



# TECHNICAL NOTE

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## ON THE LONG PERIOD LUNI-SOLAR EFFECT IN THE MOTION OF AN ARTIFICIAL SATELLITE

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## SUMMARY

Two systems of formulas are presented for the determination of the long period perturbations caused by the Sun and the Moon in the motion of an artificial satellite. The first system can be used to determine the lunar effect for all satellites. The second method is more convenient for finding the lunar effect for close satellites and the solar effect for all satellites.

Knowledge of these effects is essential for determining the stability of the satellite orbit. The basic equations of both systems are arranged in a form which permits the use of numerical integration. The two theories are more accurate and more adaptable to the use of electronic machines than the analytical developments obtained previously.

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# LIST OF SYMBOLS

$g_0$	the mean anomaly at the epoch of the satellite
$g$	the mean anomaly of the satellite for the moment $t$
$\omega$	the argument of the perigee of the satellite
$\Omega$	the longitude of the ascending node of the satellite
$i$	the inclination of the satellite orbit with respect to the equator
$e$	the eccentricity of the orbit of the satellite
$a$	the semi-major axis of the orbit of the satellite
$n$	the mean motion of the satellite
$u$	the eccentric anomaly of the satellite
$p = a(1 - e^2)$	
$g'_0$	the mean anomaly at the epoch of the disturbing body
$g'$	the mean anomaly of the disturbing body at the moment $t$
$\omega'$	the argument of the perigee of the orbit of the disturbing body, but with respect to the ecliptic of the fixed epoch
$\Omega'$	the longitude of the ascending node of the orbit of the disturbing body with respect to ecliptic
$i'$	the inclination of the orbit of the disturbing body toward ecliptic
$e'$	the eccentricity of the orbit of the disturbing body
$a'$	the semi-major axis of the orbit of the disturbing body
$n'$	the mean motion of the disturbing body
$R$	the disturbing function
$\mathbf{r}$	the position vector of the satellite
$\mathbf{u}^0$	the unit vector directed from the earth's center to the disturbing body

<b>i, j, k</b>	the basic system of unit vectors in the equatorial system of coordinates
<b>P</b>	the unit vector directed from the earth's center toward the perigee of the satellite
<b>R</b>	the unit vector standing normally to the orbit plane of the satellite in the direction of the angular momentum
<b>Q</b>	the unit vector standing normally to <b>P</b> and <b>R</b> , $Q = R \times P$
<b>P'</b>	the unit vector directed from the earth's center toward the perigee of the disturbing body
<b>R'</b>	the unit vector standing normally to the orbit plane of the disturbing body in the direction of the angular momentum
<b>Q'</b>	a unit vector standing normally to <b>P'</b> and <b>R'</b> , $Q' = R' \times P'$
<b>h</b>	the unit vector along the line of nodes
<b>F</b>	the disturbing force
<b>F<sub>0</sub></b>	the disturbing force <b>F</b> averaged over the revolution of one body
<b>M<sub>0</sub></b>	the momentum of <b>F</b> averaged over the revolution of one body
<b>M<sub>00</sub></b>	the momentum of the disturbing force <b>F</b> averaged over the revolutions of both bodies

# ON THE LONG PERIOD LUNI-SOLAR EFFECT IN THE MOTION OF AN ARTIFICIAL SATELLITE

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## INTRODUCTION

Two systems of formulas are presented herein for determining lunar and solar long period effects of the first order in the motion of an artificial satellite. The first method is based on the theory originally developed by Gauss (Reference 1) for a numerical treatment of the very long period effects in planetary motion, which was found to be applicable to the case of artificial satellites. The second method is valid for close satellites and is based on the development of the disturbing function in terms of Legendre polynomials.

Knowledge of these long period effects is essential in determining the stability of the orbit and the lifetime of the satellite. As an example, Kozai (Reference 2) found that the solar and lunar perturbations have shortened the lifetime of Explorer VI (1959 Delta) by a factor of ten. The long period lunar effect can be deduced by averaging the perturbations with respect to the mean anomaly of the Moon and with respect to the mean anomaly of the satellite. The first averaging process is performed analytically, the second process numerically. The long period solar effect is obtained by averaging the perturbations with respect to the mean anomaly of the satellite only. Both methods result in the numerical integration of the equations for the variation of elements.

