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MEMORANDUM

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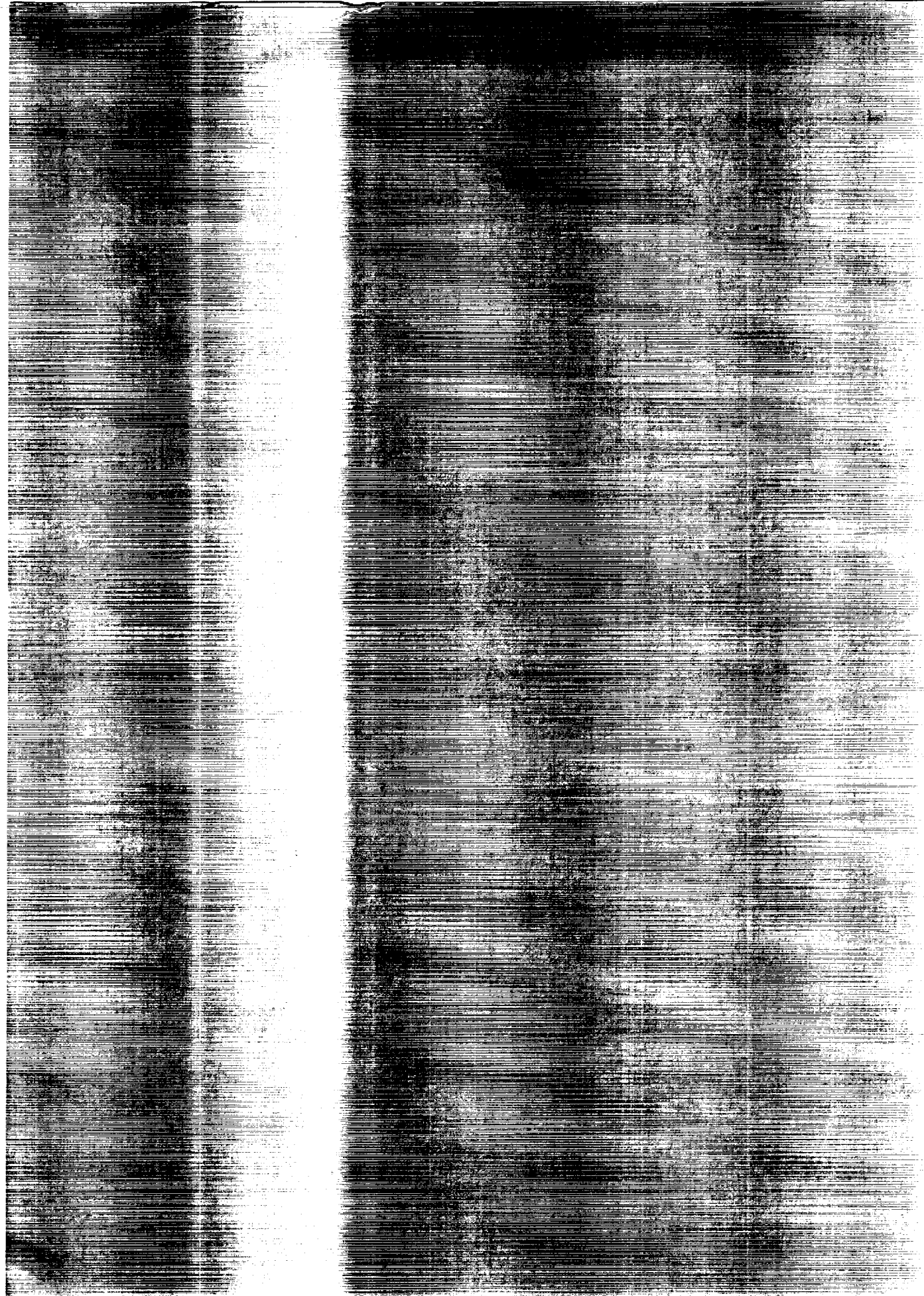
BOUNDARY-LAYER FLOWS

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SUMMARY

An appraisal is made of existing solutions of magnetohydrodynamic boundary-layer equations for stagnation flow and flat-plate flow, and some new solutions are given. Since an exact solution of the equations of magnetohydrodynamics requires complicated simultaneous treatment of the equations of fluid flow and of electromagnetism, certain simplifying assumptions are generally introduced. The full implications of these assumptions have not been brought out properly in several recent papers. It is shown in the present report that for the particular law of deformation which the magnetic lines are assumed to follow in these papers a magnet situated inside the missile nose would not be able to take up any drag forces; to do so it would have to be placed in the flow away from the nose. It is also shown that for the assumption that potential flow is maintained outside the boundary layer, the deformation of the magnetic lines is restricted to small values. The literature contains serious disagreements with regard to reductions in heat-transfer rates due to magnetic action at the nose of a missile, and these disagreements are shown to be mainly due to different interpretations of reentry conditions rather than more complicated effects.

In the present paper the magnetohydrodynamic boundary-layer equation is also expressed in a simple form that is especially convenient for physical interpretation. This is done by adapting methods to magnetic forces which in the past have been used for forces due to gravitational or centrifugal action. The simplified approach is used to develop some new solutions of boundary-layer flow and to reinterpret certain solutions existing in the literature. An asymptotic boundary-layer solution representing a fixed velocity profile and shear is found. Special emphasis is put on estimating skin friction and heat-transfer rates.

INTRODUCTION

Several recent papers deal with reduction in skin friction and heat-transfer rates due to the influence of magnetic fields on electrically

conducting flows along flat plates (ref. 1) and near stagnation points (refs. 2 to 6). The reason for this strong interest is that the electrical conductivities produced in the high-temperature region near the nose of reentering blunt bodies appear sufficiently high to cause such effects. This is especially so if the possibility of seeding the flow with easily ionizable materials is considered.

Since an exact treatment of magnetohydrodynamic flows would require a simultaneous solution of the equations of fluid motion and of the electromagnetic equations, simplifying assumptions are generally introduced. In the understandable rush to obtain numerical results the full implications of these assumptions have not been properly brought out. As a result, in some cases it is unclear which problem is actually being solved and to what approximation the results are correct. An attempt is made in this paper to clarify this situation. It seems also that insufficient effort has been made to present the problems in a form that lends itself most readily to physical interpretations of skin friction and heat-transfer rates of a great variety of boundary-layer flows. A simplified approach is developed in this paper to reinterpret existing results from a novel viewpoint and also to develop some new solutions. For purposes of orientation, the problems treated are outlined in the following paragraphs.

In reference 1 the calculations for boundary-layer flows along flat plates are simplified by assuming that the magnetic field induced by the motion of the flow across the original magnetic field is small compared with the originally imposed magnetic field. In references 2 and 4, which deal with stagnation flow, the deformation of the magnetic lines due to the induced magnetic effects is included in the considerations. Since general solutions would be very hard to obtain, the simplifying assumption is made that the deformation of the magnetic lines is restricted to a particular law. Furthermore, the magnetic lines are subjected to an especially simple boundary condition at the surface of the blunt body.

It is found in the present paper that for the particular deformation and boundary conditions of the magnetic lines, a magnet situated in the blunt nose will not be able to take up any forces and thus its field cannot offer resistance to the flow. On the contrary, the magnet would have to be situated in the flow if it is to take up forces.

In reference 2 and in one example in reference 4 the additional assumption is introduced that the stagnation flow outside the boundary layer remains of the potential type even when deformation of the magnetic field is included in the considerations.

It is shown in the present paper, however, that for the particular law of deformation chosen, the induced magnetic effects or the deformation of the lines must be small if potential flow is to be maintained

under action of the magnetic force. The results for the assumed potential flow thus apply only to small induced effects or deformation of the magnetic lines, contrary to what is implied by the numerical analysis based on the complete equations that include the induced effects. In the present paper another particular law of deformation is pointed out which permits potential flow to exist for large deformations of the magnetic lines. In contrast, it also is shown that the assumption of small deformation of magnetic lines does not always guarantee the existence of potential flow under action of a magnetic force.

The order of magnitude of the induced effects is governed by the so-called magnetic Reynolds number $R_M = \sigma\mu UL$, where σ is the electrical conductivity, μ is the magnetic permeability, and U and L are a velocity and a length characteristic of the given problem. The manner in which the equations in reference 2 are made dimensionless suggests that R_M is based on the boundary-layer thickness. The information that the induced effects are small is thus actually included in the equations of reference 2, but in such an indirect manner that it is not immediately evident there.

Apparently a misinterpretation of the assumptions and results in reference 2 has led to an attempt in reference 5 to make a numerical comparison of the results for the shear based on the assumption of small induced effects with those from reference 2 based on supposedly complete induced effects. Such a comparison seems unnecessary since the assumption of small induced effects is implicitly contained in reference 2. The small difference of 1/400 between the results for the shears in these two references is further evidence that the approaches are basically the same.

In reference 6 the calculation of the reduction in heat-transfer rate due to the magnetic field is based on the assumption of small induced effects, and for reentry conditions a reduction of only 5 percent was obtained as compared with 28 percent in reference 3. In reference 6 the boundary conditions at the shock ahead of the blunt body are also included; for simplicity, both body and shock are assumed to be spherical. In contrast, in reference 3 (based on ref. 2) the assumption is made that the pressure distribution of the stagnation flow remains constant and is independent of the magnetic-field strength. This assumption is offered in reference 6 as the major reason for the differences in results for the reduction in heat-transfer rates. It is shown in the present paper that the difference can, however, be mainly attributed to different numerical values of the magnetic parameter $\sigma B^2 L/\rho u$ (where B is the magnetic induction and ρ the density).

