NASA TN D-456

NASA TN D-456

IN-71

TECHNICAL NOTE

D-456

DIFFUSION OF SOUND WAVES IN A TURBULENT ATMOSPHERE

By Richard H. Lyon

Massachusetts Institute of Technology

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

WASHINGTON September 1960

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

TECHNICAL NOTE D-456

DIFFUSION OF SOUND WAVES IN A TURBULENT ATMOSPHERE

By Richard H. Lyon

SUMMARY

The directional and frequency diffusion of a plane monochromatic sound wave in statistically homogeneous, isotropic, and stationary turbulence is analyzed theoretically. The treatment is based on the diffusion equation for the energy density of sound waves, using the scattering cross section derived by Kraichnan for the type of turbulence assumed here.

A form for the frequency-wave number spectrum of the turbulence is adopted which contains the pertinent parameters of the flow and is adapted to ease of calculation. A new approach to the evaluation of the characteristic period of the flow is suggested. This spectrum is then related to the scattering cross section.

Finally, a diffusion equation is derived as a small-angle scattering approximation to the rigorous transport equation. The rate of spread of the incident wave in frequency and direction is calculated, as well as the power spectrum and autocorrelation for the wave.

INTRODUCTION

The fluctuation of acoustic signals in media of randomly varying index of refraction has been considered by several investigators from nearly as many points of view. On the basis of geometrical or ray acoustics, Bergmann (ref. 1) and Obukhov (ref. 2) have evaluated the fluctuations in phase and amplitude of a spherical acoustic wave emanating from a point source. Skudrzyk (ref. 3) has also estimated these quantities and obtained the probability distribution of the signal from a geometrical scattering point of view and has tabulated his results. Mintzer (refs. 4, 5, and 6), using single and double scattering formulations, also calculated similar quantities. One of the beauties of Obukhov's approach is that he was able to obtain both Mintzer's and Bergmann's results as limiting forms of a single derivation.

The present work proposes to solve somewhat the same problem from a multiple scattering point of view by formulating the transport equation for the acoustic energy density spectrum in turbulent flow. The diffusion takes place both in frequency and direction, since the flow varies in time as well as space. The directional diffusion of a sound wave has been previously considered by Lighthill (ref. 7), who proceeded from a diffusion equation which appears to have been inspired by random walk considerations.

The present investigation was conducted at the Acoustics Laboratory of the Massachusetts Institute of Technology under the sponsorship and with the financial assistance of the National Advisory Committee for Aeronautics.

W 1 2 7

SYMBOLS

A,B	parameters for hypothetical spectrum
b	rms frequency variation
c	wave in undisturbed medium
$d\bar{k}$	volume element in k space
$d\bar{x}$	volume element in x space
$d(\Phi)$	element of solid angle
E	wave number and spatial density of acoustic energy
$\mathcal{E}$	acoustic energy
$\bar{\mathcal{J}}$	acoustic intensity
$\bar{I}$	wave number and spatial density of acoustic intensity
I_o	intensity of unscattered sound wave
$J_o(x)$	Bessel's function of zero order
$\bar{K}$	scattering vector and turbulent wave number
$\bar{k}$	wave vector

k_o	wave number where $W(K)$ has its maximum
L	correlation length
M	Mach number of turbulent fluctuations
m	summation index
$\bar{n},\bar{n}'$	unit vectors in direction of $\bar{k}$ and $\bar{k}'$, respectively
$Pm(\xi)$	Legendre polynomial
$P(n,\omega)$	power spectrum of scattered wave
$Q(\bar{k})$	total scattering cross section per unit volume
r	frequency variable
t	time
U	space-time transform of $\bar{u}$
$\bar{u}$	turbulent velocity field
V	volume of scattering region
$W(K)$	isotropic spectrum for time-independent turbulence
$w(K,\Omega)$	isotropic spectrum for time-dependent turbulence
$X(\omega)$	time transform of acoustic condensation
$\bar{x}$	position vector
z	distance over which sound has propagated in atmosphere
α	inverse of k_o
β	scale factor in $W(K)$
$\Gamma(x)$	gamma function
γ	normalizing constant
Δ	relative frequency shift from ω_o
$\delta(x)$	Dirac delta function
ϵ	dissipation rate for turbulent energy

μ	parameter in hypothetical spectrum
η	distance variable
θ	total angle of deviation from original direction
κ	transform variable
λ	characteristic frequency of turbulence
ν	kinematic viscosity
ξ	$\cos\theta$
ρ	relative frequency shift from ω
ρ_o	ambient density of atmosphere
$\rho_{1,2}$	branch points of integrand
τ	delay time
σ	differential scattering cross section
$\bar{\sigma}$	vector from $\bar{x}'$ to x
ϕ	scattering angle
Φ_{ij}	spectral function for homogeneous turbulence
ψ	azimuthal angle
$\psi(\tau)$	autocorrelation function for acoustic signal
Ω	frequency of turbulence
ω	frequency associated with wave number $\bar{k}$
ω_o	frequency of unperturbed sound wave

Superscripts:

$(\ )'$	pertaining to scattered quantity
$(\bar{\ })$	vector

Subscripts:

$\langle \cdot \rangle_{av}$	ensemble average
$\langle \cdot \rangle_{\sigma}$	averaged over differential cross section σ
p	propagation

