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EXTENDING THE LORENTZ TRANSFORMATION TO MOTION WITH

VARIABLE VELOCITY

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SUMMARY

The problem considered is that of rectilinear motion with variable velocity. The paper gives, by an elementary construction, a system of coordinates which is conformal in the vicinity of the axis of motion. By a particular choice of the scale relation, such restricted conformal transformations can be made to reduce to the Lorentz transformation everywhere in the case of uniform velocity and locally in the case of variable velocity.

INTRODUCTION

The use of Lorentz transformations when the velocity v is a function of the time leads to a nonuniform correspondence of events between accelerated coordinate systems. A uniform correspondence can be achieved, however, by a simple generalization of the Lorentz formula to a transformation of the type known as conformal.

Extension of the theory of relativity by conformal transformations was considered many years ago by H. Bateman (ref. 1). Such transformations preserve a constant velocity of light even during accelerated motions. It seems, however, that such transformations, if applied to the whole of space, are consistent only with special types of motion. In the present paper an extension of the theory by conformal transformations of the simplest possible type, restricted to the vicinity of the axis of motion, is suggested. Such restricted conformal transformations can be made to reduce to the Lorentz transformation everywhere in the case of constant velocity, and locally in the case of variable velocity.

The paper gives an elementary (though somewhat indirect) derivation of the coordinates from the wave pattern formed by electromagnetic signals. In view of the simplicity of the results it seemed unnecessary to translate them into the more concise language of differential geometry.

CONSTRUCTION OF A CONFORMAL COORDINATE SYSTEM BY MEANS OF INTERFERING WAVE SIGNALS

In the Michelson-Morley experiment, an electromagnetic oscillator and a mirror are arranged on a rigid reference body B so as to produce an interfering wave system. Since the waves returned by the mirror are of the same frequency as those emitted by the oscillator, a system of standing waves results.

We may produce the same wave system by means of two oscillators at x_p and x_q on the x axis in an A system which moves relatively to B. Here the waves returned by the upstream oscillator may have a different frequency than those emitted by the downstream oscillator. The interference pattern will then drift in the direction of the line joining the two oscillators. Such an interference pattern moving at the velocity v is of course a standing wave in the B system. The difference in frequency is then related to the Doppler shift as observed from system A.

If we imagine Einstein's clock synchronization experiments (ref. 2) to be carried out continuously, then these experiments will again create a standing wave system in which the phase of a clock is identified with the phase of the electromagnetic wave and the length of a measuring rod (or the spacing between the clocks) is identified with the wave length.

Figure 1 shows an x,t diagram of the clock experiments interpreted as an interfering wave pattern. For convenience the x and t scales are chosen so that the velocity of light is unity. The rectangular x,t axes are those of system A. Maxima and minima of the downstream moving waves are along lines sloping to the left with increasing t, or along lines $x + t$ = constant. The upstream moving waves are identified similarly with the lines $x - t$ = constant.

The maxima and minima of the standing waves in the B system are diagonals of the rectangles formed by the lines $x - t$ = constant and $x + t$ = constant. The Lorentz transformation is, in these terms,

$$\left.\begin{aligned} x' + t' &= \sqrt{\frac{1-v}{1+v}}\,(x+t) \\ x' - t' &= \sqrt{\frac{1+v}{1-v}}\,(x-t) \end{aligned}\right\} \qquad (1)$$

In system B (x',t') we may isolate two clocks C_1 and C_2. A signal from C_1 at t_1' arrives at C_2 at the time t_2' and is reflected back to C_1, arriving there at the time t_3'. By setting C_2 to the time $t_2' = \frac{1}{2}(t_1' + t_3')$ we express the constancy of the velocity of light in the moving system.

Now it is clear that the velocity of light will remain constant and that the Michelson-Morley experiment would show this result in any coordinate system constructed from interfering waves of both families. Hence as the equation of such a transformation we may write, disregarding for the moment y and z,

$$\left.\begin{aligned} x' + t' &= F(x+t) \\ x' - t' &= G(x-t) \end{aligned}\right\} \tag{2}$$

Figure 2 shows a curvilinear coordinate system constructed in this way. By starting with the Doppler variation of the frequencies received at the points x_p and x_q we obtain a network of lines at 45^o, but with variable spacing. The interference pattern then forms a family of "sloshing" waves with the velocity of the wave crests variable from point to point. In a coordinate system tied to such wave crests the result of the Michelson-Morley experiment is always the same. It is a simple matter to verify that the coordinate system of figure 2 preserves a constant velocity of light.