The simplified approach to magnetohydrodynamic boundary-layer flows is based on the assumption that potential flow is maintained outside the

boundary layer under action of the magnetic force. In that case the pressure can be split into a dynamic and a magnetic part. This idea of splitting the pressure is actually borrowed from approaches developed in the past for flows subject to mechanical mass forces such as gravitational or centrifugal force. The analysis of boundary-layer flows is further simplified by dividing them into two major types. In one type the pressure distribution (dynamic plus magnetic) is assumed to be constant and independent of the magnetic field; in the other, the velocity distribution in the flow outside the boundary layer is assumed to be constant. The second type has received only very brief treatment in the literature.

SYMBOLS

$\vec{B}$, $\vec{b}$	magnetic induction
b	scale factor
$\vec{E}$	electric-field intensity
$\vec{H}$	magnetic intensity, $\vec{B}/\mu$
$\vec{j}$	electric-current density
m	magnetic parameter, $\sigma B_u{}^2 / \rho u_p$
m_1	magnetic parameter, $\sigma B_u{}^2 / \rho$
μ	magnetic permeability
ν	kinematic viscosity
R_M	magnetic Reynolds number, $\mu\sigma UL$
γ	magnetic parameter, $1/\mu\sigma\nu$
ρ	density
p	pressure
σ	electrical conductivity
t	time

$\vec{w}$ velocity vector

u, v   velocity components in x- and y-directions

x, y   Cartesian coordinates

U   characteristic velocity

L   characteristic length

θ angle extended from body axis

r   nose radius

u_N defined by $u = x u_N{}'(y)$ and $v = -u_N(y)$

Subscripts:

D   hydrodynamic

M   magnetic

P   potential

o   undisturbed stagnation flow

b   along body

st   stagnation point

u   uniform

$(\sim)$ dimensionless quantity; used only when designation is not obvious

∞ free stream

Superscripts:

$(\vec{\ })$ vector

$(\ )'$, $(\ )''$, $(\ )'''$ derivatives

EQUATION OF MOTION AND OHM'S LAW

For the steady flows under consideration, the equation of motion with the inclusion of magnetic forces is

$$\vec{w} \cdot \nabla \vec{w} = -\frac{1}{\rho}\nabla p + \frac{1}{\rho}(\vec{j} \times \vec{B}) + \nu\nabla^2\vec{w} + \frac{1}{3}\nu\nabla\nabla \cdot \vec{w} \qquad (1)$$

Ohm's law for fluid moving in a magnetic field is

$$\vec{j} = \sigma(\vec{E} + \vec{w} \times \vec{B}) \qquad (2)$$

The two terms in the parentheses of equation (2) indicate that for the interaction of a moving fluid with a magnetic field, essentially two types of electromotive forces exist by means of which charges may be separated and a closed current produced.

For simple flow analysis, those cases are of special interest in which the motionally induced electromotive force (represented by the electric intensity $\vec{w} \times \vec{B}$ in eq. (2)) is alone responsible for current closing, and the static electromotive force (represented by the electric intensity $\vec{E}$) can be ignored.

Stagnation flow is one of the cases in which $\vec{E}$ can actually be ignored for physical reasons. Because of the flow symmetry (the applied magnetic field is assumed to be uniform) the motionally induced currents on each side of the center plane or axis flow in opposite directions and have exactly the same value. For axially symmetric stagnation flow the currents simply flow in circles.

The boundary-layer flow along a flat plate does not have this symmetry of stagnation flow. The currents thus require for closure an outside wire connection. If this outside connection has a resistance, work has to be performed in overcoming this resistance, which is made possible through the existence of a static electromotive force (E ≠ 0). If the special artifice of a short-circuiting outside wire is used, the existence of a static electromotive force is no longer necessary for current flow. Due to the artificiality of this three-dimensional wire connection caution is necessary in defining the two-dimensional boundary-value problem.

For simplicity the condition E = 0 is used in the magnetohydro-dynamic boundary-layer equations throughout this paper. Ohm's law is

thus used in the form

$$\vec{j} = \sigma(\vec{w} \times \vec{B})$$ (3)

MAGNETIC-FIELD DEFORMATION

Mathematical Development and Solution

of Differential Equation

The induced current given by Ohm's law (eq. (3)) has a magnetic field around it which is determined by Ampere's law

$$\vec{j} = \nabla \times \vec{H}$$ (4)

where $\vec{H}$ represents the total magnetic intensity. It is convenient to express the magnetic induction $\vec{B} = \mu\vec{H}$ in Ohm's law in the form $\vec{B_u} + \vec{b}$, where $\vec{B_u}$ is the original imposed magnetic field and $\vec{b}$ the induced field. The induced magnetic field, and thus the deformation of the magnetic lines, can be obtained by equating the current from equations (3) and (4):

$$\nabla \times \vec{b} = \mu\sigma\vec{w} \times (\vec{B_u} + \vec{b})$$ (5)

If it is taken for granted that the original imposed magnetic field B_u is produced only by currents flowing outside the fluid, $\nabla \times B_u$ is zero and equation (5) can be written in the form

$$\nabla \times \vec{B} = \mu\sigma(\vec{w} \times \vec{B})$$ (6)

or the equivalent form, with $\vec{B} = \mu\vec{H}$,

$$\nabla \times \vec{H} = \mu\sigma(\vec{w} \times \vec{H})$$ (7)

or in dimensionless form

$$\nabla \times \vec{\underset{\sim}{H}} = R_M(\vec{\underset{\sim}{w}} \times \vec{\underset{\sim}{H}}) \tag{8}$$

The quantity $R_M = \mu\sigma UL$ is the magnetic Reynolds number, a dimension-less measure of the induced effects, and large values of R_M yield large deformations of the magnetic lines. The velocity U and the length L are characteristic of the particular problem under study.

In references 2 and 4 a particular solution of the differential equation (8) for the deformation of the magnetic field is found by postulating a law which the deformation is to follow. This law is

$$\underset{\sim}{H}_y = -\underset{\sim}{h}(\underset{\sim}{y}) \qquad \underset{\sim}{H}_x = \underset{\sim}{x}\underset{\sim}{h}'(\underset{\sim}{y}) \tag{9}$$

According to equation (8) the actual deformation of the magnetic field depends, also, on the velocity distribution. Since the velocity vector appears in the equation of motion (1) as well as in equation (8), a simultaneous solution of these two equations would be necessary to find the actual deformation of the magnetic lines. Since this solution is difficult to obtain, certain simplifying assumptions have been made for the effect of the magnetic force on the velocity distribution. In the subsequent development the designation $\sim$ for dimensionless quantities is omitted for convenience.

In reference 4 it is assumed that the velocity distribution of the viscous stagnation flow is of the form

$$u = xf'(y) \qquad v = -f(y) \tag{10a}$$

In reference 2, the velocity distribution is of the same form but expressed in different symbols, as follows:

$$U = xu'(y) \qquad V = -u(y) \tag{10b}$$

In the present paper the notation for the velocities in equation (10a) is adopted. To avoid confusion between the symbol u as used in equations (10b) and in the present paper, the notation of equations (10b) is changed herein to

$$u = xu_N{}'(y) \qquad\qquad v = -u_N(y) \qquad\qquad (10c)$$

For potential flow f' (or $u_N{}'$) is constant or unity so that

$$u = x \qquad \text{and} \qquad u = -y \qquad\qquad (11)$$

where the quantities are in dimensionless form.

It is shown in detail in the next section of the present paper that the assumption of potential flow actually limits the deformation of the magnetic lines to small values. The general trend of deformation calculated in reference 4 on the basis of this simplifying assumption seems, however, correct. This trend is verified in a subsequent section of the present paper with the aid of physical reasoning and a brief check based on the equation for small deformation of magnetic lines. In fairness it should be stated that in reference 4, in a subsequent numerical development of boundary-layer flow for large R_M, the assumption of potential flow is not used.

The deformation of the magnetic field is determined in reference 4 by substitution of the potential-flow velocity distribution (eqs. 11), together with the postulated law of deformation of the magnetic lines of equation (9), into equation (8). Equation (8) is first written in Cartesian coordinates:

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = R_M(uH_y - vH_x) \qquad\qquad (12)$$

Substitution of equations (9) and (10) into equation (12) yields

$$h'' = R_M(f'h - h'f) \qquad\qquad (13)$$

or for potential flow, where $f' = 1$,

$$h'' = R_M(h - h'y) \qquad\qquad (13a)$$

Equation (13a) is solved in reference 4 subject to the boundary conditions

$$h(0) = -1 \qquad h'(0) = 0 \qquad (14)$$

where (0) corresponds to conditions at the wall where $y = 0$. The solution in reference 4 yields magnetic lines that, for large values of $R_M y^2$, become tangent to the fluid streamlines as indicated in sketch (a), which corresponds to figure 4 in reference 4 and is repeated here for convenience.
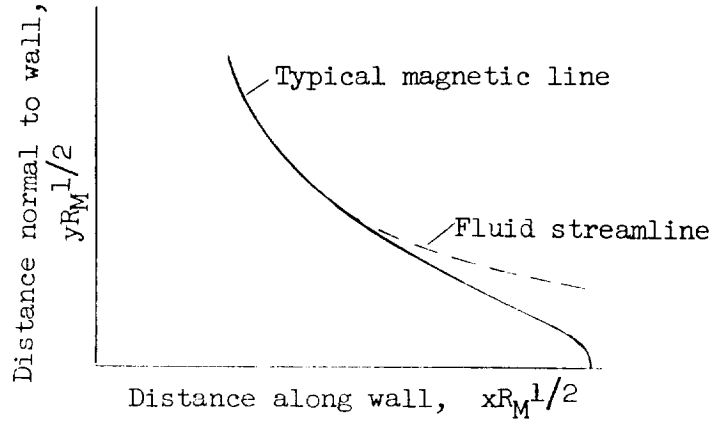
Sketch (a)

The trend of deformation of the magnetic lines is verified as follows for the condition that the product $h'y$ is small compared with h. (This condition, which corresponds to small deformations, is shown later to be required for existence of potential flow.) Equation (13) is reduced to

$$h'' - R_M h = 0 \qquad (15)$$

A solution of this equation subject to the boundary conditions at the wall in equations (14) gives

$$h = -\cosh \sqrt{R_M} y \qquad (16)$$

and

$$h' = -\sqrt{R_M} \sinh \sqrt{R_M} y \qquad (17)$$

or, substituting from equation (9) for the y and x components of the magnetic field,

$$H_y = \cosh \sqrt{R_M}\, y \tag{18}$$

and

$$H_x = -x\sqrt{R_M}\, \sinh \sqrt{R_M}\, y \tag{19}$$

verification is thus obtained for the increasingly negative slope of the magnetic lines.