The interval of integration depends upon the proximity of the satellite to the earth and upon the secular changes of the node and perigee produced by the earth's oblateness. In a normal case the interval of integration will be of the order of several days. For more distant satellites it can be of the order of one month or more. The choice of such a large interval is impossible if Cowell's method of integration in the rectangular coordinates is used.

For more distant satellites it was found that the development of the disturbing function into series of Legendre polynomials converges so slowly that it is impossible to include all important long period terms in an analytical development. A large orbital inclination might also contribute substantially to the slowness of the convergence. In such a case, as for example  $a = 10$  earth radii,  $e = 0.8$ , the analytical development obtained by Musen and Bailie (Reference 3) becomes incomplete. These circumstances gave rise to the investigation of the possible use of the Gaussian method for determining the lunar long period effects in the motion of an artificial satellite.

Halphen's form (Reference 4, with corrections by Goriachev, Reference 5) of the Gaussian method was found to be the most convenient, partly because it is very adaptable to the use of electronic computers. Some necessary modifications were made to include the basic perturbations of the motion of the Moon. Also, the Goursat transformation (Reference 6) and the Euler summability process were employed to speed up the convergence of the hypergeometric series. This method is valid for all values of  $e$ ,  $i$  and  $a/a'$ .

The development in terms of Legendre polynomials can be used if the lunar perturbations for close satellites or the solar perturbations for all satellites are to be determined. The basic equations of these two cases are arranged in the form which permits the use of the numerical integration and includes all the significant long period terms. Theoretically, these equations are completely equivalent to the extensive analytical developments obtained by Kozai (Reference 2) and by Musen and Bailie (Reference 3) but they have a more compact and symmetrical form and are more adaptable to the use of electronic machines. Thus the formulas given here represent, from a practical point of view, a substantial improvement over the previous methods.

## LUNAR DISTURBING FORCE IN THE MOTION OF A DISTANT ARTIFICIAL SATELLITE

Let  $g_0, \omega, \Omega, i, e, a, n$  be the osculating elements of the artificial satellite referred to the earth's equator, and  $g'_0, \omega', \Omega', i', e', a', n'$  be



the elements of the Moon referred to the fixed ecliptic. We assume that there is no sharp commensurability between the mean motions  $n$  and  $n'$ . It is sufficient to take into account the secular changes in the  $\omega'$ ,  $\Omega'$  and to neglect the periodic effects and the influence of the precession. Let  $\mathbf{P}$  be the unit vector directed from the center of the earth to the perigee of the orbit of the satellite,  $\mathbf{R}$  be the unit vector standing normally to the orbit plane and

$$\mathbf{Q} = \mathbf{R} \times \mathbf{P}.$$

Let  $\mathbf{P}'$ ,  $\mathbf{Q}'$ ,  $\mathbf{R}'$  be the corresponding vectors of the orbit of the Moon. Putting

$$A_1(\alpha) = \begin{bmatrix} +1 & 0 & 0 \\ 0 & +\cos \alpha & -\sin \alpha \\ 0 & +\sin \alpha & +\cos \alpha \end{bmatrix},$$

$$A_3(\alpha) = \begin{bmatrix} +\cos \alpha & -\sin \alpha & 0 \\ +\sin \alpha & +\cos \alpha & 0 \\ 0 & 0 & +1 \end{bmatrix},$$

we have for the components of  $\mathbf{P}$ ,  $\mathbf{Q}$ ,  $\mathbf{R}$  and  $\mathbf{P}'$ ,  $\mathbf{Q}'$ ,  $\mathbf{R}'$  in the equatorial system:

$$[\mathbf{P}, \mathbf{Q}, \mathbf{R}] = A_3(\Omega) \cdot A_1(i) \cdot A_3(\omega), \quad (1)$$

$$[\mathbf{P}', \mathbf{Q}', \mathbf{R}'] = A_1(\epsilon) \cdot A_3(\Omega') \cdot A_1(i') \cdot A_3(\omega'). \quad (2)$$