If we denote

$$\left.\begin{aligned} F'(x+t) &= f \\ G'(x-t) &= g \end{aligned}\right\} \tag{3}$$

then the velocity v of B relative to A is given by

$$\left(\frac{dx}{dt}\right)_{x'=\text{const}} = v = \frac{g-f}{g+f} \tag{4}$$

and is variable from point to point. We require that if B moves at v relative to A, then A moves at the velocity $-v$ relative to B. The calculation shows

$$\left(\frac{dx'}{dt'}\right)_{x=\text{const}} = \frac{f-g}{f+g} = -v \tag{5}$$

For the law of composition of velocities we need three coordinate systems A, B, C. If B moves at v_1 relative to A, and C moves at v_2 relative to B, then we write

$$\left.\begin{aligned} v_1 &= \frac{g_1-f_1}{g_1+f_1} \\ v_2 &= \frac{g_2-f_2}{g_2+f_2} \end{aligned}\right\} \tag{6}$$

There results

$$v_3 = \frac{g_1g_2-f_1f_2}{g_1g_2+f_1f_2} \tag{7}$$

for the velocity of C relative to A. The latter formula reduces to the law of composition of velocities in special relativity, namely,

$$v_3 = \frac{v_1+v_2}{1+v_1v_2} \tag{7a}$$

along each single line $x' =$ constant. Hence this law depends only on the constancy of the velocity of light.

DETERMINATION OF THE SCALE FACTOR fg

If we select one of the curvilinear x' lines as the origin of the primed system, then we may identify this line as the world track of a particle B in the A system. For the rate of increase of the proper time along this track we compute

$$(dt')_{x'=\text{const}} = \sqrt{1-v^2}\,\sqrt{fg}\,dt \tag{8}$$

This relation leads to the well-known time paradox of special relativity. Similarly, for a distance paradox, we may write

$$(dx')_{t'=\text{const}} = \sqrt{1-v^2}\,\sqrt{fg}\,dx \tag{9}$$

In our case the displacements following a variable motion are not the same for all clocks of the moving system, but are functions of position. However, if a system moves but later comes to rest, the residual space and time displacements eventually become the same for all clocks within light cones originating at the disturbance.

Thus far the functions F and G, or their derivatives f and g, are undetermined functions. If we adopt a certain curvilinear line in the x',t' system as the path of a particle, or as the spatial origin of the moving system, then the functions f and g will be partly determined by the velocity along the path, that is, through equation (4). However, since equation (4) involves only the ratio of f to g there remains a scale factor, or gauge factor, to be determined.

For the case of uniform motion the scale factor of the Lorentz transformation is so adjusted that the direct and the inverse transformations indicate the same change of x and t dimensions. The transformation and its inverse are then indistinguishable, except for the sign of the velocity. This condition is satisfied by making the quantity fg equal to unity throughout.

In the case of variable motion such a reciprocal scale relation cannot be maintained everywhere, nor can it be maintained uniformly between any two points of the systems A and B. It becomes necessary then to make a distinction between the two coordinate systems. To make this distinction in the customary way we shall suppose that A is an inertial system. Then our transformation will approach the Lorentz transformation in the vicinity of the moving origin of B. This condition requires that $fg = 1$ along $x' = 0$ and is, together with equation (4), sufficient to determine the functions f and g.

The requirement of the reciprocal scale relation $f = 1/g$ along $x' = 0$ imposes at each instant a relation of symmetry between this single point of the B system and a succession of different points of the A system. As a consequence this relation cannot be maintained between discrete points of the two systems which become separated. Hence the relation of kinematic equivalence demanded by Milne (ref. 3) is not satisfied. Evidently Milne's relation cannot be maintained by conformal transformations that are locally tangent to the Lorentz transformation, except in the case of uniform velocity.

Figure 3 illustrates the application of the present method to a problem of accelerated rectilinear motion. If the path of a particle B is given then it is only necessary to lay off from this path lines at 45^o, spaced at equal intervals of the proper time as given by the velocity v in conjunction with the restricted theory of relativity. The intersections of these characteristic lines then determine the continuation of the coordinates throughout the x,t region.

