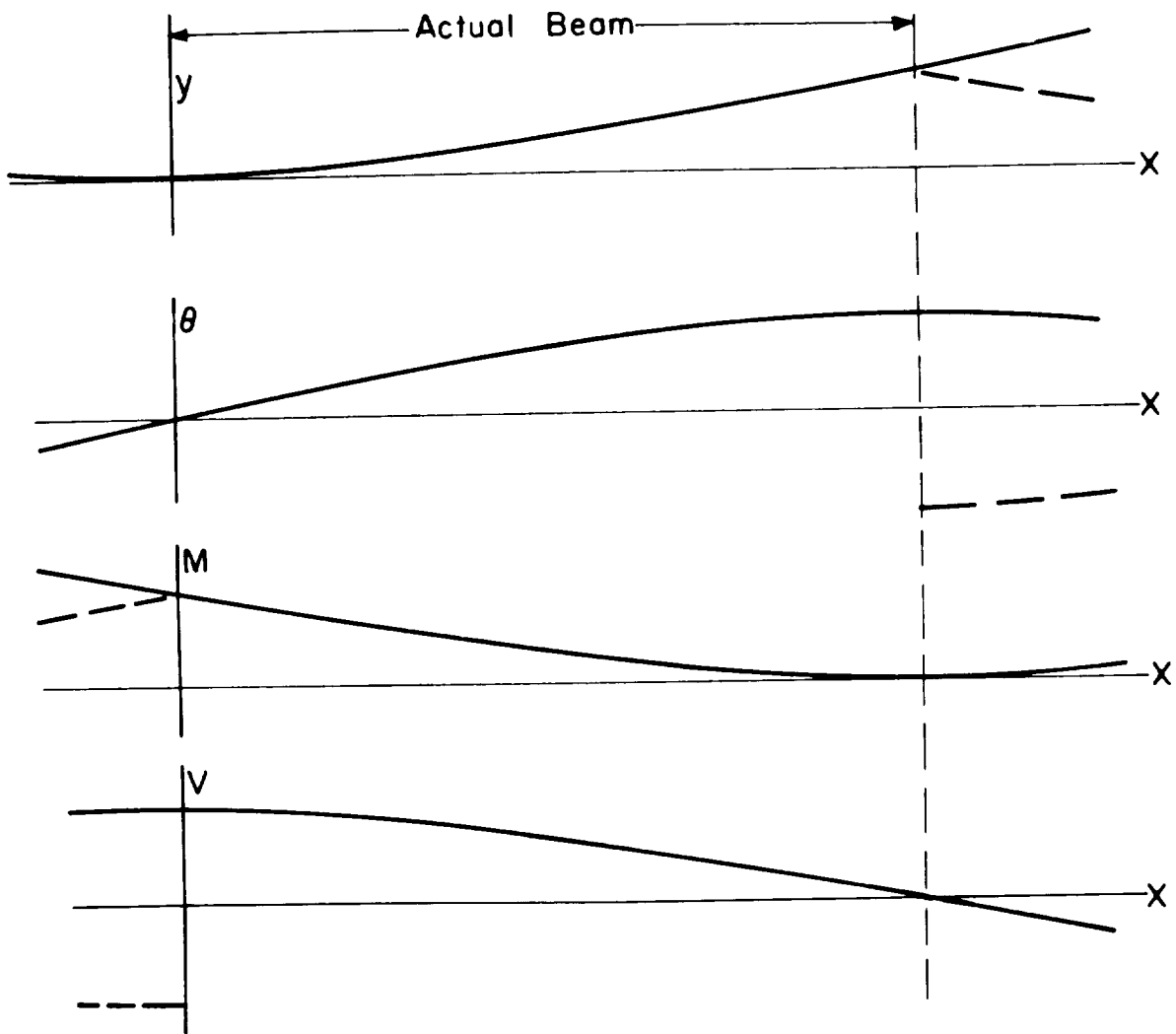
Fig. 10.   Symmetry   assumptions for a cantilever beam.   Solid
lines show assumptions used in computations.   Dashed lines
indicate deviations giving a physically consistent set of
assumptions.
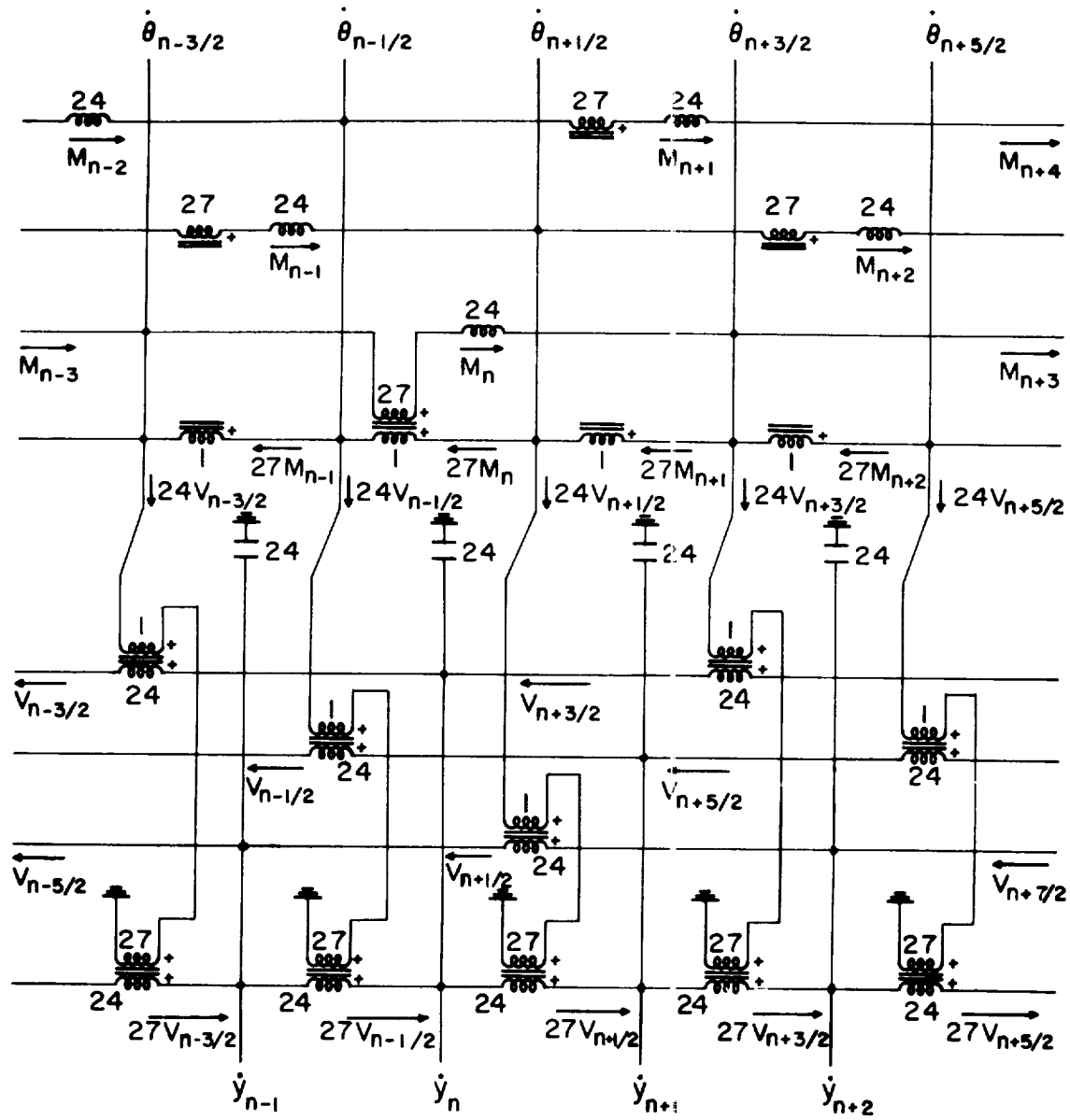
Fig. II. Passive circuit analogy for the basic h^4 approximation.
