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MEMORANDUM

NUMERICAL SOLUTION OF THE FLOW OF A PERFECT GAS OVER

A CIRCULAR CYLINDER AT INFINITE MACH NUMBER

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NUMERICAL SOLUTION OF THE FLOW OF A PERFECT GAS OVER

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SUMMARY

A solution for the two-dimensional flow of an inviscid perfect gas over a circular cylinder at infinite Mach number is obtained by numerical methods of analysis. Nonisentropic conditions of curved shock waves and vorticity are included in the solution. The analysis is divided into two distinct regions, the subsonic region which is analyzed by the relaxation method of Southwell and the supersonic region which was treated by the method of characteristics. Both these methods of analysis are inapplicable on the sonic line which is therefore considered separately. The shapes of the sonic line and the shock wave are obtained by iteration techniques.

The striking result of the solution is the strong curvature of the sonic line and of the other lines of constant Mach number. Because of this the influence of the supersonic flow on the sonic line is negligible. On comparison with Newtonian flow methods, it is found that the approximate methods show a larger variation of surface pressure than is given by the present solution.

INTRODUCTION

Calculation of the flow over a blunt leading edge is a problem of considerable difficulty. The flow is subsonic as well as supersonic so that more than one method of analysis must be used. Because of the detached shock wave and the several distinct regions of flow there is the further difficulty of undetermined boundaries. Finally the entire problem is nonlinear especially in hypersonic flow because of the large entropy changes that occur. Hence, it appears that direct numerical integration of the equations of motion for this type of problem is desirable.

Several schemes employing numerical method have been developed. Belotserkovsky (ref. 1) has derived a method based on stripwise division of the flow field and the development of special integral equations which are then solved numerically. The method is applied to a circular cylinder

for several finite Mach numbers. In the work of Garabedian (ref. 2) the equation is integrated in the realm of complex variables so that the usually elliptic-type differential equation becomes hyperbolic; hence, a method of characteristics may be used. The reason for the approach is to avoid certain difficulties of instabilities. Van Dyke (ref. 3) has devised a method of integrating from the shock wave directly which appears to have satisfactorily minimized instability difficulties. In both the latter methods the shock-wave shape is given and the body shape is produced as one of the results of the solution. A method similar to that used in reference 3 is given by Mangler and Evans (ref. 4).

In the present work the numerical method used is the relaxation method developed by Southwell and his group (ref. 5). This method is particularly effective when the differential equation being solved is an elliptical type as it is in the case of subsonic flow. Maccoll and Codd (ref. 6) calculated the subsonic portion of the flow over a wedge of large included angle using greatly simplified assumptions for the sonic line position. Drebinger (ref. 7) solved a similar problem with more accurate conditions for the sonic line obtaining a better answer but with more extensive computation. The use of the relaxation method, however, is questionable when applied to supersonic flow because the differential equation is not elliptic but hyperbolic and, in addition, the construction of suitable boundary condition is unsatisfactory. Mitchell (ref. 8) calculated the entire flow field for the supersonic flow over a square leading-edge plate where the necessary boundary conditions were obtained from experimental data; without such experimental aids it is doubtful whether the relaxation solution of the supersonic region could be obtained.

In this report the flow about a circular cylinder is calculated where the relaxation method is used for the subsonic portion of the flow and the supersonic part of the flow is computed by the well established method of characteristics. The free-stream Mach number of the flow is chosen as infinite.

SYMBOLS

c velocity of sound

C_D pressure drag coefficient

C_p local pressure coefficient

d characteristic length

F functional relation for gas density variable

G constant representing free-stream mass-flow rate

i $\sqrt{-1}$

l distance along sonic line

$\bar{l}$ $\dfrac{l}{d}$

M   Mach number

n   coordinate distance normal to streamline

$\bar{n}$ $\dfrac{n}{d}$

p   fluid pressure

$\bar{p}$ $\dfrac{p}{(p_s)_r}$

q   fluid velocity

Q   residual in relaxation process

r   radius in polar coordinates

$\bar{r}$ $\dfrac{r}{d}$.

R   gas constant

s   distance along streamline

$\bar{s}$ $\dfrac{s}{d}$

S   entropy of gas

u,v   velocity components in the x and y coordinates, respectively, or in the r and θ coordinates, respectively

u,v   coordinate variables obtained by conformal transformation of r,φ variables

w   complex variable, u + iv

x,y   Cartesian coordinates

z   complex variable, $re^{i\theta}$

4

γ ratio of specific heats of gas

δ relaxation net spacing

η angle between the sonic line and streamline

ξ distance from sonic line to chosen starting line for solution by method of characteristics

θ angle between streamlines and axis of symmetry

θ_w angle between shock wave and axis of symmetry

μ compliment of Mach angle, $\frac{\pi}{2} - \sin^{-1}\frac{1}{M}$

ν streamline variable

ρ gas density

$\bar{\rho}$ $\dfrac{\rho}{(\rho_s)_r}$

φ angular variable in polar coordinates

χ transformed density variable, $\rho^{-\frac{1}{2}}$

$\bar{\chi}$ $\dfrac{\chi}{(\chi_s)_r}$

ψ stream function

$\bar{\psi}$ $\dfrac{\psi}{G}$

Subscripts

∞ free-stream conditions

r functions of the streamlines only
(The subscript is omitted when the streamline passes through the point where the shock is normal.)

s stagnation conditions

x,y   derivatives with respect to subscript evaluated at the sonic
line

*   sonic conditions

o,1,2,3,4   identification of net point variables

METHOD OF SOLUTION

Statement of the Problem

The problem to be considered is the two-dimensional flow of an inviscid gas over a circular cylinder at the limiting velocity of infinite Mach number. The fluid is taken to be a perfect gas with $\gamma = 1.4$. This is at some variance with the suspected state of affairs at the high temperature generated in actual flight where γ is smaller and other thermal imperfections exist, but the uncertainty of the actual state makes the choice of gas properties difficult.

Outline of the Entire Solution

The complete numerical process is difficult to describe precisely. Much of it cannot be given as a step-by-step procedure as, for example, there are places where iteration takes place and, in addition, judgment plays an important part in the direction of the operations. In what follows, a general survey of the method of solution is given and at the end the more specific steps of the solution will be presented.

The first part of the solution is the subsonic region. In order to set up the relaxation procedure it is necessary to have given the conditions around a closed boundary of the field being solved. In this case the boundaries are the body surface, the axis of symmetry, the shock wave, and the sonic line as shown in the sketch. An initial guess as to the positions of the shock wave and the sonic line must be made, and the more educated the guess, the easier and shorter will be the subsequent work. All possible information should be used to obtain the initial estimate including clues from experimental results, if available. For these initial boundaries, the process of relaxation will give a solution of the subsonic flow field. This solution must be
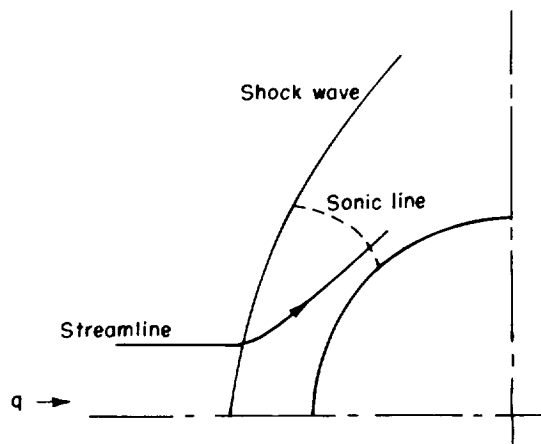
compatible with the conditions along the shock wave and sonic line. For example, the flow angle at the shock wave can be determined from the streamlines of the relaxation solution and also from the shock-wave angle. A new shock-wave angle distribution can then be calculated from the relaxation solution and a new shock-wave shape obtained by integration. The testing for compatibility of the sonic line is more difficult as it involves such things as streamline curvature and density gradients, but the principle is the same. With these improved boundaries the interior can be corrected. Before the sonic line can be regarded as accurately established, however, it is necessary to make the sonic line and the neighboring subsonic region compatible with the initial supersonic region. The Mach number gradients along a streamline, for example, should not be discontinuous through the sonic line. This step is necessary because there may be several different sonic lines compatible to the condition in the subsonic region, but not all of them compatible to the ensuing supersonic region. Beyond a certain point the disturbances in the supersonic region cannot affect the subsonic or transonic region, and the characteristic calculations can proceed as far as desired into the supersonic region.