The angle between the equator and the ecliptic is  $\epsilon$ . Designating  $u$  as the eccentric anomaly of the satellite, we have for the position vector

$$\mathbf{r} = \mathbf{P} a (\cos u - e) + \mathbf{Q} a \sqrt{1 - e^2} \sin u. \quad (3)$$

Let

$$\boldsymbol{\rho} = \frac{\mathbf{r}}{a} s + e' \mathbf{P}' \quad (4)$$

$$= \mathbf{P} s (\cos u - e) + \mathbf{Q} s \sqrt{1 - e^2} \sin u + e' \mathbf{P}',$$

where  $s$  is the parallax,

$$s = \frac{a}{a'}.$$

Put

$$\alpha = \boldsymbol{\rho} \cdot \mathbf{P}', \quad \beta = \boldsymbol{\rho} \cdot \mathbf{Q}', \quad \gamma = \boldsymbol{\rho} \cdot \mathbf{R}' .$$

The system of Halphen's formulas for the computation of auxiliary quantities can be slightly modified and rewritten in our notation:

$$K_1 = \rho^2 - 2 + e'^2, \quad (5)$$

$$K_2 = (1 - e'^2)(1 - \alpha^2) - \beta^2 - (2 - e'^2) \gamma^2, \quad (6)$$

$$K_3 = \gamma^2(1 - e'^2) \quad (7)$$

$$g_2 = \frac{4}{3} (K_1^2 - 3 K_2), \quad (8)$$

$$g_3 = \frac{4}{27} (2 K_1^3 - 9 K_1 K_2 + 27 K_3), \quad (9)$$

$$\xi = \frac{27 g_3^2}{g_2^3}. \quad (10)$$

The next step in Halphen's method is the computation of

$$\psi(\xi) = \frac{\pi}{\sqrt[4]{3}} F\left(\frac{1}{12}, \frac{5}{12}, 1, 1 - \xi\right), \quad (11)$$

$$\psi'(\xi) = -\frac{5}{144} \frac{\pi}{\sqrt[4]{3}} F\left(\frac{13}{12}, \frac{17}{12}, 2, 1 - \xi\right), \quad (12)$$

and of

$$A = \frac{\sqrt{6} \sqrt[4]{g_2}}{9 g_2^3} \frac{144}{\pi} \sqrt{\xi} \psi'(\xi), \quad (13)$$

$$B = \frac{\sqrt{2}}{\pi g_2 \sqrt[4]{g_2}} \psi(\xi). \quad (14)$$

The values of  $\xi$  will be proper fractions. The series (Equations 11 and 12) possess an algebraic branch point at  $\xi = 0$  and the convergence in the

neighborhood of this singularity is slow. To obtain more convenient formulas representing the series rather uniformly in the interval  $0 \leq \xi \leq 1$  with the accuracy of at least  $10^{-6}$ , the Goursat transformations and then the E summability process are applied. Putting

$$a = \frac{1}{12}, \quad b = \frac{5}{12}, \quad z = 1 - \xi$$

in

$$\begin{aligned} & F\left(a, b, a + b + \frac{1}{2}, z\right) \\ &= \left(\frac{1 + \sqrt{1 - z}}{2}\right)^{-2a} F\left(2a, a - b + \frac{1}{2}, a + b + \frac{1}{2}, -\frac{1 - \sqrt{1 - z}}{1 + \sqrt{1 - z}}\right), \end{aligned}$$

we deduce that

$$\psi(\xi) = \frac{\pi}{\sqrt[6]{3}} \sqrt[6]{\frac{2}{1 + \sqrt{\xi}}} F\left(\frac{1}{6}, \frac{1}{6}, 1, -w\right). \quad (15)$$