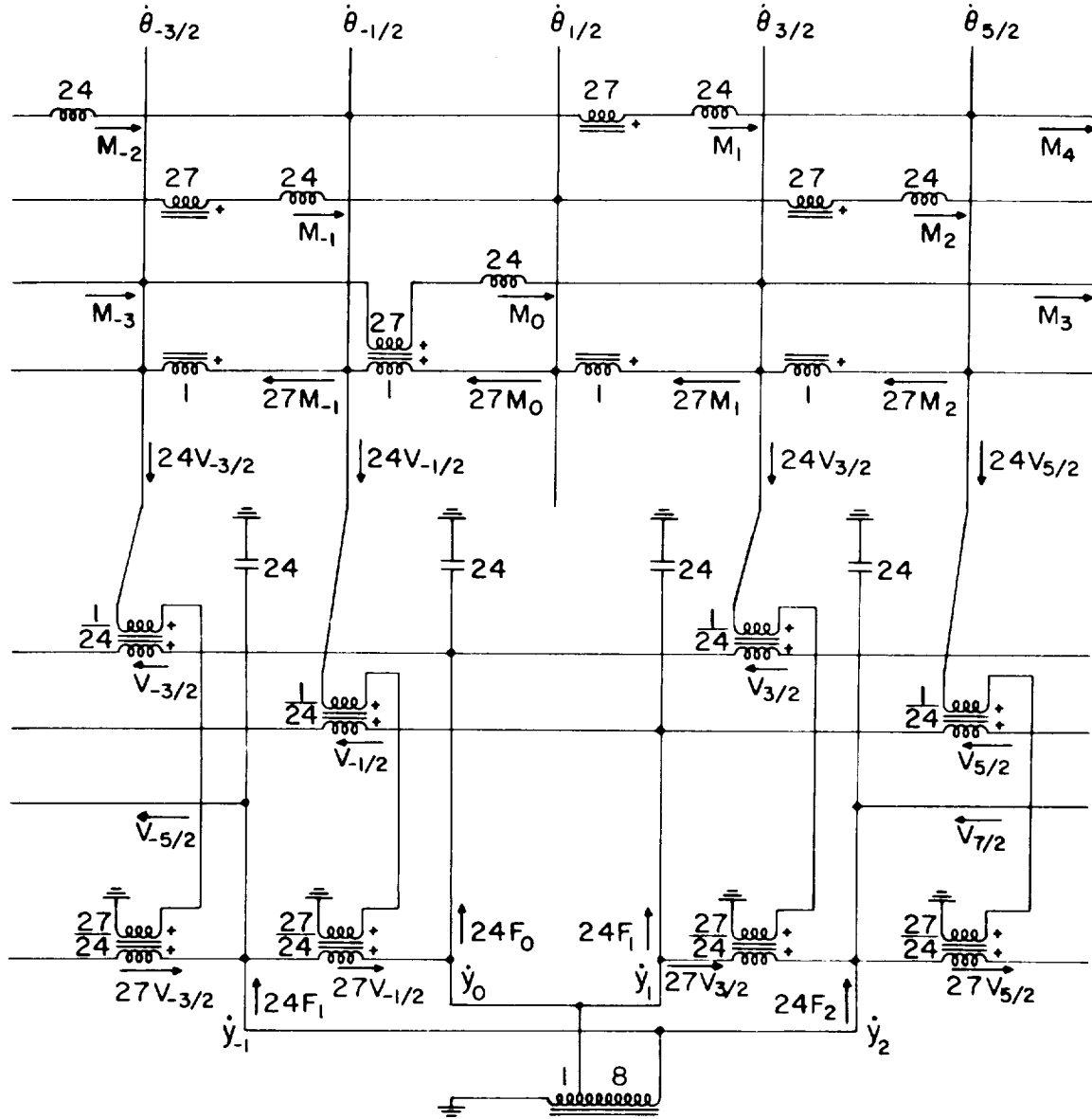
Fig. 12. Passive circuit analogy for a built-in end at station 1/2.
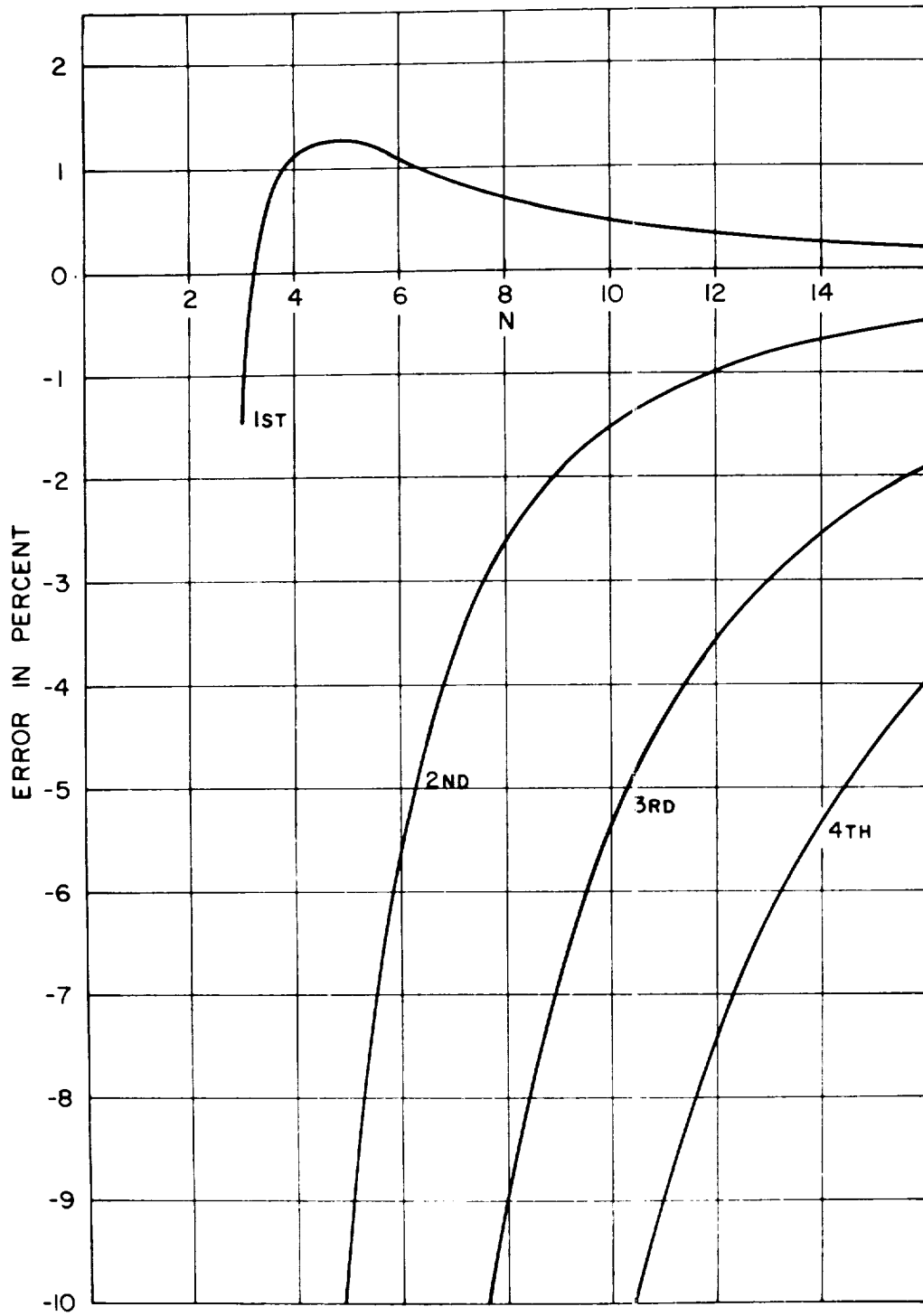
Fig. 13. Built-in, h^2 approximation.