Subsonic Region

The equations for rotational flow of a compressible fluid are given in many references (see, e.g., ref. 9). In the subsonic region the natural variable to use is the stream function which is defined so as to satisfy the continuity equation as follows:

$$\frac{\partial \psi}{\partial \varphi} = r\rho u \ , \qquad \frac{\partial \psi}{\partial r} = -\rho v \tag{1}$$

Here the polar coordinates r and φ are used since they are consistent with the shape of the body in this problem. This is then substituted into the vorticity equation producing after some transformation the required equation:

$$\frac{\partial}{\partial r}\left(\frac{r}{\rho} \frac{\partial \psi}{\partial r}\right) + \frac{\partial}{\partial \varphi}\left(\frac{1}{\rho r} \frac{\partial \psi}{\partial \varphi}\right) + \frac{r\rho}{R} \frac{\partial S}{\partial \psi} = 0 \tag{2}$$

A relation between the density, ρ, and the stream function will be needed and can be obtained from the equation for the mass-flow rate.

$$(\rho q)^2 = (\rho u)^2 + (\rho v)^2 = \left(\frac{\partial \psi}{\partial r}\right)^2 + \left(\frac{1}{r} \frac{\partial \psi}{\partial r}\right)^2 \tag{3}$$

The method of relaxation is applied by Drebinger and Mitchell in somewhat different manners (refs. 7 and 8, respectively). The present procedure combines features of both approaches. Equation (2) is transformed slightly by the introduction of a new variable.

$$\chi = \rho^{-\frac{1}{2}} \tag{4}$$

giving the result,

$$\frac{\partial^2}{\partial r^2}\left(\chi\psi\right) + \frac{1}{r}\frac{\partial}{\partial r}\left(\chi\psi\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}\left(\chi\psi\right) - \psi\left(\frac{\partial^2\chi}{\partial r^2} + \frac{1}{r}\frac{\partial\chi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\chi}{\partial\varphi^2}\right) + \frac{rp}{R\chi}\frac{\partial S}{\partial\psi} = 0 \tag{5}$$

Since entropy is constant along a streamline, the variation of the fluid properties along a streamline is regarded as isentropic, and for the rth streamline, the Bernoulli's equation can be expressed as

$$\left(\frac{q}{c_s}\right)^2 = \frac{2}{\gamma-1}\left[1 - \frac{p}{(p_s)_r}\frac{\chi^2}{(\chi_s)_r{}^2}\right] \tag{6}$$

where subscript s refers to stagnation conditions. Note that $(p_s)_r$ and $(\chi_s)_r$ are constant along a given streamline, that is, they are functions of ψ only. The stagnation sound velocity, c_s, on the other hand, is a constant for the entire flow. If equations (6) and (3) are combined and the pressure terms are eliminated by the isentropic gas law, an expression relating the stream function to the density variable can be obtained:

$$\frac{2}{\gamma-1}\frac{c_s{}^2}{\chi^4}\left\{1 - \left[\frac{\chi}{(\chi_s)_r}\right]^{2(1-\gamma)}\right\} = \left(\frac{\partial\psi}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial\psi}{\partial\varphi}\right)^2 \tag{7}$$

Following Mitchell, a new variable is introduced defined by

$$v_r = \frac{G}{(\rho_s)_r c_s d} \tag{8}$$

where G is a free-stream mass-flow constant and d is a characteristic length. It is easily seen that v_r is a function only of the stream function variable. Equations (5) and (7) are now made dimensionless by introduction of the variables

$$\bar{\chi} = \frac{\chi}{(\chi_s)_r} \ , \qquad \bar{p} = \frac{p}{(p_s)_r} \ , \qquad \bar{r} = \frac{r}{d} \ , \qquad \bar{\psi} = \frac{\psi}{G} \tag{9}$$

There is then obtained from equations (5) and (7):

$$\frac{\partial^2(\bar{\chi}\bar{\psi})}{\partial\bar{r}^2} + \frac{1}{\bar{r}}\frac{\partial(\bar{\chi}\bar{\psi})}{\partial\bar{r}} + \frac{1}{\bar{r}^2}\frac{\partial^2(\bar{\chi}\bar{\psi})}{\partial\varphi^2} - \bar{\psi}\left(\frac{\partial^2\bar{\chi}}{\partial\bar{r}^2} + \frac{1}{\bar{r}}\frac{\partial\bar{\chi}}{\partial\bar{r}} + \frac{1}{\bar{r}^2}\frac{\partial^2\bar{\chi}}{\partial\varphi^2}\right) +$$

$$\bar{\chi}\left(\frac{\partial \log \nu_r}{\partial\bar{r}}\frac{\partial\bar{\psi}}{\partial\bar{r}} + \frac{1}{\bar{r}^2}\frac{\partial \log \nu_r}{\partial\varphi}\frac{\partial\bar{\psi}}{\partial\varphi}\right) + \frac{\bar{p}}{\gamma\nu_r^2\bar{\chi}R}\frac{\partial S}{\partial\bar{\psi}} = 0 \tag{10}$$

$$\bar{\chi}^{-4}\left[1 - \bar{\chi}^{2(1-\gamma)}\right] = F(\bar{\chi}) = \frac{\nu_r^2(\gamma-1)}{2}\left[\left(\frac{\partial\bar{\psi}}{\partial\bar{r}}\right)^2 + \left(\frac{1}{\bar{r}}\frac{\partial\bar{\psi}}{\partial\varphi}\right)^2\right] \tag{11}$$

For numerical purposes equation (10) can be put in more convenient form by use of the relation derived from the shock-wave equation (23)[1]

$$\frac{\nu_r}{\nu} = \frac{\rho_s}{(\rho_s)_r} = e^{\frac{S_r-S}{R}}$$

where the omission of the subscript r refers to the values for the streamline originating from the normal shock. Then by means of such transformations as

$$\frac{\partial \log \nu_r}{\partial\bar{r}} = \frac{\partial \log(\nu_r/\nu)}{\partial\bar{r}} = \frac{1}{R}\frac{\partial S}{\partial\bar{\psi}}\frac{\partial\bar{\psi}}{\partial\bar{r}} \tag{12}$$

and elimination of $\bar{p}$ by the isentropic gas law, equation (10) becomes

$$\frac{\partial^2(\bar{\chi}\bar{\psi})}{\partial\bar{r}^2} + \frac{1}{\bar{r}}\frac{\partial(\bar{\chi}\bar{\psi})}{\partial\bar{r}} + \frac{1}{\bar{r}^2}\frac{\partial^2(\bar{\chi}\bar{\psi})}{\partial\varphi^2} - \bar{\psi}\left(\frac{\partial^2\bar{\chi}}{\partial\bar{r}^2} + \frac{1}{\bar{r}}\frac{\partial\bar{\chi}}{\partial\bar{r}} + \frac{1}{\bar{r}^2}\frac{\partial^2\bar{\chi}}{\partial\varphi^2}\right) +$$

$$\left\{\bar{\chi}\left[\left(\frac{\partial\bar{\psi}}{\partial\bar{r}}\right)^2 + \left(\frac{1}{\bar{r}}\frac{\partial\bar{\psi}}{\partial\varphi}\right)^2\right] + \frac{1}{\gamma\nu_r^2\bar{\chi}^{(2\gamma+1)}}\right\}\frac{1}{R}\frac{\partial S}{\partial\bar{\psi}} = 0 \tag{13}$$

The corresponding equation in Mitchell's paper (ref. 8) is apparently in error as it omits the first part of the last term. It may have been assumed by Mitchell that the omitted term could be neglected for low Mach numbers, but no explanation or indication was given in the paper.

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[1]Because of the interrelated nature of the parts of this problem, referencing to subsequent equations could not readily be avoided.

It is convenient to work the relaxation method in a Cartesian net; hence, the first step is to transform the polar net to a rectangular one by the conformal transformation
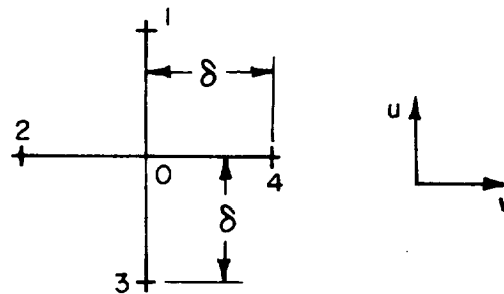
$$w = u + iv = \log z = \log \bar{r} + i\varphi \tag{14}$$

With these transformations, $u = \log \bar{r}$ and $v = \varphi$, equations (13) and (11) become, respectively,

$$\frac{\partial^2(\bar{\chi}\bar{\psi})}{\partial u^2} + \frac{\partial^2(\bar{\chi}\bar{\psi})}{\partial v^2} - \bar{\psi}\left(\frac{\partial^2\bar{\chi}}{\partial u^2} + \frac{\partial^2\bar{\chi}}{\partial v^2}\right) + \left\{\bar{\chi}\left[\left(\frac{\partial\bar{\psi}}{\partial u}\right)^2 + \left(\frac{\partial\bar{\psi}}{\partial v}\right)^2\right] + \frac{\bar{r}^2}{\gamma v_r{}^2\bar{\chi}(2\gamma+1)}\right\} \frac{1}{R}\frac{\partial S}{\partial\bar{\psi}} \tag{15}$$

$$F(\bar{\chi}) = \frac{(\gamma-1)v_r{}^2}{2\bar{r}^2}\left[\left(\frac{\partial\bar{\psi}}{\partial u}\right)^2 + \left(\frac{\partial\bar{\psi}}{\partial v}\right)^2\right] \tag{16}$$

Consider now a uniform net of points over the coordinate system covering the flow region with a uniform spacing of δ between the points, as shown in the sketch. Now, if the values of the variables are used at, for example, points 4, 0, and 2, difference formulas can be obtained which will approximate the derivatives at point 0. These difference formulas can be used to write equations (15) and (16) in the following forms:

$$\sum_{j=1}^{4} \bar{\chi}_j(\bar{\psi}_j-\psi_0) + \left\{\bar{\chi}_0[(\psi_1-\psi_3)^2 + (\psi_2-\psi_4)^2] + \frac{\delta^2 r_0}{\gamma v_{r_0}{}^2\bar{\chi}_0(2\gamma+1)}\right\}\frac{1}{R}\frac{\partial S}{\partial\bar{\psi}} = Q \tag{17}$$

$$F(\chi_0) = \frac{\gamma-1}{2}\frac{v_{r_0}{}^2}{r_0{}^2\delta^2}[(\bar{\psi}_1-\bar{\psi}_3)^2 + (\bar{\psi}_2-\bar{\psi}_4)^2] \tag{18}$$

There will be a set of these equations at every point in the net; and together with the given values of $\bar{\psi}$ along the boundaries, it is theoretically possible but obviously impractical to solve all these equations simultaneously. The relaxation method is an effective approach for finding a solution to a set of equations like this. The first step is to assume values of the unknown variables, $\bar{\psi}$, at all points. Then at each point the density variable $\bar{\chi}$ is calculated from equation (18) which

enables the value Q, which is called the residue, to be calculated from equation (17). It can be seen that the required solution has been achieved if $Q = 0$ at all the points simultaneously. Since a correct initial guess of the stream function is obviously quite unlikely, the problem becomes one of reducing the Q's to negligibly small amounts by modification of the values of ψ at all the points with equation (17) as a guide. The means and techniques by which these modifications are made form the substance of the relaxation method. The analogy from which the method gets its name is that of an elastic net with ψ being an arbitrary displacement and Q the unbalanced force at each mode of the net. The unbalanced forces may then be reduced by "relaxing" the displacements. The amount of computation that can be saved by the use of special techniques as well as systematic procedures is at times remarkable and the reader is referred to papers already mentioned and especially to more general works (refs. 5 and 10).