Thus far our considerations have been restricted to the single space direction x parallel to the direction of the velocity. If the constancy of the velocity of light is to be maintained in all directions, then the differential form of the transformation must remain.

$$-ds'^2 = \lambda^2(x,y,z,t)(dx^2+dy^2+dz^2-dt^2) \tag{10}$$

Generalization of the theory of relativity by transformations of the group satisfying equation (10) was considered by Bateman (ref. 1) and more recently by Infeld and Schild (ref. 4), Littlewood (ref. 5), E. L. Hill (ref. 6), and others. It seems that the transformations of this group are essentially limited in such a way that only certain motions are consistent with a constant velocity of light throughout space. However, if we restrict our attention to a small region in the vicinity of the x axis, we may write

$$\left.\begin{aligned} y' &= \sqrt{fg}\, y \\ z' &= \sqrt{fg}\, z \end{aligned}\right\} \tag{11}$$

and then

$$(-ds'^2)_{y^2+z^2\to o} = fg(dx^2+dy^2+dz^2-dt^2) \tag{12}$$

Hence a constant velocity of light can be maintained in the vicinity of the axis throughout a variable motion.

It is interesting to note (see fig. 3) that the coordinate distortions or "gravitational waves" associated with the acceleration of system B ultimately propagate away from the origin with the velocity of light. The interpretation of these waves, as well as other dynamical questions, will require an extension of the considerations given herein.

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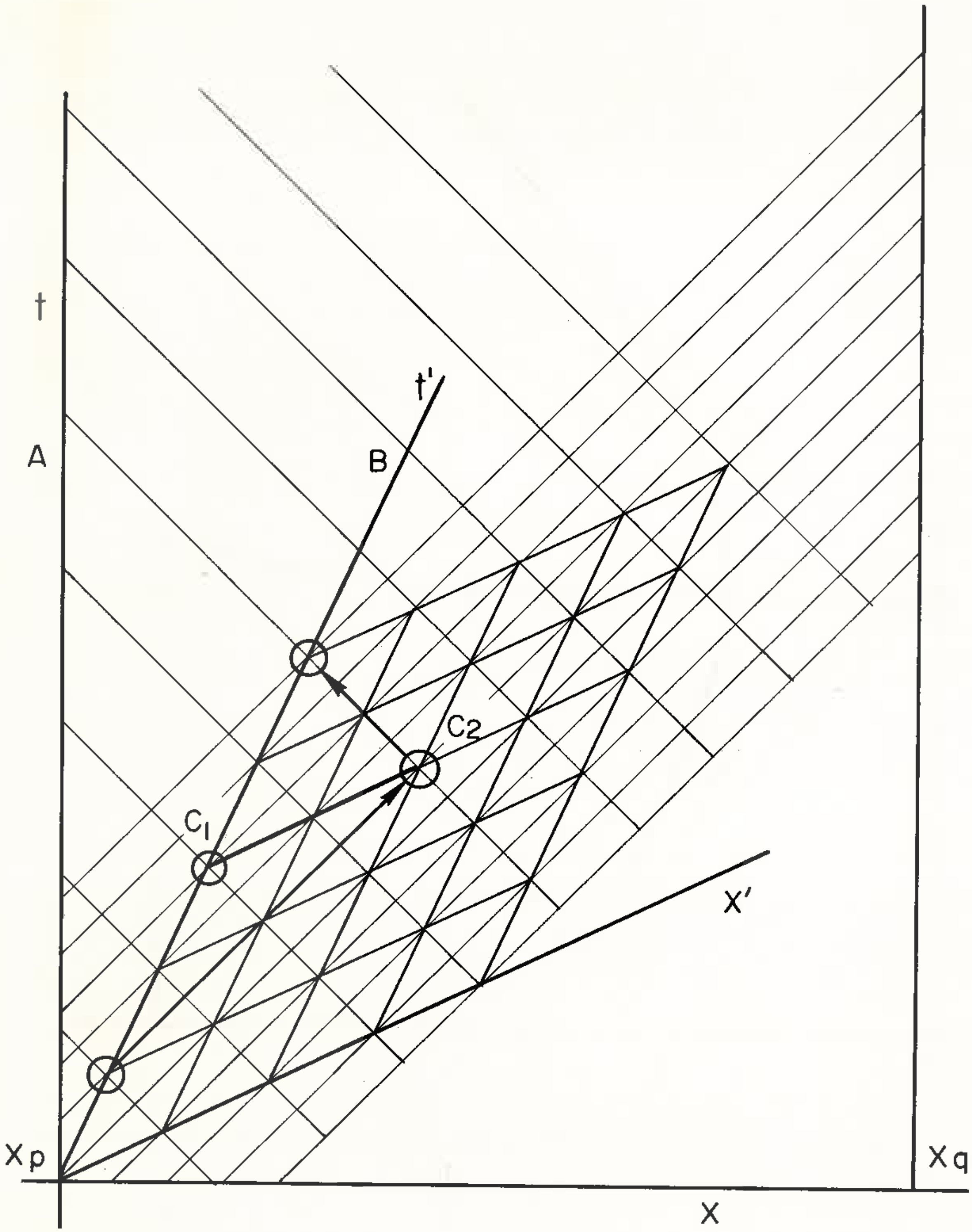