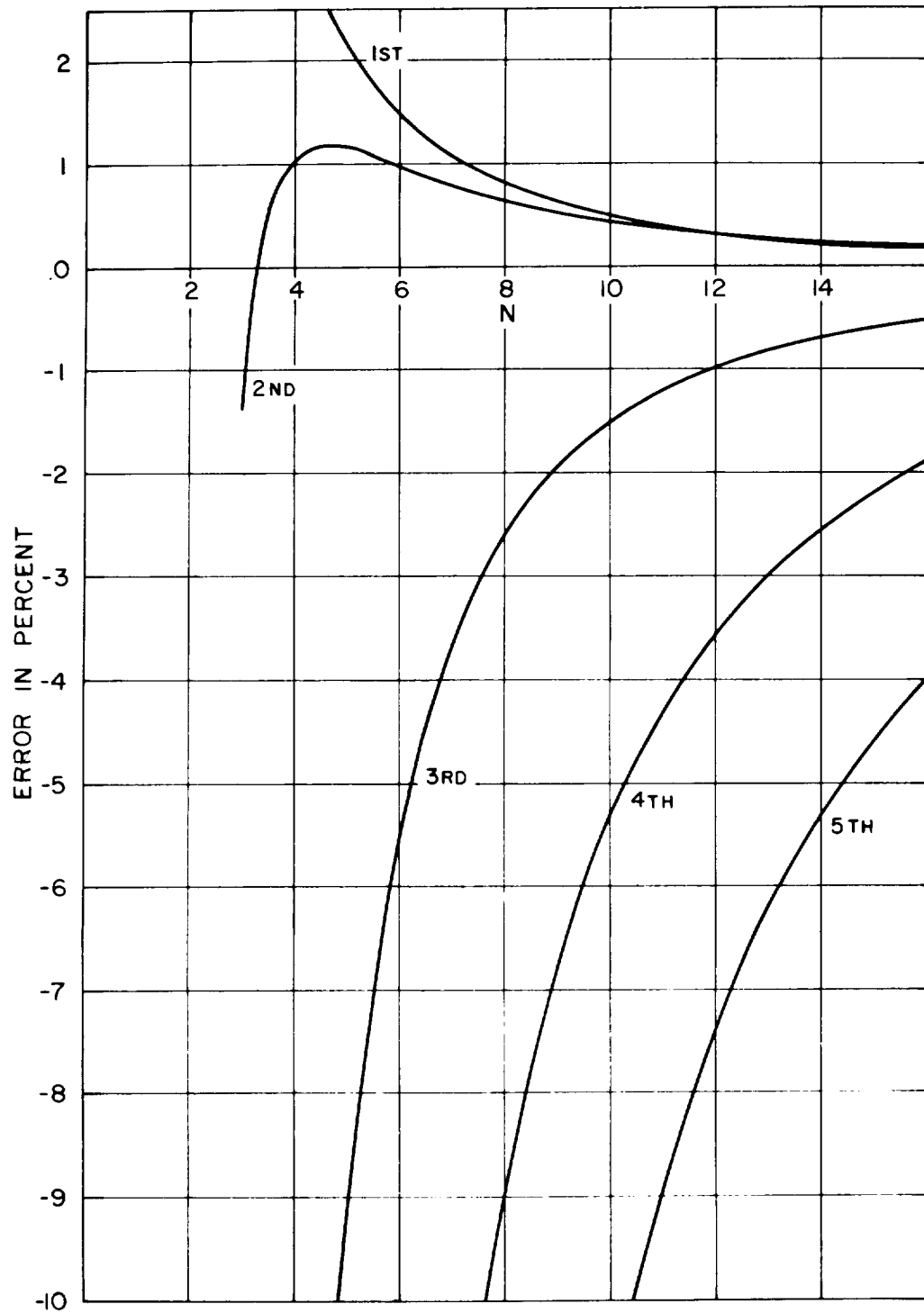
Fig. 14. Cantilever, h^2 approximation.
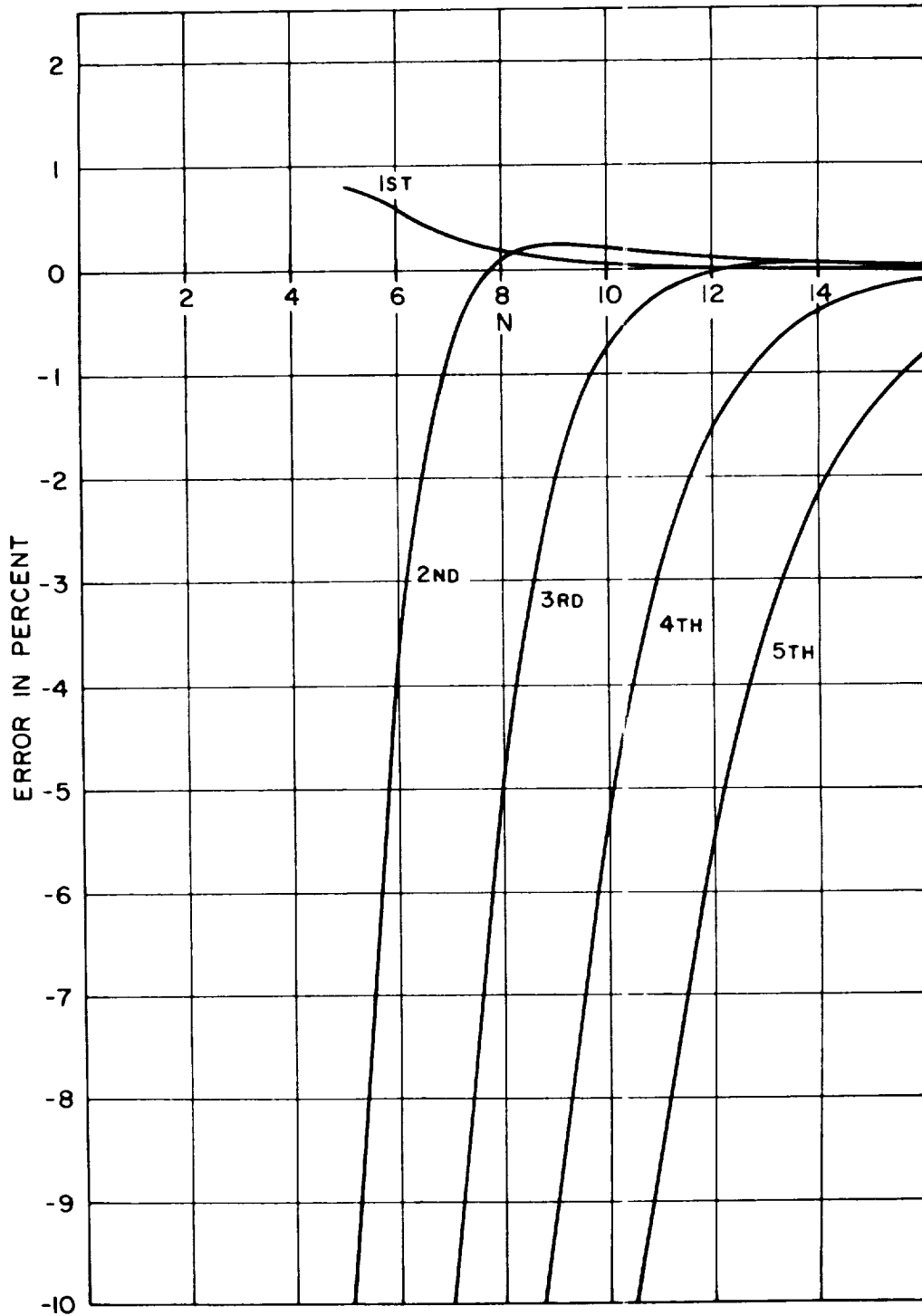
Fig. 15. Built-in, h^4 approximation.