TRANSPORT EQUATION

The spectral energy density $E(\bar{x},\bar{k})$ of a sound field is defined in such a manner that in the volume $d\bar{x}$ and in the range of wave numbers $d\bar{k}$, (i.e., the wave number $\bar{k}$ falls in the range $\bar{k}$, $\bar{k} + d\bar{k}$) the acoustic energy is

$$d\mathcal{E} = E(\bar{x},\bar{k})d\bar{x}\ d\bar{k} \tag{1}$$

A beam of intensity $d\bar{\mathcal{I}}$ and the wave number in $d\bar{k}$ has energy scattered into beams with wave numbers in $d\bar{k}'$ at a rate given by $|d\bar{\mathcal{I}}|\sigma(\bar{k}/\bar{k}')d\bar{k}'$. In the plane waves being considered, the intensity is related to the energy density by

$$d\bar{\mathcal{I}} = c\,\frac{\bar{k}}{k}\,E(\bar{x},\bar{k})d\bar{x}\ d\bar{k} \equiv \bar{I}(\bar{x},\bar{k})d\bar{x}\ d\bar{k} \tag{2}$$

where $\bar{I}(\bar{x},\bar{k})$ is an "intensity spectral density" and c is the wave speed in the undisturbed medium. It is assumed that the rate of change of energy in time is totally accounted for by scattering and propagation, and such effects as viscous, thermal, and relaxation loss mechanisms are neglected.

The energy loss rate due to propagation alone is given by

$$\left[\frac{\partial E(\bar{x},\bar{k})}{\partial t}\right]_p = -\mathrm{div}\ \bar{I} \tag{3}$$

In the volume $d\bar{x}$, the conversion of $\bar{k}$ waves to $\bar{k}'$ waves results in

$$\left[\frac{\partial E(\bar{x},\bar{k})}{\partial t}\right]_{\bar{k}\to\bar{k}'} = -\int_{\bar{k}'} \left|\bar{I}(\bar{x},\bar{k})\right| \sigma(\bar{k}/\bar{k}')d\bar{k}' \tag{4}$$

and the conversion of $\bar{k}'$ waves to $\bar{k}$ waves gives

$$\left[\frac{\partial E(\bar{x},\bar{k})}{\partial t}\right]_{\bar{k}'\to\bar{k}} = \int_{\bar{k}'} \left|\bar{I}(\bar{x},\bar{k}')\right| \sigma(\bar{k}'/\bar{k})d\bar{k}' \qquad (5)$$

The integration in equation (4) may be performed immediately if the total cross section is introduced

$$Q(\bar{k}) = \int_{\bar{k}'} \sigma(\bar{k}/\bar{k}')d\bar{k}' \qquad (6)$$

If the acoustic source has come into statistical equilibrium with the field, then the total rate of change of energy vanishes. Thus, adding equations (3), (4), and (5) and setting the result equal to zero gives the transport equation

$$\text{div } \bar{I} = -cE(\bar{x},\bar{k}) + c\int_{\bar{k}'} d\bar{k}'\sigma(\bar{k}'/\bar{k})E(\bar{x},\bar{k}') \qquad (7)$$

Now it is necessary to evaluate $\sigma(\bar{k}/\bar{k}')$ for turbulent scattering.

Scattering Cross Section $\sigma(\bar{k}/\bar{k}')$

The expression for the time-dependent cross section used here is essentially that derived by Kraichnan (ref. 8). Let the directions of $\bar{k}$ and $\bar{k}'$ be defined by the unit vectors $\bar{n}$ and $\bar{n}'$, respectively. The turbulent velocity field is described by a Fourier transform,

$$U(\bar{K},\Omega) = \left(\frac{1}{2\pi}\right)^2 \int_V d\bar{x} \int_{-\infty}^{\infty} dte^{-i(\bar{K}\cdot\bar{x}-\Omega t)}\bar{u}(\bar{x},t) \qquad (8)$$

Referring to appendix A and reference 8, the reader will see that the appropriate form for σ is

$$\sigma(\bar{k}/\bar{k}') = \frac{2\pi\omega^2}{c^3}(\bar{n}\cdot\bar{n}')^2 \left\langle\left|U_n\right|^2\right\rangle_{av} \qquad (9)$$

where U_n is $U \cdot \bar{n}$ and ω is the frequency of the incident sound. The ensemble averaged factor in equation (9) must be related to the one-dimensional spatial spectrum for isotropic turbulence. This spectrum, as described by Batchelor (ref. 9), is denoted by $W(K)$. The kinetic energy and the rate of energy dissipation per unit mass are (ref. 9)

$$\frac{1}{2}\left\langle u^2\right\rangle = \int_0^\infty W(K)\,dK \tag{10}$$

and

$$\epsilon = 2\nu\int_0^\infty K^2W(K)\,dK \tag{11}$$

For time-dependent turbulence, the spectrum $w(K,\Omega)$ is introduced which must satisfy

$$W(K) = \int_0^\infty w(K,\Omega)\,d\Omega \tag{12}$$

The mean flow is assumed to vanish and the autocorrelation at a point is assumed of the form

$$\begin{aligned}\left\langle u_1(t)u_1(t+\tau)\right\rangle_{av} &= \left\langle u_2(t)u_2(t+\tau)\right\rangle_{av}\\ &= \left\langle u_3(t)u_3(t+\tau)\right\rangle_{av}\\ &= \frac{1}{3}\left\langle u^2\right\rangle_{av} e^{-\lambda|\tau|}\end{aligned} \tag{13}$$

with a frequency spectrum

$$\frac{1}{3}\left\langle u^2\right\rangle \frac{\lambda}{\pi\left(\lambda^2+\Omega^2\right)}$$