Putting

$$a = \frac{13}{12}, \quad b = \frac{17}{12}, \quad z = 1 - \xi$$

in

$$\begin{aligned} & F\left(a, b, a + b - \frac{1}{2}, z\right) \\ &= (1 - z)^{-\frac{1}{2}} \left(\frac{1 + \sqrt{1 - z}}{2}\right)^{1 - 2a} F\left(2a - 1, a - b + \frac{1}{2}, a + b - \frac{1}{2}, -\frac{1 - \sqrt{1 - z}}{1 + \sqrt{1 - z}}\right) \end{aligned}$$

we have

$$\sqrt{\xi} \psi'(\xi) = -\frac{5}{144} \frac{\pi}{\sqrt[6]{3}} \left(\frac{2}{1 + \sqrt{\xi}}\right)^{\frac{7}{6}} F\left(\frac{1}{6}, \frac{7}{6}, 2, -w\right), \quad (16)$$

where

$$w = \frac{1 - \sqrt{\xi}}{1 + \sqrt{\xi}}.$$

The series (Equations 15 and 16) are alternating and the E-process can be applied to speed up their convergence. The formula of the E-process for hypergeometric series takes the form

$$F(a, b, c, -x) = \sum_{k=0}^N (-1)^k \frac{(a, k)}{(1, k)} \frac{(b, k)}{(c, k)} x^k$$

$$+ \lim_{m \rightarrow \infty} \sum_{j=0}^m (-1)^{N+j} \frac{(a, N+j)}{(1, N+j)} \frac{(b, N+j)}{(c, N+j)} x^{N+j} \sum_{p=j}^m \frac{1}{2^{p+1}} \binom{p}{j}, \quad (17)$$

where

$$(q, k) = q(q+1) \cdots (q+k-1).$$

Applying Equation 17 to the series (Equations 15 and 16) and putting  $N = 3$ ,  $m = 19$ , we deduce the two following expressions in which the coefficients are rapidly decreasing as the power of  $w$  increases.

$$\psi(\xi) = \left( \frac{2}{1 + \sqrt{\xi}} \right)^{\frac{1}{6}} \quad \frac{144}{\pi} \sqrt{\xi} \psi'(\xi) = \left( \frac{2}{1 + \sqrt{\xi}} \right)^{\frac{7}{6}}$$

$\times \left( + 2.3870942 \right.$ $- 0.0663082 w$ $+ 0.0225632 w^2$ $- 0.0117691 w^3$ $+ 0.0073743 w^4$ $- 0.0051060 w^5$ $+ 0.0037250 w^6$ $- 0.0027325 w^7$ $+ 0.0019070 w^8$ $- 0.0011936 w^9$ $+ 0.0006337 w^{10}$ $- 0.0002710 w^{11}$ $+ 0.0000884 w^{12}$ $- 0.0000205 w^{13}$ $+ 0.0000030 w^{14}$ $- 0.0000002 w^{15} \left. \right) ;$	$\times \left( - 3.7991784 \right.$ $+ 0.3693646 w$ $- 0.1556119 w^2$ $+ 0.0889726 w^3$ $- 0.0586828 w^4$ $+ 0.0419870 w^5$ $- 0.0313364 w^6$ $+ 0.0233758 w^7$ $- 0.0165247 w^8$ $+ 0.0104483 w^9$ $- 0.0055933 w^{10}$ $+ 0.0024083 w^{11}$ $- 0.0007898 w^{12}$ $+ 0.0001837 w^{13}$ $- 0.0000268 w^{14}$ $+ 0.0000018 w^{15} \left. \right) .$
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The next step is to compute

$$K_4 = 9K_3 - K_1 K_2 ,$$

$$K_5 = K_1 (K_1 K_2 - 3K_3) - 2K_2^2 ,$$

and

$$a_{11} = K_4 (\alpha^2 - 1) + K_5 + \frac{3}{2} g_2 K_3 ,$$

$$a_{22} = K_4 (\beta^2 - 1 + e'^2) + K_5 + \frac{3}{2} \frac{g_2 K_3}{1 - e'^2} ,$$

$$a_{33} = K_4 \gamma^2 + K_5 + \frac{3}{2} g_2 [\alpha^2 (1 - e'^2) + \beta^2 - (1 - e'^2)] ,$$