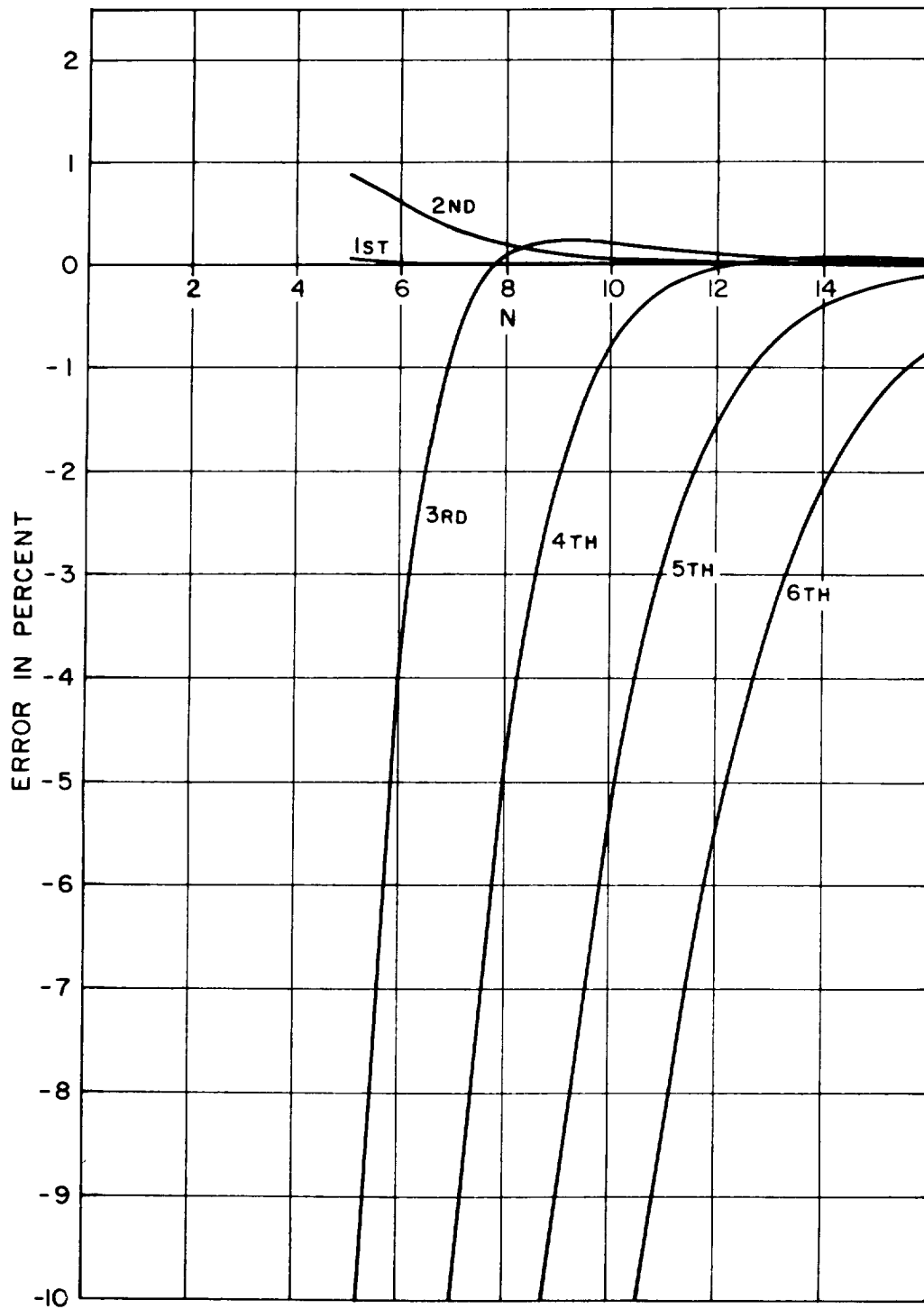
Fig. 16.   Cantilever, h^4 approximation.
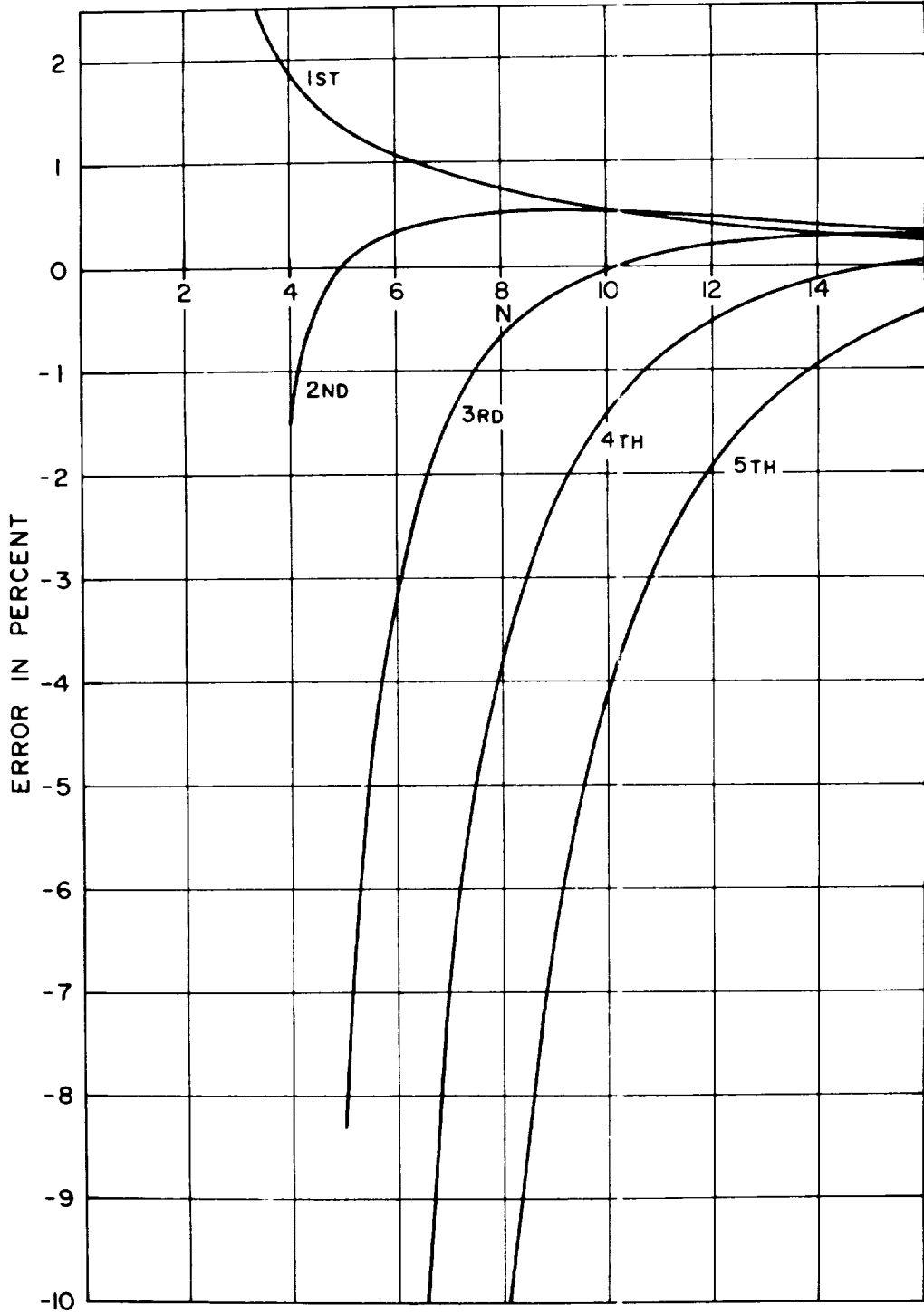
Fig. 17. Built-in, using symmetry.
