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MEMORANDUM

TWO-DIMENSIONAL, SUPERSONIC, LINEARIZED FLOW WITH HEAT ADDITION

By Harvard Lomax

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SUMMARY

Calculations are presented for the forces on a thin supersonic wing underneath which the air is heated. The analysis is limited principally to linearized theory but nonlinear effects are considered. It is shown that significant advantages to external heating would exist if the heat were added well below and ahead of the wing.

INTRODUCTION

Efficient propulsion by a ram-jet engine which brings the air to a low subsonic speed for ignition and burning is, at present, limited to the Mach numbers below which the stagnation temperatures do not present prohibitive cooling problems. One method of propulsion that diminishes the cooling problem and still makes use of the surrounding air is to add heat directly to the external supersonic flow. The object of this paper is to study the latter effect and to find heat distributions which are in some sense optimum. Specifically, calculations are presented for the forces on a thin supersonic wing underneath which the air is heated. The analysis is based on linear theory so the flow disturbances must be small. However, the importance of certain nonlinear effects are discussed.

Other authors (see refs. 1, 2, and 3) have analyzed exterior heating by means of linear theory. The conclusion reached in these papers, as well as in reference 4, is that no substantial gain in airplane range performance can be expected from small amounts of heating in a thin (on the order of the wing thickness) layer beneath the wing at supersonic (M_∞ = 3 to 5) speeds. The analysis presented herein is in basic agreement with such a conclusion but it shows that this kind of heating, while perhaps the most practical at present, is far from optimum. It is true that the optimum heating in a supersonic stream may be attainable only by some novel form of energy release such as electromagnetic radiation, but the attractiveness of external heat addition appears to be limited only by the extent to which it can be controlled.

SYMBOLS

a	speed of sound
c	wing chord
$\bar{c}_p$	specific heat at constant pressure
$\bar{c}_v$	specific heat at constant volume
C_D	drag coefficient, $\frac{D}{q_\infty c}$
C_L	lift coefficient, $\frac{L}{q_\infty c}$
C_p	pressure coefficient, $\frac{p-p_\infty}{q_\infty}$
C_x	net force coefficient in free-stream direction
D	drag
G, G_o, G_1	integrals of heating function (see eqs. (52) and (56))
h_f	maximum depth of heated region
K	see equation (76)
L	lift
M	Mach number
p	local pressure
P	power
P_c	power coefficient, $\frac{P}{q_\infty U_\infty c}$
q	dynamic pressure, $\frac{1}{2}\rho U^2$
Q_v	heat added per unit volume per unit time
$\bar{Q}_v$	dimensionless heat term (see eq. (27))
r	length of heated region measured from trailing edge
R	gas constant
t	wing thickness

T	stagnation temperature
T_h	thrust from added power
U	total velocity in the x direction
u,v,w	perturbation velocities in x,y,z directions
$\overline{V}$	dimensionless wing volume, $\frac{\text{volume}}{c^2}$, (unit span)
$\vec{V}$	total velocity (vector)
x,y,z	Cartesian coordinates
Z	distance from z = 0 plane to wing surface
α	wing angle of attack
β	$\sqrt{M_\infty{}^2-1}$
γ	ratio of specific heats
η	engine efficiency factor, $\left(\frac{T_h U_\infty}{P}\right)_{\text{engine}}$
λ	wing slope
μ	heat added per unit air mass
ρ	local density
τ	wing thickness ratio
φ	perturbation velocity potential

Subscripts

∞	free-stream conditions
i,f	conditions at initial and final heating stage
a,Q	velocities caused by aerodynamic and heat effects
u,l	conditions on upper and lower surface of wing
c,t	slopes attributed to wing camber and twist

BASIC ASSUMPTIONS AND EQUATIONS

Equations for Small Perturbation Flow

Neglecting viscosity and heat conduction, and assuming a perfect gas, we can write the equations of state and continuity of mass, momentum, and energy in the form

$$p = \rho RT \tag{1a}$$

$$(\vec{V} \cdot \text{grad})\rho + \rho \operatorname{div} \vec{V} = 0 \tag{1b}$$

$$\rho(\vec{V} \cdot \text{grad})\vec{V} + \text{grad } p = 0 \tag{1c}$$

$$(\vec{V} \cdot \text{grad}) \left(\overline{c}_p T + \frac{\vec{V} \cdot \vec{V}}{2} \right) = \frac{Q_V}{\rho} \tag{1d}$$

where Q_V is the heat added per unit volume per unit time. If the flow is limited to one having only small disturbances, these can be combined to form (see refs. 5 and 6)

$$(M_\infty^2 - 1)\varphi_{xx} - \varphi_{yy} - \varphi_{zz} = -Q_V \frac{(\gamma - 1)M_\infty^2}{2q_\infty} \tag{2}$$

where φ is the perturbation velocity potential and q_∞ is the free-stream dynamic pressure. The term Q_V is related to μ, the heat added per unit mass of air, by the equation

$$(\vec{V} \cdot \text{grad})\mu = \frac{Q_V}{\rho} \tag{3a}$$

which, for small disturbances, yields

$$\mu = \frac{1}{\rho_\infty U_\infty} \int_{g(y,z)}^{x} Q_V(x_1, y, z)\, dx_1 \tag{3b}$$

where $x = g(y,z)$ is the equation of the forward surface of the heated region. Similarly, the equations for pressure and density reduce to

$$\frac{p - p_\infty}{q_\infty} = -\frac{2u}{U_\infty} \tag{4}$$

$$\frac{\rho - \rho_\infty}{q_\infty} = -\frac{1}{a_\infty^2} \left[\frac{2u}{U_\infty} + (\gamma - 1) \frac{2\mu}{U_\infty^2} \right] \tag{5}$$

The basic assumptions are that the effects of viscosity and heat conduction are negligible and that the air is a perfect gas. The mechanics of the heating itself are ignored and assumed to be independent of the gas behavior. By far the most restrictive assumptions are those which limit the flow to small disturbances. Since we are interested here only in high supersonic flows, these can be expressed as

$$M_\infty \tau \ll 1 \tag{6}$$

$$\frac{(\gamma-1)\mu}{a_\infty^2} \ll 1 \tag{7}$$

where τ is a representative slope of an exposed surface. The first of these conditions is the usual one that limits the application of linear theory when applied to high Mach number flows and the second one represents the limitation on the heat intensity in such flows.

Discussion of Validity of Linearized Equations

The conditions (6) and (7) which have been imposed on flow fields to which equations (2) through (5) apply, have a clear interpretation; namely, that the wing slopes be small compared to $1/M_\infty$ and the heat addition be limited to

$$\frac{\gamma-1}{2}\frac{M_\infty^2}{q_\infty U_\infty}\int_g^x Q_v dx_1 \ll 1 \tag{8}$$

Quite possibly other special types of heating not restricted by (8) are governed by the linearized equations. Consider, for example, a one-dimensional flow in which there is no area change. In this case the exact equation (neglecting viscosity and heat conduction) can be written

$$\left[a_\infty^2 + (\gamma-1)\left(\mu + \frac{U_\infty^2 - U^2}{2}\right) - U^2\right]\frac{dU}{dx} = (\gamma-1)U\frac{d\mu}{dx} \tag{9}$$

which has the solution

$$\mu = -\frac{1}{\gamma-1}\left[a_\infty^2 + \frac{\gamma-1}{2}U_\infty^2 - \frac{U}{U_\infty}(a_\infty^2 + \gamma U_\infty^2) + \frac{\gamma+1}{2}U^2\right] \tag{10}$$

if $\mu = 0$ at $U = U_\infty$. In terms of the perturbation velocity u, the pressure and heat input can be expressed as

$$\frac{p-p_\infty}{q_\infty} = -\frac{2u}{U_\infty} \tag{11}$$

and

$$\frac{\gamma-1}{2}\frac{1}{q_\infty U_\infty}\int_g^x Q_v dx_1 = -\frac{u}{U_\infty}\left(1-\frac{1}{M_\infty^2}\right)-\frac{\gamma+1}{2}\left(\frac{u}{U_\infty}\right)^2 \tag{12}$$

From the linearized equations, on the other hand, one finds

$$(M_\infty^2-1)\varphi_{xx} = -\frac{(\gamma-1)Q_v M_\infty^2}{2q_\infty} \tag{13}$$

$$\frac{p-p_\infty}{U_\infty} = -\frac{2u}{U_\infty} \tag{14}$$

and

$$\frac{\gamma-1}{2}\frac{1}{q_\infty U_\infty}\int_g^x Q_v dx_1 = -\frac{u}{U_\infty}\left(1-\frac{1}{M_\infty^2}\right) \tag{15}$$

Clearly, in the case of one-dimensional flow, linearized theory predicts the heating effect on the pressure to a higher order than that given by expressions (6) and (7). In the one-dimensional case the linear theory is valid for supersonic Mach numbers if $u/U_\infty \ll 1$ and not $M_\infty u/U_\infty \ll 1$. We will return to this point presently.

An obvious way to check a proposed approximate theory is to apply it to examples which have been studied by more exact methods. One such set of examples is supplied in reference 4 wherein the effect of adding heat under a two-dimensional, 5-percent-thick, biconvex wing at 0° and 2° angles of attack is calculated by a nonlinear theory. The heat was added so that the stagnation temperature rose linearly from its free stream to a specified final value. Figure 1 gives a comparison of these results with those found by the linear theory[1] used in this report.

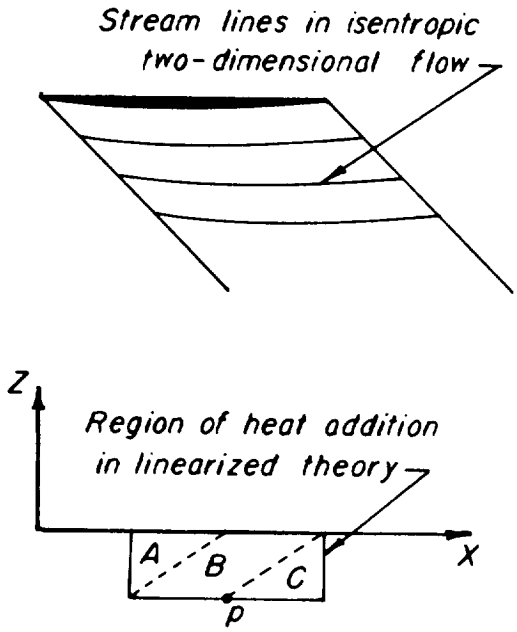
The two theories are in good agreement especially as to the effect of heat addition on the pressure. This agreement is surprising in the case of figure 1(a). One can show that

$$\frac{(\gamma-1)\mu}{a_\infty^2} = \left(\frac{M_\infty^2}{5}+1\right)\left(\frac{T_f}{T_i}\frac{x-x_i}{x_f-x_i}-1\right) \tag{16}$$

and for $x = x_f$, $M_\infty = 5$, and $T_f/T_i = 1.243$ this becomes 1.458 which certainly violates condition (7). Yet in figure 1(a) the linear theory follows the more exact one even to the trailing edge.