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Physical Interpretation of Deformation and

Effect on Location of Magnet

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Perhaps the most important concept in the physical interpretation of the results is that an electrically conducting flow more or less drags the magnetic lines along. This concept is now applied to the problem which is mathematically discussed in references 2 and 4. It is assumed there that the magnetic field imposed for zero flow conditions is normal to the plate and uniform, that is, the magnetic lines are spaced equally (sketch (b)). Now, assume that the flow is started and approaches steady stagnation flow near the blunt body nose. It is evident that the magnetic lines will be dragged away from the line of symmetry (sketch (c)).

Sketch (b)   Sketch (c)
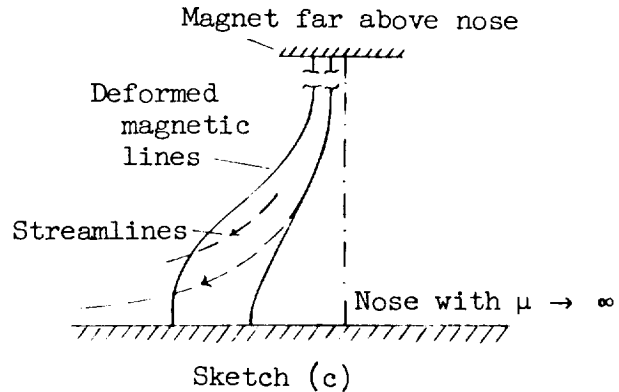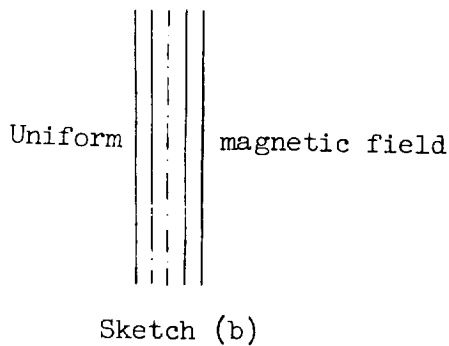
The boundary conditions for the magnetic field at the plate are given in equations (14) as $h(0) = -1$ and $h'(0) = 0$. According to equations (9) this means that the normal component H_y is constant and the

longitudinal component H_x is zero. In other words, the boundary conditions are that the magnetic field at the nose is uniform and normal to the wall after its deformation to the steady state has taken place. Sketch (c), based on the physical interpretation, evidently agrees with the mathematical results of reference 4 presented in sketch (a). Specifically, the concentration of the magnetic lines or the increase in field strength in the direction away from the nose is in agreement with the results in reference 4 expressed in the approximate form (necessary for potential flow) by equations (18) and (19).

The fact that the magnetic lines are bent inward and that the field strength increases in the direction away from the plate in sketch (c) raises the suspicion that the magnetic lines are not actually anchored in the plate but in a magnet or coil located at a large distance (approaching infinity) away from the plate. This is further evident from the following physical considerations. If the magnetic lines were anchored in the plate the plate would have to take up the magnetic shear forces caused by their deformation. Thus the magnetic lines would sweep outward, as shown in sketch (d). It should be noted that in sketch (d) it is assumed again that the magnetic field is originally uniform and normal to the nose; in contrast the magnetic lines are assumed to become normal to a wall high above the nose as they are deformed. For the practical case the original magnetic lines are of the form shown in sketch (e) and the deformed magnetic lines in the stagnation region will be swept away by the flow without approaching the vertical away from the nose. Whether a magnet situated in the outside flow is practical is not discussed here except to note that it could perhaps be useful for establishing a magnetic drag skirt if situated to the side of the stagnation region. Naturally, a magnet situated above the stagnation region as in sketches (c) and (d) would block some of the flow; the stagnation-flow streamlines are drawn only to illustrate the hypothetical problem presented in references 2 and 4.
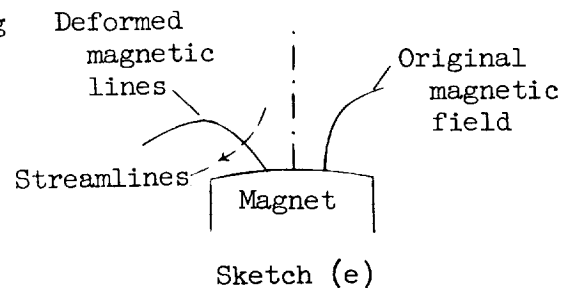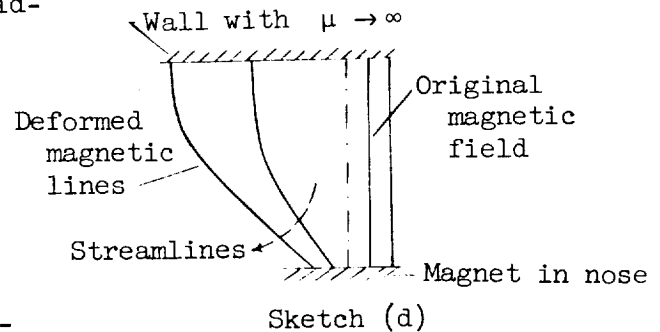
Sketch (d)

Sketch (e)

Finally, the physical meaning of the boundary conditions $h'(0) = 0$ and $h(0) = -1$ in sketches (a), (b), (c), and (d) is discussed below.

As mentioned previously, these boundary conditions signify that the magnetic lines at the wall are normal and uniformly spaced. The condition that the magnetic lines are still normal to the nose after their deformation means that the nose must have a magnetic permeability approaching infinity, so that no resistance is offered to the motion of the magnetic lines. The situation is analogous to having, say, a string rigidly suspended at large distance from the nose but able to move through the nose without resistance. As it is dragged away from its point of suspension by the flow it will be displaced from its original position at the nose, although the velocity at the nose is zero because of friction. It seems evident that the stagnation flow would sweep away the magnetic lines issuing above the nose in such a manner that their spacing normal to the nose is not uniform. The uniformity of field strength is specified by the condition that, according to the postulated deformation law (eqs. (9)), the y-component of the magnetic field is $H_y = -h(y)$ and thus is solely a function of the y-coordinate (normal to the plate). The infinite magnetic permeability of the plate plays its part only in the sense that it offers no resistance to the particular distribution of magnetic lines imposed on it and cannot exert a force. For the law of deformation and the boundary conditions stated in this section, the magnetic field at the nose is thus a result of the interaction rather than being determined before the interaction.

If the high-permeability nose were of finite dimensions so that the displacement of the lines were resisted at its ends, a magnetic pressure could build up in the flow direction. This situation furnishes an interesting counterexample to the case in which the magnetic lines are anchored in the plate by building up a magnetic shear in the direction of the plate.

For purposes of illustration it is assumed that the magnetic pressure in the flow where μ is finite builds up in a one-dimensional manner. Under such conditions the magnetic lines remain normal to the nose as they can redistribute themselves freely in the high μ nose, and the law of magnetic-field deformation is

$$H_y = h(x) \qquad\qquad H_x = 0 \qquad\qquad (20)$$

Again it is assumed that the flow is a potential stagnation flow; that is,

$$u = x \qquad\qquad v = -y \qquad\qquad (21)$$

14

Substitution of equations (20) and (21) into equation (12) yields (in dimensional form)

$$\frac{dH_y}{dx} = \mu \sigma x H_y \qquad (22)$$

for deformation of the magnetic field. Integration yields

$$H_y = H_u e^{\mu \sigma x^2/2} \qquad (23)$$

The intensity of the magnetic field thus increases in the longitudinal direction, and unless a finite plate representing the nose is considered the lines are swept to infinity. (The deviations from one-dimensional magnetic pressure due to end effects are neglected.) Since the magnetic pressure depends on H_y alone, the equation for the pressure of the stagnation flow is

$$p_t - p = \frac{\rho}{2}(x^2 + y^2) + \frac{\mu}{2}H_y^2 \qquad (24)$$

or, after substitution of equation (23),

$$p_t - p = \frac{\rho}{2}(x^2 + y^2) + \frac{\mu}{2}H_u^2 e^{\mu \sigma x^2} \qquad (25)$$

where p_t is total pressure.