The shock wave.- When shock-wave equations are written for infinite Mach number, it is necessary to use some reference point behind the shock wave to avoid infinite quantities. The simplest choice is to refer to the normal shock quantities. The shock-wave equations (ref. 11) then become

$$\tan \theta = \frac{2 \cos \theta_w \sin \theta_w}{(\gamma+1) - 2 \sin^2 \theta_w} \tag{19}$$

$$M^2 = \frac{(\gamma+1)^2 - 4\gamma \sin^2 \theta_w}{2\gamma(\gamma-1)\sin^2 \theta_w} \tag{20}$$

$$\frac{\chi}{(\chi_s)_r} = \bar{\chi} = \left(\frac{\gamma+1}{4\gamma \sin^2 \theta_w}\right)^{\frac{1}{2(\gamma-1)}} \tag{21}$$

$$\frac{S - S_r}{R} = \frac{2}{\gamma-1} \log \sin \theta_w \tag{22}$$

$$\frac{(\rho_s)_r}{\rho_s} = \frac{(p_s)_r}{p_s} = e^{-\frac{S_r - S}{R}} = (\sin \theta_w)^{-\frac{2}{\gamma-1}} \tag{23}$$

$$\frac{p}{p_s} = \left[\frac{4\gamma}{(\gamma+1)^2}\right]^{\frac{\gamma}{\gamma-1}} \sin^2 \theta \tag{24}$$

The quantities such as p_s and ρ_s where the subscript r is omitted refer to normal shock quantities which occur along the streamline, along the axis of symmetry, and along the body. The first three equations are

used to test the compatibility of the shock wave; the third and fourth equations are needed in connection with the subsonic solution; and the last equation is used to determine the shock-wave position in the characteristic calculations of the supersonic region.

The other necessary condition is the determination of the stream function along the shock wave. If the definition of the stream function is applied to the free stream, there is obtained in polar coordinates

$$\psi = \rho_\infty q_\infty r \sin \varphi \tag{25}$$

where the subscript ∞ refers to free-stream conditions. When converted to dimensionless form, this equation becomes

$$\bar{\psi} = \frac{\rho_\infty q_\infty \bar{r} \sin \varphi}{(\rho_s)_r c_s \nu_r} \tag{26}$$

It is seen that for numerical purposes ν_r can be chosen arbitrarily, the most convenient choice being the one related to the net spacing so as to simplify the constants in equations (17) and (18). If the conditions at the normal shock wave are used, then $(\rho_s)_r = \rho_s$, $\nu_r = \nu = \delta/d$ and finally, when conditions for infinite Mach number are substituted, the stream function is given by

$$\bar{\psi} = \frac{d}{\delta} \frac{\sqrt{2(\gamma-1)}}{\gamma+1} \left[1 + \frac{(\gamma-1)^2}{4\gamma} \right]^{-\frac{1}{\gamma-1}} \bar{r} \sin \varphi \tag{27}$$

From equations (22), (23), and (27) and the given shape of the shock wave, the values of $(S_r-S)/R$ and ν_r/ν can be determined as a function of ψ and then $(1/R)(\partial S/\partial \psi)$ can be found by numerical differentiation.

The sonic line.- From the definition of the stream function one can readily derive the equation for ψ along the sonic line:

$$\frac{\partial \psi}{\partial l} = (\rho_*)_r q_* \sin \eta \tag{28}$$

where the asterisk refers to sonic quantities, η is the angle from the streamline to the sonic line, and l is the distance along the sonic line. When put into dimensionless form, equation (28) becomes

$$\frac{\partial \bar{\psi}}{\partial \bar{l}} = \frac{1}{\nu_r} \frac{q_*}{c_s} \left(\frac{\rho_*}{\rho_s} \right)_r \sin \eta = \frac{1}{\nu_r} \left(\frac{\gamma+1}{2} \right)^{-\frac{\gamma+1}{2(\gamma-1)}} \sin \eta \tag{29}$$

Two other important relations are given by Drebinger in reference 7. The first one gives the slope of the sonic line:

$$\tan \eta = -\frac{\dfrac{\partial q}{\partial s}}{q\dfrac{\partial \theta}{\partial s} - \dfrac{C_*}{\gamma R}\dfrac{\partial S}{\partial n}} \qquad (30)$$

Here streamline coordinates are used, s along the streamline and n normal to it. In terms of the dimensionless variables evaluated at sonic condition this becomes

$$\tan \eta = \frac{\dfrac{\partial (q/C_*)}{\partial \bar{s}}}{\dfrac{\partial \theta}{\partial \bar{s}} - \left(\dfrac{\gamma+1}{2}\right)^{-\frac{\gamma+1}{2(\gamma-1)}}\dfrac{1}{\nu_r R}\dfrac{\partial S}{\partial \psi}} \qquad (31)$$

where

$$\bar{s} = \frac{s}{d} \qquad (32)$$

The other relation of Drebinger is given by

$$\frac{\partial \theta}{\partial \bar{l}} = \cos \eta \frac{\partial \theta}{\partial \bar{s}} \qquad (33)$$

It is seen from the above equations that the calculations of the sonic-line properties depend very largely on the properties of the surrounding flow field which makes computation of the sonic line difficult. The sonic region is indeed the most difficult part of the entire solution. Even the initially assumed sonic line is a troublesome problem. The location of the sonic point on the present body was made on the assumption that the flow direction on the body was the same as at the sonic point on the shock wave. It turned out to be a very good guess. Some idea of the shape of the sonic line can be obtained from equations (30) and (31) since estimates of quantities in these equations can be made. An additional aid in determining the sonic line is determining the sonic-line slope η at the shock wave which is known in terms of shock-wave properties and can be calculated by a rather lengthy formula derived by Drebinger in reference 7. Equation (28) is important not only to obtain a boundary condition for the subsonic solution but also to determine the distances between the shock wave and the body. It is obvious that the stream function calculated from equation (28) and the one calculated from equation (27) must be equal at the shock-wave sonic point. The location of this shock-wave point requires a trial and error method of computation. Equations (31) and (32) are used to help estimate the initial sonic-line

slope, η, and stream angle, θ. The first guess of the entropy function, ν_r, along the sonic line is liable to be quite crude at first since it is an implicit variable in this calculation. Then the stream function along the sonic line is found by integration of a suitable form of equation (29). On comparison with the stream function on the assumed shock wave, the sonic line is then elongated or shortened as required, and the variable ν_r is modified to be consistent with the expected distribution of ψ along the sonic line. The computation is repeated until the necessary agreement is obtained. It is important that this calculation be done accurately, as errors can seriously disturb the relaxation process for solving the subsonic region near the sonic line. If, for example, the gradient of ψ is too high along the sonic line, then the flow would be an erroneous supersonic one.

The supersonic region.- As was mentioned previously, the relaxation method has been used by Mitchell to extend the solution into the supersonic region. To make this extension it is necessary to assume some kind of approximate boundary in the supersonic region. Such a procedure might be adequate for low supersonic speeds, but at hypersonic speeds the large entropy gradients make it practically impossible to find boundaries of even moderate accuracy. Hence, the well-known method of characteristics is used for the present problem. The characteristic equation can be written in many different forms, the one chosen here is given in reference 12:

$$\frac{dp}{p} = \mp \frac{\gamma M^2}{\sqrt{M^2-1}} \, d\theta \tag{34}$$

where the variables are integrated along first and second families of characteristic lines given by the equation

$$\frac{dn}{ds} = \pm \tan \beta \tag{35}$$

where $\beta = \sin^{-1}\frac{1}{M}$ is the Mach angle. Because of the large entropy gradients existing in the flow, a hodograph chart is not practical so that the computation must progress as a step-by-step numerical integration

It is, of course, impossible to start the characteristic solution right on the sonic line as then the coefficient in equation (34) is infinite. Consequently it is necessary to locate some line in the supersonic region as a starting place. A means of doing this is given by Holt (ref. 13); however, his equations were developed for potential flow so to use his methods the equations must be rederived for rotational flow This derivation is given in the appendix together with a summary of the method of Holt for locating the starting line.

Summary of the Procedure

1. The shock-wave shape is obtained by integration of an assumed distribution of the shock-wave angle. When this shock wave is located as indicated in the next step, the stream function and entropy functions can be calculated from equations (27), (22), and (23).

2. The shape of the sonic line is estimated with the aid of equations (30) and (33). The stream function along the sonic line is calculated from equation (29), and the sonic-line length is obtained by matching the values of the stream function on the sonic line and on the shock wave at their intersection.