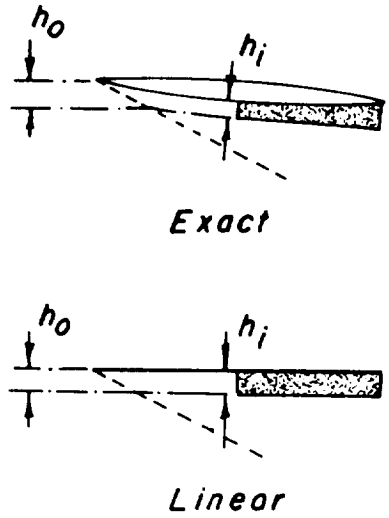
[1]The analytical form of the linearized results and a brief discussion of their theoretical development are given in the next section.

Perhaps one explanation for this degree of accuracy can be drawn from the discussion of one-dimensional flow. It is well known that, to a first order, the stream-tube area intercepted by a Mach line does not change in the region bounded by the fore and aft shocks in a two-dimensional flow, as shown in sketch (a). The linearized treatment of the heated flow contains the result that under the condition of constant heat addition, as is the case for both examples in figure 1, the stream tubes are unaffected by the heating in region A of sketch (a), and begin to diverge at a rate proportional to the distance from the wing in regions B and C. The heat added in region C, however, has no influence on the wing pressure distribution and the value of $(\gamma-1)\mu/a_\infty^2$ at p, the last point on the outer portion of the heated region that can affect the wing pressures, is about 0.66. Thus, over most of the part of the heated region that contributes to the wing pressure field either the flow is nearly one-dimensional or the magnitude of $(\gamma-1)\mu/a_\infty^2$ is small.



Sketch (a)

From another point of view, the high degree of accuracy for the linearized results shown in figure 1(a) must be considered fortuitous. Mager in reference 1, for example, uses the identical linearized theory and calculates pressures about 40 percent higher than those shown here. The reason can be explained with the aid of sketch (b). The upper part of the sketch shows the exact boundaries of the wing and heated region. The original height h_o of the stream tube which enters the heated region with height h_i is, in this case, about 40 percent greater than h_i. Mager cast his equation for pressure in terms of the power coefficient given in reference 4 which, in turn, is correctly based on h_o. Hence his calculations show the correct value for the power but high values of pressure. In this report the equations were derived in terms of the boundary conditions in the lower part of sketch (b) and predict the correct value for the pressure but a power 40 percent low.



Sketch (b)

The point of all this discussion is that figure 1(a) suggests an accuracy for the linearized theory of heat addition at Mach numbers and heat intensities which undoubtedly exists

only for special cases. In the following sections some quite general results concerning the forces developed on two-dimensional wings in heated flows are derived. In general their validity depends on the extent to which expressions (6) and (7) are true.

Solutions to the Linear Equations in Two and Three Dimensions

Obviously equation (2) is either elliptic or hyperbolic depending on whether M_∞ is less or greater than 1. By the application of Green's theorem, solutions can be written for either the subsonic or supersonic case. We will consider here only supersonic flow for which the potential can be expressed in the form

$$\varphi(x,y,z) = -\frac{1}{2\pi}\frac{\partial}{\partial x}\iint_S \Lambda\left(\sigma\frac{\partial\varphi}{\partial\nu} - \varphi\frac{\partial\sigma}{\partial\nu}\right) dS + \frac{1}{2\pi}\frac{\partial}{\partial x}\iiint_V \sigma\frac{(\gamma-1)M_\infty^2 Q_V}{2q_\infty}\,dV \tag{17}$$

where

$$\frac{\partial\varphi}{\partial\nu} = \frac{\partial\varphi}{\partial x}\nu_1 + \frac{\partial\varphi}{\partial y}\nu_2 + \frac{\partial\varphi}{\partial z}\nu_3 \tag{18}$$

and the direction cosines ν_1, ν_2, ν_3 of the conormal ν are related to those of the inwardly directed normal by

$$-n_1\beta^2 = \Lambda\nu_1\,, \qquad n_2 = \Lambda\nu_2\,, \qquad n_3 = \Lambda\nu_3 \tag{19a}$$

Λ is fixed by the condition

$$\nu_1^2+\nu_2^2+\nu_3^2 = 1 \tag{19b}$$

and σ is given by the equation

$$\sigma = \operatorname{arc\,cosh}\frac{x-x_1}{\beta\sqrt{(y-y_1)^2+(z-z_1)^2}} \tag{20}$$

The symbol V represents the volume enclosed by the surface S over which φ and $\partial\varphi/\partial\nu$ are integrated. Applications of equation (17) to supersonic wing theory when Q_V is zero are well known.

In two dimensions the solution simplifies to

$$\oint_s \Lambda\frac{\partial\varphi}{\partial\nu}\,|ds| + \iint_S \frac{(\gamma-1)M_\infty^2 Q_V}{2q_\infty}\,dS = 0 \tag{21}$$

where S is now the area enclosed within the curve s. Notice that when the integration is along a streamline $\Lambda = 1$, and when along a characteristic $\Lambda = \beta$.

In both two and three dimensions the effect of heat addition on small disturbance flow fields is identical to that of distributing simple fluid sources of strength $(\gamma-1)M_\infty^2 Q_v/2q_\infty$ throughout the flow. (This effect was pointed out in both refs. 5 and 6.)

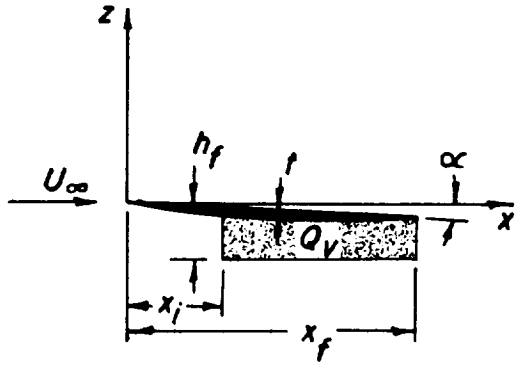
APPLICATIONS

Heating Under a Biconvex Wing

Consider the wing shown in sketch (c). Its upper surface is flat and its lower surface is given by the equation

$$Z = -\frac{4t}{c^2}x(c-x) - \alpha x \qquad (22)$$

where α is the angle of attack. If there is no heating, the linearized-theory value of the pressure coefficient on the lower surface is



Sketch (c)

$$C_p = \frac{p-p_\infty}{q_\infty} = \frac{8}{\beta}\left(\frac{t}{c}\right)\left(1 - \frac{2x}{c}\right) + \frac{2\alpha}{\beta} \qquad (23)$$

or

$$\frac{p}{p_\infty} = 1 + \frac{4\gamma M_\infty^2}{\beta}\left(\frac{t}{c}\right)\left(1 - \frac{2x}{c}\right) + \frac{\alpha M_\infty^2 \gamma}{\beta} \qquad (24)$$

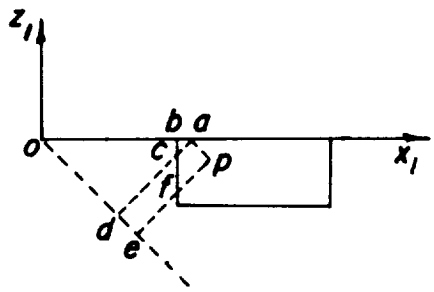
If heat is added in the region shown in the sketch in such a way that h, the stagnation enthalpy, increases linearly with x then

$$\mu = \Delta h = \bar{c}_p \frac{x-x_i}{x_f-x_i}(T_f-T_i) = \frac{1}{\rho_\infty U_\infty}\int_{x_i}^{x} Q_v dx \qquad (25)$$

where T_i and T_f are stagnation temperatures at x_i and x_f, respectively. From equation (25) one finds

$$\frac{(\gamma-1)M_\infty^2 Q_v}{2q_\infty} = \left(1 + \frac{\gamma-1}{2}M_\infty^2\right)\left(\frac{T_f}{T_i} - 1\right)\frac{U_\infty}{x_f-x_i} \qquad (26)$$

that is to say, Q_v is a constant over the entire heated region.



Sketch (d)

As an example of the way in which equation (21) can be used, consider the problem of finding the pressure at the point p indicated in sketch (d). Introducing the dimensionless heat parameter $\bar{Q}_v$ where

$$\bar{Q}_v = \frac{(\gamma-1)M_\infty^2 h_f Q_v}{2q_\infty U_\infty} \tag{27}$$

we have

$$-\int_p^a \frac{\partial\varphi}{\partial s}\,ds + \frac{1}{\beta}\int_o^a \frac{\partial\varphi}{\partial(-z_1)}\,dx_1 + \int_e^p \frac{\partial\varphi}{\partial s}\,ds + \frac{U_\infty}{\beta h_f}\iint_{paoep} \bar{Q}_v dx_1 dz_1 = 0$$

$$-\int_p^a \frac{\partial\varphi}{\partial s}\,ds + \int_a^d \frac{\partial\varphi}{\partial s}\,ds + \int_e^p \frac{\partial\varphi}{\partial s}\,ds + \frac{U_\infty}{\beta h_f}\iint_{padep} \bar{Q}_v dx_1 dz_1 = 0$$

since φ is a constant along oe. These equations give

$$2\varphi_p - \varphi_a - \frac{1}{\beta}\int_o^{x-\beta z} \frac{\partial\varphi}{\partial z_1}\,dx_1 + \frac{U_\infty}{\beta h_f}\iint_{paoep} \bar{Q}_v dx_1 dz_1 = 0 \tag{28}$$

$$2\varphi_p - 2\varphi_a + \frac{U_\infty}{\beta h_f}\iint_{padep} \bar{Q}_v dx_1 dz_1 = 0 \tag{29}$$

Eliminating φ_a, we find

$$\varphi_p = \frac{1}{\beta}\int_o^{x-\beta z} \frac{\partial\varphi}{\partial z_1}\,dx_1 - \frac{U_\infty}{2\beta h_f}\left(\iint_{pabfp} \bar{Q}_v dx_1 dz_1 + \iint_{abca} \bar{Q}_v dx_1 dz_1\right) \tag{30}$$

The first term in equation (30) gives the effect of the wing on the flow and the second term that of the heat. If $\bar{Q}_v$ is a constant, as in our example, the effect of the heat is proportional to the sum of the two areas pabfp and abca. The pertinence of these areas is at once obvious if we think of solving the problem by using an image system of sources having the wing as a plane of symmetry.

