

TECHNICAL NOTE

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DENSITY IN A PLANETARY EXOSPHERE

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SUMMARY

A discussion of the Öpik-Singer theory of the density of a planetary exosphere is presented. Their density formula permits the calculation of the depth of the exosphere. Since the correctness of their derivation of the basic formula for the density distribution has been questioned, an alternate method based directly on Liouville's theorem is given. It is concluded that the Öpik-Singer formula seems valid for the ballistic component of the exosphere; but for a complete description of the planetary exosphere, the ionized and bound-orbit components must also be included.

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INTRODUCTION

Recently Öpik and Singer (Reference 1) have developed a theory which gives the ballistic component of the neutral density distribution in a planetary exosphere. Their theory assumes that above the base of the exosphere, collisions may be entirely neglected — at least as far as an approximate calculation of the neutral density profile is concerned. The particles at the base of the exosphere are assumed to be in a truncated Maxwellian distribution, with no incoming particles at greater than escape velocities. The absence of such incoming particles implies a sink, located at some distance from the planet, which prevents the eventual build-up of a full Maxwellian distribution above the escape level; and this prevents an extension of the barometric law beyond the base of the exosphere. The bound-orbit and ionized components of the exosphere are omitted in the original Öpik-Singer development (however, see Reference 2). Other theories of the exosphere have been developed by Johnson and Fish (Reference 3) and by Chamberlain (Reference 4).

This paper will present several comments on the Öpik-Singer theory of the exosphere. First, since the correctness of their derivation of the basic formula for the density distribution (Equation 12 of Reference 1) has been questioned (Reference 5), it is worthwhile to present an alternate derivation of their formula for the density, based directly on Liouville's theorem. Second, a relatively simple analytic expression for the density distribution can be obtained which replaces the numerical integration required in the Öpik-Singer theory. The density formula permits a calculation of the depth of the exosphere, that is, the number of particles per unit area in a column extending from the base of the exosphere to infinity. This quantity is relevant to calculations of the escape of a planetary atmosphere.

EQUATIONS FOR THE DENSITY OF THE EXOSPHERE

The formula for the density $\rho(r)$ can be derived directly from the one-particle form of Liouville's theorem, which states that the density of particles in phase space, $f(r, \vec{v})$ is constant along particle trajectories:

$$f(\vec{r}, \vec{v}) = f(\vec{R}, \vec{v}_0), \quad (1)$$

where \vec{v} is the velocity of a particle at position \vec{r} and \vec{v}_0 is the velocity that the same particle had at the base of the exosphere, located on a sphere at \vec{R} . In the absence of collisions, \vec{v}, \vec{r} and \vec{v}_0, \vec{R} are related by the conservation of energy and of angular momentum:

$$v = \sqrt{v_0^2 - \frac{2MG}{R}(1 - Y)}, \quad (2a)$$

$$v \sin \theta = v_0 Y \sin \theta_0 \quad (2b)$$

where M is the mass of the planet, G is the gravitational constant, and Y is R/r . The angles θ and θ_0 are the angles the trajectory makes with the radius vector passing through the center of the planet. These angles are defined with respect to the orbital plane. We shall assume in the following discussion that the density and temperature at the base of the exosphere are constants and therefore independent of the angular co-ordinates of \vec{R} . The spacial density $\rho(r)$ is then

$$\rho(r) = \int f(\vec{r}, \vec{v}) d\vec{v}. \quad (3)$$

Equation 1 then allows us to write:

$$\rho(r) = \int f(\vec{R}, \vec{v}_0) d\vec{v}. \quad (4)$$

In Equations 3 and 4, the range of integration $d\vec{v}$ extends over all velocity space compatible with Equations 2a and 2b; that is, only over those orbits intersecting the spherical surfaces at \vec{R} and \vec{r} . In order to evaluate the integral in Equation 4 we introduce the Jacobian, $J(v, \theta/v_0, \theta_0)$, which transforms the integration over $d\vec{v}$ to one over $d\vec{v}_0$:

$$\rho(r) = \int v^2 \sin \theta f(\vec{R}, \vec{v}_0) J\left(\frac{v, \theta}{v_0, \theta_0}\right) dv_0 d\theta_0. \quad (5)$$

The Jacobian may be evaluated by using Equations 2a and 2b:

$$J \left(\frac{\mathbf{v}, \theta}{v_0, \theta_0} \right) = \left(\frac{v_0}{v} \right)^2 Y \frac{\cos \theta_0}{\sqrt{1 - \left(\frac{v_0 Y}{v} \right)^2 \sin^2 \theta_0^2}} \quad (6)$$

Again using Equations 2a and 2b, we eliminate v and θ from Equation 5 and use Equation 6:

$$\rho(r) = Y^2 \int f(\vec{R}, \vec{v}_0) \frac{v_0^3 \cos \theta_0 \sin \theta_0 dv_0 d\theta_0}{\sqrt{v_0^2 (1 - Y^2 \sin^2 \theta_0) - \frac{2MG}{R} (1 - Y)}} \quad (7)$$

Öpik and Singer's Equation 12 may be obtained from Equation 7 if we replace $f(\vec{R}, \vec{v}_0)$ by a truncated Maxwellian distribution which omits incoming particles with velocities greater than escape velocity.