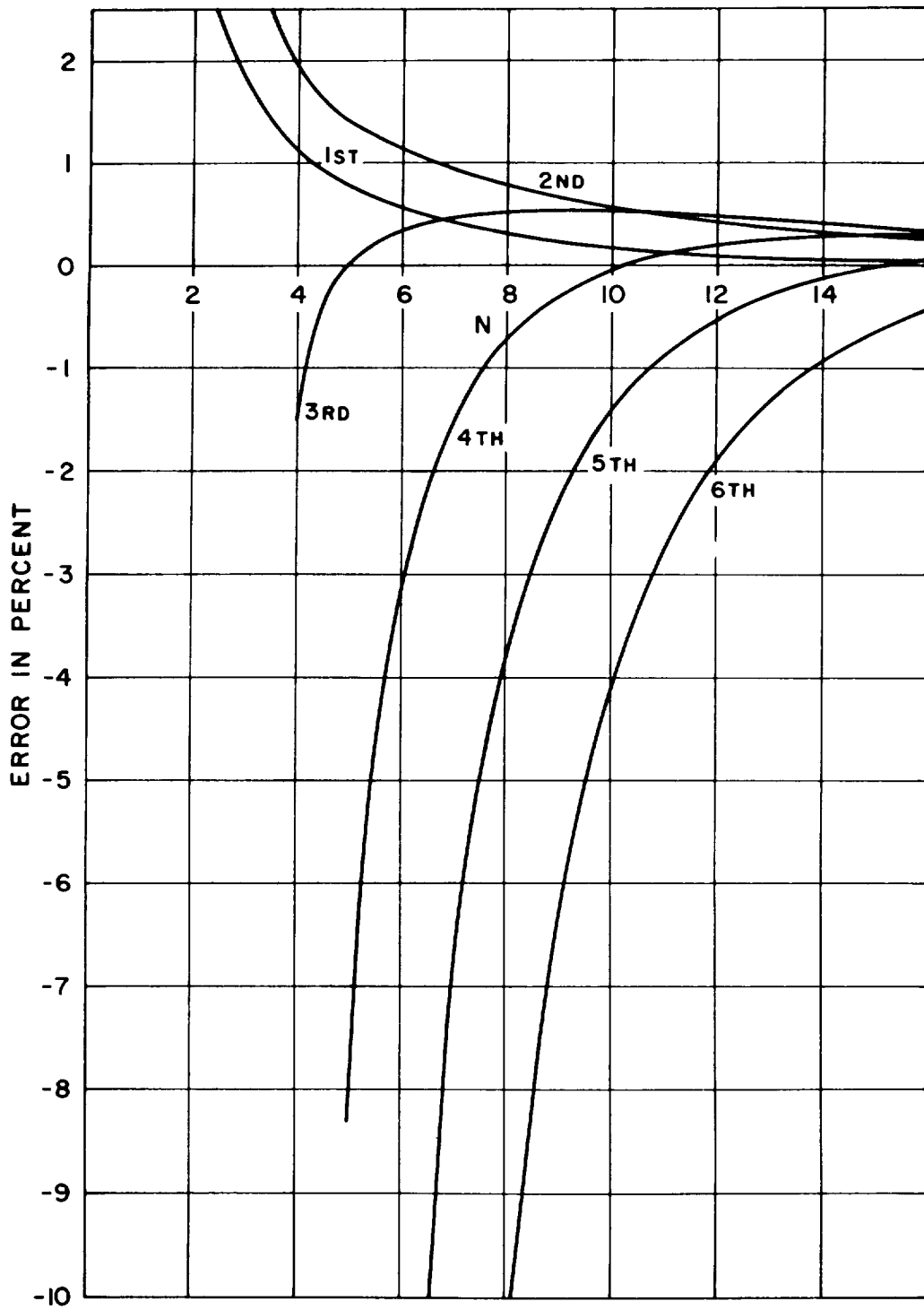
Fig. 18. Cantilever, using symmetry.
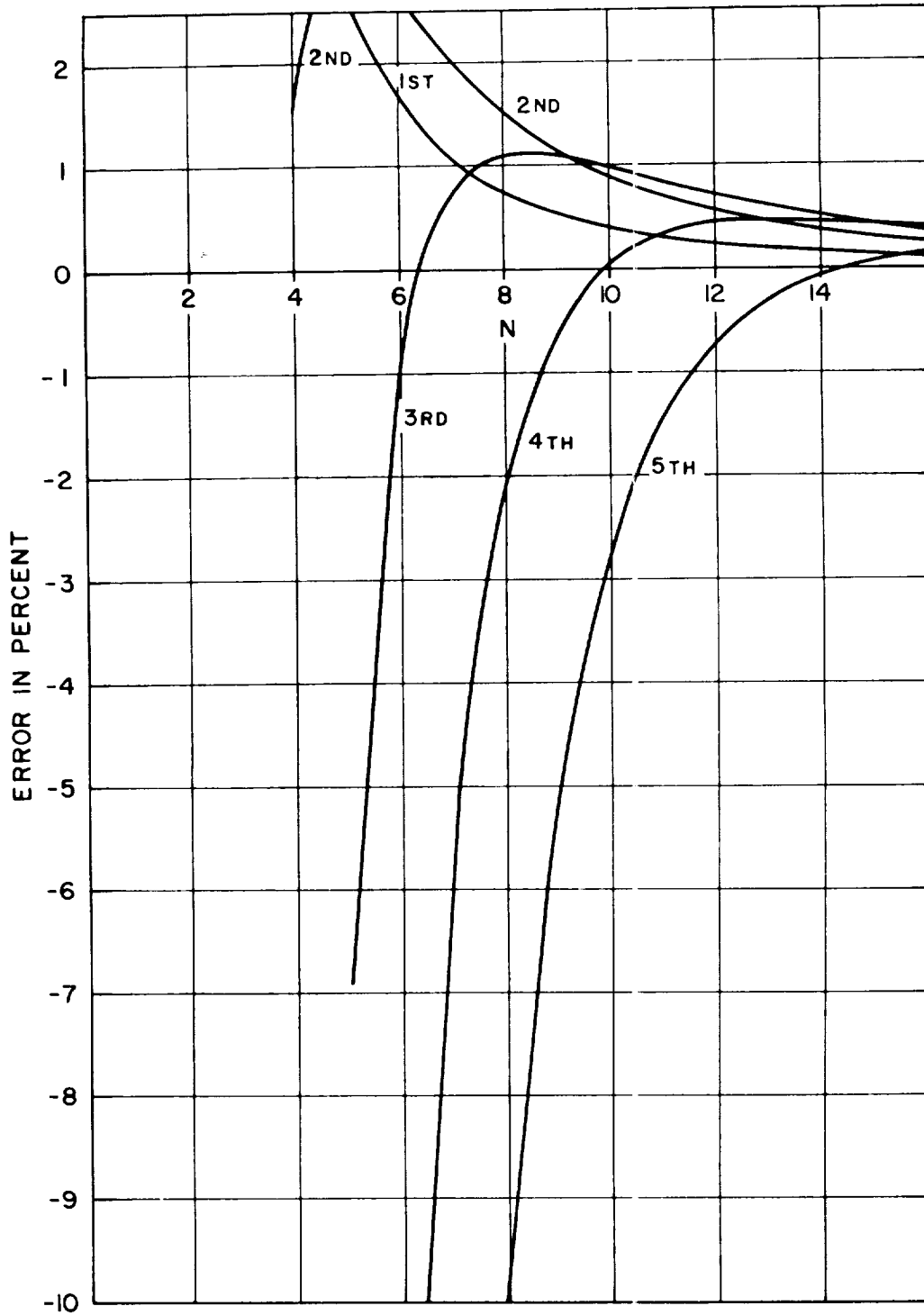
Fig. 19. Built-in, passive circuit.