The pressure coefficient on the lower wing surface is given by equation (4) and, using equation (26) for Q_v, is

$$C_p = \frac{8}{\beta}\left(\frac{t}{c}\right)\left(1 - \frac{2x}{c}\right) + \frac{2\alpha}{\beta} + \frac{2}{\beta^2}\left(1 + \frac{\gamma-1}{2} M_\infty^2\right)\left(\frac{T_f}{T_i} - 1\right)\left(\frac{x-x_i}{x_f-x_i}\right) \tag{31}$$

This equation is valid for $x_i < x < x_i + \beta h_f$, that is until the forward Mach line from p intersects the lower front corner of the heated region. For $x_i + \beta h_f < x < x_f$, one can show

$$C_p = \frac{8}{\beta}\frac{t}{c}\left(1 - \frac{2x}{c}\right) + \frac{2\alpha}{\beta} + \frac{2}{\beta}\left(1 + \frac{\gamma-1}{2}M_\infty^2\right)\left(\frac{T_f}{T_i} - 1\right)\frac{h_f}{x_f - x_i} \qquad (32)$$

and similar results can be written for $x > x_f$.

In figure 1 the above results from the linearized theory are compared with those from the more exact theory given in reference 4. The close agreement between the two theories has been discussed in a preceding section.

We can estimate the significance of the result of external heating by studying its effect on airplane range performance. From the form of the Breguet range equation

$$\text{Range} = \left(\frac{T_h U_\infty}{P}\right)\left(\frac{L}{D}\right)\left(\begin{matrix}\text{fuel heat energy} \\ \text{per unit weight}\end{matrix}\right) \ln \frac{\text{weight full}}{\text{weight empty}} \qquad (33)$$

let us consider the combined term

$$\left(\frac{LU_\infty}{P}\right)_{D=T_h} = \left(\frac{T_h U_\infty}{P}\right)\left(\frac{L}{D}\right) \qquad (34)$$

where

T_h total thrust

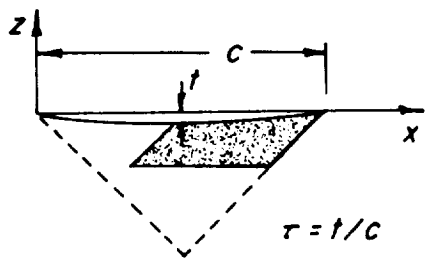
D total drag

L total lift

P total power input

In the following, we make the assumption that maximum range is obtained when $(LU_\infty/P)_{D=T_h}$ is a maximum and will, therefore, refer to this quantity as the range efficiency factor.

As a simple example let the wing, again, as in sketch (c), be flat on top and biconvex on the bottom. For simplicity, however, Q_v is now taken to be a constant over the rear half of the lower surface in a region



Sketch (e)

bounded fore and aft by Mach lines as shown in sketch (e). The pressure on the lower surface of this wing is given by equation (32), from which the sum of the forces in the free-stream direction[2] on the upper and lower surfaces is

$$C_x = C_{D_o} + \frac{32}{3\beta}\tau^2 + \frac{4}{\beta}\alpha^2 - \frac{1}{\beta}\bar{Q}_v(2\tau-\alpha) \tag{35}$$

and C_{D_o} is the net airplane drag exclusive of the wing wave drag. The lift is assumed to come entirely from the wing angle of attack and the heating, so

$$C_L = \frac{L}{q_\infty c} = \frac{4\alpha}{\beta} + \frac{1}{\beta}\bar{Q}_v \tag{36}$$

Finally the power is given by

$$P = \iint Q_v dx\, dz \tag{37}$$

so

$$\frac{P}{q_\infty U_\infty c} = \frac{\bar{Q}_v}{(\gamma-1)M_\infty^2} \tag{38}$$

Let us now evaluate the performance of the wing and external heating in terms of the parameter $(LU_\infty/P)_{C_x=0}$ where $C_x = 0$ implies that the total thrust and drag are equal. First it is clear that C_L and P are independent of τ, so τ should be chosen to minimize C_x in equation (35). As a result

$$\tau = \frac{3\bar{Q}_v}{32} \tag{39}$$

and

$$\beta C_x = \beta C_{D_o} + 4\alpha^2 + \bar{Q}_v\alpha - \frac{3\bar{Q}_v^2}{32} \tag{40}$$

[2]In keeping with linearized theory the angle between the normal and the vertical is neglected in evaluating the lift and drag.

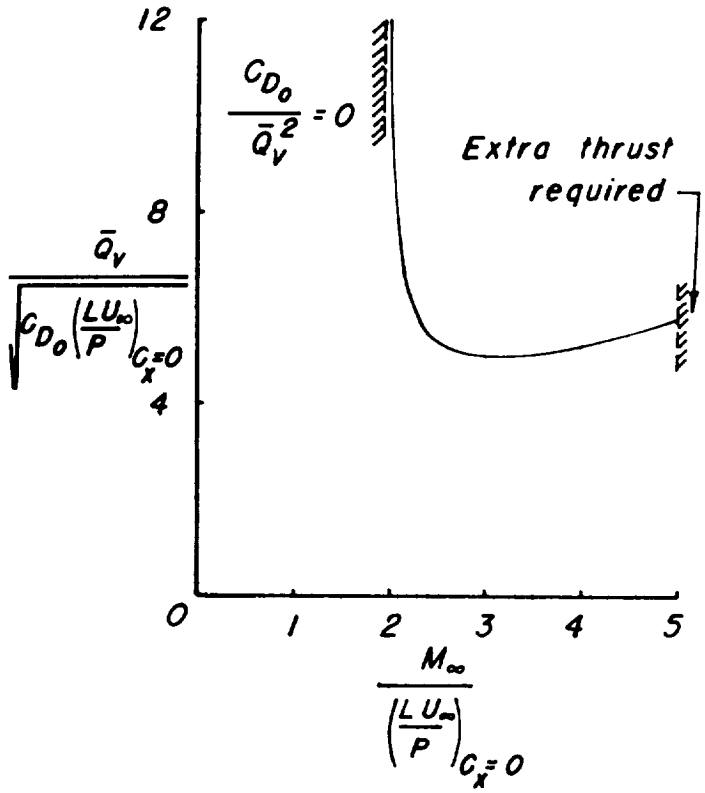
For a given power, that is, a given value of $\bar{Q}_v$, equation (40) can be solved for α under the condition that $C_X = 0$. For optimum $(LU_\infty/P)_{C_X=0}$ one finds

$$\alpha = -\frac{1}{8}\left(\bar{Q}_v - \sqrt{\frac{5}{2}\bar{Q}_v^{\,2} - 16\beta C_{D_o}}\right) \tag{41}$$

$$C_L = \frac{\bar{Q}_v}{2\beta}\left(1 + \sqrt{\frac{5}{2} - 16\,\frac{\beta C_{D_o}}{\bar{Q}_v^{\,2}}}\right) \tag{42}$$

$$\left(\frac{LU_\infty}{P}\right)_{C_X=0} = \frac{\gamma-1}{2}\,\frac{M_\infty^{\,2}}{\beta}\left(1 + \sqrt{\frac{5}{2} - 16\,\frac{\beta C_{D_o}}{\bar{Q}_v^{\,2}}}\right) \tag{43}$$

The results for $(LU_\infty/P)_{C_X=0}$ are summarized in sketch (f) under the assumption that $\sqrt{M_\infty^{\,2}-1} = M_\infty$. The maximum value of $(LU_\infty/P)_{C_X=0}$ obtained



Sketch (f)

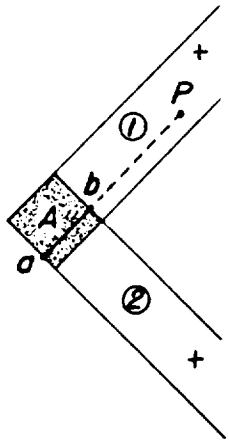
by this particular method is seen, from equation (43), to be $[(\gamma-1)M_\infty^{\,2}/2\beta](1+\sqrt{5/2})$ or about $0.5\ M_\infty$. This forms the left-hand boundary

in the sketch and represents either an infinite heating rate or a zero value of C_{D_o}. The right-hand boundary can not be exceeded without providing some extra thrust by an engine to satisfy the requirement $C_x = 0$.

Arbitrary Heating in Two-Dimensional Fields

The preceding section contains an analysis of a specific wing section heated in a specific way. There is no reason to suppose, however, that the particular section and type of heating studied were optimum from the point of view of, say, fixed volume and maximum $(LU_\infty/P)_{C_x=0}$. We will show, in fact, they are not. In order to prove this, let us now derive the simple expression for the lift and drag on arbitrary, planar, two-dimensional wings in fields with arbitrary heat addition.

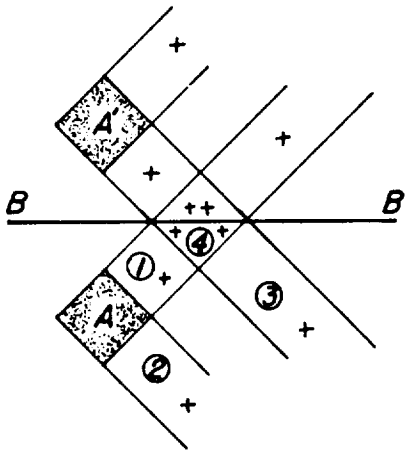
Evaluating planar wing velocity fields by heat images.- It has been determined that a region of heat addition disturbs a linearized flow in exactly the same manner as if the field were filled with fluid sources of strength $(U_\infty/h_f)\bar{Q}_v$. Thus if, as in the upper half of sketch (g), region A is a region of heat addition, the value of u and w at P would be



$$w = \frac{1}{2M_\infty}\int_a^b \left(\frac{U_\infty}{h_f}\bar{Q}_v\right) dl$$

$$u = \pm\frac{w}{\beta}$$

The 1/2 results from the fact that the source strength is proportional to 1/2 the stream deflection in the upgoing or downgoing curves.