LIMITATIONS OF THE MAGNETIC-FIELD CONFIGURATION

DUE TO ASSUMPTION OF POTENTIAL FLOW

General Development

In view of the great simplification obtained for the nonmagnetic case by dividing the flow into a viscous rotational boundary-layer region and a nonviscous potential outside flow, it is desirable to

state the conditions under which such a division is possible in the magnetohydrodynamic case. For that purpose, the equation of motion (1) is rewritten in such a form that a distinction between potential, rotational, and viscous flow is possible. Using the vector relations

$$\vec{w} \cdot \nabla \vec{w} = \nabla \frac{w^2}{2} - \vec{w} \times (\nabla \times \vec{w})$$

and

$$\nabla^2 \vec{w} = \nu(\nabla \cdot \vec{w} - \nabla \times \nabla \times \vec{w})$$

gives

$$\frac{1}{\rho}(\vec{j} \times \vec{B}) = \nabla \frac{w^2}{2} + \frac{1}{\rho}\nabla p - \vec{w} \times \nabla \times \vec{w} + \nabla \times \nabla \times \vec{w} - \frac{4}{3}\nu(\nabla \cdot \vec{w})$$

For potential flow,

$$\frac{1}{\rho}(\vec{j} \times \vec{B}) = \nabla \frac{w^2}{2} + \frac{1}{\rho}\nabla p \qquad (26)$$

Equation (26) shows that if the velocity distribution is to have a potential, $(\vec{j} \times \vec{B})/\rho$ must be expressible as a gradient or the equivalent condition

$$\nabla \times \frac{\vec{j} \times \vec{B}}{\rho} = 0 \qquad (27)$$

must exist.

It is interesting to note that in order to satisfy the condition of potential flow, the velocity term in equation (26) has to be kept free of ρ, since potential flow refers to the potential of the velocity rather than its product with ρ. As a consequence the condition for potential flow requires that the curl of $(\vec{j} \times \vec{B})/\rho$ be zero. Since the induced currents produce Joule heat the density of the flow could vary if compressibility effects were taken into account. It is evident that the conditions for the existence of magneto-gas-dynamic potential flow will be even more restrictive than those for magnetohydrodynamic potential flow, where the density is assumed to be constant. The case

of constant density is discussed in the present paper, rather than special flows that include density variations. It can be also seen from the relations at the beginning of this section that when the magnetic force does not have a potential, the resulting curl would have to be absorbed by the velocity field since it is capable of rotational behavior while the scalar pressure field is not. The same is also true in reverse - a rotational flow will produce a magnetic force that is no longer derivable from a potential.

A few remarks concerning the terminology of potential flow under action of magnetic forces would seem in order. The criticism has been made that the word "potential" implies that the forces are part of a conservative system, and since the currents produce Joule heat this requirement of conservation is not satisfied. This definition of potential may have been the first one historically, but in more recent usage the existence of a potential depends on the fact that the curl of the particular quantity, be it force or velocity, is zero. This is immediately evident from the fact that the electric tank is used to simulate potential flow, and in addition the very viscous Hele-Shaw flow is used to simulate potential flow. Perhaps the most outstanding example in this connection is the potential flow around a body with moving walls to prevent boundary-layer formation (ref. 7). Thus the conditions for potential flow are exactly fulfilled without the necessity of using zero viscosity. Note, especially, that the dissipation function does not become zero when the curl of the velocity distribution is set equal to zero.

<div align="center">Application to Law of Deformation of Magnetic Lines</div>

<div align="center">for Stagnation Flow</div>

For constant-density flow the existence criterion given in equation (27) for potential flow can be written as

$$\nabla \times (\vec{j} \times \vec{B}) = 0 \tag{28}$$

The current density $\vec{j}$ is expressed by Ampere's law:

$$\vec{j} = \nabla \times \vec{H} \tag{29}$$

where $\vec{H} = \vec{B}/\mu$. Substitution of equation (29) into equation (28) results in

$$\nabla \times \left[(\nabla \times \vec{H}) \times \vec{H} \right] = 0 \qquad (30)$$

Expressed in Cartesian coordinates, the existence criterion of equation (28) is

$$\frac{\partial}{\partial x}\left[H_x\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)\right] + \frac{\partial}{\partial y}\left[H_y\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)\right] = 0 \qquad (31)$$

The law postulated in reference 2 for the deformation of the magnetic field is

$$H_x = xh'(y) \qquad\qquad H_y = -h(y) \qquad (32)$$

Note that the magnetic intensity H has a dimension, whereas in reference 2 it is used in dimensionless form; this does not affect the derivation.

If the scale constant is equal to unity, the velocity distribution for potential stagnation flow is

$$u = x \qquad\qquad v = -y \qquad (33)$$

Substitution of equations (32) for the magnetic-field deformation into equation (31) for the existence criterion yields

$$-\frac{\partial}{\partial x}\left(H_x \frac{\partial H_x}{\partial y}\right) - \frac{\partial}{\partial y}\left(H_y \frac{\partial H_x}{\partial y}\right) = 0$$

or

$$\frac{\partial}{\partial x}(x^2 h'h'') - \frac{\partial}{\partial y}(hh''x) = 0$$

and

$$(hh'')' - 2h'h'' = 0$$

Finally,

$$hh''' - h'h'' = 0 \qquad (34)$$

For subsequent use, equation (34) is integrated, where the relation

$$\frac{hh''' - h'h''}{h^2} = \left(\frac{h''}{h}\right)' \tag{35}$$

is used. According to equation (34), $hh''' - h'h'' = 0$. Therefore

$$\left(\frac{h''}{h}\right)' = 0 \tag{36}$$

and with integration,

$$h'' - Ch = 0 \tag{37}$$

where C is a constant.

It is important to note that a differential equation for h can be obtained also by substitution of the magnetic transformation, equations (32), into the electromagnetic relation obtained by equating the current $\vec{j}$ from Ampere's law, equation (29), with the current $\vec{j}$ from Ohm's law, equation (2). Since for the stagnation flow the electric intensity $\vec{E} = 0$ is zero, the result is equation (8) (repeated here for convenience):

$$\vec{j} = \nabla \times \vec{H} = R_M(\vec{w} \times \vec{H}) \tag{38}$$

Since the solution for the magnetic field $\vec{H}$ or the magnetic parameter h is unique, the differential equation obtained by substituting equations (32) and (33) into equation (38) would have to agree with the differential equation (37). The resulting differential equation is already given by equation (13a), which is repeated here for convenience:

$$h'' - R_M h + R_M h'y = 0 \tag{39}$$

To justify the requirement that equations (37) and (39) be the same, C must equal R_M and the product $h'y$ must be negligibly small; that is,

$$h'' - R_M h(0) = 0 \tag{40}$$

Equation (40) is the same as equation (15) which was previously used to analyze the deformation of the magnetic lines for small values of $h'y$. The expression for h' obtained by integration of equation (15) is $h' = -\sqrt{R_M} \sinh\left(\sqrt{R_M}\,y\right)$ according to equation (17). It can be seen that even the use of small y values does not always guarantee small values of h' and that R_M must also be restricted to small values if h' is to approach zero.

The physical significance of this situation is found by recalling that differentiation of equations (32) shows that $h' = \partial H_x/\partial x = -\partial H_y/\partial y$. Thus h' gives the changes in the x- and y-components of the magnetic field. The restriction of $R_M = \sigma \mu UL$ to small values can be accomplished by keeping σ, U, and L small individually or together if h' is to be restricted to truly small values. The components $H_x = h'x$ and $H_y = -h$ of the magnetic field are thus restricted to small deviations from their values at the nose surface where $H_x = 0$ and $H_y = -1$, since the magnetic lines are assumed to be perpendicular and uniformly spaced. This is especially so if the characteristic length L and velocity U of the flow region are small, so that the deviations are restricted to the neighborhood of the stagnation point.

Order of Magnitude of Induced Magnetic Effects

due to Restriction of Potential Flow

It is shown in the preceding section that the induced effects in reference 2 are restricted to small values because of the assumption of potential flow outside the boundary layer. However, no explicit statement is made there concerning this restriction; as a matter of fact, the numerical analysis is based on the complete equations that include the induced effects. A search was made, however, to determine whether this restriction is contained in the simplifying assumptions used. For that purpose it was necessary to determine how the magnetic Reynolds number (eqs. (4) to (8)), which is characteristic of the induced effects, is defined in reference 2.

The clue to the magnitude of $R_M = \mu \sigma UL$ is given by the manner in which the significant quantities are made dimensionless. In equation (20) of reference 2, which gives the dimensionless representation

of the magnetic-field deformation, the quantity $\gamma = 1/\mu\sigma\nu$ makes its appearance. Further, the scale factor b of potential stagnation flow is included in the dimensionless approach. The fact that the kinematic viscosity ν and b are used has a special meaning since the boundary-layer thickness for stagnation flow is of the order of the constant $\sqrt{\nu/b}$. Though R_M is not directly defined in reference 2, it is evidently of the form $R_M = \mu\sigma b L^2$, where L represents the boundary-layer thickness (the scale constant b has the dimension $1/t$). The magnitude of R_M is evidently very small, since $\mu = 4\pi \times 10^{-7}$ volt-sec/amp-m and ν for high-temperature air near the missile nose is, say, 10^{-4} m^2/sec, and the value for σ is no higher than 250 ohms/m (apparently seeding with easily ionizable materials is not considered). As a result the induced effects will be small and the restrictive conditions for potential flow outside the boundary layer are satisfied.

Comparison of Calculations for Small Induced Effects

Made in Reference 5 With Results in Reference 2

In reference 5, results of calculations based on the assumption of small induced effects are compared with results from reference 2. Such a comparison seems unnecessary since reference 2 actually contains the assumption of small induced effects, though in a hidden manner. The small difference of 1/400 obtained in reference 5 between the shears based on machine computations and the calculations in reference 2 attest to this fact; in principle the two approaches are actually the same. The comparison made in reference 5 of the results in the two papers for a range of magnetic force parameters $\sigma B^2/\rho b$ shows very small differences; this is to be expected since reference 2 in essence also assumes small induced magnetic fields.

Limitations of Magnetic-Field Configurations for

Small Deformations of Magnetic Lines

So far the limitations due to potential flow have been discussed only for large deformations of the magnetic lines. The fact that for the particular deformation law used in reference 2 potential flow could be shown to exist for small deformations does not mean that potential flow always exists when deformations are small. Perhaps a better insight can be obtained into this situation (which was previously discussed as the limiting case for a particular deformation law) by rewriting the criterion for incompressible potential flow (eq. (28)) in a different form.