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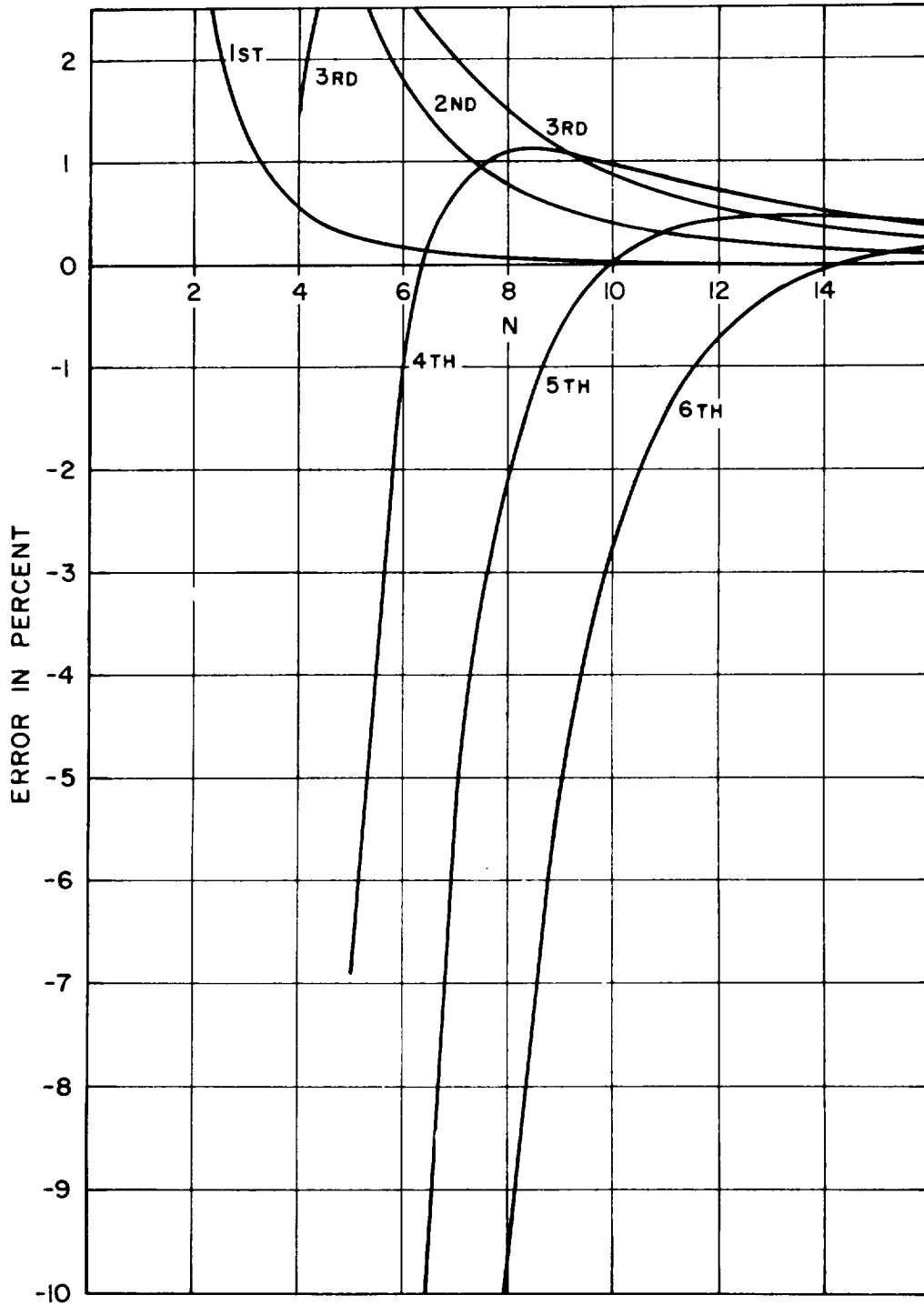
Fig. 20. Cantilever, passive circuit.
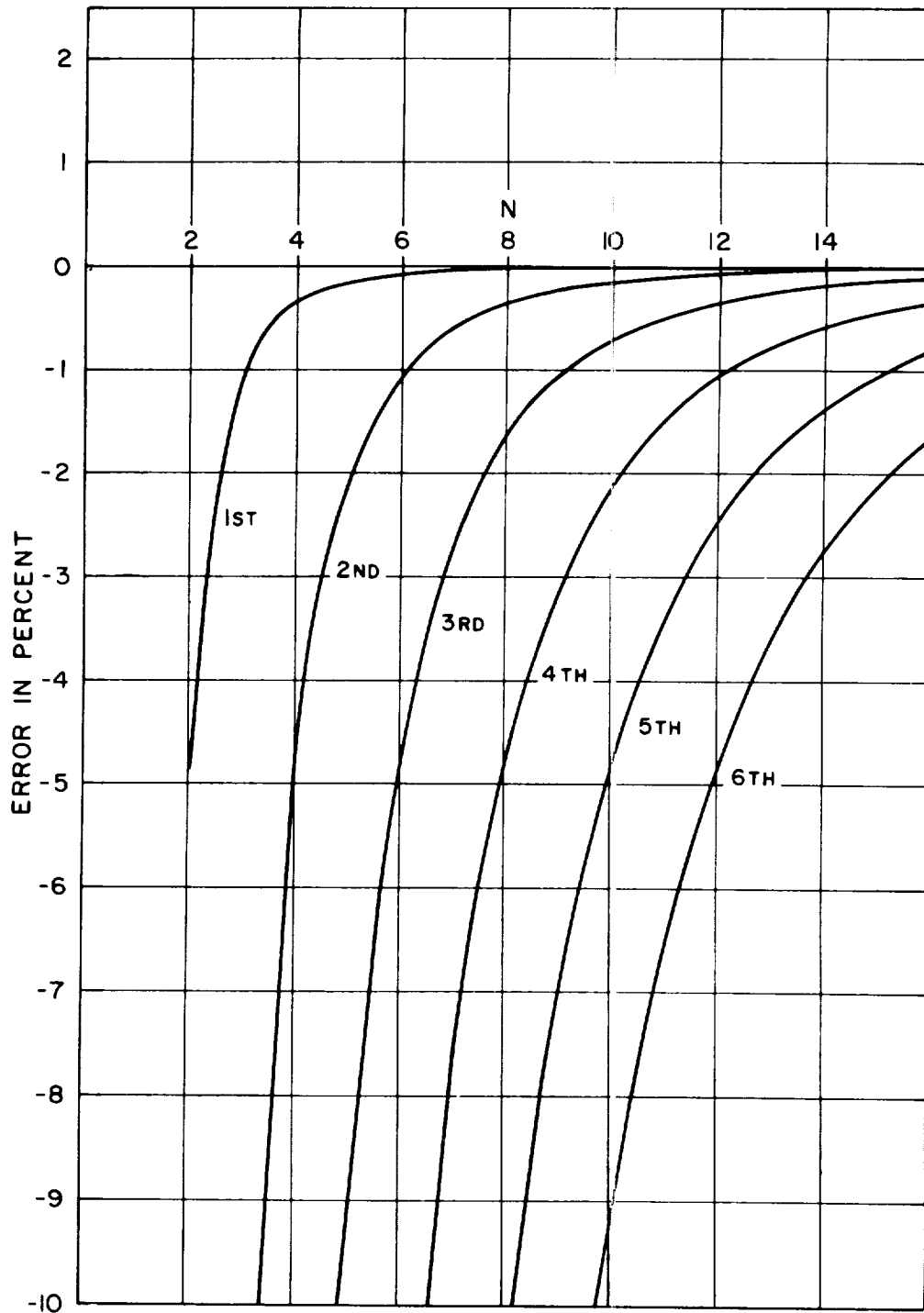
Fig. 21.   Pinned, h^4 approximation.
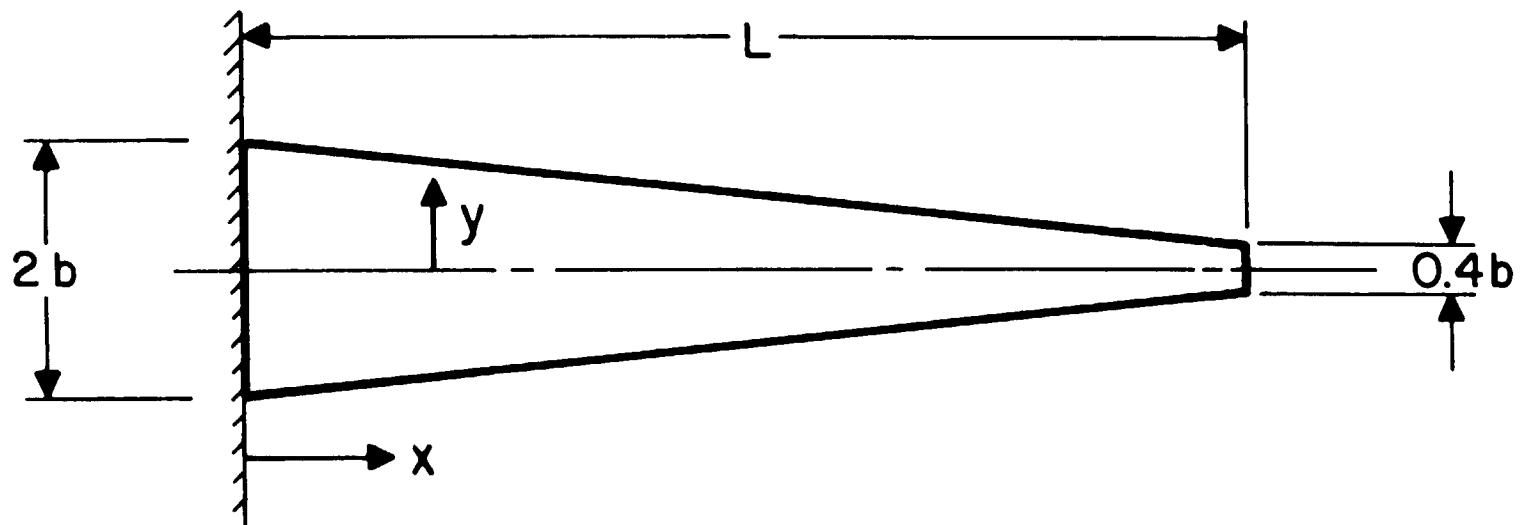
Fig. 22. Tapered cantilever beam.