Sketch (g)

Now place a wing in the plane BB above the heated region A. The boundary condition, insofar as the heat is concerned, is simply that the value of w along BB is zero; the distortion of the field caused by the wing's own shape can be superimposed on this solution for the complete determination of the disturbance field. It is clear then, that the effect of the heat addition beneath a wing is found by placing an image of A above the wing, A' in sketch (g), and discarding the wing Obviously, the pressures in waves 1, 2, and 3 in the sketch are the

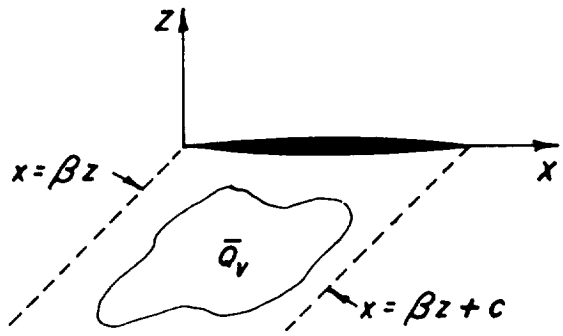
same as if a single region of heat addition were present, but the pressure in region 4 is the same as that for the two heated areas.

Forces on wing obtained from surface pressure distributions.- Using the concepts developed in the preceding section, let us introduce the following definitions:

u_a, w_a the velocities induced by the wing with no heating in the field

u_Q, w_Q the velocities induced beneath the wing by the heated region and its image above the wing without the wing being present

Then assuming that the heat is applied entirely within the fore-waves from the leading and trailing edges as in sketch (h), one can write the expressions for lift, power, and drag developed beneath the wing in the form (on the wing $u_a = -w_a/\beta$



Sketch (h)

$$\frac{U_\infty L}{q_\infty} = -2\int_0^c (u_Q+u_a)dx \tag{44}$$

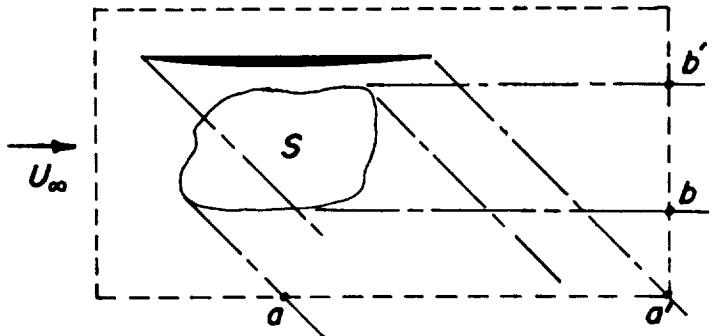
$$\frac{P}{q_\infty} = \frac{-2\beta}{(\gamma-1)M_\infty^2}\int_0^c u_Q dx \tag{45}$$

$$\frac{U_\infty^2 D}{q_\infty} = 2\beta\int_0^c (u_a^2+u_a u_Q)dx \tag{46}$$

These expressions will be used in a subsequent section to derive optimum airfoil shapes for given heat fields.

Forces obtained from momentum flux.- Although equations (44) through (46) are sufficient to analyze the performance of a two-dimensional wing in the presence of heat, it is instructive to re-examine the derivation of the drag on the basis of the momentum crossing an enclosing surface. This derivation is much more involved mathematically but it serves at least the following two purposes: one, it illustrates the existence of a new form of "momentum force" in heated flows appearing in addition to the usual wave drag and being measured in the wake of the heated region; and the other, it is capable of generalization to three dimensions.

Consider the system illustrated in sketch (i). The control surface is represented by the dashed lines. Linear theory yields no disturbances



Sketch (i)

along the lines counterclockwise from b' to a and from a' to b. The momentum drag in the wave from a to a' is well known in its two-dimensional linearized form to be simply

$$\frac{U_\infty{}^2 D_{wave}}{q_\infty} = 2\beta \int_a^{a'} (u_a+u_{(})^2 dx \tag{47}$$

A comparison of equations (47) and (46) immediately shows that another force having a similar order of magnitude must exist in the system. This force is caused by the change in density (see eq. (5)) behind a heated region.

One can show by retaining the lowest order magnitude terms contributing to the drag that the linearized value of the wake drag in three dimension is given by

$$D_{wake} = \rho_\infty \iint \left\{ \frac{1}{2}\left[(M_\infty{}^2-1)u^2+v^2+w^2 \right] + \left[\frac{(\gamma-1)u\mu M_\infty}{a_\infty} \right] + \left[\frac{(\gamma-1)M_\infty}{a_\infty} \int_{-\infty}^{x} u \frac{\partial \mu}{\partial x_1} dx_1 \right] \right\} dS \tag{48}$$

where the dS integration is made over an entire yz plane infinitely far behind the wing. In a two-dimensional flow the induced velocities behind the trailing-edge wave vanish so the first two bracketed terms in equation (48) vanish. The third term, however, depends upon the disturbance in the heated region itself. Hence, in two dimensions

$$D_{wake} = \rho_\infty(\gamma-1) \frac{M_\infty}{a_\infty} \int_b^{b'} dz_1 \int_{-\infty}^{x} u \frac{\partial \mu}{\partial x_1} dx_1$$

or

$$\frac{D_{wake}}{q_\infty} = \frac{2}{h_f U_\infty} \iint_S (u_Q + u_a)\bar{Q}_v dx_1 dz_1 \tag{49}$$

where S is the area of heat addition shown in sketch (i) (and does not include the heat image area).

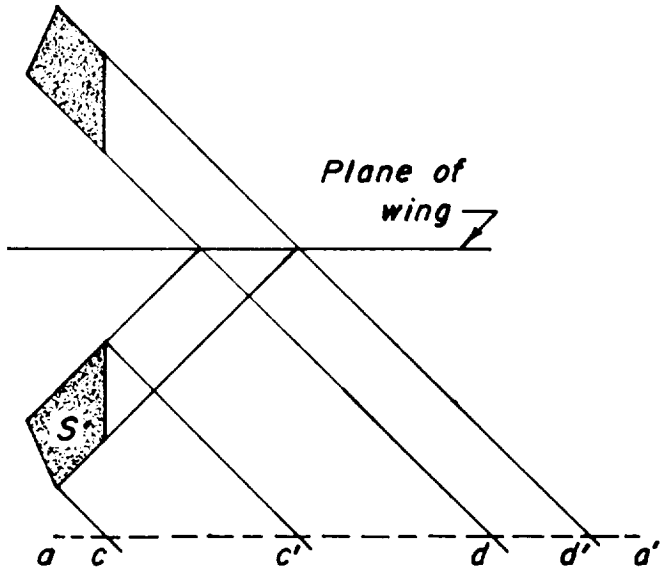
The total drag is the sum of equations (47) and (49). This total drag must be zero when the wing is a flat plate at zero angle of attack, that is, when $u_a = 0$. Hence

$$\frac{2\beta}{U_\infty^2} \int_a^{a'} u_Q^2 dx_1 + \frac{2}{h_f U_\infty} \iint_S \bar{Q}_v u_Q dx_1 dz_1 = 0$$

and

$$\frac{U_\infty^2 D}{q_\infty} = 2\beta \int_a^{a'} (u_a^2 + 2u_a u_Q) dx_1 + 2 \frac{U_\infty}{h_f} \iint_S u_a \bar{Q}_v dx_1 dz_1 \tag{50}$$

The value of u_a beneath the wing is constant along the downgoing characteristics. If, in the second term on the right-hand side of equation (50), we change the coordinate system so that one of the coordinates is parallel to a characteristic line and the free stream is the direction of the other, $\bar{Q}_v$ can be integrated along the characteristics. Using equation (30), one can show



Sketch (j)

$$\frac{2U_\infty}{h_f} \iint_S u_a \bar{Q}_v dx_1 dz_1 = -2\beta \int_c^{c'} 2u_a u_Q dx_1$$

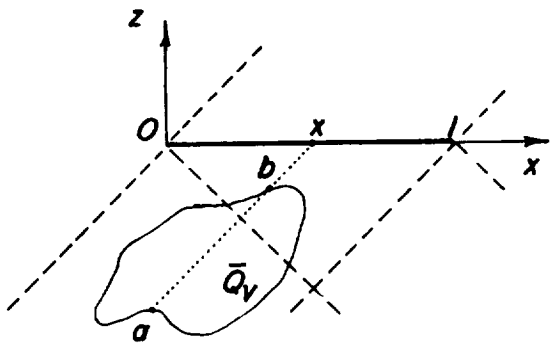
and the final expression for the drag is

$$\frac{U_\infty^2 D}{q_\infty} = 2\beta \int_a^{a'} u_a^2 dx_1 + 4\beta \int_c^{d'} u_a u_Q dx_1 \tag{51}$$

where the lengths cc' and dd' are defined in sketch (j).

Equation (51) gives the same value of drag as equation (46) since u_Q along cc' is half its magnitude along the wing as illustrated in sketch (j).

Optimum Wings in Heated Supersonic Fields



Sketch (k)

Consider the unit chord wing and heat field shown in sketch (k). The heat is added entirely within the fore waves from the wing leading and trailing edges since heat added outside this region will not influence the wing and would not, therefore, exist in an optimum combination. Consider, also, only cases where heat is added beneath the wing. Then equations (44) through (46) can be expressed in the form

$$C_D = \frac{2\beta}{U_\infty^2} \int_0^1 \left[u_l^2(x) + u_u^2(x) - \frac{U_\infty}{\beta} u_l(x) G(x) \right] dx$$

$$C_L = \frac{-2}{U_\infty} \int_0^1 \left[u_l(x) - u_u(x) - \frac{U_\infty}{\beta} G(x) \right] dx$$

$$P_c = \frac{2}{(\gamma-1)M_\infty^2} \int_0^1 G(x) dx$$

where

$$G(x) = \frac{1}{h_f} \int_a^b \bar{Q}_V(x-\beta z, z) dz \tag{52}$$

and a and b are the extremities of the heated region along the characteristic line passing through x on the airfoil. Introducing the fact that $u_l/U_\infty = \lambda_l/\beta$ and $u_u/U_\infty = -\lambda_u/\beta$ where λ_l and λ_u are the lower and upper airfoil slopes and introducing the notation

$$\left.\begin{aligned} \lambda_u &= \lambda_c+\lambda_t \\ \lambda_l &= \lambda_c-\lambda_t \end{aligned}\right\} \tag{53}$$

λ_c being the slope of the airfoil camber line and λ_t being the upper surface slope of the symmetrical thickness distribution, one can show

$$\beta C_x = \beta C_{D_o}+2\int_o^1 [2\lambda_c^2+2\lambda_t^2-(\lambda_c-\lambda_t)G(x)]dx \tag{54a}$$

$$C_L = -\frac{2}{\beta}\int_o^1 [2\lambda_c-G(x)]dx \tag{54b}$$

$$P_c = \frac{2}{(\gamma-1)M_\infty^2}\int_o^1 G(x)dx \tag{54c}$$

The coefficient C_x is the net force in the stream direction and is positive when a net drag results. The coefficient C_{D_o} is the net drag to be overcome by the heat introduced under the wing exclusive of the wing wave drag. In the absence of any other form of propulsion it represents the friction drag, induced drag, and the wave drag of the other airplane parts. Our object is again to seek a maximum value of LU_∞/P when $C_x = 0$.