Figure 1.- Interference pattern formed by clock signals in Lorentz coordinates.

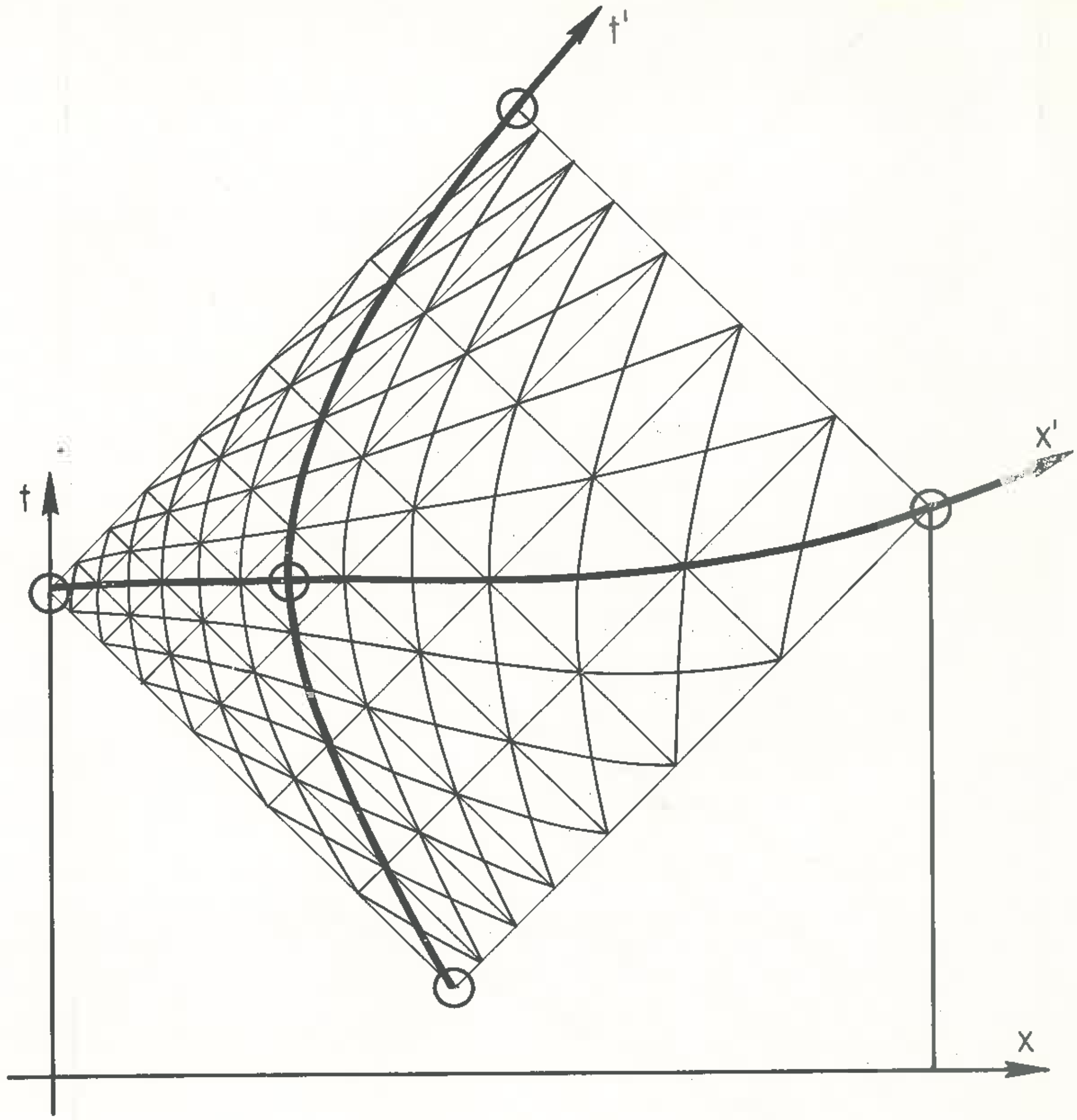

Figure 2.- Coordinates in which the velocity of light is constant.