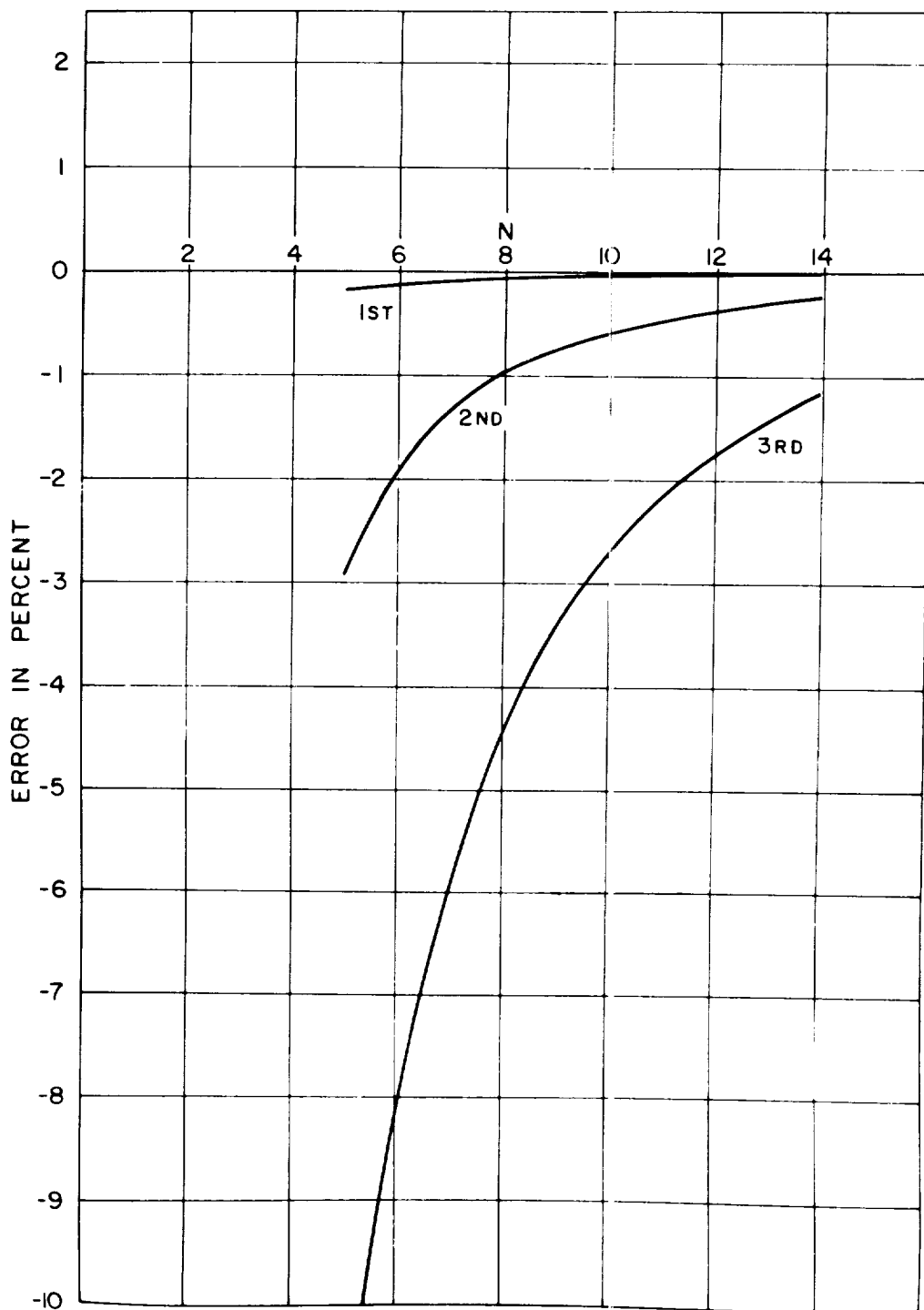
Fig. 23. Tapered cantilever, h^2 approximation.
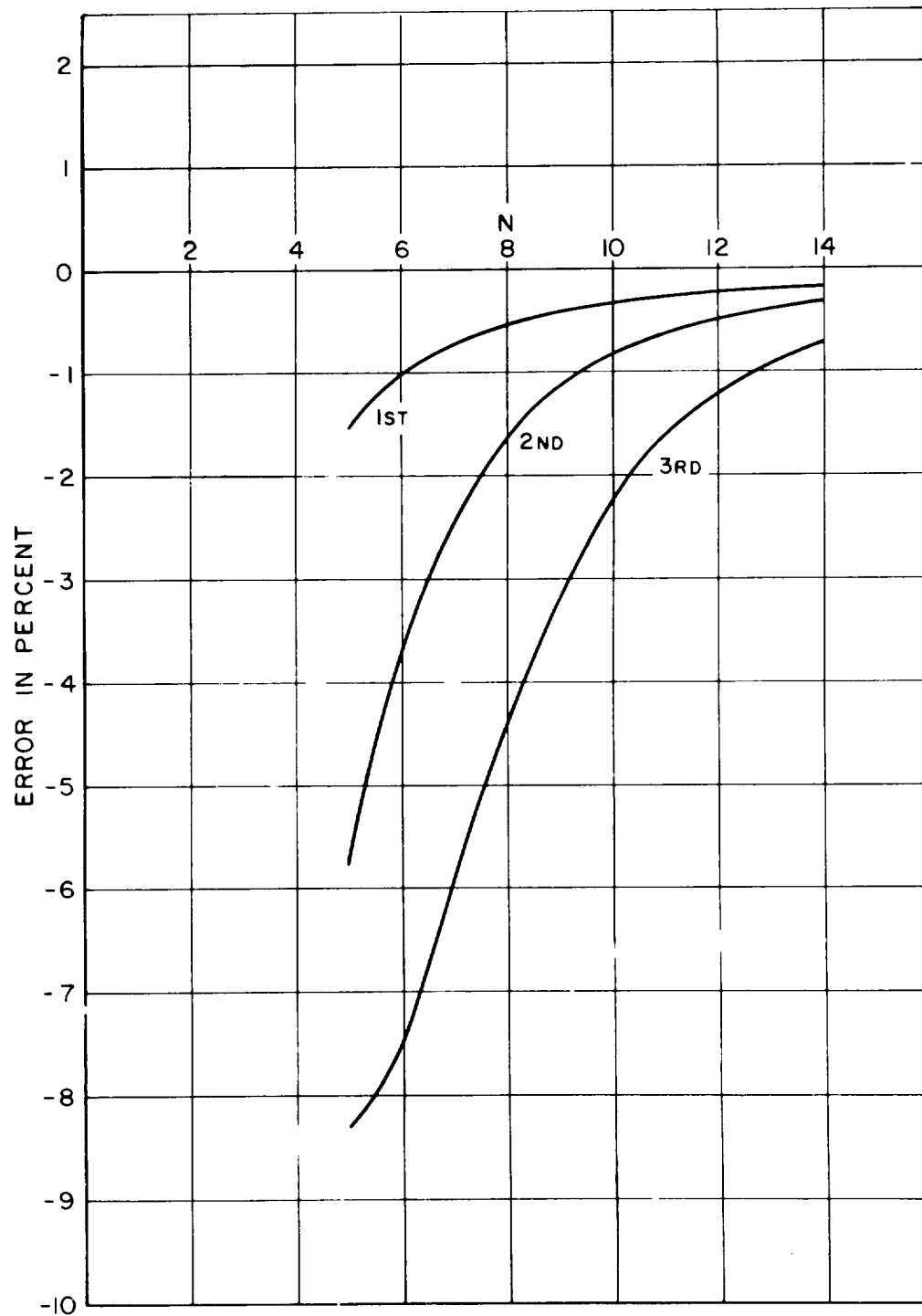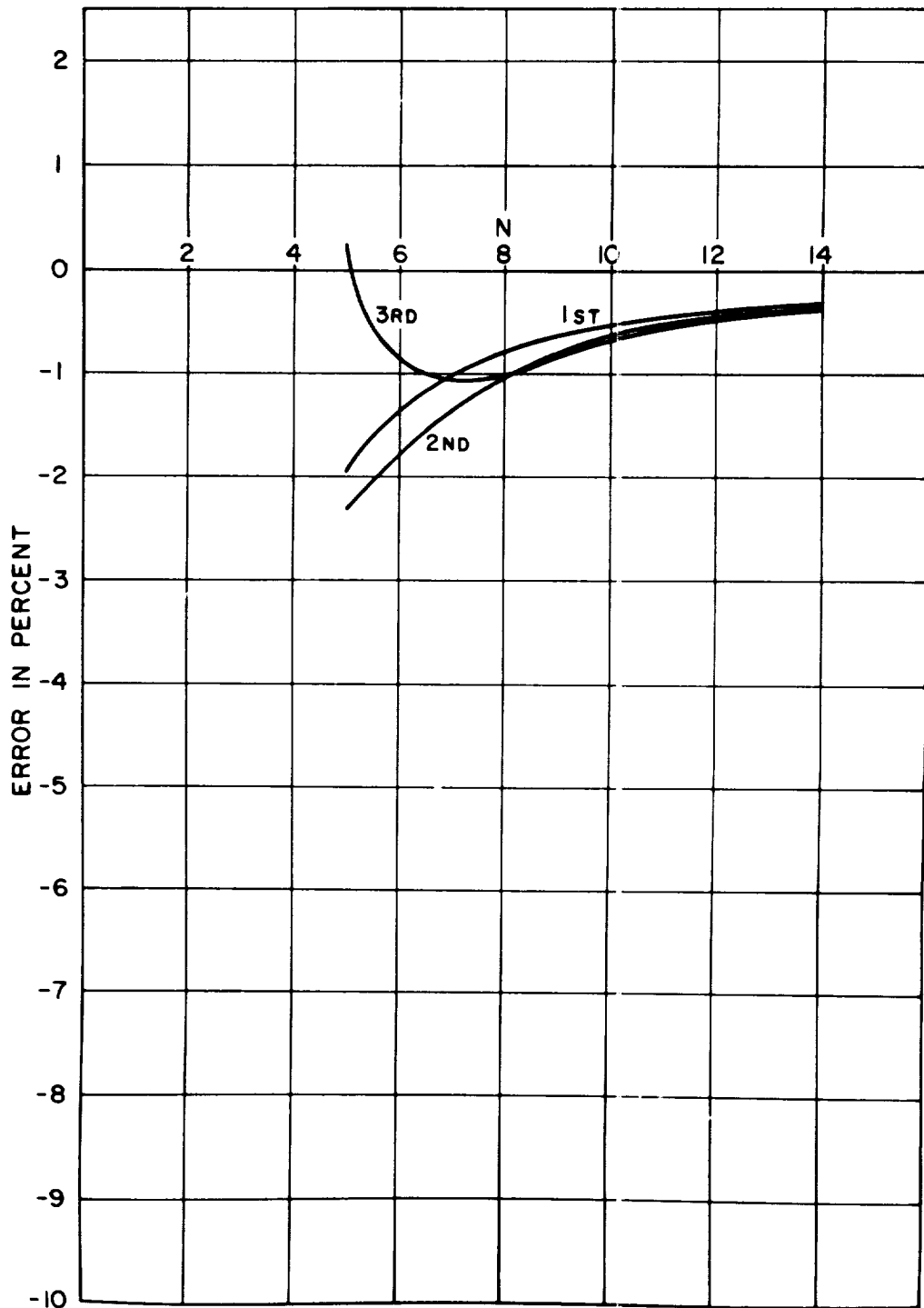
Fig. 24. Tapered cantilever, h^4 approximation.