Since C_L and P are not affected by λ_t, we can start by finding the thickness that minimizes C_x. If we require a net volume $\overline{V}$ (dimensionless, equal to (volume/c^2 times unit span)) and insist that the airfoil close, this leads to the shape

$$\lambda_t(x) = \frac{1}{2}\left(6\overline{V}-3G_1+2G_o\right) - \frac{3x}{2}\left(4\overline{V}-2G_1+G_o\right) - \frac{1}{4}G(x) \tag{55}$$

where

$$G_o = \int_o^1 G(x)dx \tag{56a}$$

$$G_1 = \int_o^1 xG(x)dx \tag{56b}$$

Then if λ_c is optimized by minimizing the quantity

$$I = C_x + \Lambda C_L$$

where Λ is a constant, there results

$$\lambda_c = \frac{1}{2}\Lambda + \frac{1}{4}G(x) \tag{57}$$

and

$$\beta C_x = \beta C_{D_o} + \Lambda^2 + 6\bar{V}(2\bar{V} + G_o - 2G_1) + G_o^2 - 3G_oG_1 + 3G_1^2 - \frac{1}{2}\int_o^1 G^2(x)dx \tag{58a}$$

$$C_L = \frac{1}{\beta}(G_o - 2\Lambda) \tag{58b}$$

$$P_c = \frac{2G_o}{(\gamma-1)M_\infty^2} \tag{58c}$$

If heating is absent, equation (55) gives the familiar result that the closed airfoil having a minimum drag for a fixed volume has a biconvex section. The optimum section shape with heating depends, obviously, on the nature of the heat addition, but even if $G(x)$ is constant over a given portion, the airfoil is, in general, a combination of a double-wedge and biconvex section.

In the interesting case when $\bar{V}$ itself is chosen to minimize C_x (under the supposition, which must be verified, that the resulting section is real), there results

$$\bar{V} = -\frac{1}{4}G_o + \frac{1}{2}G_1 \tag{59}$$

$$Z_t = \frac{x}{4}G_o - \frac{1}{4}\int_o^x G(x_1)dx_1 \tag{60}$$

$$Z_c = \frac{x}{2}\Lambda + \frac{1}{4}\int_o^x G(x_1)dx_1 \tag{61}$$

where $z = Z_t(x)$ and $z = Z_c(x)$ are the equations of the upper surface of the thickness distribution and of the camber line, respectively. Obviously a real airfoil exists if

$$x\int_o^1 G(x)dx > \int_o^x G(x)dx \tag{62}$$

for $0 \leq x \leq 1$. The equations for the drag, lift, and power become

$$\beta C_x = \beta C_{D_o} + \Lambda^2 - \frac{1}{2}\int_o^1 G^2(x)dx + \frac{1}{4} G_o^2 \tag{63a}$$

$$C_L = \frac{1}{\beta}(G_o - 2\Lambda) \tag{63b}$$

$$P_c = \frac{2G_o}{(\gamma-1)M_\infty^2} \tag{63c}$$

and, further, the range efficiency factor becomes

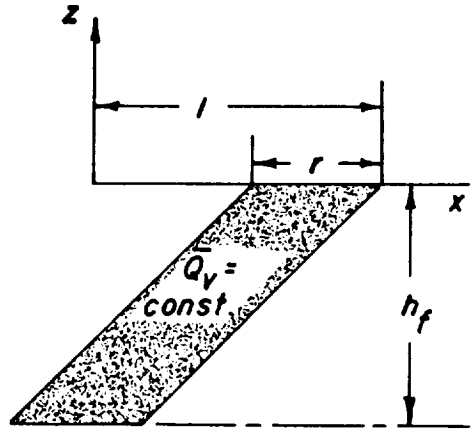
$$\left(\frac{LU_\infty}{P}\right)_{C_x=0} = \frac{(\gamma-1)M_\infty^2}{2\beta}\left\{1 + \sqrt{\frac{2\int_o^1 G^2(x)dx - 4\beta C_{D_o}}{\left[\int_o^1 G(x)dx\right]^2} - 1}\right\} \tag{64}$$

External Heating Only

Let us study the case when $\bar{Q}_v$ is a constant over the area illustrated in sketch (l). Then setting $\sqrt{M_\infty^2-1} \approx M_\infty$

$$\left(\frac{LU_\infty}{P}\right)_{C_x=0} = \frac{M_\infty}{5}\left(1 + \sqrt{\frac{2}{r} - 1 - \frac{4}{r^2}\frac{M_\infty C_{D_o}}{\bar{Q}_v^2}}\right) \tag{65}$$

An inspection of equation (65) yields the principal message of linear theory with regard to heating in a supersonic stream. Notice that no matter how small r becomes $\bar{Q}_v$ can be made large enough (and to stay within the bounds of linear theory, this means that the depth of the region can be made long enough) so that $r\bar{Q}_v$ is fixed. The term $2/r$ is then dominant and the theory predicts indefinitely large values of $(LU_\infty/P)_{C_x=0}$.

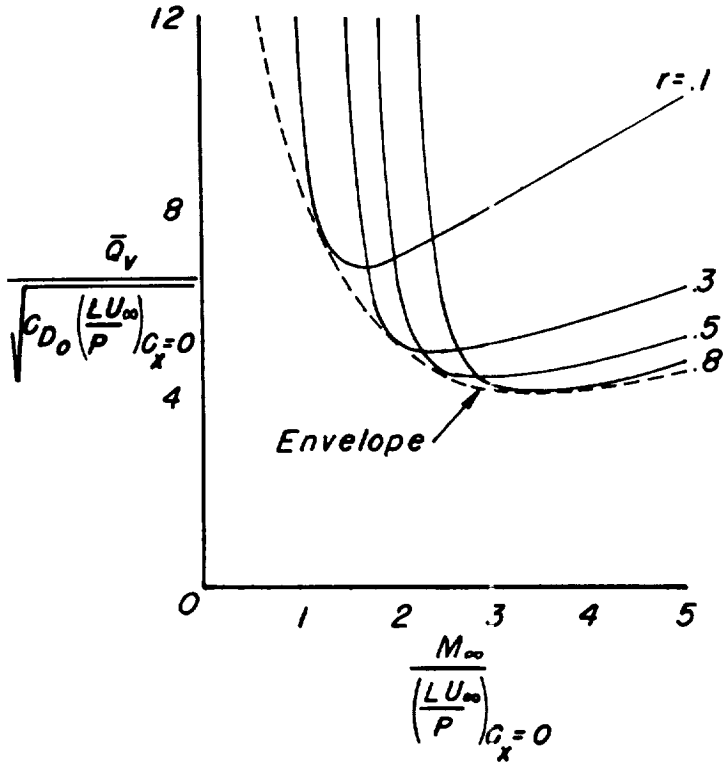


Sketch (l)

The physical reason for this is quite apparent. Two-dimensional linear theory predicts no attenuation of

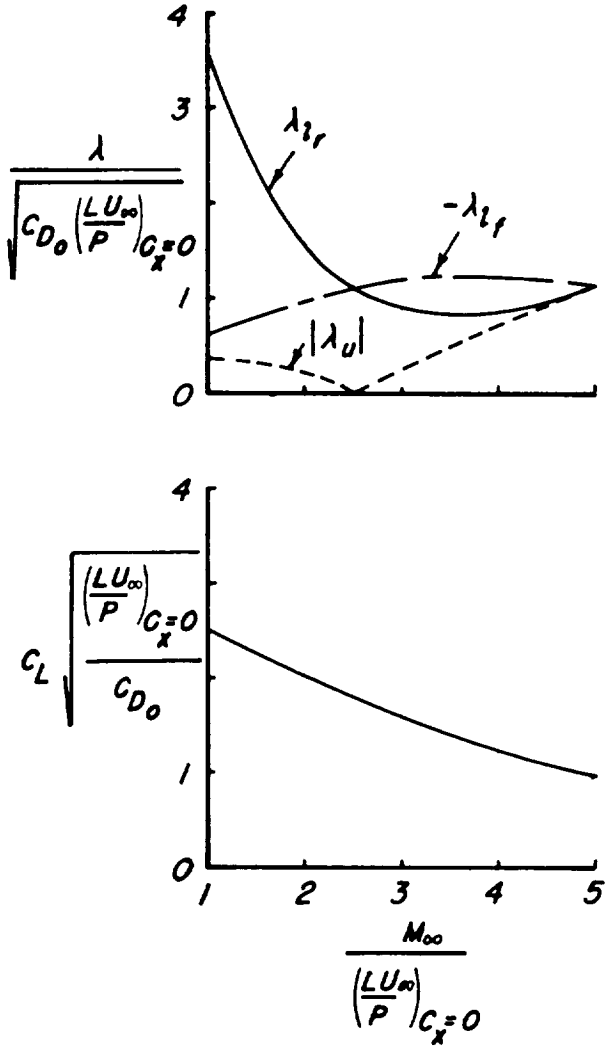
pressure along Mach waves. The efficiency of a unit heat source in producing pressure on the bottom surface of the wing is, therefore, independent of the source position so long as its upward going Mach wave strikes the wing. In the prediction of an optimum, then, the theory will lead to the extreme condition of placing all the heat source on the Mach line intersecting the point of maximum slope on the lower surface. Obviously this result is absurd from a practical standpoint, since the linear model would no longer give a valid estimate of the physical flow from several points of view. It gives, however, the trend of the first-order terms in forming a criterion for optimum heating condition in supersonic flow and provides a good starting point for the study of higher order effects. A detailed study of a particular case is presented in a subsequent section.