If it is assumed that the frequency and wave number spectra may be separated, the equation can immediately be written

$$w(K,\Omega) = W(K)\frac{\lambda}{\pi\left(\lambda^2+\Omega^2\right)} \tag{14}$$

where equations (12) and (13) were used. A convenient form of W to choose for the integrations to be performed is

$$W(K) = \alpha\beta K^4 e^{-4\alpha K} \tag{15}$$

where $1/\alpha = k_0$ is the wave number for the maximum of the spectrum; that is, it is the wave number of the energy-bearing eddies. This spectrum is chosen since it has the known K^4 behavior at small wave numbers (ref. 9) with the energy-bearing maximum and allows analytic evaluation of the integrals to follow. Using equation (10),

$$\beta = \frac{45\alpha^4}{48}\langle u^2\rangle \tag{16}$$

which ties the parameters α and β to important quantities in the description of the turbulent field.

W
1
2
7

It is shown in appendix B that $\left\langle |U_n|^2 \right\rangle$ may be related to the spectrum equation (14) by the relation

$$\left\langle \left|U_n(\overline{K},\Omega)\right|^2 \right\rangle_{av} = \frac{\cos^2 \phi/2}{8\pi K^2}\, w(K,\Omega) \tag{17}$$

where ϕ is the angle between incident and scattered wave vectors.

The characteristic frequency of the turbulence by equations (13) and (14) is λ. This may be approximately evaluated from parameters of the flow by setting

$$\lambda \approx \frac{u}{L} \tag{18}$$

where $u = \sqrt{\langle u^2\rangle}$ and L is the correlation length of the turbulence (mean eddy size). Lighthill (ref. 7) gives a formula for this length which is used in this investigation. It is

$$L = \frac{3\pi}{2\langle u^2\rangle}\int_0^\infty \frac{W(K)}{K}\, dK = \frac{3\pi\alpha}{4} \tag{19}$$

for the spectrum equation (15).

There is another possible method for the evaluation of λ. In an equilibrium turbulent field there is a balance between the randomly created vorticity and its dissipation as viscous loss. This is reminiscent of the equilibrium state of a harmonic oscillator (with damping) when excited by random noise. For the oscillator, one obtains a correlation function $\sim e^{-\lambda|\tau|}$ where λ is given by the ratio of the energy lost per second to twice the average energy stored. Making this analogous to the turbulence problem,

$$\lambda = \frac{\epsilon}{\langle u^2 \rangle} \tag{20}$$

In terms of the spectrum equation (15), this is

$$\lambda = \frac{15\nu}{8\alpha^2} \tag{21}$$

which is lower by a factor of 10 or so than experimental results which support the value given by equation (18). It is believed that the failure of equation (21) is due primarily to equation (15) and not to the idea expressed by equation (20). The assumed spectrum cuts off too sharply for large wave numbers and hence gives too low a value for ϵ. Any spectrum which gives the $K^{-5/3}$ range predicted by equilibrium theory, however, appears to be too complex to integrate in expressions which arise later in the paper. Hence, equation (15) will be used recognizing its limitations at large wave numbers.

A possible spectrum is $W \sim K^4\left(1 + AK^{-17/3\mu} + BK^{-11/\mu}\right)^{\mu}$, where $\mu < 0$ but arbitrary. This spectrum contains the necessary K^4, $K^{-5/3}$, and K^{-7} ranges, but integrations involving this factor are seen to be rather formidable.

DERIVATION OF DIFFUSION EQUATION

Referring now to equations (9), (14), (15), and (17) and to reference 8,

$$\sigma(\bar{k}/\bar{k}') = \frac{\omega^2 \alpha\beta\lambda}{4c^3} K^2 \frac{e^{-4\alpha K}}{\lambda^2 + \Omega^2} (\bar{n} \cdot \bar{n}')^2 \cos^2(\phi/2) \tag{22}$$

where $K = |\bar{K}| = |\bar{k} - \bar{k}'|$ and $\Omega = |\omega - \omega'|$. The frequency shift is Ω and the magnitude of the scattering vector K. It is seen that

$$\sigma(\bar{k}/\bar{k}') \approx \sigma(\bar{k}'/\bar{k}) \tag{23}$$

if the change in frequency upon scattering is not too great.

An attempt was made to simplify equation (7) by some appropriate assumptions. If the incident acoustic wavelength is smaller than the eddy size, then the scattering will be strongly forward. Also, the period of the sound wave is usually much less than the period of the turbulence ($\omega \gg \Omega$). Thus, $\sigma(\bar{k}/\bar{k}')$ is nonzero for only small values of $\bar{k} - \bar{k}'$, and $E(\bar{x},\bar{k}')$ can be expanded around $(\bar{x},\bar{k})$ in a Taylor's series.