3. The subsonic region is then solved by the method of relaxation. The results of this solution can be used in steps 1 and 2 to improve the shapes and positions of the shock waves and the sonic line, and the subsonic region can be corrected to agree with the improved boundaries. This is repeated until the boundaries and the subsonic solution are in accord.

4. A starting line for the method of characteristics is located away from the sonic line by the method developed by Holt. With equations (34) and (35) the characteristics solution is started on this line with the body surface and the shock wave with equation (24) used as boundary conditions.

5. When the supersonic solution for the Mach number gradients along the body surface has proceeded sufficiently, the streamlines, the shock wave, or other suitable lines are examined for continuity through the sonic line. The sonic point on the body is changed (usually by a very small amount) to satisfy this requirement.

6. With the sonic line finally established the supersonic calculations are continued as far as is desired.

It should be emphasized again that these steps should be considered more as a guide than a fixed set of rules. In step 5, for example, an alternative method of handling the Mach number gradients is to determine the flow angle distribution to give the required Mach number gradients and use this information to relocate the sonic line.

Comments on the calculations.- The relaxation computations were tedious but fairly straightforward except near the sonic line where, because of the nature of the equation, the relaxation process breaks down and certain trial and error calculations are necessary. The shock wave was easily corrected, and the subsonic solution needed no more than a small adjustment to correct for the new shock wave.

As was mentioned previously, the location of the sonic line was the most critical and difficult part of the entire calculation. The equations that have been given to test the compatibility of the sonic line are difficult to apply because the solution near the sonic line is a little uncertain. However, to aid the establishment of the proper sonic line, there is a striking property of the subsonic solution which could be called a strong convergence effect. If the density variable $\bar{X}$ is plotted for $\bar{r}$ = constant, for example, there is observed a strong tendency for the values to extrapolate toward the correct sonic line. When this effect was discovered much computation time was saved. This effect appears to be a hypersonic flow effect and seems to indicate the high stability of such flows.

The starting line for the supersonic flow calculations was chosen as the $M = 1.016$ line (Mach angle of $80°$) which made the change of Mach number quite large with flow angle. This sensitivity has the advantage of revealing errors in the sonic line very quickly. The location of the sonic line on the body, for instance, was determined quite closely from the difference in flow direction between the sonic point on the body and the sonic point on the shock wave by making these directions compatible with the Mach number gradients through the sonic line as determined from the characteristic calculations.

RESULTS

The principle results showing the streamlines and the lines of constant Mach number are given in figure 1. The most noteworthy feature of this figure is the strong curvature of the constant Mach number lines. This curvature becomes more pronounced as one goes farther into the supersonic region. Figure 2 which shows the distribution of entropy and related quantities along the shock wave is supplementary to figure 1. In figure 3 the isoclines and isobars are shown, and in figure 4 the lines of constant vorticity are given. Figure 5 shows the results of plotting the streamlines on the hodograph plane. It will be noticed that the streamlines overlap in the vicinity of the sonic region which means that the hodograph is not single-valued over the flow field. This could have been anticipated from the work of Goldstein and Lighthill (ref. 14).

In figure 6 the pressure coefficient along the body is shown together with experimental results determined at a Mach number of 6.86 (ref. 15). Also shown for comparison are the pressure-coefficient curves determined from the Newtonian flow theory

$$C_p = 2 \sin^2\theta \tag{36}$$

and from the equation

$$C_p = \frac{(\gamma+1)^{\frac{\gamma+1}{\gamma-1}}}{2^{\frac{2}{\gamma-1}}\gamma^{\frac{\gamma}{\gamma-1}}} \sin^2\theta \qquad (37)$$

which is the Newtonian flow theory modified to agree at the stagnation point with the present solution. The third curve is the Newtonian impact theory corrected for centrifugal effects by Grimminger's method (Case 5 in ref. 16).

The pressure drag coefficient for the fore part of the cylinder based on projected area can be obtained from the solution by numerical integration. The table below shows the drag coefficient as given by different theories.

Source	C_D
Present solution	1.26
Equation (36)	1.33
Equation (37)	1.23
Grimminger (ref. 16)	1.20
Penland (ref. 15)	1.27

For the purpose of continuing the supersonic solution along any chosen subsequent body contour, figures 7 and 8 present the necessary data along the coordinate lines $\varphi = 80^\circ$ and $\varphi = 90^\circ$.

DISCUSSION

Because the method of calculation is composed of so many parts, some of which are interrelated and some of which are not, the isolation of the sources of errors is difficult. The part of most concern is probably the relaxation solution. As in all methods in which the differentials are replaced by difference equations, there is an error introduced by these approximations depending upon the net size. The net size used in this problem is equivalent to an angular increment of 2.5° and is based on previous experience. As a check on the errors involved, a portion of the flow field near the sonic point of the shock wave was recalculated with a net size of one-half that of the complete solution. The differences in the stream function between the two solutions were less than 0.2 percent, indicating that the net size chosen is about optimum for the amount of work that is involved.

Errors in the subsonic solution also have an important effect on shock-wave shape. This effect is revealed primarily in the shock-wave standoff distance, that is, the distance between the shock wave and the stagnation point. For this reason, it is of interest to compare the standoff distance obtained in the present solution with that obtained by

others. In figure 9, the ratio of standoff distance to cylinder radius is plotted as a function of $1/M_\infty^2$. The present value of 0.372 for infinite Mach number appears to be consistent with the values for Mach numbers of 3, 4, and 5 obtained by Belotserkovsky in reference 1. However, the present value is 6.5 percent less than that obtained by Garabedian in reference 2 (0.398 at infinite Mach number) and 3.9 percent less than that obtained by Van Dyke in unpublished results using the method of reference 3 (0.387 at infinite Mach number). Thus, there are some differences in the standoff distances given by the various methods which are as yet unresolved. During the course of the present computations, however, it was found that the subsonic flow sufficiently removed from the shock wave is little affected by small changes in the shock-wave shape. This finding would indicate that a small error in the shock-wave shape does not have a noticeable effect on the flow on or near the body surface.

The most striking feature of the flow field found in the present results is the strong curvature of the constant Mach number lines. The Mach number decreased not only in proceeding out from the body but also when moving in from the shock wave. The former phenomenon is well known and is a result of the rapid expansion of the flow caused by the curvature of the body. The latter phenomenon seems to be specific to hypersonic flow, and a discussion as to its cause in the supersonic region would be instructive. In the application of the method of characteristics it was found that an important effect of the strong entropy gradients is a pronounced curvature of the streamline towards the direction of increasing entropy which is toward the body. This means that the flow is curving away from the shock wave and therefore the Mach number is increasing more rapidly than the Mach number in the region closer to the body.

A significant result of the strong curvature of the lines of constant Mach number is the small interaction between the supersonic and subsonic flow regions. The limiting characteristic is the last characteristic in the supersonic region that starts from the body and still touches the sonic line. Between this limiting characteristic and the sonic line is a supersonic region where disturbances and flow changes can still feed back into the subsonic region of flow. In the present problem this interaction region is quite small indicating negligible interaction effect as was also indicated during the course of the computation. It should be noted here that this result is for a body that has no slope discontinuity. In the case where the body has a sharp shoulder, such as a square-nosed body, the situation may be quite different. It is suspected that the interaction region in this case will have significant importance in the determination of the shape of the sonic line.

The comparison with the data of Penland (ref. 15) for a circular cylinder at $M = 6.86$ shows differences as high as 12 percent. A good part of these differences is, however, believed to be experimental error.

The Newtonian flow theory overestimates the pressure near the stagnation point and underestimates the pressure at the $\varphi = 90^\circ$ point. Since the greatest part of the pressure drag is due to the region near the stagnation point, Newtonian flow overestimates the total-pressure drag. When modification to Newtonian flow is made for centrifugal forces, as in Grimminger's theory (ref. 16), the resulting correction is so large that it gives the smallest pressure drag in spite of the high stagnation pressure coefficient.

<div align="center">CONCLUDING REMARKS</div>

A solution by numerical methods of the hypersonic flow at $M = \infty$ of a perfect compressible inviscid fluid over a cylinder has been calculated. With this body, which has no slope discontinuity on its surface, there is negligible interaction between the supersonic flow and the subsonic flow near the body because of the strongly curved shape of the sonic line. The method of analysis based on Newtonian flow is applicable to a problem of this type only in that it shows the correct trends of pressures along the body. The approximate methods show larger variation in pressure than are given by the present solution.