The result is

$$\nabla \times (\vec{j} \times \vec{B}) = \vec{B} \cdot \nabla\vec{j} - \vec{j} \cdot \nabla\vec{B} + \vec{B}\nabla \cdot \vec{j} + \vec{j}\nabla \cdot \vec{B} = 0 \qquad (41)$$

or, since $\nabla \cdot \vec{j} = \nabla \cdot B = 0$:

$$\nabla \times (\vec{j} \times \vec{B}) = \vec{B} \cdot \nabla\vec{j} - \vec{j} \cdot \nabla\vec{B} = 0 \qquad (42)$$

For the case of stagnation flow treated in references 2, 4, and 5, where a uniform magnetic field $\vec{B}_u$ is imposed and the induced field $\vec{b}$ is small, $\nabla\vec{B} = \nabla(\vec{B}_u + \vec{b})$ becomes $\nabla\vec{B}_u$, which is zero, and the term $\vec{j} \cdot \nabla\vec{B}$ is also zero. (For the two-dimensional case $\vec{j} \cdot \nabla\vec{B}$ is zero even for arbitrary induced fields $\vec{b}$ since the current is always normal to the plane in which the vector $\vec{B} = \vec{B}_u + \vec{b}$ acts.) Now, it can be shown that for the case of small deformations the current $\vec{j}$ is a function of x only. For the deformation law $H_y = -h(y)$ and $H_x = xh'(y)$, the current $\vec{j} = \nabla \times H$ is, according to equation (13a), (15), and (40), represented by h''. According to equation (40),

$$h'' = R_M h(0)$$

where $h(0) = -H_y$ is the original magnetic field. For the present problem this equation serves only to indicate that the current depends solely on the longitudinal or u-component of the velocity. According to to Ohm's law in the proper dimensions, the current is thus

$$\vec{j} = \sigma u \vec{B}_b \qquad (43)$$

Since for potential stagnation flow $\vec{u}$ is proportional to x, the current $\vec{j}$ is also proportional to x. Since the undeformed magnetic field $\vec{B}_u$ is normal to x, the dot product $\vec{B}_u \cdot \Delta\vec{j}$ is zero. Thus for this case substitution into equation (42) shows that the condition of potential flow is satisfied for the complete stagnation flow (as far as the induced effects can be considered small), in agreement with previous results. The inclusion of the boundary conditions behind the shock of course introduces new problems.

It is evident that the existence of potential flow is not, in general, guaranteed for any complete flow pattern together with a more or less arbitrary magnetic field, even if the deformations of the latter are small. It should, however, generally be possible to maintain potential flow when the magnetic lines are normal to the flow and the curvature of the streamlines is small. The restriction of the direction of magnetic lines and the curvature of the streamlines is used in reference 1 for the somewhat different purpose of making the magnetic force solely dependent on the longitudinal velocity. It is shown in a subsequent development of the present paper that the use of potential flow outside the boundary layer simplifies the derivation of the boundary-layer equations and helps very much in their physical interpretation.

A Finite Magnetic-Field Deformation Which is Valid

for the Complete Potential Stagnation Flow

A previous section contains a discussion of a magnetic-field configuration where $H_y = h(x)$ and $H_x = 0$, that corresponds to a one-dimensional magnetic pressure gradient along a finite plate or nose with high magnetic permeability. Since $\vec{\nabla j}$ is perpendicular to $\vec{B}$ and $\vec{\nabla B}$ is perpendicular to $\vec{j}$, according to equation (42) the field will be curl-free for finite deformations also. From the present discussion and from previous comments, however, it is evident that the existence of potential under finite deformation of the magnetic lines is the exception rather than the rule.

Effect of Limitations due to Potential Flow on Equation of

Motion for Stagnation Flow

In equation (25) of reference 2 the equation of motion is presented in the dimensionless form

$$u_N''' + u_N u_N'' - (u')_N^2 + 1 + \frac{h^2(0)}{\gamma} + (h')^2 - hh'' = 0 \qquad (44)$$

This equation is used in reference 2 in conjunction with equation (20) of that reference,

$$\gamma h'' + u_N h' - u_N' h = 0 \qquad (45)$$

which also corresponds to equation (13) of the present paper for the deformation of magnetic lines. Since for potential flow outside the boundary layer yh' has to be small, and thus also $u(y)h'$, inside the boundary layer (where y and h' approach zero values at the wall) equation (45) is reduced to

$$\gamma h'' - u_N'h = 0 \tag{46}$$

Substituting equation (46) into the equation of motion (44) and using the previously shown condition that h' itself is small gives

$$u_N''' + u_N u_N'' - (u_N')^2 + 1 + \frac{h^2(0)}{\gamma}(1 - u') = 0 \tag{47}$$

This equation of motion for magnetohydrodynamic boundary-layer flow is in agreement with the boundary-layer equation developed in a subsequent section specifically for small deformations of the magnetic lines (eq. (55)).

SIMPLIFIED ANALYSIS OF BOUNDARY-LAYER FLOW

For the potential flow outside the boundary layer the equation of motion can be written in the form of equation (26), which is repeated here for convenience:

$$\frac{1}{\rho}(\vec{j} \times \vec{B}) = \nabla\frac{w^2}{2} + \frac{1}{\rho}\nabla p$$

If the magnetic force has a potential, it is possible to use simplifying techniques developed long ago for mechanical forces having a potential, such as gravitation or centrifugal force. For example, in reference 8 (pp. 115-116) it is shown that for flow subject to a gravitational force the pressure can be split into two parts, one due to dynamic action and a static gravitational pressure p_{gr} due to the weight of the fluid defined as

$$p_{gr} = -\rho gh + \text{Constant} \tag{48}$$

It is also stated that this simplification cannot be applied to compressible flows because the changes in density affect both static and dynamic action. A detailed discussion in reference 8 shows that the combined pressure consisting of the dynamic and the static gravitational parts must satisfy the boundary conditions, for example, at a free surface.

In an analogous manner, the pressure can be split for the case of magnetic forces that have a potential. Equation (26) becomes

$$\frac{1}{\rho}(\vec{j} \times \vec{B}) = \nabla\frac{w^2}{2} + \frac{1}{\rho}\nabla(p_D + p_M) \tag{49}$$

where the magnetic-pressure gradient is now balanced by the magnetic force, or

$$\vec{j} \times \vec{B} = \nabla p_M \tag{50}$$

It is perhaps of interest to note that, in contrast to the gravitational pressure (eq. (49)), the magnetic pressure (eq. (50)) does not depend on the density. However, it was shown in a previous section (see eq. (27) and following material) that the existence of a magnetic force potential would be an extremely rare occurrence if compressibility were accounted for, since the Joule heat produces density variations. Thus the simplicity of equation (50) alone is not enough to permit pressure splitting for compressible flow.

It was shown previously that the maintenance of potential flow outside the boundary layer is, in most cases, possible only when the deformation of the magnetic lines is small, but that even for this situation potential flow need not always exist. The fact was also brought out that potential flow should be possible when the magnetic lines are normal to the flow and the curvature of the streamlines is small. Under such conditions

$$\vec{j} \times \vec{B} = \sigma\left[\vec{B}(\vec{B} \cdot \vec{w}) - \vec{w}(\vec{B} \cdot \vec{B})\right] \tag{51}$$

becomes

$$\frac{1}{\rho}(\vec{j} \times \vec{B}) = -\frac{\sigma B_u^2}{\rho}\vec{u} \tag{52}$$

The above conditions are also specified in reference 1, but without mention of the fact that with them the simplifications of magnetohydrodynamic potential flow are introduced.

With the above conditions the magnetohydrodynamic boundary-layer equation can be written in the form

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{1}{\rho}\frac{\partial}{\partial x}(p_D + p_M) + m_1 u = v\frac{\partial^2 u}{\partial y^2} \qquad (53)$$

where

$$m_1 = \frac{\sigma B_u^2}{\rho}$$

In the potential flow outside the boundary layer the magnetic force is balanced by the magnetic pressure gradient, or

$$\frac{1}{\rho}\frac{\partial p_M}{\partial x} + m_1 u_P = 0 \qquad (54)$$

Equation (53) without the magnetic terms is the conventional boundary-layer equation. It is usually assumed that the pressure gradient $\partial p_M/\partial x$ acts uniformly across the boundary layer. An alternate form of the equations can be obtained by replacing $(\partial p_M/\partial x)/\rho$ with $-m_1 u_P$ from equation (54) in equation (53), with the result

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{1}{\rho}\frac{\partial p_D}{\partial x} + m_1(u - u_P) = v\frac{\partial^2 u}{\partial y^2} \qquad (55)$$

It is immediately evident that equation (55) is of the same form as equation (47), which was obtained by limiting the approach in reference 2 to small deformation of the magnetic lines:

$$u_N''' + u_N u_N'' - (u_N')^2 + 1 + \frac{h^2(0)}{\gamma}(1 - u_N') = 0 \qquad (56)$$

For comparison, equation (55) is regrouped:

$$\nu\frac{\partial^2 u}{\partial y^2} - v\frac{\partial u}{\partial y} - u\frac{\partial u}{\partial x} - \frac{1}{\rho}\frac{\partial p_D}{\partial x} + m_1(u_P - u) = 0 \qquad (57)$$

In comparing equations (56) and (57) it must be realized that equation (56) is written in dimensionless form while equation (57) is not. The dimensionless parameter $h^2(0)/\gamma$ used in references 2 and 3, however, agrees in essence with the dimensionless expression mx obtained by multiplying m_1 by x and dividing by u, whereby

$$mx = \frac{\sigma B_u^2}{\rho u}x$$

or, for stagnation flow where $u = bx$,

$$mx = \frac{\sigma B_u^2}{\rho b}$$

In the following paragraphs the general scope of the problem is outlined. A more specific analysis of a few selected cases is given in the appendix with the aid of the split-pressure approach. Basic trends in skin friction and heat-transfer rates are also discussed there.