Since the diffusion of a plane wave of frequency ω_0 with a wave vector parallel to the z-axis is to be considered, it is expected that the function E will depend only on z, θ, and $\Delta = \dfrac{\omega - \omega_0}{\omega_0}$, where θ is the polar angle from the z-axis. If $\xi = \cos\theta$, then

W127

$$\left\langle E(z,\xi',\Delta')\right\rangle_\sigma = E(z,\xi,\Delta) + \left\langle(\xi' - \xi)\right\rangle_\sigma \frac{\partial E}{\partial \xi} + \left\langle(\Delta' - \Delta)\right\rangle_\sigma \frac{\partial E}{\partial \Delta}$$

$$+ \frac{1}{2}\left\langle(\xi' - \xi)\right\rangle_\sigma^2 \frac{\partial^2 E}{\partial \xi^2} + \frac{1}{2}\left\langle(\Delta' - \Delta)\right\rangle_\sigma^2 \frac{\partial^2 E}{\partial \Delta^2}$$

$$+ \left\langle(\xi' - \xi)(\Delta' - \Delta)\right\rangle_\sigma \frac{\partial^2 E}{\partial \xi\, \partial \Delta} + \ldots \qquad (24)$$

where $\left\langle f(\xi',\Delta')\right\rangle_\sigma \equiv \int_{\cup k'} f(\xi',\Delta')\sigma(\bar{k}/\bar{k}')d\bar{k}'$. Putting equation (24) into equation (7), gives

$$\xi\frac{\partial E}{\partial z} = \left\langle(\xi' - \xi)\right\rangle_\sigma \frac{\partial E}{\partial \xi} + \left\langle(\Delta' - \Delta)\right\rangle_\sigma \frac{\partial E}{\partial \Delta} + \frac{1}{2}\left\langle(\xi' - \xi)\right\rangle_\sigma^2 \frac{\partial^2 E}{\partial \xi^2}$$

$$+ \frac{1}{2}\left\langle(\Delta' - \Delta)\right\rangle_\sigma^2 \frac{\partial^2 E}{\partial \Delta^2} + \left\langle(\Delta' - \Delta)(\xi' - \xi)\right\rangle_\sigma \frac{\partial^2 E}{\partial \Delta\, \partial \xi} + \ldots \qquad (25)$$

This is the diffusion equation which will be used. With a knowledge of σ, it remains to calculate the unknown coefficients.

If $\rho = \Delta' - \Delta$, using the law of cosines

$$K = \left[k^2\rho^2 + 4k^2\sin^2(\phi/2) - 4k^2\rho\,\sin^2(\phi/2)\right]^{1/2} \qquad (26)$$

Now examine the relative magnitudes of these terms. Consider the ratio $k\rho/2k\,\sin(\phi/2)$. Now $k\rho \approx \lambda/c = \lambda M/u$, where λ is a characteristic frequency of the turbulence and M is the Mach number of the turbulent fluctuations. The factor $2k\,\sin(\phi/2) \approx 1/l$, where l is a characteristic length for the flow. Then

$$\frac{k\rho}{2k\,\sin(\phi/2)} \approx \frac{\lambda M}{u/l} \approx M \ll 1 \qquad (27)$$

substituting for equation (18). Also, $\rho \approx \lambda/\omega_o \ll 1$. This means equation (26) can be expanded as

$$K \approx 2kx\left(1 - \frac{1}{2}\rho + \frac{1}{8x^2}\rho^2\right) \tag{28}$$

and $x = \sin(\phi/2)$. Using equation (28) in (22) gives

$$\sigma(\bar{k}/\bar{k}') \approx \frac{\alpha\beta\lambda}{c}k^4x^2(1 - x^2)(1 - 2x^2)^2(\lambda^2 + \omega^2\rho^2)^{-1}\exp\left[-8\alpha kx\left(1 - \frac{1}{2}\rho + \frac{1}{8x^2}\rho^2\right)\right] \tag{29}$$

COEFFICIENTS IN DIFFUSION EQUATION

To calculate the "averaged" coefficients in equation (25) considering $\langle\xi' - \xi\rangle_\sigma$

$$\langle\xi' - \xi\rangle_\sigma = \int d\bar{k}'\sigma(\bar{k}/\bar{k}')(\xi' - \xi)$$

The evaluation of this integral using equation (29) is carried out in appendix C. Since ξ is independent of Ω, it is possible to calculate this average with the simplified spectrum equation (15). The result is shown to be

$$\langle\xi' - \xi\rangle_\sigma = \left(\frac{5\pi M^2}{32\alpha}\right)\xi \tag{30}$$

where $M^2 \equiv \langle u^2\rangle/c^2$.

The calculation of $\langle(\xi' - \xi)^2\rangle_\sigma$ is performed in appendix D, and again the simpler spectrum equation (15) can be utilized. This simplification occurs because $w(K,\Omega)$ is chosen to be a product of 2 factors, one independent of K and the other, of Ω. (It is probable that the actual isotropic spectrum which would be determined from the equations of motion would not factor in this manner, since it is seen from equation (18) that the periods and eddy sizes are related.) The result here is

$$\langle(\xi' - \xi)^2\rangle_\sigma = \left(\frac{5\pi M^2}{32\alpha}\right)\left(1 - \xi^2\right) \tag{31}$$

The evaluation of $\langle \rho \rangle_\sigma$ is not so straightforward as in the previous cases. However, there is no physical basis for assuming that there will be any change of frequency of the wave. In appendix E, it is shown that although the expression (22) does predict some slight average frequency shift, this shift will be very small, and can be neglected if compared to its rms value. Accordingly, let

$$\langle \rho \rangle_\sigma = 0 \tag{32}$$

which also premits the assumption that $\langle \rho(\xi' - \xi) \rangle_\sigma = 0$.