Ames Research Center
    National Aeronautics and Space Administration
        Moffett Field, Calif., Nov. 25, 1958

APPENDIX

HOLT'S METHOD OF SONIC-LINE ANALYSIS MODIFIED FOR

ROTATIONAL FLOW

The coordinate system used here is the same as that used by Holt; namely, the x axis coincides with the streamlines with the flow direction given by positive x, and the origin is on the sonic line. The starting equations are the same ones used by Holt except, in this case, they become

$$u\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \frac{c^2}{\gamma R} \frac{\partial S}{\partial y} \quad \text{(rotation)} \tag{A1}$$

$$(u^2 - c^2) \frac{\partial u}{\partial x} + uv\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + (v^2 - c^2) \frac{\partial v}{\partial y} + \frac{2c^2}{R} v \frac{\partial S}{\partial y} = 0 \quad \text{(continuity)} \tag{A2}$$

$$c^2 = \frac{\gamma+1}{2} c_*{}^2 - \frac{\gamma-1}{2} (u^2 + v^2) \quad \text{(energy)} \tag{A3}$$

The double Taylor expansion of the velocity components,

$$\left.\begin{aligned}
u &= c_* + xu_x + yu_y + \frac{1}{2} (x^2u_{xx} + 2xyu_{xy} + y^2u_{yy}) + \cdots \\
v &= xv_x + yv_y + \frac{1}{2} (x^2v_{xx} + 2xyv_{xy} + y^2v_{yy}) + \cdots
\end{aligned}\right\} \tag{A4}$$

are substituted in equations (A1) through (A3). Here the notation u_x represents the derivative of u with respect to x evaluated at the origin of this coordinate system. From the coefficients of the powers of x and y, the following relations between the derivatives are found on the sonic line.

$$v_x - u_y = \frac{c_*}{\gamma R} \frac{\partial S}{\partial y} \tag{A5}$$

$$v_y = 0 \tag{A6}$$

$$v_{yy} = (\gamma+1)u_x u_y \tag{A7}$$

$$v_{xy} - u_{xy} = - \frac{u_x}{R} \frac{\partial S}{\partial y} \tag{A8}$$

$$v_{xy} - u_{yy} = - \frac{u_y}{R} \frac{\partial S}{\partial y} \tag{A9}$$

$$v_{xy} = \frac{\gamma+1}{c_*} (u_x{}^2 + v_x{}^2) - (\gamma-1) \frac{v_x u_y}{c_*} \tag{A10}$$