It appears convenient to divide the great variety of boundary-layer flows into two main types. For one the distribution of the pressure (hydrodynamic plus magnetic) is assumed to be constant, whereas for the other the velocity distribution in the potential flow outside the boundary layer is assumed to be constant and independent of the magnetic-field strength. The value of such a classification depends on whether these types, singly or in a combined form, lead to physically realizable boundary conditions. For example, for the boundary-layer flow along a flat plate it should be possible to maintain constant pressure independently of the field strength, if certain conditions are satisfied. If a magnetic field extending a finite distance above the plate and the resulting changes in the flow direction are considered small (generally necessary for potential flow), it should be possible to transmit a constant pressure to the plate. Another example in which the assumption of constant pressure distributions has been used is that of stagnation flow behind the bow wave of a blunt body in hypersonic flow (ref. 2). While, as shown in reference 6, the assumption of constant pressure distribution

is not exact, it is shown in a subsequent section that the assumption provides a valuable approximation.

When the velocity distribution of the potential flow is maintained independently at the magnetic-field strength, the boundary-layer flow requires a magnetic pressure gradient that balances the magnetic force. The maintenance of the velocity distribution may require the obvious boundary condition of a pump, or may arise indirectly from more involved boundary conditions. In the literature (ref. 1) this type of problem has been treated only for the case in which a given initial flow acceleration outside the boundary layer is maintained against a magnetic force. The case in which the velocity distribution of the potential flow is maintained is also of interest for certain aspects of stagnation flow. Details are given in the appendix.

In one case, the maintenance of a velocity distribution results in the rare situation of a boundary layer that approaches a fixed velocity profile (and fixed shear) in an asymptotic manner. This happens when a constant velocity is maintained in the potential flow outside the boundary layer. The equations for this and the previously mentioned problems are given in the appendix.

REASONS FOR DIFFERENCES IN RESULTS FOR REDUCTION

IN HEAT-TRANSFER RATES

As noted in the introduction, the reduction in heat-transfer rates (and skin friction) due to the magnetic field is given as 28 percent in reference 3, whereas in reference 6 a reduction of only 5 percent is obtained (in both papers the induced effects are small). The difference is ascribed in reference 6 to the simplifying assumption made in reference 3 (based on ref. 2) that the pressure distribution in the flow behind the bow wave is maintained independently of the magnetic-field strength. In reference 6, where the proper boundary conditions at the shock are used (for a spherical shock corresponding to a spherical body) it is implied that the difference in reduction of heat-transfer rates is mainly based on the inadequacy of the assumption of constant pressure distribution. However, it is shown below, by substituting values for reentry conditions into the equations used in that paper, that the differences in the results are in the main due to a different choice in magnetic parameter rather than due to the assumption concerning the pressure distribution.

According to equation (13) of reference 6, the pressure variation in the incompressible flow near the stagnation point is given by

$$p_b - p_{st} = -\frac{4}{3}\left(1 - \frac{1}{4}S_b\right)(1 + \lambda)\sin^2\theta \tag{58}$$

The parameters S_b and λ are defined by

$$S_b = \frac{\sigma B_b r_b}{\rho_\infty u_\infty} \tag{59}$$

and

$$\lambda = \frac{\sigma B_b^2}{\rho(\partial v_\theta/\partial\theta)_{st}} \tag{60}$$

where $(\partial v_\theta/\partial\theta)_{st}$ represents the velocity distribution near the stagnation point and is given by equation (12b) of reference 6:

$$\left(\frac{\partial v_\theta}{\partial\theta}\right)_{st} = \left[\frac{8}{3}k\left(1 - \frac{1}{4}S_b\right)\right]^{1/2} \tag{61}$$

with $k = \rho_\infty/\rho$. It is stated in reference 6 that λ corresponds to the parameter used in references 2 and 4. If this were true λ would have to be dimensionless, but according to equation (61) $\partial v_\theta/\partial\theta$ does not have the dimension velocity/length which would be necessary to make λ dimensionless. To judge from the development in reference 6, λ should apparently be divided by u_∞ and multiplied by r_b, or

$$\lambda = \frac{\sigma B_b^2}{\left(\rho\frac{\partial u}{\partial x}\right)_{st}} = \frac{\sigma B_b^2 r_b}{\left(\rho\frac{\partial v}{\partial\theta}\right)_{st} u_\infty} \tag{62}$$

The quantity S_b is assumed to be given, since B_b is assumed not to vary during the interaction; r_b and σ are known, as well as ρ_∞ and u_∞ corresponding to the conditions ahead of the shock. The parameter λ for the stagnation region was to be calculated. For the present purpose it is most convenient to establish the ratio of S_b and λ. According to equations (59) and (62), with $k = \rho_\infty/\rho$,

$$\frac{S_b}{\lambda} = \frac{\rho\frac{\partial u}{\partial x}}{\frac{\rho_\infty u_\infty}{r_b}} = \frac{\rho}{\rho_\infty}\left[\frac{8}{3}\frac{\rho_\infty}{\rho}\left(1 - \frac{1}{4}S_b\right)\right]^{1/2} = \left[\frac{8}{3}\frac{\rho}{\rho_\infty}\left(1 - \frac{1}{4}S_b\right)\right]^{1/2} \tag{63}$$

In reference 6 the assumption is made that for rentry conditions S_b is 2/3 and the density ratio ρ/ρ_∞ is about 15. Substitution of these values into equation (63) gives $S_b/\lambda = 5.77$.

This value for S_b/λ is used in calculating the deviation from the condition of constant pressure distribution by means of equation (58). Since without the magnetic field the pressure distribution is

$$p_b - p_{st} = -\frac{4}{3}\sin^2\theta$$

the relative deviation from the pressure difference that would exist without the magnetic field is

$$\Delta p = \lambda\left(1 - \frac{1}{4}S_b\right) - \frac{1}{4}S_b \tag{64}$$

When $S_b = 2/3$ and $S_b/\lambda = 5.77$, Δp is only 0.07. To make certain that this small relative deviation of the pressure is not just accidental, slightly higher values of S_b (say $S_b = 1$) were investigated ($S_b = 2/3$ is perhaps a little conservative for reentry conditions); some smaller values of ρ/ρ_∞ were also used, with the result that the relative pressure deviations still remained small. Furthermore, the value of λ was verified by using the "modified Newtonian theory" for the value of $(\partial u/\partial x)_{st}$ without the magnetic field:

$$\left(\frac{du}{dx}\right)_{st} = \frac{1}{r_b}\sqrt{\frac{1}{\rho}(p_{st} - p_\infty)} \tag{65}$$

Small values for Δp were again obtained.

The smallness of the deviations seems further plausible in view of the fact that Δp goes through zero for the range of values of S_b and

S_b/λ investigated (see also physical interpretation in the appendix). Specifically, for the assumed case of constant pressure distribution $(\Delta p = 0)$, equation (64) yields a value of 5 for S_b/λ when $S_b = 2/3$.

Evidently λ in reference 6 is equivalent to the significant parameter $h^2(0)/\gamma$ in reference 3, since both refer to conditions near the stagnation point. For $S_b = \frac{2}{3}$ and $S_b/\lambda = 5$, $\lambda = \frac{2}{3}\frac{1}{5} = \frac{2}{15}$. This value for λ is to be compared with the value for $h^2(0)/\gamma = \sigma B^2/\rho b$ in reference 3, which is 4.9. It is evident from figure 1 in reference 3 that the high reduction in heat-transfer rate is essentially due to the extreme values of the magnetic parameter, which would correspond to reentry conditions at altitudes higher (smaller ρ) than that at which the maximum heating rate occurs, unless seeding with easily ionizable materials (higher σ) is considered. For a more realistic value of $h^2(0)/\gamma$ or λ the reductions in heat-transfer rates obtained in references 3 and 6 should approximately agree.

REMARKS ON PRACTICAL APPLICATION OF MAGNETIC FIELDS

TO REENTRY PROBLEM

In addition to producing a reduction in heat-transfer rates, the magnetic field has the function of increasing the drag. The resulting deceleration decreases the total heat transfer to the missile (ref. 9). It seems clear from the previous section that the application of a magnetic field near the nose will not yield sufficiently large effects unless seeding with easily ionizable materials or some other means of boosting the electric conductivity is considered. Effects due to variable conductivity are not considered here. Actually, since the drag due to the magnetic force is proportional to the velocity, it could have been more or less expected that attempts to obtain drag increase in the low-velocity region near the nose would not be successful.

The use of magnetic drag skirts at some distance from the nose looks more promising. Such drag skirts could also reduce the velocities, skin friction, and heat-transfer rates near the nose in a manner similar to that of a blunt body with a concave nose. Lift control through magnetic fields also offers an interesting possibility. In this connection the effects of a magnetic field due to a thin solenoid extended into the flow (ref. 10) are of interest.

CONCLUSIONS

1. The law of deformation of the magnetic lines assumed in some of the existing literature on magnetohydrodynamic stagnation flow corresponds to magnetic lines which diverge as they approach the missile nose. The magnetic lines are dragged along by an electrically conducting flow. If this condition is to hold true the magnetic lines must have roots in a magnet situated away from the nose rather than one in the nose. The drag forces are taken up by this magnet situated away from the nose, or, in other words, the magnetic field is not assigned at the nose but is a result of the interaction. The boundary conditions specified in this literature, that the magnetic lines are equally spaced at the missile nose and normal to it, reinforce these conclusions. Their fulfillment requires that the magnetic permeability of the nose approach infinite values and thus offer no resistance to the displacement of magnetic lines.