Let us look at equation (65) in a different light and find the value of r which requires the minimum heat input for a given $(LU_\infty/P)_{C_x=0}$. It is apparent from sketch (m) that these minima exist. Considering this r



Sketch (m)

to be optimum, we can construct the curves shown in sketch (n) where λ_{l_f} and λ_{l_r} are the slopes of the front and rear portions of the lower surface, respectively. Under the assumption that $\bar{Q}_v$ is a constant over the region shown in sketch (l), these curves give the lift and power



Sketch (n)

coefficients, the extent of heating, and the surface slopes for the condition of minimum heating required to produce a given $(LU_\infty/P)_{C_x=0}$ at a given Mach number.

Engine Combined With External Heating

If we seek to find the effect of combining engine thrust with the lift and thrust provided by external heating, we can return to equations (54) in which C_x is not zero and the power coefficient is increased by C_x/η where

$$\eta = \left(\frac{T_h U_\infty}{P}\right)_{\text{engine}}$$

is the engine alone efficiency parameter. Combining equation (55) with (54a) and optimizing the volume, one can write

$$\beta C_X = \beta C_{D_O} + 2\int_O^1 [2\lambda_c{}^2 - \lambda_c G(x)]dx + \frac{1}{4} G_O{}^2 - \frac{1}{4}\int_O^1 G^2(x)dx \tag{67a}$$

$$\beta C_L = -2\int_O^1 [2\lambda_c - G(x)]dx \tag{67b}$$

$$P_c = \frac{2}{(\gamma-1)M_\infty{}^2}\int_O^1 G(x)dx + \frac{1}{\eta} C_X \tag{67c}$$

For simplicity, again assume $\bar{Q}_V$ is a constant over the area shown in sketch (l), and, further, that

$$\lambda_c = \begin{cases} -\alpha_O , & 0 < x < 1 - r \\ -\alpha_1 , & 1 - r < x < 1 \end{cases} \tag{68}$$

where α_O and α_1 are constants. Now optimizing

$$\frac{LU_\infty}{P} = \frac{4\alpha_O(1-r)+4\alpha_1 r - 2r\bar{Q}_V}{\frac{2\beta r\bar{Q}_V}{(\gamma-1)M_\infty{}^2} + \frac{\beta C_X}{\eta}}$$

for a fixed

$$\beta C_X = \beta C_{D_O} + 4\alpha_c{}^2(1-r) + 4\alpha_1{}^2 r + 2\bar{Q}_V\alpha_1 r - \frac{1}{4}\bar{Q}_V{}^2 r(1-r)$$

gives

$$\frac{LU_\infty}{P} = \frac{r\bar{Q}_V + \sqrt{r(2-r)\bar{Q}_V{}^2 + 4\beta(C_X - C_{D_O})}}{\frac{2\beta r\bar{Q}_V}{(\gamma-1)M_\infty{}^2} + \frac{\beta C_X}{\eta}} \tag{69}$$

When C_X, which now represents the thrust which must be provided by the engine, is zero, this reduces to equation (65). For zero external heating it becomes

$$\left(\frac{LU_\infty}{P}\right)_O = \frac{2\sqrt{\beta(C_X - C_{D_O})}}{\left(\frac{\beta C_X}{\eta}\right)} \tag{70}$$

which is a maximum for $C_x = 2C_{D_o}$; being, in that case

$$\left(\frac{LU_\infty}{P}\right)_o = \frac{\eta}{\sqrt{\beta C_{D_o}}} \tag{71}$$

If we seek the value of C_x which makes LU_∞/P in equation (69) a maximum, there results for $\beta \approx M_\infty$

$$\frac{C_x}{2C_{D_o}} = \left(2 - \frac{2}{r}\right)\xi^2 + 5\xi\frac{\left(\frac{LU_\infty}{P}\right)_o}{M_\infty} + 1 - \xi\sqrt{\left(2 - \frac{2}{r}\right)\xi^2 + 10\xi\frac{\left(\frac{LU_\infty}{P}\right)_o}{M_\infty} + 1} \tag{72}$$

$$\frac{LU_\infty}{P} = \frac{\left(\frac{LU_\infty}{P}\right)_o}{\sqrt{\left(2 - \frac{2}{r}\right)\xi^2 + 10\xi\frac{\left(\frac{LU_\infty}{P}\right)_o}{M_\infty} + 1} - \xi} \tag{73}$$

where

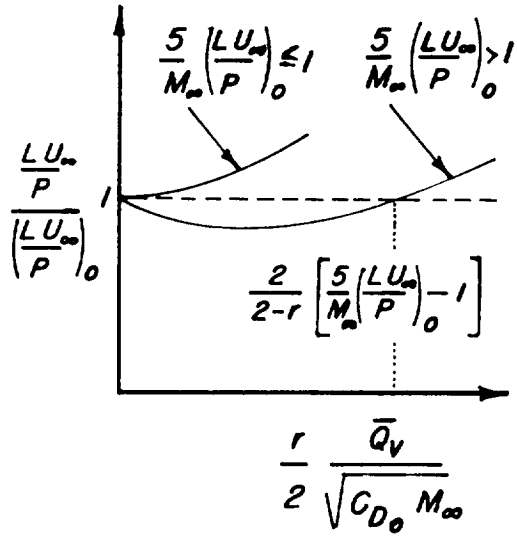
$$\xi = \frac{r}{2}\frac{\bar{Q}_v}{\sqrt{M_\infty C_{D_o}}} \tag{74}$$

and $(LU_\infty/P)_o$ has been defined in equation (71) as the maximum range efficiency factor for zero external heat addition.

Sketch (o) illustrates the general nature of the results. If $(5/M_\infty)(LU_\infty/P)_o > 1$ (for an aerodynamic lift-drag ratio of 6 and an engine efficiency of 0.3, $(LU_\infty/P)_o = 1.8$) a small amount of external heating decreases the range efficiency factor. However, the efficiency can be increased if enough heat is supplied so that

$$\frac{\bar{Q}_v}{\sqrt{M_\infty C_{D_o}}} > \frac{4}{r(2-r)}\left[\frac{5}{M_\infty}\left(\frac{LU_\infty}{P}\right)_o - 1\right] \tag{75}$$

If $(5/M_\infty)(LU_\infty/P)_o \leq 1$, LU_∞/P can be increased by even the smallest amount of external heat addition.



Sketch (o)

Let us set

$$K = \frac{\left(\frac{LU_\infty}{P}\right)}{\left(\frac{LU_\infty}{P}\right)_o} \tag{76}$$

and seek the value of r which requires the minimum external heat to produce a given K. We find for $(5K/M_\infty)(LU_\infty/P)_o > 1$

$$r = \frac{\sqrt{K^2-1}}{\sqrt{K^2-1} + \frac{5K}{M_\infty}\left(\frac{LU_\infty}{P}\right)_o - 1} \tag{77a}$$

$$\frac{\bar{Q}_v}{\sqrt{C_{D_o}\left(\frac{LU_\infty}{P}\right)_o}} = \frac{2}{K}\sqrt{\frac{M_\infty}{\left(\frac{LU_\infty}{P}\right)_o}}\left[\sqrt{K^2-1} + \frac{5K}{M_\infty}\left(\frac{LU_\infty}{P}\right)_o - 1\right] \tag{77b}$$

$$\frac{C_x}{C_{D_o}} = \frac{2}{K^2}\left\{1 - \left[\frac{5K}{M_\infty}\left(\frac{LU_\infty}{P}\right)_c - 1\right]\sqrt{K^2-1}\right\} \tag{77c}$$

and for $(5K/M_\infty)(LU_\infty/P)_o < 1$

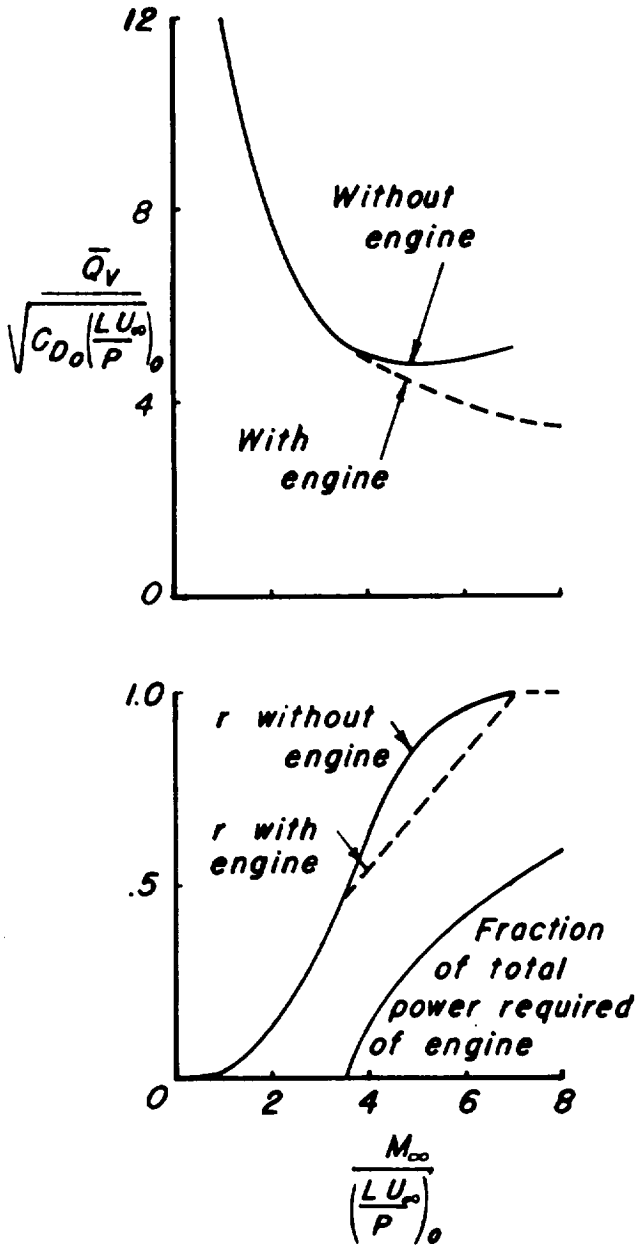
$$r = 1 \tag{78a}$$

$$\frac{\bar{Q}_v}{\sqrt{C_{D_o}\left(\frac{LU_\infty}{P}\right)_o}} = \frac{2}{K}\left\{\frac{5K}{M_\infty}\left(\frac{LU_\infty}{P}\right)_o - 1 + \sqrt{\left[\frac{5K}{M_\infty}\left(\frac{LU_\infty}{P}\right)_o - 1\right]^2 + K^2 - 1}\right\}\sqrt{\frac{M_\infty}{\left(\frac{LU_\infty}{P}\right)_o}} \tag{78b}$$

$$\frac{C_x}{C_{D_o}} = \frac{2}{K^2}\left\{1 - \left[\frac{5K}{M_\infty}\left(\frac{LU_\infty}{P}\right)_o - 1\right]^2 - \left[\frac{5K}{M_\infty}\left(\frac{LU_\infty}{P}\right)_o - 1\right]\sqrt{\left[\frac{5K}{M_\infty}\left(\frac{LU_\infty}{P}\right)_o - 1\right]^2 + K^2 - 1}\right\} \tag{78c}$$