$$a_{12} = a_{21} = K_4 \alpha \beta ,$$

$$a_{23} = a_{32} = \left( K_4 - \frac{3}{2} g_2 \right) \beta \gamma ,$$

$$a_{31} = a_{13} = \left[ K_4 - \frac{3}{2} g_2 (1 - e'^2) \right] \gamma \alpha ,$$

$$a'_{11} = \alpha^2 - 1 - \frac{1}{3} K_1 ,$$

$$a'_{22} = \beta^2 - 1 + e'^2 - \frac{1}{3} K_1 ,$$

$$a'_{33} = \gamma^2 - \frac{1}{3} K_1 ,$$

$$a'_{12} = a'_{21} = \alpha \beta ,$$

$$a'_{23} = a'_{32} = \beta \gamma ,$$

$$a'_{31} = a'_{13} = \gamma \alpha .$$

Then the matrix is formed with the elements

$$A_{ij} = a_{ij} A + a'_{ij} B .$$

Introducing the dyadic (a matrix),

$$\Phi = \begin{bmatrix} + A_{11}\mathbf{P}'\mathbf{P}' & + A_{12}\mathbf{P}'\mathbf{Q}' & + A_{13}\mathbf{P}'\mathbf{R}' \\ + A_{21}\mathbf{Q}'\mathbf{P}' & + A_{22}\mathbf{Q}'\mathbf{Q}' & + A_{23}\mathbf{Q}'\mathbf{R}' \\ + A_{31}\mathbf{R}'\mathbf{P}' & + A_{32}\mathbf{R}'\mathbf{Q}' & + A_{33}\mathbf{R}'\mathbf{R}' \end{bmatrix}, \quad (18)$$

we can represent the "disturbing force," averaged with respect to the mean anomaly of the Moon, in the form

$$\mathbf{F}_0 = -\frac{2km'}{a'^3} \Phi \cdot \mathbf{r}; \quad (19)$$

and the averaged momentum

$$\mathbf{M}_0 = -\frac{2km'}{a'^3} \mathbf{r} \times \Phi \cdot \mathbf{r}. \quad (20)$$

In addition, in the system adopted in this paper, we have to compute the vector (Reference 7)

$$\mathbf{K}_0 = \left(1 + \frac{r}{p}\right) \mathbf{F}_0 - \frac{1}{rp} \mathbf{r} \mathbf{r} \cdot \mathbf{F}_0. \quad (21)$$

The vectors  $\mathbf{M}_0$  and  $\mathbf{K}_0$  must be averaged along the orbit of the satellite:

$$(\mathbf{M}_{00}) = \frac{1}{2\pi} \int_0^{2\pi} \mathbf{M}_0 dg = \frac{1}{2\pi} \int_0^{2\pi} \mathbf{M}_0 \frac{r}{a} du, \quad (22)$$

$$(\mathbf{K}_{00}) = \frac{1}{2\pi} \int_0^{2\pi} \mathbf{K}_0 dg = \frac{1}{2\pi} \int_0^{2\pi} \mathbf{K}_0 \frac{r}{a} du. \quad (23)$$

This averaging is done numerically, by giving the values  $0^\circ, 30^\circ, 60^\circ \dots, 330^\circ$  to  $u$ , computing values of  $\mathbf{M}_0(r/a)$  and  $\mathbf{K}_0(r/a)$  for each of these angles and then forming the arithmetical means.

## THE FORM OF VARIATION OF CONSTANTS FOR THE LONG PERIOD LUNAR EFFECTS

Designating the area "constant" by  $c$ , we have

$$c = \sqrt{p} R, \quad (24)$$

$$p = a(1 - e^2), \quad (25)$$

and

$$\frac{dc}{dt} = \mathbf{r} \times \mathbf{F}. \quad (26)$$

Let  $\mathbf{h}$  be the unit vector along the line of nodes and let  $(\mathbf{i}, \mathbf{j}, \mathbf{k})$  be the unit vectors of the equatorial system. Then

$$\mathbf{h} = \mathbf{i} \cos \Omega + \mathbf{j} \sin \Omega, \quad (27)$$

$$\mathbf{R} = \mathbf{h} \times \mathbf{k} \sin i + \mathbf{k} \cos i. \quad (28)$$