2. For the magnetic deformation law and the boundary conditions stated in conclusion 1, the assumption that potential flow is maintained outside the boundary layer is a good approximation only if the induced magnetic effects or the deformation of the magnetic lines is negligibly small. This conclusion is contrary to the implications of the numerical analysis in reference 2 which is based on the complete equations that include the induced effects. However, the magnetic Reynolds number $R_M = \sigma\mu UL$ which is representative of the induced effects is implicitly based on the boundary-layer thickness; thus the results in reference 2 are actually restricted to small induced effects.

3. The comparison of the calculations for small induced effects in reference 5 with those for supposed full induced effects seems unnecessary, since the restriction to small induced effects is implicitly contained in reference 2. This is also brought out by the fact that the results for the shears in the two references differ by only 1/400.

4. A law of deformation of the magnetic lines exists for which potential flow is maintained when induced effects are large.

5. The restriction to small induced magnetic effects does not always guarantee the existence of potential flow under action of a magnetic force.

6. The reduction in heat-transfer rates (and skin friction) due to the magnetic field is given as 28 percent in reference 3, whereas in reference 6 a reduction of only 5 percent is obtained. The difference is ascribed in reference 6 to the simplifying assumption made in reference 3 that the pressure distribution behind the bow wave is maintained

independently of the magnetic-field strength. However, it is shown in the present paper that the major reason for the differences in heat-transfer rates is a different choice of a magnetic parameter that depends on reentry conditions. The reentry conditions used in reference 6 are in better correspondence with the maximum heat-transfer rates which are to be reduced by the magnetic field.

7. For potential flow outside the boundary layer the pressure can be split into a magnetic and a dynamic part. This splitting leads to considerable simplification in the presentation of magnetohydrodynamic boundary-layer effects.

8. The division of boundary-layer flows into two types, where the pressure or the velocity distribution in the potential flow is maintained independently of the field strength, leads to a useful classification.

9. For the case of constant pressure distribution, shear and heat-transfer rates are reduced, whereas for the case of constant velocity distribution, they are increased.

10. A comparison of the magnetic force effects with the better known effects of pressure gradients shows that the decreases and increases in shear and heat-transfer rates are smaller for the magnetic case.

11. For flow along a flat plate with constant velocity maintained under action of a magnetic force, a rare case occurs in which the velocity profile of the boundary layer approaches asymptotically a fixed shape and the shear approaches a fixed value.

Langley Research Center,
    National Aeronautics and Space Administration,
        Langley Field, Va., December 30, 1958.

APPENDIX

APPLICATION OF THE SIMPLIFIED APPROACH TO A VARIETY OF

MAGNETOHYDRODYNAMIC BOUNDARY-LAYER FLOWS

General Remarks

It is shown in the body of the paper that for potential flow outside the boundary layer the equations can be presented in the simplified form (eq. (53)):

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + \frac{1}{\rho} \frac{\partial}{\partial x}(p_D + p_M) + m_1 u = \nu \frac{\partial^2 u}{\partial y^2} \tag{A1}$$

The magnetic force $m_1 u_P$ is held in balance by the pressure gradient $\partial p_M / \partial x$ according to equation (54):

$$\frac{1}{\rho} \frac{\partial p_M}{\partial x} + m_1 u_P = 0 \tag{A2}$$

Equations (A1) and (A2) can be combined to yield the alternate form

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + \frac{1}{\rho} \frac{\partial p_D}{\partial x} + m_1(u - u_P) = \nu \frac{\partial^2 u}{\partial y^2} \tag{A3}$$

The problems arising from the somewhat artificial nature of the two-dimensional boundary conditions, especially for the magnetic field, are not discussed here.

Boundary Layer Along a Flat Plate

With pressure distribution maintained.- The assumption that the zero pressure gradient is maintained independently of the magnetic-field strength is conveniently substituted into equation (A1). As a result, the equation reduces to

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + m_1 u = \nu \frac{\partial^2 u}{\partial y^2} \tag{A4}$$

The boundary condition in the potential flow, where (according to the usual boundary-layer assumption) $\nu \frac{\partial^2 u}{\partial y^2}$ and $\nu \frac{\partial u}{\partial y}$ approach zero, is

$$u \frac{\partial u}{\partial x} = -m_1 u \qquad (A5)$$

The split-pressure approach is advantageous in that it shows that both the hydrodynamic and the magnetic pressure gradient can vary, since

$$\frac{\partial}{\partial x}\left(p_D + p_M\right) = 0 \qquad (A6)$$

Equation (A4) is similar in structure with the well-known equation for boundary-layer flow under unfavorable pressure gradient. The essential difference is that the pressure gradient acts uniformly across the boundary layer, whereas the magnetic force $m_1 u$ decreases with the reduction of the velocity toward the wall.

In this connection it should be pointed out that the velocity profile in sketch (f) of reference 1 which has negative velocities, or reverse flow, is not correct since the magnetic force approaches zero at the wall and, in addition, the smoothing effect of the viscosity is important. As an increasing amount of boundary-layer material approaches zero speed, continuity requires that the flow be lifted from the wall, but reverse flow cannot occur.

It is also evident from the structure of the magnetic force that the reduction in skin friction will be less than that for the corresponding pressure gradient. Estimates of skin-friction reduction due to the magnetic force could be developed on this basis.

A simple comparison of the reduction in heat-transfer rates for the two cases is also possible through the comparative deviations from the Reynolds analogy when the static electric intensity $\vec{E}$ is zero. The reason is that under such conditions the total energy (including Joule heating) is constant under the action of the magnetic force. The constancy of total energy under these conditions has been derived in several papers (refs. 1, 11, 12, and 13). Since the deviations from the Reynolds analogy for the cases of pressure gradients are well established for conditions of constant total energy (ref. 14), the corresponding trend for the magnetic case can be readily established. Like the reduction in skin friction, the reduction in heat-transfer rates can be expected to be less for the magnetic case than for the corresponding pressure-gradient case.

<u>With steady velocity distribution maintained in the potential flow.-</u>
For the case in which steady velocity distribution is maintained in the
potential flow, it is convenient to use equation (A3). As $y \to \infty$ and
$u \to u_P$ the potential-flow boundary condition is

$$u_P \frac{\partial u_P}{\partial x} + \frac{1}{\rho} \frac{\partial p_D}{\partial x} = 0 \qquad (A7)$$

since the terms $v_P\left(\partial u_P / \partial y\right)$ and $\nu\left(\partial^2 u_P / \partial y^2\right)$ are zero. In order to
apply the boundary condition of equation (A7) to equation (A3), the latter
is rewritten, using $u(\partial u / \partial x) = \partial\left(u^2/2\right)/\partial x$, in the following form:

$$\frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) - \frac{\partial}{\partial x}\left(\frac{u_P^2}{2}\right) + \frac{\partial}{\partial x}\left(\frac{u_P^2}{2}\right) + \frac{1}{\rho}\frac{\partial p_D}{\partial x} + v\frac{\partial u}{\partial y} + m_1\left(u - u_P\right) = \nu\frac{\partial^2 u}{\partial y^2} \qquad (A8)$$

where, according to equation (A7), $\frac{\partial}{\partial x}\left(\frac{u_P^2}{2}\right) + \frac{1}{\rho}\frac{\partial p_D}{\partial x}$ is equal to zero.

For a given potential-flow velocity distribution the steady-flow
equation thus becomes, with use of the boundary conditions,

$$\frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) - \frac{\partial}{\partial x}\left(\frac{u_P^2}{2}\right) + v\frac{\partial u}{\partial y} + m_1\left(u - u_P\right) = \nu\frac{\partial^2 u}{\partial y^2} \qquad (A9)$$

For a flat plate when the potential-flow velocity is held constant (that
is, $\partial\left(u_P^2/2\right)/\partial x = 0$) equation (A9) reduces to

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + m_1\left(u - u_P\right) = \nu\frac{\partial^2 u}{\partial y^2} \qquad (A10)$$

It is again of interest to compare the magnetic-force term
$m_1\left(u - u_P\right)$ with a pressure gradient. Since $u_P > u$, the magnetic-
force term corresponds to a favorable pressure gradient. This corre-
spondence is evidently due to the fact that when the velocity distri-
bution is assumed to be constant the magnetic pressure gradient which
balances the magnetic force $-m_1 u$ has to "drag" the flow along. The
effect of the magnetic force $m_1\left(u - u_P\right)$ on the shear increase is evi-
dently smaller than the effect of a corresponding pressure gradient.
As a matter of fact, the magnetic force is even zero in the potential
flow but increases toward the wall. However, as the wall is approached
the retarding effects of the viscosity oppose the trend toward increase
in $\partial u / \partial y$ and the shear.

36

For the flat-plate boundary layer with a constant velocity, it can be shown that a constant shear (and velocity distribution) is asymptotically approached. The reason is that the shear diminishes for the flat-plate boundary layer without a magnetic field, whereas the magnetic force acts similarly to a favorable pressure gradient in increasing the shear. An equilibrium state is attained given by

$$m_1\left(u - u_P\right) = \nu \frac{\partial^2 u}{\partial y^2} \tag{A11}$$

With the application of the boundary conditions that $u \to 0$ as $y \to 0$ and $u \to u_P$ as $y \to \infty$, the solution of equation (A11) is

$$u = u_P\left(1 - e^{-\sqrt{\frac{m_1}{\nu}}y}\right) \tag{A12}$$

which represents the shape of the asymptotic velocity profile.