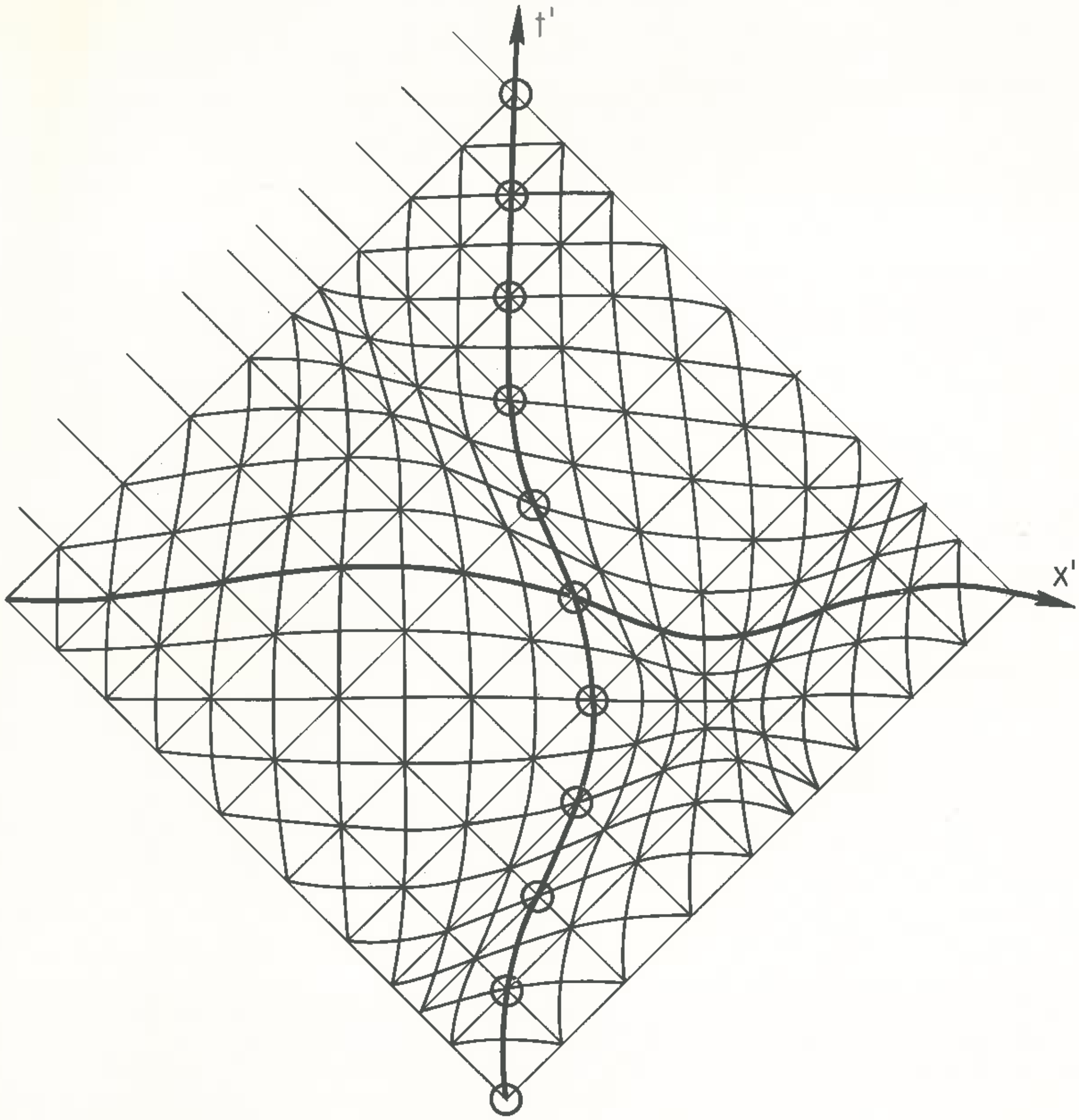

Figure 3.- Continuation of Lorentz transformation along straight characteristic lines.