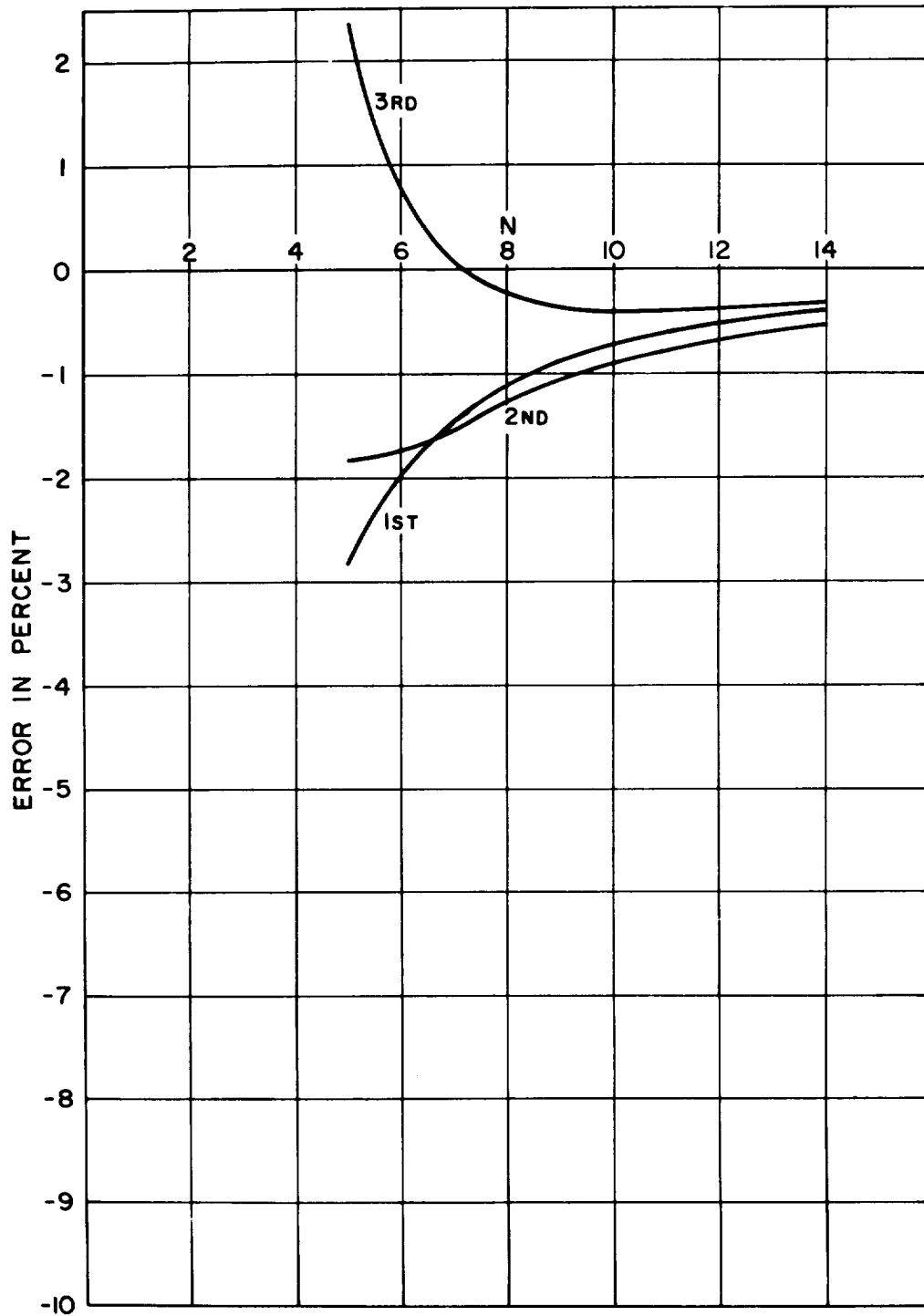
Fig. 25. Tapered cantilever, using symmetry.

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Fig. 26. Tapered cantilever, passive circuit.