With an initial acceleration maintained in the potential flow.- Reference 1 treats the case in which an initial acceleration is maintained in the potential flow, but in a perhaps less direct manner than the present split-pressure approach. For the initial acceleration the unsteady term $\partial u/\partial t$ has to be included in equation (A3) but the inertia terms $u(\partial u/\partial x)$ and $v(\partial u/\partial y)$ can be neglected. As a result, equation (A3) becomes

$$\frac{\partial u}{\partial t} + \frac{1}{\rho} \frac{\partial p_D}{\partial x} + m_1\left(u - u_P\right) = \nu \frac{\partial^2 u}{\partial y^2} \tag{A13}$$

For potential flow outside the boundary layer where $y \to y_\infty$, $u \to u_P$, and $\nu\left(\partial^2 u/\partial y^2\right) = 0$,

$$\frac{\partial u_P}{\partial t} + \frac{1}{\rho} \frac{\partial p_D}{\partial x} = 0 \tag{A14}$$

and the pressure gradient $\left(\partial p_M/\partial x\right)/\rho$ is balanced by the magnetic force $-m_1 u_P$ according to equation (A2). In order to apply the boundary condition of equation (A14) to equation (A13), the term $\partial u_P/\partial t$ is conviently subtracted and added to give

$$\frac{\partial u}{\partial t} - \frac{\partial u_P}{\partial t} + \frac{\partial u_P}{\partial t} + \frac{1}{\rho} \frac{\partial p_D}{\partial x} + m_1\left(u - u_P\right) = \nu \frac{\partial^2 u}{\partial y^2} \tag{A15}$$

where, according to equation (A14), $\dfrac{\partial u_P}{\partial t} + \dfrac{1}{\rho}\dfrac{\partial p_D}{\partial x}$ is equal to zero.

Since u_P is a function of time only, $\partial^2 u/\partial y^2$ can be written as $\partial^2(u - u_P)/\partial y^2$ and thus equation (A15) becomes

$$\frac{\partial(u - u_P)}{\partial t} + m_1(u - u_P) = \nu\,\frac{\partial^2(u - u_P)}{\partial y^2} \qquad (A16)$$

If the relative velocity $u - u_P$ is designated as Δu, equation (A16) seems to agree with the initial stages of boundary-layer flow over an accelerated flat plate when the magnetic field is fixed in the fluid (a case treated in ref. 1). This agreement is mentioned also at the bottom of page 20 of reference 1. Equation (A13) of the present paper agrees with the third equation on page 20 in reference 1, which is stated but not derived. The fourth equation on page 20 states that when $y = \infty$, $\partial u/\partial t = \partial^2 u/\partial y^2 = u = 0$. According to equation (A16) of the present paper, however, this condition would apply when $(u - u_P)$ or Δu becomes zero. The last equation on page 20 of reference 1 states that at $y = \infty$, $-m_1 u_\infty + \dfrac{1}{\rho}\dfrac{\partial p}{\partial x} = 0$ where the subscript ∞ seems identical with the P used in the present paper for potential flow. This statement in reference 1 seems in contrast with the fourth equation of page 20 (ref. 1), where u is zero at $y = \infty$.

Boundary-Layer Flow Near a Stagnation Point

In reference 2 the boundary-layer equation is solved under the assumption that the pressure distribution is maintained under action of the magnetic force. Since for stagnation flow a pressure distribution

$$(p_t - p)_{D,o} = \frac{\rho}{2}\,b_o^2 x \qquad (A17)$$

exists without the magnetic field, a pressure gradient

$$\frac{1}{\rho}\left(\frac{\partial p_D}{\partial x}\right)_o = -b_o^2 x$$

has to be maintained rather than the zero pressure gradient of flat-plate flow. Thus

$$\frac{1}{\rho}\left(\frac{\partial p_D}{\partial x}\right)_o = -b_o{}^2 x = \frac{1}{\rho}\frac{\partial p_D}{\partial x} + \frac{1}{\rho}\frac{\partial p_M}{\partial x} \qquad (A18)$$

where, according to equation (A2),

$$\frac{1}{\rho}\frac{\partial p_M}{\partial x} + m_1 u_P = 0$$

The new hydrodynamic pressure gradient is adjusted to the change in scale of the potential stagnation flow from b_o to b, that is, the gradient becomes

$$\frac{1}{\rho}\frac{\partial p_D}{\partial x} = -b^2 x$$

The magnetic pressure gradient $(\partial p_M/\partial x)/\rho = -m_1 u_P$ undergoes a similar scale adjustment. In detail, since $m_1 = \sigma B_u{}^2/\rho$ and $u_P = bx$, $m_1 u_P$ can be written in a more convenient form:

$$m_1 u_P = \frac{\sigma B_u{}^2}{\rho} u_P = \frac{\sigma B_u{}^2}{\rho b} b^2 x \qquad (A19)$$

Equation (A18) can thus be expressed in the form

$$\frac{1}{\rho}\left(\frac{\partial p_D}{\partial x}\right)_o = -b_o{}^2 x = \frac{1}{\rho}\frac{\partial p_D}{\partial x} + \frac{1}{\rho}\frac{\partial p_M}{\partial x} = -b^2 x\left(1 + \frac{\sigma B_u{}^2}{\rho b}\right) \qquad (A20)$$

there follows

$$\frac{b}{b_o} = \left(1 + \frac{\sigma B_u{}^2}{\rho b}\right)^{-\frac{1}{2}} \qquad (A21)$$

The scale reduction in equation (A21) agrees with that in reference 2 $\left(\text{where } \sigma B_u{}^2/\rho b \text{ is called } h^2(0)/\gamma\right)$ with the difference that b_o instead of b is used in $\sigma B_u{}^2/\rho b$ in the reference; this means that an additional linearization is contained in reference 2. (In ref. 2 $h^2(0)/\gamma$ is somewhat differently defined, but in ref. 3 it is the same as in the present paper.)

The pressure gradient $(\partial p_D/\partial x)_o/\rho$ of the nonmagnetic flow is thus split under action of the magnetic force into a gradient $\partial p_D/\partial x = -b^2 x$ due to scale reduction of the nonmagnetic flow and a gradient $(\partial p_M/\partial x)/\rho$

that balances a scale-reduced magnetic force to maintain the new veloc-
ity distribution at the reduced scale. The effect on the shear is thus
also split into two parts. One is a reduction due to the nonmagnetic
scale-reduction effect of the stagnation flow. The other (as is clear
from the discussion of the flat-plate case) is a shear increase at the
scale-reduced velocity distribution. Although, according to equa-
tion (A18), the reduction in pressure gradient due to scale reduction

$$\left(\frac{\partial p_D}{\partial x}\right)_0 - \frac{\partial p_D}{\partial x} = \frac{\partial p_M}{\partial x}$$

equals the favorable pressure gradient $\partial p_M/\partial x$ that maintains the con-
stant velocity distribution (at reduced scale), the shear increase due
to this pressure gradient will have to be smaller than that due to scale
reduction. The reason is, as noted before, that the shear increase due
to the magnetic pressure gradient at constant velocity distribution is
smaller than that due to the corresponding favorable pressure gradient.
This fact is strongly in evidence in the numerical results of reference 2.

Similar arguments can be made for the reduction in heat-transfer
rate, which consists of one part based on a nonmagnetic scale reduction
of the stagnation flow and a smaller increase corresponding to the
increase in skin friction at reduced velocities. The basic results for
the deviation from Reynolds analogy (ref. 14) for flows with pressure
gradients apply with proper modifications.

For stagnation flow, when the original velocity distribution is
maintained in the potential outside flow the pressure gradient $\left(\partial p_D/\partial x\right)_0$
is also maintained (in contrast to the combined pressure gradient
$\partial(p_D + p_M)/\partial x$). As a result, according to equations (A1), (A2), and (A3),
the magnetic force $m_1 u_p$ will have to be balanced by the pressure gra-
dient $\partial p_M/\partial x$. The resulting shear increase is now not reduced by a scale
reduction as was the case for the unchanged pressure gradient. In the
actual case, however, the tendencies toward constant pressure distribu-
tion should predominate.

It may be of interest to note that in the calculations of refer-
ence 6, where the proper boundary conditions at the shock are used, the
effect of the magnetic force on the pressure gradient near the stagna-
tion point may be a small decrease, a small increase, or no change.
These small variations were taken as supporting evidence in the present
paper that the pressure gradient is essentially maintained. The case
in which the pressure gradient is reduced by the magnetic force with a
resulting local drag increase has so far not been mentioned, but it can
be readily constructed by starting with equation (A18) and assuming that

the sum of the pressure gradients $\frac{\partial p_D}{\partial x} + \frac{\partial p_M}{\partial x}$ is smaller than the original pressure gradient $\left(\partial p_D / \partial x\right)_o$. Finally, it should be noted that the possibility of a pressure-difference increase near the stagnation point implied by calculations in reference 6 suggests a small local drag decrease due to the magnetic force. In principle, a local drag decrease is not impossible since the magnetic effects are similar to frictional or throttling actions. Of course, in a transonic flow such as that around a blunt-nosed body, a local drag decrease does not mean that the total drag decreases; as a matter of fact, local considerations offer very incomplete information concerning the drag.

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National Aeronautics and Space Administration.
SOME BASIC ASPECTS OF MAGNETOHYDRODY-
NAMIC BOUNDARY-LAYER FLOWS.   Robert V.
Hess.   April 1959.   42p.
(NASA MEMORANDUM 4-9-59L)

In several recent papers on magnetohydrodynamic
stagnation flow, the meaning of the simplifying
assumptions used has been improperly interpreted.
It is shown that for the particular law of deformation
of the magnetic field postulated in some papers the
magnet would have to be situated in the flow away
from the body to take up the induced forces resisting
the flow.   Conditions for the existence of magneto-
hydrodynamic potential flow are also discussed.
Certain large differences in heat-transfer rates
found in the literature are shown to be mainly due to
different interpretations of reentry conditions rather
than due to more subtle reasons.   A simplified
approach for boundary-layer flow is developed.

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