Typical results arising from the application of equations (77) and (78) are shown in sketch (p). The results apply to the case in which the range efficiency parameter is to be increased 40 percent above its maximum value obtainable with no external heating. Two possibilities are considered; one, no engine, and the other, a combination of engine and external heat. In both cases r, the extent of heating, is chosen to minimize $\bar{Q}_v$. For

values of $M_\infty/(LU_\infty/P)_o$ below about 3.46 the magnitude of $\bar{Q}_v$ is lower if no engine at all is used. Above 3.46 less external heat is required if an engine is used and the percent of total power required of the engine is shown in the lower part of the sketch.



Sketch (p)

Application to a Particular Example

In order to evaluate the rather general results presented in the preceding sections, let us consider in some detail a particular example. Let us consider the case of external heating only and fix at the outset the following parameters

$$\left.\begin{aligned} M_\infty &= 6 \\ C_{D_o} &= 0.0034 \\ (LU_\infty/P)_{C_x=0} &= 2.4 \end{aligned}\right\} \qquad (79)$$

The value of C_{D_o} is not excessively low when one considers the reduction of the turbulent skin-friction coefficient with Mach number. The value of $(LU_\infty/P)_{C_x=0}$ represents an equivalent conventional airplane combination of engine $T_h U_\infty/P = 0.4$ and aerodynamic $L/D = 6$. These values are what might be expected[3] at the present time of ram-jet airplane performance at $M_\infty = 6$. Therefore, if the values in equations (79) could be attained for a flying vehicle, they would make this kind of heat addition warrant some consideration.

The values assumed in equations (79) when used in conjunction with sketches (m) and (n) represent the following conditions

[3]It is assumed that the boundary layer is for the most part turbulent.

$$\left.\begin{aligned} r &= 0.5 \\ C_L &= 0.067 \\ P_c &= 0.028 \\ \lambda_u &= 0 \\ \lambda_{l_f} &= -0.1 \\ \lambda_{l_r} &= 0.1 \\ \bar{Q}_v &= \left(\frac{\gamma-1}{a_\infty^2}\frac{\partial\mu}{\partial x}\right) h_f = 0.40 \end{aligned}\right\} \tag{80}$$

The lift and power coefficients are reasonable and the lower surface slopes are less than 6° with reference to the free stream. The term involving the heating is the really critical factor.

A principal assumption underlying the linearized theory is that $(\gamma-1/a_\infty^2)(\partial\mu/\partial x)$ is small. For the value of $\bar{Q}_v$ given above, this means that h_f must be of the order of a wing chord for the above figures to be first-order estimates of the heating effect recorded in sketches (m) and (n) under the conditions imposed by equations (79). The length of the heated column is not so surprising if we remember that we are attempting to drive an airplane with an unswept 5-percent-thick wing through the air at a Mach number of 6 with an equivalent L/D of around 6 and we are attempting to do this by adding only a small amount of heat along any given streamline. That such efficiencies are even theoretically possible is due, as has been pointed out, to the fact that the heat is added to a two-dimensional supersonic stream so that pressure disturbances along a characteristic line accumulate with little loss in intensity.

A very valid question remains as to whether the low heating rates we have considered are optimum even if they are practical. They were considered first, not only because linear theory demanded it, but also because small disturbance fields are, in conventional aerodynamics, generally the most efficient. Quite possibly it is preferable for the heat to be intense enough to produce higher order disturbances; this point is discussed briefly in the next section.

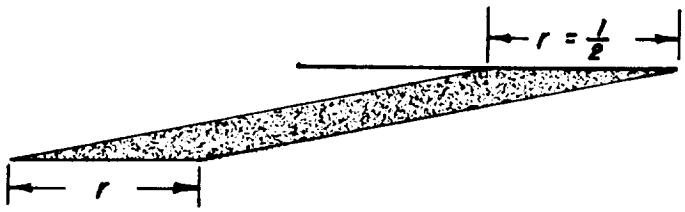
Some Remarks on Nonlinear Effects

Estimations.- Although linear theory can be depended upon, for most practical purposes, to give reliable values for the disturbances near a thin wing if these disturbances are very small relative to the free stream, there is always the question as to its dependability when the

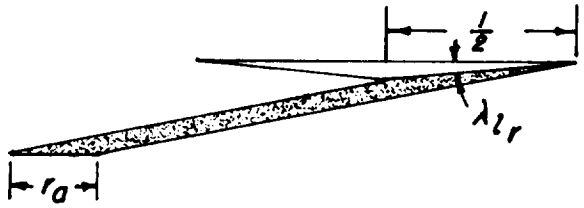
disturbances are those produced by a vehicle actually intended to fly. In problems as simple as the ones we have been studying, estimates of what are probably the principal errors can be made for particular cases. For example, let us examine more closely the true nature of the flow, the linearized version of which is described by equations (79) and (80).

The discussion is divided into three parts; the area of heating, the pressures caused by the wing shape, and the pressures caused by the heat addition. Quite clearly the very fact that the nonlinear aspects of the flow field are being considered means that these three points can not really be considered independently; at this point, however, it seems reasonable to expect that their independent effects deviate more from linear theory than their interdependent effects.[4] This matter can really only be settled by solving the exact equations for the given boundary conditions.

Consider now the actual region of heat addition for the wing in our example. Because we have applied our boundary conditions in the $z = 0$ plane (as is usual in linearized theory) the assumed region of heating is that shown in the upper part of sketch (q). In a real flow, however, the



Linear-theory boundary conditions



Exact boundary conditions

Sketch (q)

width of the heated region required to influence the rear half of the wing is reduced to that shown in the lower half of sketch (q). In our example $r_a \approx 0.4r$. Now according to linear theory, the value of the pressure coefficient on the wing due to the heating is

$$C_p \approx \frac{2}{M_\infty}\left(\frac{\gamma-1}{a_\infty^2}\frac{\partial \mu}{\partial x}\right) h_f$$

[4]Except, perhaps, for the possibility of separation in the heated region.

and the power coefficient is

$$\frac{P}{q_\infty U_\infty c} = \frac{2}{(\gamma-1)M_\infty^2}\left(\frac{\gamma-1}{a_\infty^2}\frac{\partial\mu}{\partial x}\right)h_f r$$

In other words, the pressures, that is, the lift and thrust, are unaffected by shortening the region, but the power required to produce them is decreased to 0.4 of the value based on the linearized boundary conditions.

The pressure coefficients on the bottom of the double-wedge section described in equation (80) flying at $M = 6$ in an unheated flow are, by linear theory

$$C_p = 0.033, \quad \text{front half}$$

$$C_p = -0.033, \quad \text{rear half}$$

On the other hand, simple wave theory (i.e., Prandtl-Meyer expansion for both compression and expansion) gives

$$C_p = 0.048, \quad \text{front half}$$

$$C_p = -0.024, \quad \text{rear half}$$

Notice that the drag is increased by the amount

$$\Delta C_o = \left[(\Sigma|C_p|)_{swt} - (\Sigma|C_p|)_{linear}\right]\tau = 0.0003$$

(and this can be interpreted as reducing the C_{D_o} in equation (79) to 0.0031) and the lift is also increased by the amount

$$\Delta C_L = \frac{1}{2}\left[(\Sigma C_p)_{swt} - (\Sigma C_p)_{linear}\right] = 0.012$$

Finally, consider the pressure produced by the heat addition. The error incurred by the use of linearized theory in this region is difficult to evaluate since an exact solution for the particular example under consideration is not available. A very rough estimate of the error is made in the following discussion.

First, if μ is assumed to be independent[5] of $\vec{V}^2$, the pressure coefficient can be written

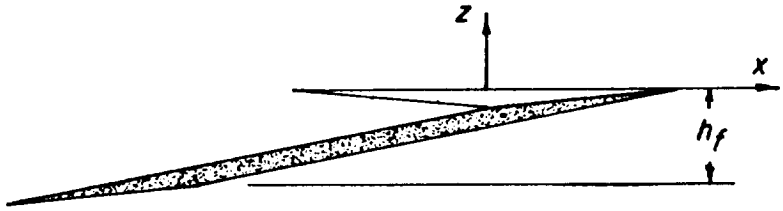
[5]For this example, the effect of μ is, in any event, not large.