With these expressions only the x derivatives of the velocity components need be known to obtain the remaining derivatives.

The other important expression of Holt changed by the introduction of entropy gradients is the sonic-line slope which is given by

$$dq = u_x dx + u_y dy = 0$$

along sonic line giving

$$\tan \eta = \frac{dy}{dx} = - \frac{u_x}{u_y} = \frac{-u_x}{v_x - \frac{c_*}{\gamma R} \frac{\partial S}{\partial y}} \tag{A11}$$

The coefficient v_x is evaluated from the streamline curvature

$$v_x = c_* \left(\frac{\partial \theta}{\partial S}\right)_* \tag{A12}$$

The method proposed by Holt for starting the supersonic solution is essentially to establish a line of known Mach number greater than 1 by extrapolation. This is done by fixing a line a normal distance ξ from the sonic line, and then the coordinates of a point on the line are found by

$$\frac{x}{\sin \eta} = - \frac{y}{\cos \eta} = \xi \tag{A13}$$

The distance ξ can be obtained from the expression

$$\frac{q^2}{c^2} = \sec^2 \mu \tag{A14}$$

where

$$\mu = \frac{\pi}{2} - \sin^{-1} \frac{1}{M} \tag{A15}$$

If the terms of higher order than x and y are expanded and neglected, equation (A14) can be put into explicit form and can be expressed in terms of the coefficients of equations (A4) as

$$\xi = \frac{\mu^2}{(\gamma+1)\sqrt{\left(\dfrac{u_x}{c_*}\right)^2 + \left(\dfrac{u_y}{c_*}\right)^2}} \tag{A16}$$

If M is chosen close enough to 1, ξ can be made small enough to make error negligible. The flow direction on the new line can then be determined from the flow direction on the sonic line with the aid of equations (A3) and (A4).

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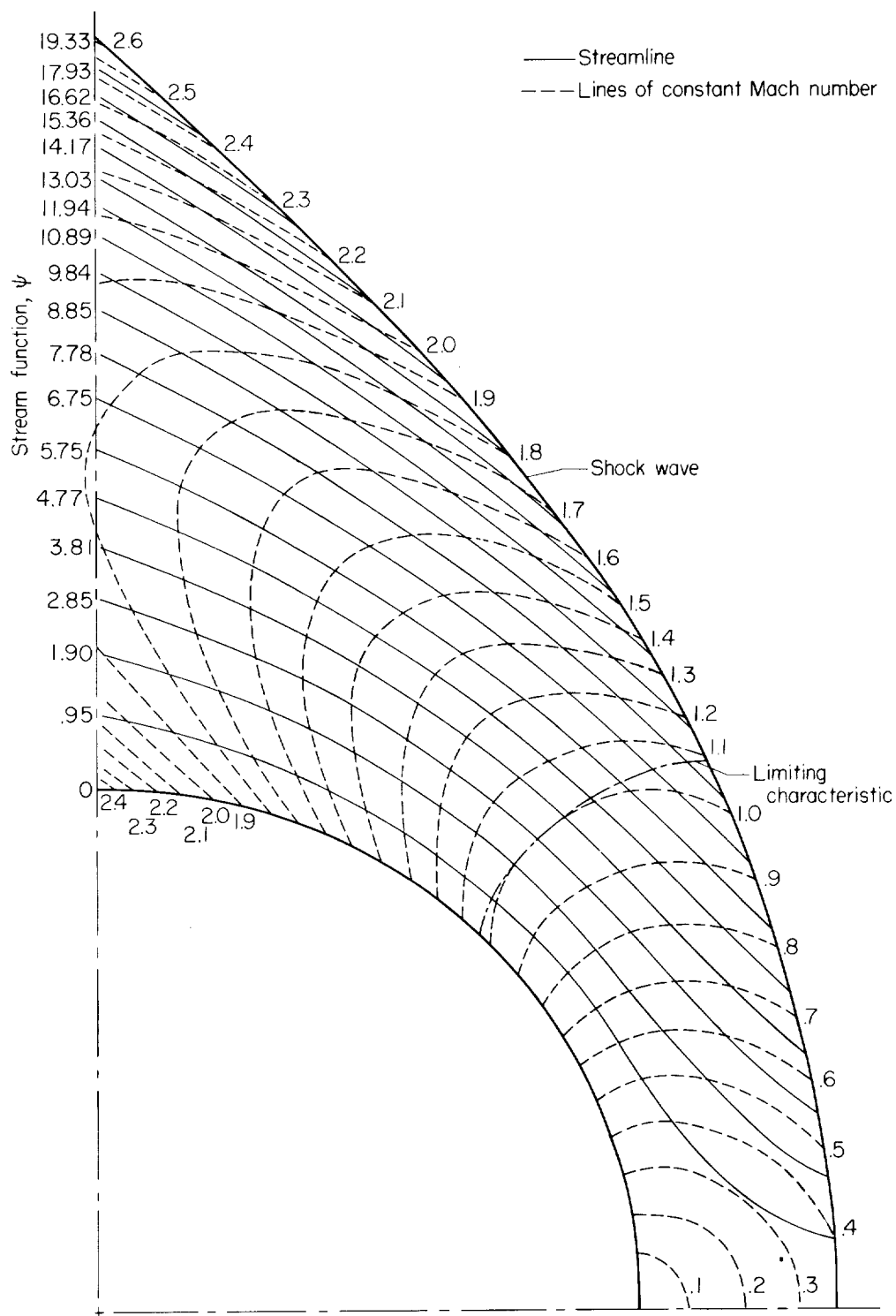
Figure 1.- Flow field over circular cylinder.
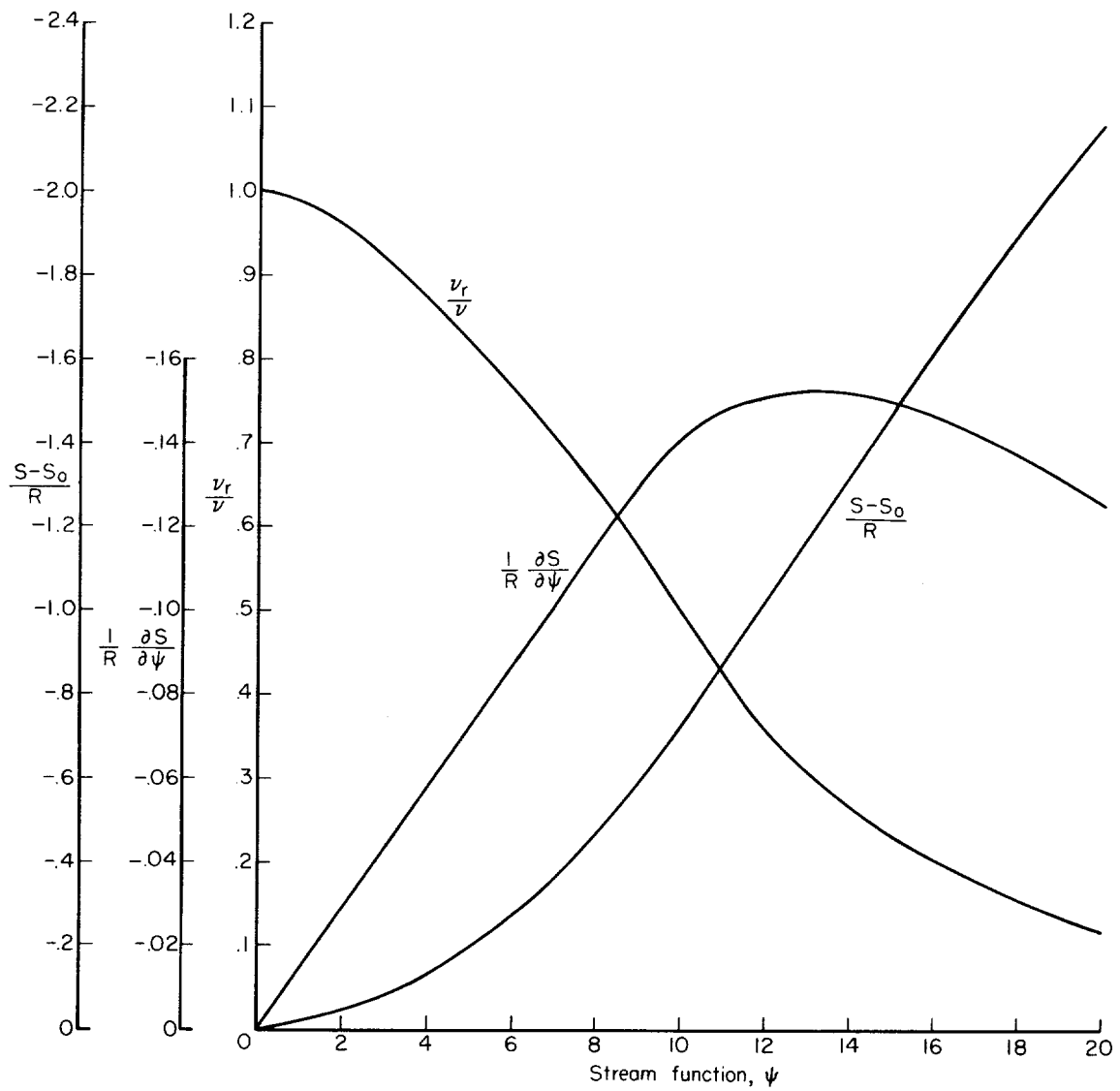
Figure 2.- Fluid properties behind the shock wave.
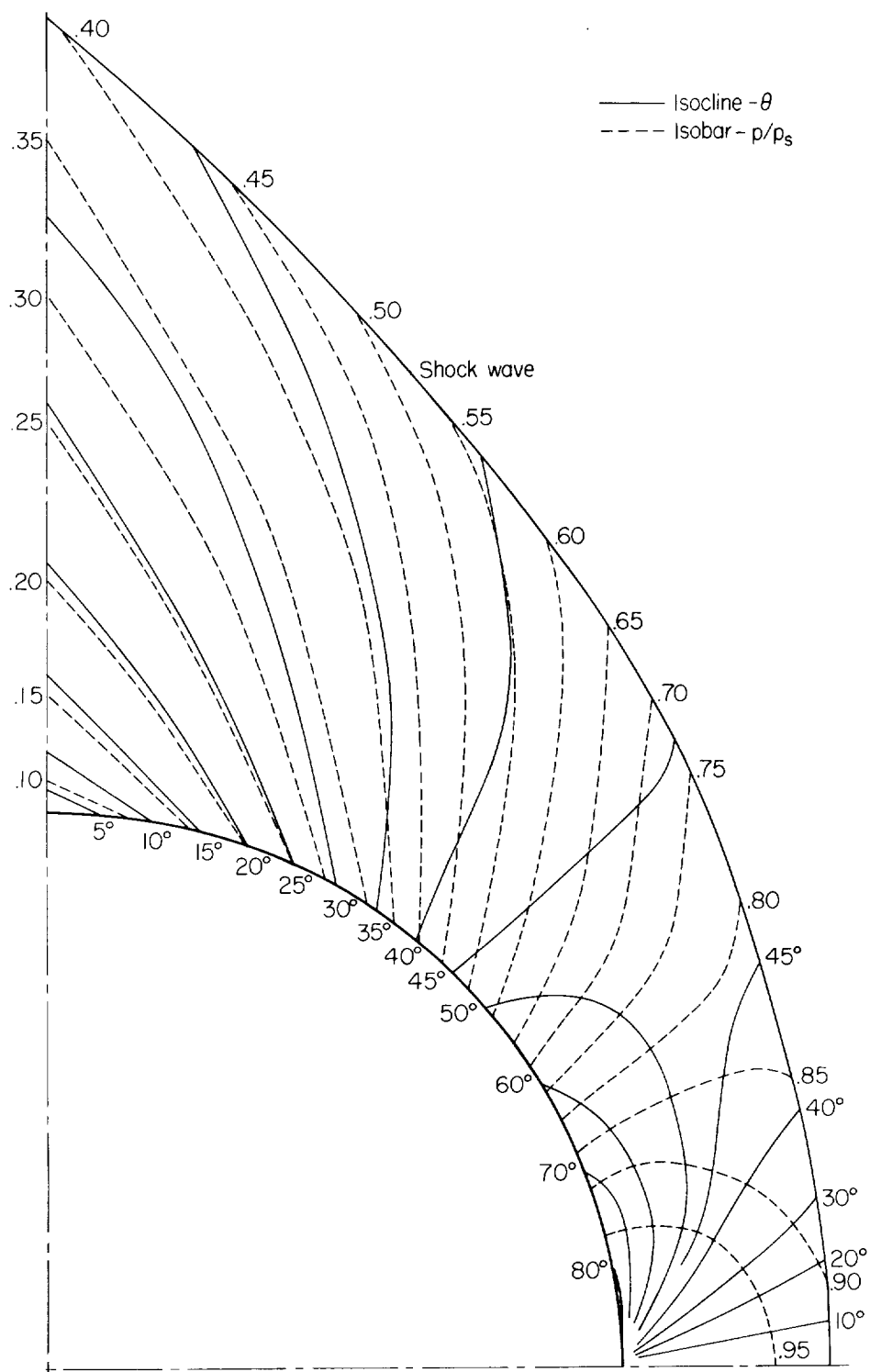
Figure 3.- Lines of constant flow angle and constant pressure.