The evaluation of $\langle \rho^2 \rangle_\sigma$ can be carried forth without difficulty, and in appendix F it is found to be

W127

$$\langle \rho^2 \rangle_\sigma = \frac{\sqrt{2}\pi^2\Gamma(9/2)}{192} \frac{\lambda}{c} M^2 \tag{33}$$

Notice that according to this result the rms frequency deviation should be proportional to $\lambda^{1/2}$ instead of λ as one might suppose.

SOLUTION OF DIFFUSION EQUATION

By putting the calculated coefficients into the diffusion equation (25), after making the proper definitions, one obtains

$$\xi \frac{\partial E}{\partial \eta} = \xi \frac{\partial E}{\partial \xi} + \frac{1}{2}\left(1 - \xi^2\right)\frac{\partial^2 E}{\partial \xi^2} + \frac{1}{2}\frac{\partial^2 E}{\partial r^2} \tag{34}$$

where $\eta = \dfrac{5\pi M^2 z}{32\alpha}$, $r = \dfrac{\Delta}{\gamma}$, and $\dfrac{\gamma^2}{2} = \dfrac{\pi\alpha\lambda\Gamma(9/2)}{15\sqrt{2}c}$

This equation must be solved with the condition that there is a monochromatic plane wave at $z = 0$. The energy density at $z = 0$ will be of the form $E = B\delta(1 - \xi)\delta(r)$, where B is to be determined. Then, from equation (1), the energy density becomes

$$\varepsilon(z = 0) = \int E(0,\bar{k})d\bar{k} = 2\pi B k_o^3 \int_0^1 d\xi\,\delta(1 - \xi) \int_{-\infty}^{\infty} d(r\gamma)\delta(r)$$

or, $I_o c^{-1} = \pi k_o^3 \gamma B$, where I_o is the intensity at $z = 0$. Then,

$$E(0,\bar{k}) = \frac{I_o}{\pi k_o^3 \gamma c}\, \delta(r)\delta(1 - \xi) \qquad (35)$$

Equation (34) can be solved rather easily if the distances z are such that the initial wave has spread only slightly in direction. In this region then, let the term on the left-hand side of equation (34) reduce to $\partial E/\partial \eta$. The solution in then immediately

$$E(\bar{x},\bar{k}) = \sum_{m=0}^{\infty} \int_{-\infty}^{\infty} d\kappa e^{-i\kappa r} P_m(\xi) A_m(\kappa) e^{-\eta\left(\frac{\kappa^2}{2} + \frac{1}{2}m(m+1)\right)} \qquad (36)$$

The boundary condition equation (35) defines $A_m(\kappa)$ to be

$$A_m(\kappa) = \frac{I_o}{4\pi^2 k_o^3 \gamma c}(2m + 1) \qquad (37)$$

Finally then,

$$E(\eta,\xi,r) = \frac{I_o}{4\pi^2 k_o^3 \gamma c}\left(\frac{2\pi}{\eta}\right)^{1/2} e^{-\Delta^2/2\eta} \sum_{m=0}^{\infty} (2m + 1) P_m(\xi) e^{-m(m+1)\eta/2} \qquad (38)$$

which reduces to Lighthill's result (ref. 7) for time-independent scattering. The separation of the frequency and angular diffusion processes into factors in equation (38) is, of course, a result of the form of the turbulent spectrum and the vanishing of the term $\langle \rho(\xi' - \xi) \rangle_\sigma$.

This equation may be made more transparent by considering the diffusion to have progressed only slightly. Then the angular deviations are small and it is possible to substitute

$$\frac{\partial}{\partial \xi}\left(1 - \xi^2\right)\frac{\partial}{\partial \xi} \rightarrow \frac{\partial^2}{\partial \theta^2} + \frac{1}{\theta}\frac{\partial}{\partial \theta} \qquad (39)$$

for which the general solution is

$$E(\bar{x},\bar{k}) = \int_0^{\infty} \beta d\beta J_o(\beta\theta) e^{-\frac{1}{2}\beta^2 \eta} \int_{-\infty}^{\infty} d\kappa e^{i\kappa r - \frac{1}{2}\kappa^2 \eta} A(\kappa,\beta) \qquad (40)$$

where $A(\kappa,\beta)$ is evaluated by requiring that $E(\bar{x},\bar{k})$ correspond to a plane monochromatic wave at $z = 0$ of intensity I_0. The result of this gives

$$E(\bar{x},\bar{k}) = \frac{A}{2\pi}\int_0^\infty \beta d\beta J_0(\beta\theta) e^{-\frac{1}{2}\beta^2\eta} \int_{-\infty}^{\infty} d\kappa e^{i\kappa r-\kappa^2\eta/2} \qquad (41)$$

and using the formula for the Fourier transform of a normal frequency function and Weber's first exponential formula for the Bessel function integral, gives

$$E(\bar{x},\bar{k}) = \frac{I_0}{k_0{}^3\gamma c\left(2\pi^3\eta^3\right)^{1/2}} \exp\left[-\frac{\left(\theta^2 + r^2\right)}{2\eta}\right] \qquad (42)$$

W 127

a normal distribution in both frequency and direction with rms values $= \sqrt{\eta}$. Thus, the rms deviation in angle varies with the Mach number of the fluctuations and with the square root of both the mean eddy size and the distance z. In addition to these, the relative frequency deviation also depends on the square root of the eddy frequency but is independent of the eddy size.