It follows from Equations 27 and 28 that

$$\frac{d\mathbf{R}}{dt} = \mathbf{h} \sin i \frac{d\Omega}{dt} + \mathbf{h} \times \mathbf{R} \frac{di}{dt}, \quad (29)$$

and from Equation 26 that

$$\mathbf{h} \sin i \frac{d\Omega}{dt} + \mathbf{h} \times \mathbf{R} \frac{di}{dt} - \frac{e}{1 - e^2} \mathbf{R} \frac{de}{dt} + \frac{1}{\sqrt{a}} \mathbf{R} \frac{d\sqrt{a}}{dt} = \frac{1}{\sqrt{p}} \mathbf{r} \times \mathbf{F}. \quad (30)$$

The long period effects are deduced from the last equation by applying the double process of averaging and taking into account the fact that  $da/dt = 0$ :

$$\sin i \frac{d\Omega}{dt} = \frac{1}{\sqrt{p}} \mathbf{h} \cdot \mathbf{M}_{00}, \quad (31)$$

$$\frac{di}{dt} = \frac{1}{\sqrt{p}} \mathbf{M}_{00} \cdot \mathbf{h} \times \mathbf{R}, \quad (32)$$

$$\frac{de}{dt} = - \frac{\sqrt{1 - e^2}}{e \sqrt{a}} \mathbf{R} \cdot \mathbf{M}_{00}. \quad (33)$$

It is of interest to notice that for long period effects, the averaged momentum is the only function of the disturbing force contained in the Equations 31 through 33. Equations 31 and 32 represent a particular case of equations deduced by Makarova (Reference 8) for computing the special perturbations of minor planets. From Reference 7,

$$\mathbf{h} \frac{di}{dt} + \mathbf{k} \frac{d\Omega}{dt} + \mathbf{R} \frac{d\omega}{dt} = \frac{1}{\sqrt{p}} \mathbf{rR} \cdot \mathbf{F} - \frac{\sqrt{p}}{e} \mathbf{RP} \cdot \mathbf{K}, \quad (34)$$

where

$$\mathbf{K} = \left(1 + \frac{r}{p}\right) \mathbf{F} - \frac{1}{rp} \mathbf{rr} \cdot \mathbf{F}. \quad (35)$$

We deduce from Equation 34 the long period effect in  $\omega$ , and by taking Equation 31 into consideration obtain

$$\frac{d\omega}{dt} = -\frac{\sqrt{p}}{e} \mathbf{P} \cdot \mathbf{K}_{00} - \frac{\cot i}{\sqrt{p}} \mathbf{h} \cdot \mathbf{M}_{00}, \quad (36)$$

where

$$\mathbf{P} \cdot \mathbf{K}_{00} = \frac{1}{2\pi} \int_0^{2\pi} \mathbf{P} \cdot \mathbf{K}_0 \frac{r}{a} du.$$

Equation 36 is identical in form to the equation for planetary perturbations, deduced by Makarova on the basis of different considerations from Musen's theory (Reference 7).

The system (Equations 31, 32, 33 and 36) is to be integrated numerically; and the interval of integration can be of the order of several days.

## THE LUNAR PERTURBATIONS OF A CLOSE SATELLITE AND THE SOLAR PERTURBATIONS OF ALL SATELLITES

The computation of the lunar perturbations of a close satellite and of the solar perturbations for all satellites can be accomplished by developing the disturbing function into series of Legendre polynomials. For this purpose the discussion shall be limited to the second Legendre polynomial, and the disturbing function takes the form



$$R = \frac{km'r^2}{r'^3} \left( \frac{3}{2} \cos^2 H - \frac{1}{2} \right), \quad (37)$$

where  $H$  is the angle between  $\mathbf{r}$  and  $\mathbf{r}'$ . Let  $\mathbf{u}^0$  be the unit vector in the direction of  $\mathbf{r}'$ . We have  $\mathbf{u}^0$  with the accuracy up to  $e'$  in the periodic terms:

$$\mathbf{u}^0 = \mathbf{P}' (-e' + \cos g' + e' \cos 2g') + \mathbf{Q}' (\sin g' + e' \sin 2g'), \quad (38)$$

and Equation 37 becomes

$$R = \frac{km'}{2r'^3} (3\mathbf{r} \cdot \mathbf{u}^0 \mathbf{u}^0 \cdot \mathbf{r} - r^2). \quad (39)$$