$$\zeta = C_s \frac{\bar{p}}{\gamma \nu_r} \frac{1}{R} \frac{\partial S}{\partial \bar{\psi}}$$

Shock wave

$\frac{\zeta}{.98}$

.95

.90

.80

.60

.40

.20
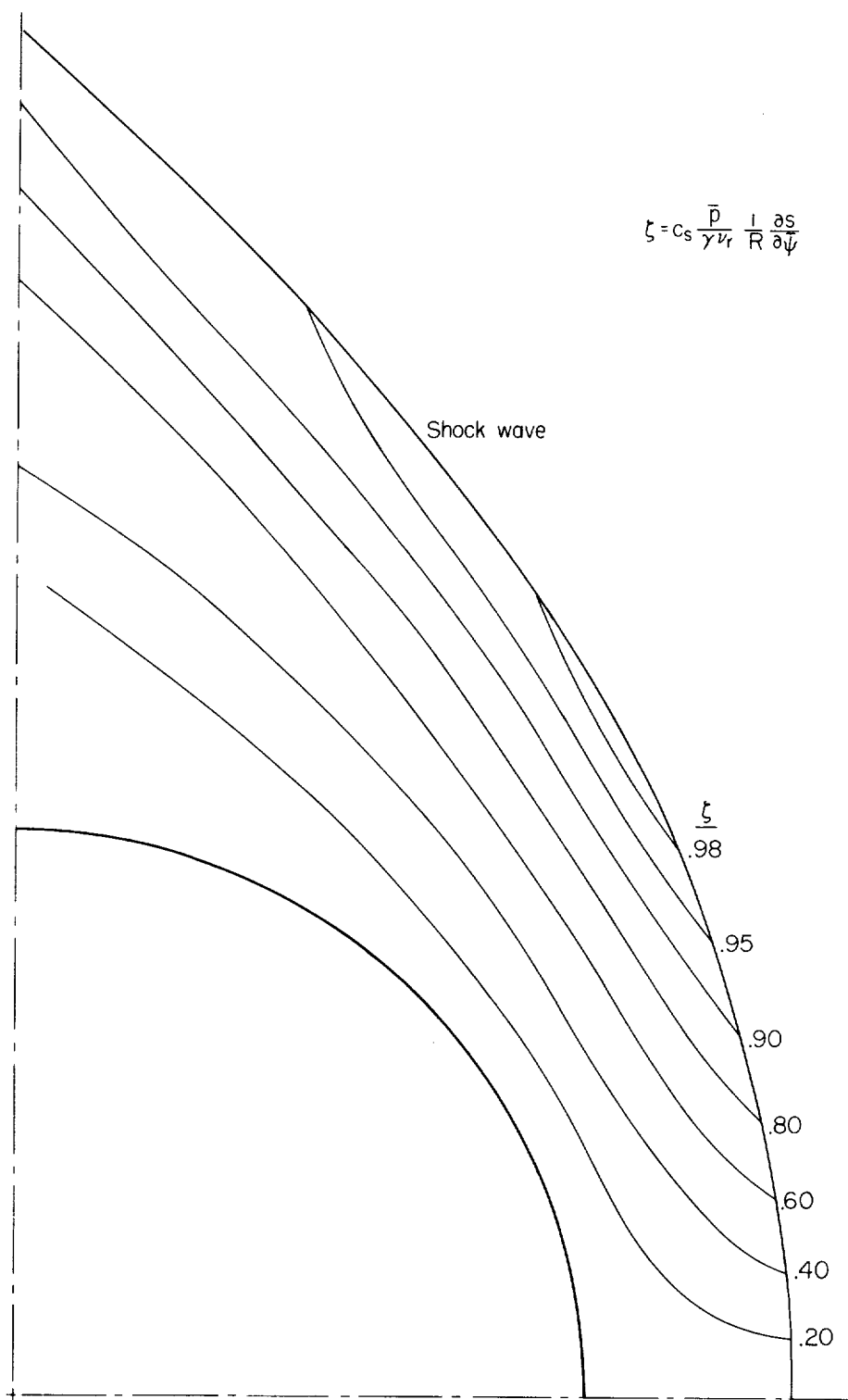
Figure 4.- Vorticity distribution in flow field.
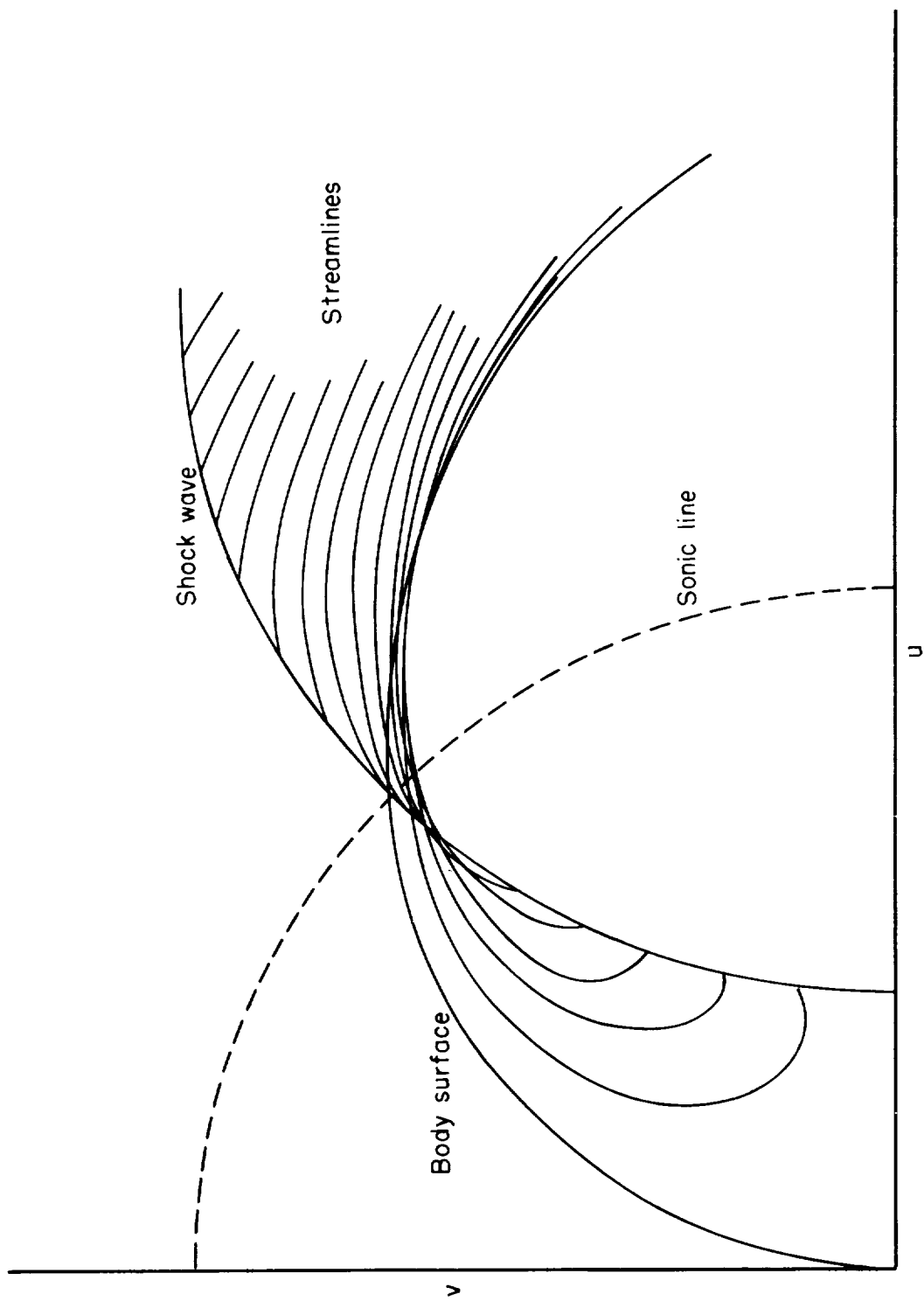
Figure 5.- Hodograph of flow field.
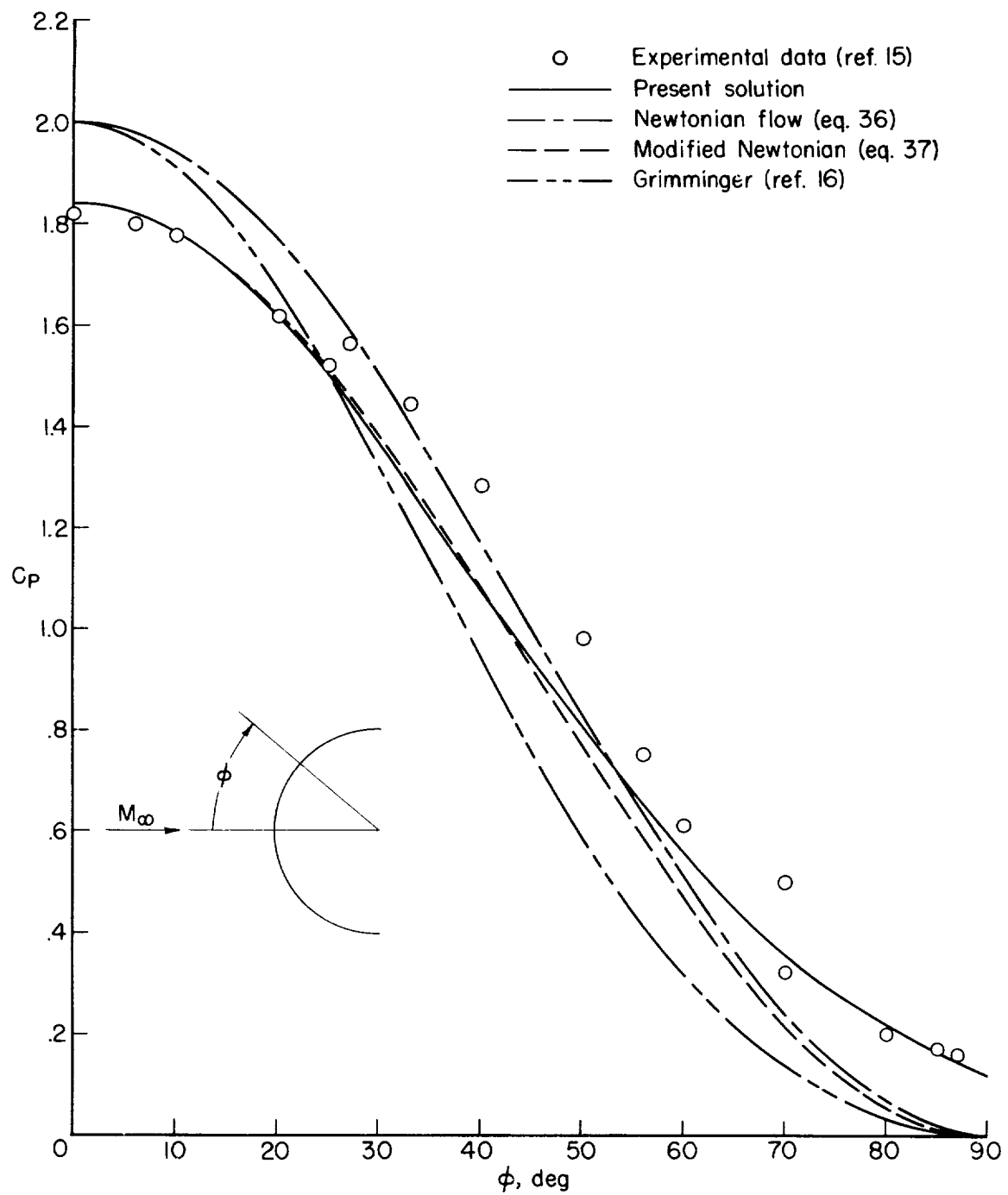