If the wave number part of the probability distribution equation (42) is now interpreted as the power spectrum of a wave, then it is possible to calculate the autocorrelation function of the signal at any position z. The normalized autocorrelation function is then

$$\psi(\tau) = \int_0^\infty E(\bar{x},\bar{k})\cos\omega\tau \, d\bar{k} \Big/ \int_0^\infty E(\bar{x},\bar{k})d\bar{k} \qquad (43)$$

and for the above spectrum equation (42),

$$\psi(\tau) = \cos\omega_0\tau \exp\left[-b^2(\omega_0\tau)^2\right] \qquad (44)$$

where $b^2 = \nu^2\eta = M^2\pi^2\lambda\Gamma(9/2)z/192\sqrt{2}c$, and the assumed relative "spread" of the spectrum is slight. At $z = 0$, there is the cosine correlation function of a pure sine wave. For distances z, there is a loss in correlation for large values of delay τ.

CONCLUDING REMARKS

It has been shown how the nature of a plane monochromatic sound wave may be changed by passage through a turbulent atmosphere. Perhaps it is well to pause and review the assumptions and then see how useful the results may be.

The Lighthill and Kraichnan scattering formulae are valid when the Mach number of the turbulence is low (<< 1). In addition, it will generally be true that the sound particle velocity will be small compared with the turbulent fluctuation velocities. This means that the sound-turbulence interaction is assumed larger than the nonlinear acoustic effects. The sound generated by atmospheric turbulence usually will have frequencies much lower than those which are of interest in propagating through the atmosphere. The interaction or scattering phenomenon may then be considered to predominate in the audible frequency range.

In evaluating the integrals in the appendices (C, D, E, and F) the assumption has been made that $k_0 L \gg 1$, which means that the acoustic wavelength is small compared with the turbulent eddy size. This is not too severe a restriction since the scale of atmospheric turbulence is usually fairly large, of the order of the height above ground. In addition, however, it was required that the scattering volume have dimensions large compared to L. This means that propagation distances z should be large compared to L, the scale of turbulence.

With these restrictions, then, the transport-diffusion treatment of propagation in an isotropic turbulent atmosphere can predict the frequency spectrum and angular distribution of the sound wave fronts in a probabilistic sense. The frequency distribution is interpreted as a power spectrum and the autocorrelation of the wave form is obtained by Fourier inversion. Unfortunately the phase information is lost in the analysis.

It is probable that the most significant failing of this work is its omission of the effects of thermal scattering which must occur when atmospheric turbulence mixes regions of different temperature. Unfortunately, the measurement of rapid temperature fluctuations in the atmosphere is still in a rather rudimentary state.

Massachusetts Institute of Technology,
Cambridge, Mass., March 26, 1958.

APPENDIX A

RELATION BETWEEN SPECTRUM AND CROSS SECTION

In reference 8, the power spectrum of the acoustic condensation ξ scattered into the direction $\bar{n}'$ from $\bar{n}$ is given by

$$P(\bar{n},\omega) \equiv |\bar{x}|^2 \left|X(\omega)\right|^2 = 4\pi \frac{I_o}{\rho_o c} \frac{\omega'^4}{c^8} (\bar{n} \cdot \bar{n}')^2 U_n(\bar{K},\Omega)^2 \qquad \text{(A1)}$$

when $\bar{K} = \bar{k}' - \bar{k}$, $\Omega = \omega' - \omega$, and $|\bar{x}|$ is the distance from the scattering region. The symmetrical time transform of ξ is $X(\bar{x},\omega)$. The energy contained in a frequency interval $d\omega'$ and a cone of solid angle $d\Phi'$ is thus

$$\frac{1}{2} P(\bar{n}',\omega')\rho_o c^3 d\Phi' d\omega' = \frac{1}{2} P(\bar{n}',\omega') \frac{\rho_o c^6}{\omega'^2} d\bar{k}' \qquad \text{(A2)}$$

Since by definition of the differential scattering cross section this must also be $I_o \sigma(\bar{k}/\bar{k}')d\bar{k}'$, it must follow that

$$\sigma(\bar{k}/\bar{k}') = 2\pi \frac{\omega'^2}{c^3} (\bar{n} \cdot \bar{n}')^2 U_n(\bar{K},\Omega)\Big|^2 \qquad \text{(A3)}$$

APPENDIX B

SPECTRUM FOR ISOTROPIC TURBULENCE

From equation (8),

$$\left\langle U_i(\bar{K},\Omega)U_j(\bar{K},\Omega)\right\rangle_{av} = \frac{1}{(2\pi)^4}\int d\bar{x}\ d\bar{x}'dt\ dt'e^{i\bar{K}(\bar{x}-\bar{x}')+i\Omega(t-t')}R_{ij}(\bar{\sigma},\tau) \tag{B1}$$

where

$$\bar{\sigma} = \bar{x} - \bar{x}'$$

$$\tau = t - t'$$

and

$$R_{ij}(\bar{\sigma},\tau) = \left\langle u_i(\bar{x},t)u_j(\bar{x}',t')\right\rangle_{av}$$