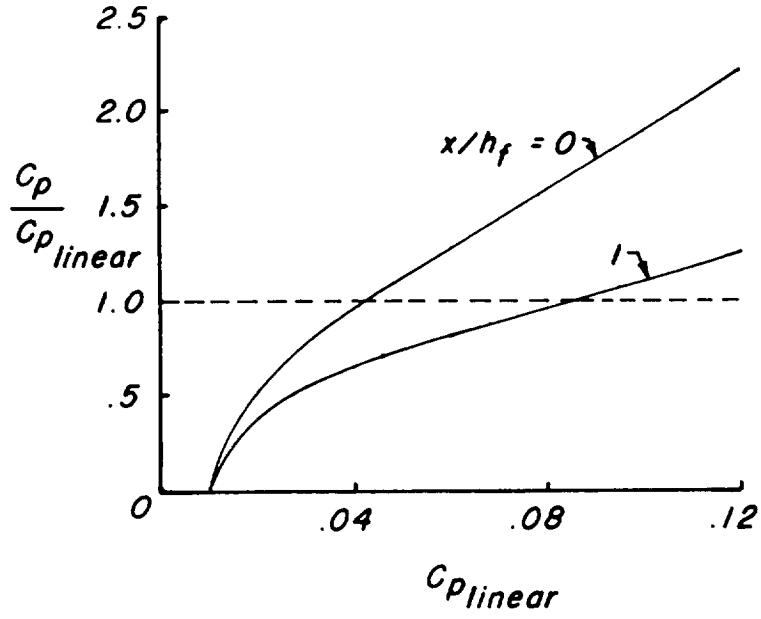
$$C_p = \frac{2}{\gamma M_\infty^2}\left\{\left[1 - \frac{\gamma-1}{2} M_\infty^2 \frac{\left(\frac{w}{U_\infty}\right)^2 + \frac{2u}{U_\infty}}{1 + \frac{\gamma-1}{a_\infty^2}\mu}\right]^{\frac{\gamma}{\gamma-1}} - 1\right\}$$

or, on the wing, according to equations (79) and (80)

$$C_p = 0.0397\left[\left(1+7.2\,\frac{C_{p_{linear}}-0.01}{1+0.4\,\frac{x}{h_f}}\right)^{3.5} - 1\right]$$

where $C_{p_{linear}}$ is the value of the pressure coefficient given by linearized theory and x is the distance into the heated region. Sketch (r) shows the results. The two trends are clearly indicated: one,





Sketch (r)

at some point on the wing the actual pressure is greater than that given by linear theory, and as the linearized value increases (i.e., as the u perturbation on the wing increases) this discrepancy increases; the other, as the heated region is penetrated beyond such a point (or as its depth is decreased) the actual pressure falls relative to its linear value.

In order to make use of sketch (r), we must estimate the value of u along the surface of the wing. If we use linear theory for the heat sources but take into account the fact that the "image plane" is tilted, there results

$$\frac{u_{heat}}{U_\infty} = -\frac{\overline{Q}_v - \lambda l_r}{\beta}$$

So, using equation (80) one finds

$$\left(C_{p_{linear}}\right)_{heat} \approx 0.1$$

It is impossible to tell what the nonlinear effects on this calculation would be without a much more sophisticated analysis. It is almost certain that the value 0.1 is much too low because the entire effect of the heat and reflection is to compress the air, and the linear theory is already 30 percent too low in an unheated flow when it predicts a value of 0.033 at $M_\infty = 6$. It seems conservative, then to estimate $(C_{p_{linear}})_{heat}$ at 0.13.

One can easily show that the value of u induced on the rear surface by the wing alone yields the expression

$$\left(C_{p_{linear}}\right)_{wing} \approx -0.03$$

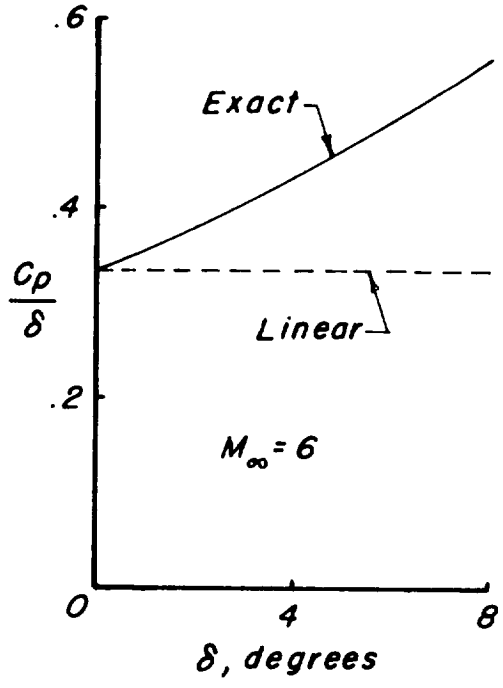
Hence, by these approximations, $C_{p_{linear}}$ on the lower rear surface is about 0.1, which, according to sketch (r), gives an average value for C_p of about 0.14 there.

The following table summarizes the above crude corrections to linear theory for the airfoil section and heating rates ($h_f \approx 0.5c$) given in equations (79) and (80):

	P_c	C_L	C_{D_o}	$\frac{LU_\infty}{P}$
Linear theory	0.028	0.067	0.0034	2.4
Plus correction for size of heated region	.0112	.067	.0034	6.0
Plus correction for actual wing pressure	.0112	.079	.0031	7.0
Plus correction for actual heating pressure	.0112	.094	.0047	8.4

Discussion.- According to the table the linear theory is extremely conservative for the particular example studied. When the aforementioned corrections are applied, not only is the allowable excess drag (C_{D_o}) increased by 38 percent but the equivalent L/D (relative to a ram-jet efficiency $T_h U_\infty / P = 0.4$) is increased to 21. The reliability of these calculations has been repeatedly qualified. They clearly indicate, however, that this kind of heating, if it could be practically obtained, is worthy of study, and really quantitative theoretical estimates must include nonlinear effects.

As was mentioned at the end of the last section, the optimum heating intensity is likely to produce disturbances well above those dealt with by linear theory. The reason for this can be demonstrated as follows. The effect of the heating is to increase the pressure of the air. In one respect adding a given amount of heat to the field corresponds as far as the pressure field is concerned to the insertion of a wedge-shaped airfoil in the flow. It is well known that the pressure rise per degree for a wedge is greater, the greater the angle of stream deflection (see sketch (s)). In the usual case of aerodynamic design this means, in a general way, that surface slopes should be reduced as much as possible to minimize the drag. In the case of heat addition, however, the opposite is true. Of course, no net force acts on a heat source whose only use is to provide as high a pressure as possible. Therefore, inasmuch as a heat source is like a fluid source the heat rate should not be of a low intensity.



Sketch (s)

Unfortunately the effect shown in sketch (s) is not the only nonlinear effect caused by increasing the heat

rate. Another important effect is to increase the local speed of sound which, in turn, decreases the pressure. An estimate of this result has been presented in sketch (r) and is illustrated by the reduction in C_p for a fixed $C_{p_{linear}}$ when x/h_f goes from 0 to 1. Some optimum relation between wing-surface and heating-region geometry and heating intensity is indicated and it appears probable that the intensity required will be of a magnitude to produce higher order disturbances in the flow.

Ames Research Center
National Aeronautics and Space Administration
Moffett Field, Calif., Oct. 10, 1958

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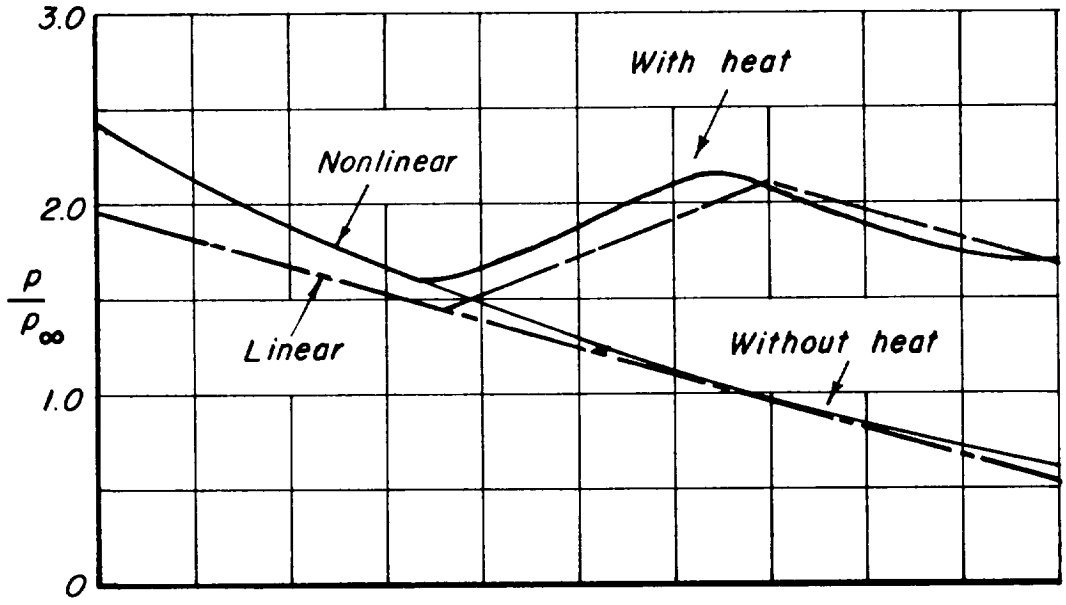
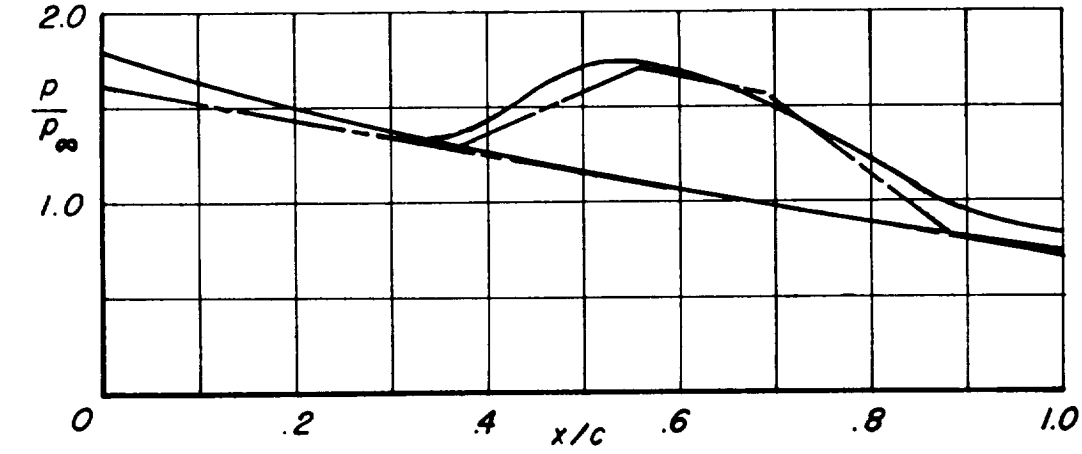


(a) $M_\infty = 5$, $x_i/c = 0.356$, $x_f/c = 1.0$



(b) $M_\infty = 3$, $x_i/c = 0.356$, $x_f/c = 0.690$

Figure 1.- Pressures on lower surface of 5-percent-thick biconvex wing with and without heat addition; $T_f/T_i = 1.243$, $\alpha = 2^\circ$, $h_f/c = 0.07$.