Figure 6.- Pressure coefficient on surface of cylinder.
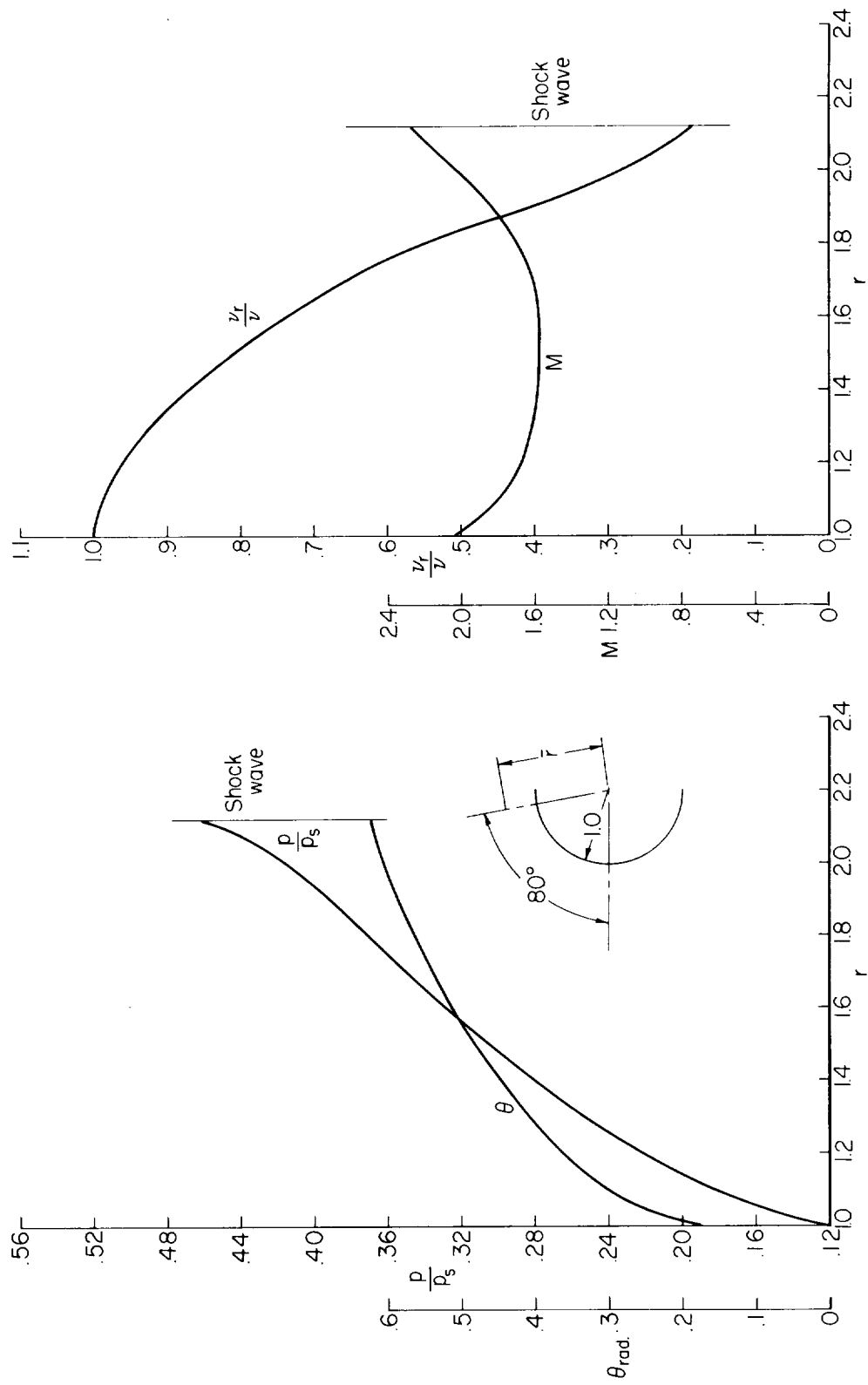
Figure 7.- Flow properties along 80° radial line.
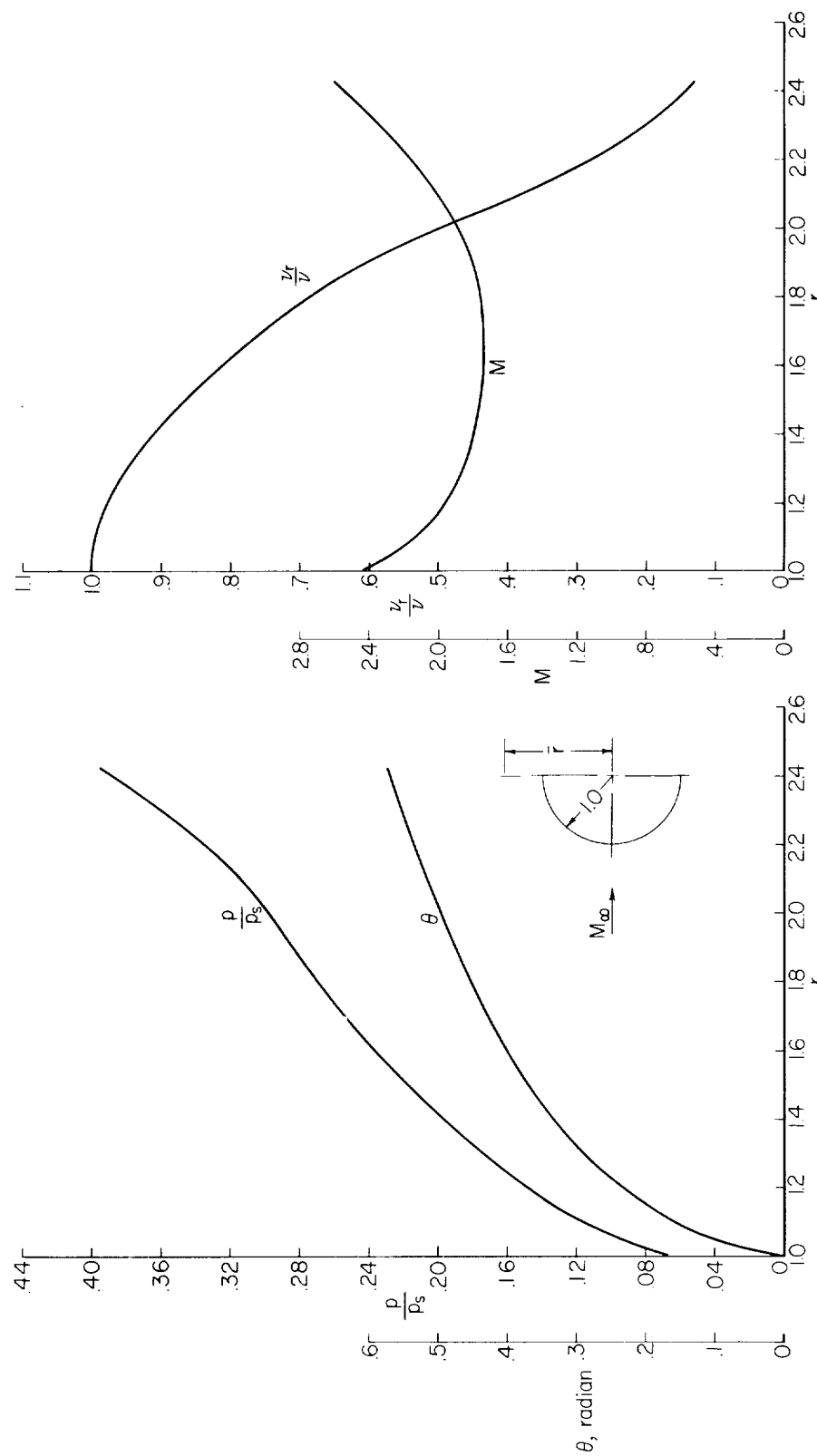
Figure 8.- Flow properties along 90° radial line.
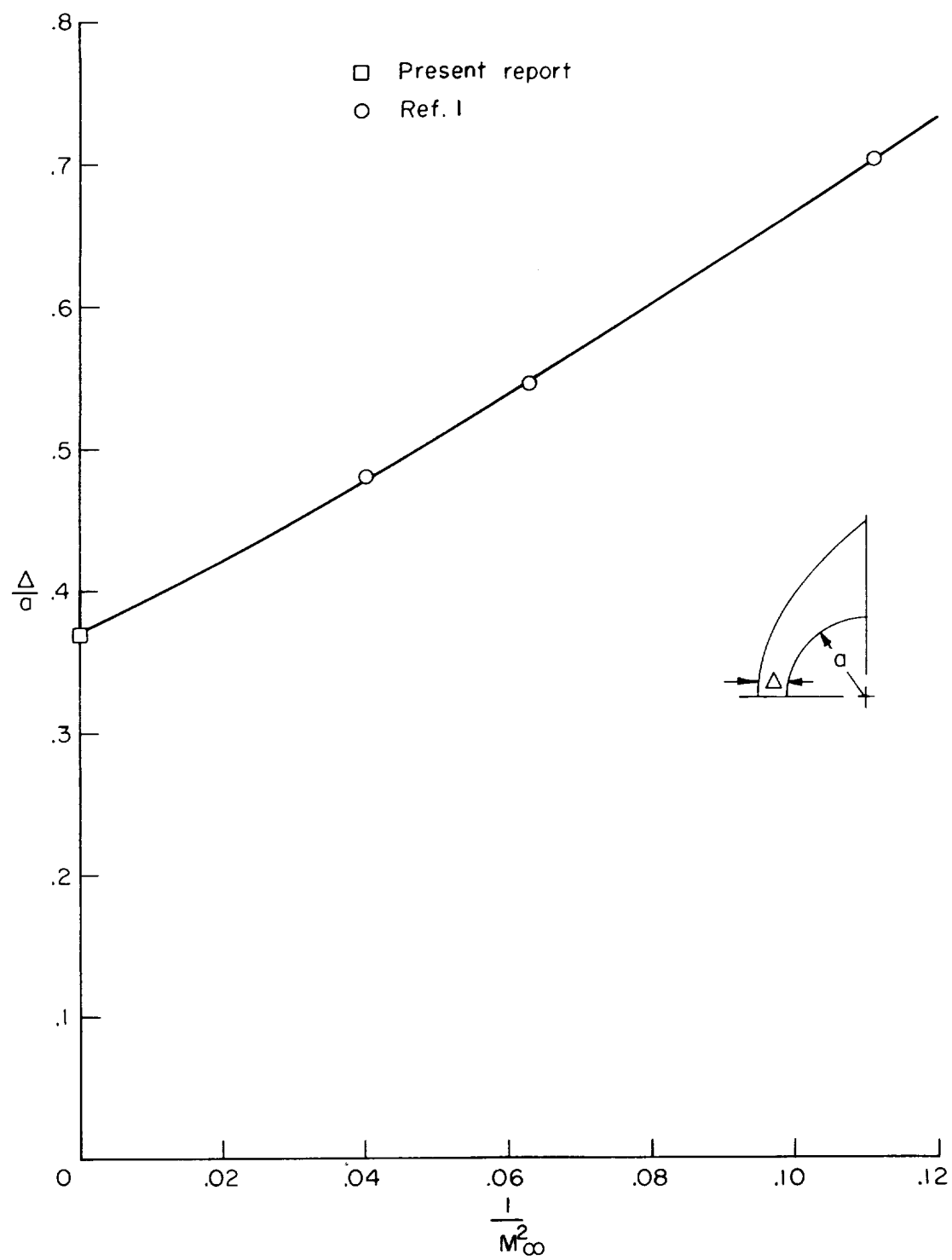
Figure 9.- Shock-wave standoff distance for cylindrical body.

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