Now, setting

$$\Phi_{ij}(\bar{K},\Omega) \equiv \frac{\left\langle U_iU_j\right\rangle_{av}}{VT} = \frac{1}{(2\pi)^4}\int d\bar{\sigma}\ d\tau e^{-i(\bar{K}\cdot\bar{\sigma}-\Omega\tau)}R_{ij}(\bar{\sigma},\tau) \tag{B2}$$

then it must follow by inversion that

$$R_{ij}(\bar{\sigma},0) = \int d\bar{K}e^{i\bar{K}\cdot\bar{\sigma}}\left[\int_{-\infty}^{\infty} d\Omega\Phi_{ij}(\bar{K},\Omega)\right] \tag{B3}$$

where the quantity in brackets is the $\Phi_{ij}(\bar{K})$ introduced by Batchelor (ref. 9, equation 2.4.3). Accordingly, using reference 9,

$$\begin{aligned}\int_{-\infty}^{\infty} d\Omega\Phi_{ij}(\bar{K},\Omega) &= \Phi_{ij}(\bar{K}) \\ &= \frac{W(K)}{4\pi K^4}\left(K^2\delta_{ij} - K_iK_j\right) \\ &= \frac{1}{8\pi K^4}\left(K^2\delta_{ij} - K_iK_j\right)\int_{-\infty}^{\infty} W(K,\Omega)d\Omega\end{aligned} \tag{B4}$$

for isotropic turbulence. The association may then be made

$$\Phi_{ij}(\bar{K},\Omega) = \frac{1}{8\pi K^4}\left(K^2\delta_{ij} - K_iK_j\right)w(K,\Omega) \tag{B5}$$

whence follows

$$\frac{\left\langle |U_n|^2 \right\rangle_{av}}{VT} = \Phi_{nn}(\bar{K},\Omega) = \frac{w(K,\Omega)}{8\pi K^2}\cos^2\frac{\phi}{2} \tag{B6}$$

where ϕ is the angle between $\bar{k}$ and $\bar{k}'$. To seek the scattering per unit volume and per second, set $V = T = 1$ in equation (B6).

APPENDIX C

COSINE DRIFT COEFFICIENT

Select the direction of $\bar{k}$ as a polar axis with an azimuthal angle ψ and a polar angle θ with respect to the z-axis. The polar and azimuthal angles of $\bar{k}'$ with respect to $\bar{k}$ are ϕ and β, respectively. Using the identity

$$\cos \theta' = \cos \theta \cos \phi + \sin \phi \sin \theta \cos(\beta - \psi) \tag{C1}$$

the integral can be written as

$$\left\langle \xi' - \xi \right\rangle_\sigma = \frac{4k^6 \alpha\beta}{c^2} \int_0^1 dx x^3 (1 - 2x^2)^2 (1 - x^2) e^{-8\alpha kx} \int_0^{2\pi} d\beta \Big\{ \cos \theta \left(-2x^2\right) + \sin \theta \cos(\beta - \psi) \Big\} \tag{C2}$$

where $x = \sin \phi/2$. The reader should consult Lighthill (ref. 7) for a discussion of the integral over ϕ or x. For typical spatial turbulence spectra, assuming $k/k_o \gg 1$, the exponential diminishes very rapidly, and the upper limit of integration may be allowed to extend to infinity. The integration over β becomes $-4\pi x^2 \cos \theta$, and the integration over x is then

$$\left\langle \xi' - \xi \right\rangle_\sigma \approx - \frac{16\pi k^6 \alpha\beta}{c^2} \xi \int_0^\infty dx x^5 e^{-8\alpha kx}$$

or

$$\left\langle \xi' - \xi \right\rangle_\sigma = - \frac{5\pi\xi}{32\alpha} M^2 \tag{C3}$$

APPENDIX D

COSINE VARIANCE

Using expression (C1) and the spectrum of equation (15), one can write

$$\left\langle (\xi' - \xi)^2 \right\rangle_\sigma = \frac{4k^4\alpha\beta}{c^2} \int_0^1 dx x^3 \left(1 - 2x^2\right)^2 \left(1 - x^2\right) e^{-8\alpha k x_x} \int_0^{2\pi} d\beta \left\{ -2x \cos\theta + 2x\left(1 - x^2\right)^{1/2} \sin\theta \cos(\beta - \psi)^2 \right\} \qquad \text{(D1)}$$

W
1
2
7

The integral over β contains $8\pi x^4 \cos^2\epsilon$ and $2\pi x^2\left(1 - x^2\right)\sin^2\theta$. The x^4 terms may be ignored compared to the x^2 terms (since the interest is in the range $x \ll 1$) and this yields

$$\left\langle (\xi' - \xi)^2 \right\rangle_\sigma \approx \frac{8\pi k^6\alpha\beta}{c^2}\left(1 - \xi^2\right) \int_0^1 x^5 \left(1 - 2x^2\right)^2 \left(1 - x^2\right) e^{-8\alpha k x} \qquad \text{(D2)}$$

or

$$\left\langle (\xi' - \xi)^2 \right\rangle_\sigma = \frac{5\pi\left(1 - \xi^2\right)}{32\alpha} M^2 \qquad \text{(D3)}$$

APPENDIX E

FREQUENCY DRIFT

From equations (26) and (22), the expression for $\langle \rho \rangle_\sigma$ is proportional to

$$\frac{1}{2}\int_{-\infty}^{\infty}\left(\lambda'^2 + \rho^2\right)^{-1} e^{-\mu\left(\rho^2-4\rho x^2+4x^2\right)^{1/2}} \rho \, d\rho \qquad \text{(E1)}$$

where $x = \sin(\phi/2)$ and $\lambda' = \lambda/\omega$. The branch points of the integrand are located at

$$\rho_{\substack{1\\2}} = \frac{4x^2 \pm \sqrt{16x^4 - 16x^2}}{2} = \pm i2 \sin\frac{\phi}{2}\, e^{\pm i\phi/2} \approx \pm i2 \sin(\phi/2) \qquad \text{(E2)}$$

since the scattering angle is small. This makes $\langle \rho \rangle_\sigma$ proportional to

$$\int_{-\infty}^{\infty}\left(\lambda'^2 + \rho^2\right)^{-1} e^{-\mu\left(\rho^2+4x^2\right)^{1/2}} \rho \, d\rho \qquad \text{(E3)}$$

which must vanish. Hence $\langle \rho \rangle_\sigma \approx 0$.