The "disturbing force" to be used in connection with the variation of elements will be

$$\mathbf{F} = \text{grad } R = \frac{km'}{a'^3} \left( \frac{a'}{r'} \right)^3 (3 \mathbf{u}^0 \mathbf{u}^0 \cdot \mathbf{r} - \mathbf{r}) \quad (40)$$

and the momentum of  $\mathbf{F}$  takes the form

$$\mathbf{M} = \frac{3km'}{a'^3} \left( \frac{a'}{r'} \right)^3 \mathbf{r} \times \mathbf{u}^0 \mathbf{u}^0 \cdot \mathbf{r}. \quad (41)$$

Substituting

$$\mathbf{r} = \mathbf{P} r \cos f + \mathbf{Q} r \sin f$$

into Equation 41, yields:

$$\begin{aligned} \mathbf{M} = & \frac{3km'}{2a'^3} \left( \frac{a'}{r'} \right)^3 \left[ \mathbf{P} \times \mathbf{u}^0 \mathbf{u}^0 \cdot \mathbf{P} \frac{r^2}{a^2} (1 + \cos 2f) \right. \\ & + \mathbf{P} \times \mathbf{u}^0 \mathbf{u}^0 \cdot \mathbf{Q} \frac{r^2}{a^2} \sin 2f + \mathbf{Q} \times \mathbf{u}^0 \mathbf{u}^0 \cdot \mathbf{P} \frac{r^2}{a^2} \sin 2f \\ & \left. + \mathbf{Q} \times \mathbf{u}^0 \mathbf{u}^0 \cdot \mathbf{Q} \frac{r^2}{a^2} (1 - \cos 2f) \right]. \quad (42) \end{aligned}$$

Taking

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{r^2}{a^2} dg = 1 + \frac{3}{2} e^2, \quad \frac{1}{2\pi} \int_0^{2\pi} \frac{r^2}{a^2} \cos 2f dg = + \frac{5}{2} e^2,$$

we obtain the averaged value of  $M_0$ :

$$M_0 = \frac{3km'a^2}{2a'^3} \left( \frac{a'}{r'} \right)^3 \left[ (1 + 4e^2) \mathbf{P} \times \mathbf{u}^0 \mathbf{u}^0 \cdot \mathbf{P} + (1 - e^2) \mathbf{Q} \times \mathbf{u}^0 \mathbf{u}^0 \cdot \mathbf{Q} \right]. \quad (43)$$

Let  $\phi$ ,  $\psi$ , and  $\theta$  be angles which the basic vectors  $\mathbf{P}$ ,  $\mathbf{Q}$ , and  $\mathbf{R}$  form with the direction to the Moon; hence

$$\cos \phi = \mathbf{P} \cdot \mathbf{u}^0, \quad \cos \psi = \mathbf{Q} \cdot \mathbf{u}^0, \quad \cos \theta = \mathbf{R} \cdot \mathbf{u}^0. \quad (44)$$

Replacing  $M_{00}$  in Equations 31, 32 and 33 by Equation 43, and considering

$$\begin{aligned} \mathbf{h} \cdot \mathbf{P} \times \mathbf{u}^0 &= + \cos \theta \sin \omega, \\ \mathbf{h} \cdot \mathbf{Q} \times \mathbf{u}^0 &= + \cos \theta \cos \omega, \\ (\mathbf{h} \times \mathbf{R}) \cdot (\mathbf{P} \times \mathbf{u}^0) &= + \cos \theta \cos \omega, \\ (\mathbf{h} \times \mathbf{R}) \cdot (\mathbf{Q} \times \mathbf{u}^0) &= - \cos \theta \sin \omega, \\ \mathbf{R} \cdot \mathbf{P} \times \mathbf{u}^0 &= + \cos \psi, \\ \mathbf{R} \cdot \mathbf{Q} \times \mathbf{u}^0 &= - \cos \phi, \\ \mathbf{h} &= \mathbf{P} \cos \omega - \mathbf{Q} \sin \omega, \end{aligned}$$