The integration in Equation 7 can be performed to give $\rho(r)$ in terms of known functions:

$$\rho(r) = \rho_0(R) \left[e^{-\alpha(1-Y)} \left(1 - \frac{1}{2} \operatorname{erf} \sqrt{\alpha Y} \right) - \sqrt{1-Y^2} e^{-\frac{\alpha}{1+Y}} \left(1 - \frac{1}{2} \operatorname{erf} \sqrt{\frac{\alpha Y}{1+Y}} \right) - \sqrt{\frac{\alpha Y}{\pi}} \left(1 - \sqrt{1-Y} \right) e^{-\alpha} \right] \quad (8)$$

where

$$\alpha = R/H,$$

$$H = \text{scale height},$$

$$\rho_0(R) = \text{density at the critical level, and}$$

$$\operatorname{erf} x = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-Y^2} dY.$$

CONCLUDING REMARKS

When considering the above distribution, we first notice that as $\alpha \rightarrow \infty$, the barometric law for the density variation is obtained. Secondly, we observe that $\rho(R)$ is not equal to $\rho_0(R)$, the density just below the base of the exosphere. The reason for this is that we have

omitted all incoming particles with velocities greater than escape velocity. This discontinuity results from arbitrarily neglecting all collisions above the escape level, and a more realistic theory would replace the sharp boundary of the escape level by a diffuse zone above which collisions gradually become less likely.

The density profile, Equation 8, is characterized by a parameter $\alpha = R/H$, but not in too transparent a way. This parameter determines the extent of the exosphere, and in order to gain some insight into this matter, we have calculated the quantity d :

$$d = \int_R^{\infty} \rho(r) dr. \quad (9)$$

In the conventional theory of escape of a planetary atmosphere, d is taken to be equal to $\rho_0 H$, with H being the scale height at the base of the exosphere. In Figure 1, the ratio $d/\rho_0 H$ is plotted as a function of α . For orientation, we may note that for the earth's exosphere $\alpha \approx 4.5$ for atomic hydrogen and $\alpha \approx 72$ for atomic oxygen, provided we take a temperature of 1500°K at the base of the exosphere. In the limit as α approaches zero, H approaches infinity, d becomes independent of α , and ρ becomes proportional to $1/r^2$. For large values of α , d approaches $\rho_0 H$; in other words, there is one scale height of atmosphere above the escape level. This result has been used by various investigators (Reference 6) in their studies of the escape of atmospheres, but their justification for its

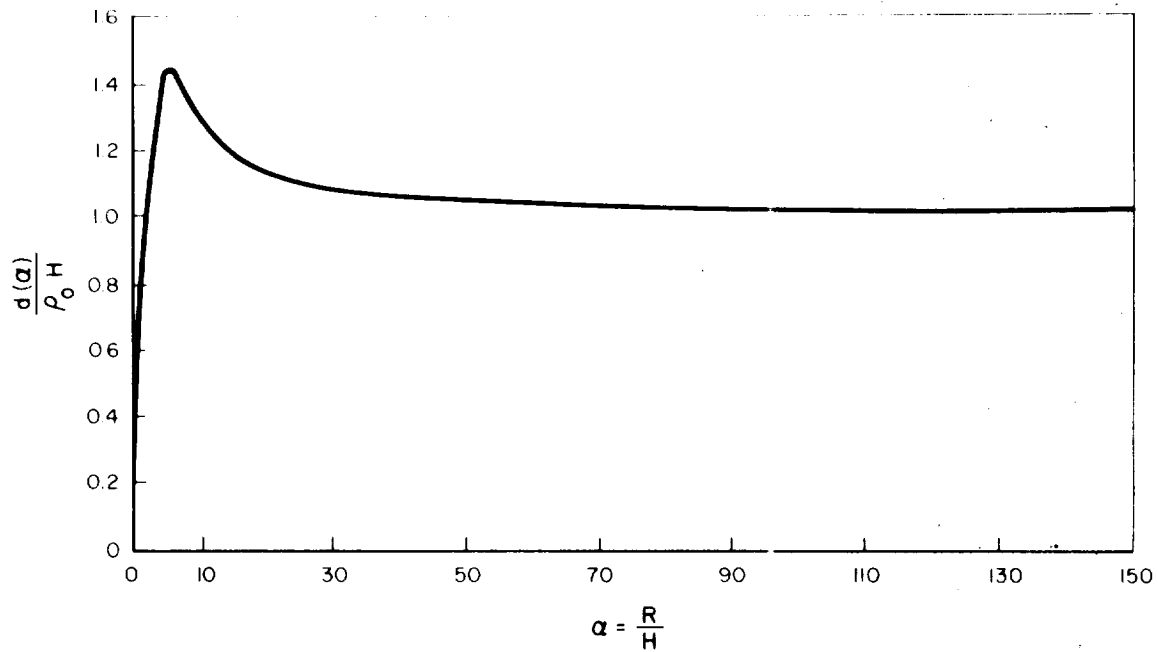


Figure 1 - Variation of the ratio $d/\rho_0 H$ with the parameter α . The integrated neutral exosphere density is given by d in Equation 8. In the conventional theory of the escape of a planetary atmosphere, d is set equal to $\rho_0 H$.

use applies to stellar rather than to planetary atmospheres. The Öpik-Singer theory seems to provide a valid justification for its use in the case of the ballistic component of the exosphere; however, it should be stressed that the ionized and bound-orbit components must be included for a complete description of a planetary exosphere. For example, Singer (Reference 7) has suggested that in the earth's exosphere the dominant components above 1800 kilometers may be O^+ and H^+ . Öpik and Singer have recently published a supplemental discussion of the bound orbit component. Their findings indicate that this component is unimportant in the earth's exosphere, but may be important in the exospheres of the major planets (Reference 2).

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