APPENDIX F

FREQUENCY VARIANCE

Again, from equations (22) and (29), $\left\langle \rho^2 \right\rangle_\sigma$ can be expressed as

$$\left\langle \rho^2 \right\rangle_\sigma \approx \frac{8\pi\alpha\beta\lambda k^7}{c}\int_0^1 dx\,x^3\left(1 - x^2\right)\left(1 - 2x^2\right)^2 e^{-\xi\alpha k x}\int d\rho\left(\lambda^2 + \omega^2\rho^2\right)^{-1} e^{-\frac{\alpha k}{x}\rho^2} \qquad \text{(F1)}$$

W
1
2
7

To evaluate this, calculate

$$\int_{-\infty}^{\infty} d\rho\, e^{-\sigma^2\rho^2}\left(\lambda^2 + \omega^2\rho^2\right)^{-1}$$

where $\sigma^2 = \alpha k/x$. The integrand diminishes with ρ under the primary influence of the denominator. Thus the exponential is treated as a perturbation on this and written as follows

$$e^{-\sigma^2\rho^2} \approx 1 - \sigma^2\rho^2 \approx \frac{1}{1 + \sigma^2\rho^2}$$

assuming convergence for large ρ. Thus it is that one integral takes the form

$$\frac{1}{\omega^2\sigma^2}\int_{-\infty}^{\infty} \frac{d\rho}{\left(\lambda^2\omega^{-2} + \rho^2\right)\left(\sigma^{-2} + \rho^2\right)} = \frac{1}{\omega^2\sigma^2}\, 2\pi i\left\{\frac{1}{2i\frac{\lambda}{\sigma}\left(\frac{1}{\sigma^2} - \frac{\lambda^2}{\omega^2}\right)} + \frac{1}{2i\frac{1}{\sigma}\left(-\frac{1}{\sigma^2} + \frac{\lambda^2}{\omega^2}\right)}\right\}$$

$$= \frac{\pi}{\lambda(\omega + \sigma\lambda)}$$

Now,

$$\int \rho^2 d\rho\, \frac{e^{-\sigma^2\rho^2}}{\lambda^2 + \omega^2\rho^2} = -\frac{d}{d\sigma^2}\left\{\frac{\pi}{\lambda(\omega + \sigma\lambda)}\right\} \approx \frac{\pi}{2\sigma\omega^2} \qquad \text{(F2)}$$

if it is assumed that $\frac{\omega}{\lambda} \gg \sigma$. Thus, putting this result into the integration over x, the result becomes

$$\left\langle \rho^2 \right\rangle_\sigma = \frac{4\pi^2 \alpha\beta\lambda k^7}{(\alpha k)^{1/2}\omega^2 c} \int_0^1 x^{7/2} e^{-8\alpha k x} dx$$

$$= \frac{\sqrt{2}\pi^2 \Gamma(9/2)}{192} \frac{\lambda}{c} M^2 \tag{F3}$$

REFERENCES

1. Bergmann, P. G.: Propagation of Radiation in Medium With Random Inhomogeneities. Phys. Rev., vol. 70, no. 7, 8, Oct. 1946, pp. 486-492.

2. Obukhov, A. M.: Isvestia Akad. Nauk, SSSR, Ser. Geofiz., vol. 17, 1953, p. 155.

3. Skudrzyk, E.: Scattering in an Inhomogeneous Medium. Jour. Acous. Soc. Am., vol. 29, no. 1, Jan. 1957, pp. 50-60.

4. Mintzer, D.: Wave Propagation in a Randomly Inhomogeneous Medium. I. Jour. Acous. Soc. Am., vol. 25, no. 5, Sept. 1953, pp. 922-927.

5. Mintzer, D.: Wave Propagation in a Randomly Inhomogeneous Medium. II. Jour. Acous. Soc. Am., vol. 25, no. 6, Nov. 1953, pp. 1107-1111.

6. Mintzer, D.: Wave Propagation in a Randomly Inhomogeneous Medium. III. Jour. Acous. Soc. Am., vol. 26, no. 2, Mar. 1954, pp. 186-190.

7. Lighthill, M. J.: On the Energy Scattered From the Interaction of Turbulence With Sound or Shock Waves. Proc. Cambridge Phil. Soc., vol. 49, pt. 13, July 1953, pp. 531-551.

8. Kraichnan, R. H.: The Scattering of Sound in a Turbulent Medium. Jour. Acous. Soc. Am., vol. 25, no. 6, Nov. 1953, pp. 1096-1104.

9. Batchelor, G. K.: The Theory of Homogeneous Turbulence (Ch. 3 and 4). Cambridge Univ. Press (London), 1953.

W
1
2
7