we obtain

$$\begin{aligned} \sin i \frac{d\Omega}{dt} &= \frac{3km'a^2}{2a'^3} \left( \frac{a'}{r'} \right)^3 \frac{\cos \theta}{\sqrt{p}} \left[ (1 + 4e^2) \cos \phi \sin \omega + (1 - e^2) \cos \psi \cos \omega \right]; \quad (45) \end{aligned}$$

$$\frac{di}{dt} = \frac{3km'a^2}{2a'^3} \left( \frac{a'}{r'} \right)^3 \frac{\cos \theta}{\sqrt{p}} \left[ (1 + 4e^2) \cos \phi \cos \omega - (1 - e^2) \cos \psi \sin \omega \right]; \quad (46)$$

$$\frac{de}{dt} = - \frac{15}{2} e \sqrt{1 - e^2} \frac{k}{\sqrt{a}} \frac{m'a^2}{a'^3} \left( \frac{a'}{r'} \right)^3 \cos \phi \cos \psi. \quad (47)$$

Setting the mass of the earth equal to 1, we have

$$\frac{k}{\sqrt{a}} = na,$$

and the system (Equations 45 through 47) becomes

$$\begin{aligned} \sin i \frac{d\Omega}{dt} &= \\ &= \frac{3m'n}{2\sqrt{1-e^2}} \left(\frac{a}{a'}\right)^3 \left(\frac{a'}{r'}\right)^3 \left[ (1+4e^2) \cos \phi \sin \omega + (1-e^2) \cos \psi \cos \omega \right] \cos \theta; \quad (48) \end{aligned}$$

$$\begin{aligned} \frac{di}{dt} &= \\ &= \frac{3m'n}{2\sqrt{1-e^2}} \left(\frac{a}{a'}\right)^3 \left(\frac{a'}{r'}\right)^3 \left[ (1+4e^2) \cos \phi \cos \omega - (1-e^2) \cos \psi \sin \omega \right] \cos \theta; \quad (49) \end{aligned}$$

$$\frac{de}{dt} = - \frac{15 m' n e \sqrt{1-e^2}}{2} \left(\frac{a}{a'}\right)^3 \left(\frac{a'}{r'}\right)^3 \cos \phi \cos \psi. \quad (50)$$

In addition we have, with sufficient accuracy,

$$\left(\frac{a'}{r'}\right)^3 = (1-e'^2)^{-\frac{3}{2}} + 3e' \cos g';$$

and for  $d\omega/dt$  an equation analogous to Equation 36,

$$\frac{d\omega}{dt} = - \frac{\sqrt{p}}{e} \mathbf{P} \cdot \mathbf{K}_0 - \cos i \frac{d\Omega}{dt}, \quad (51)$$

where

$$\mathbf{K}_0 = \frac{1}{2\pi} \int_0^{2\pi} \left[ \left(1 + \frac{r}{p}\right) \mathbf{F} - \frac{1}{rp} \mathbf{F} \cdot \mathbf{r} \mathbf{r} \right] dg. \quad (52)$$

Substituting

$$\mathbf{F} = \frac{km'}{r'^3} (3 \mathbf{u}^0 \mathbf{u}^0 \cdot \mathbf{r} - \mathbf{r})$$

and with the solar perturbations for all satellites. This system contains all the significant long period terms. In addition to the long period terms, the lunar part also contains the terms having periods of the order of one month. The interval of integration for the solar perturbations can be taken to be approximately one month. However, for the lunar perturbations it would be preferable to take the interval to be of the order of days, to obtain a smoother curve and to avoid ambiguity in interpreting the results.

## CONCLUSION

The methods described here are more accurate and more adaptable to the use of electronic computers than the analytical development obtained previously. These methods can be used in investigations connected with the stability of orbits and for the separation of known and unknown long period effects in observations. It is recommended that in practical use the analytical development be replaced by the semi-analytical solution given here.

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