SPECTRAL DENSITY OF LASER BEAM

SCINTILLATION IN WIND TURBULENCE:

I. THEORY*

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Abstract

The temporal spectral density of the log-amplitude scintillation of a laser beam wave due to a spatially dependent vector-valued crosswind (deterministic as well as random) is evaluated. The path weighting functions for normalized spectral moments are derived, and offer a potential new technique for estimating the wind velocity profile. The Tatarskii-Klyatskin stochastic propagation equation for the Markov turbulence model is used with the solution approximated by the Rytov method. The Taylor "frozen-in" hypothesis is assumed for the dependence of the refractive index on the wind velocity, and the Kolmogorov spectral density is used for the refractive index field.

1. Introduction

This paper is in two parts — this first part deals exclusively with theory. The second part will be devoted to computational/experimental results.

The study of the log-amplitude scintillation spectrum of a coherent beam in wind turbulence was initiated largely in the late-1960 to early-1970 period spurred by the application to "remote sensing" — see [7–15]. We are particularly interested in the case where the wind velocity varies along the propagation path — as in "wind-shear," of importance in flight systems. Most of the work involves calculating the space-time correlation function for a spherical wave or a plane wave. The "frozen-in" Taylor hypothesis is used to account for the (cross) wind, and the Kolmogorov spectral density is used for the refractive index field. Lee and Harp [8] calculate the correlation function for the spherical wave while Ishimaru [2] calculates the spectral density for a constant wind velocity, in the plane wave case. In this paper we calculate the temporal spectral density for an arbitrary spatially varying vector wind velocity for the general beam wave case.

To estimate the wind velocity profile Barakat and Buder [14], following the early work of Lawrence *et al.* [9], use the "slope method." The space-time correlation function for the spherical wave is shown to be stationary in time and space and further the time derivative at time zero is a linear functional of the magnitude of the velocity. The corresponding "path weighting function" is calculated, resulting in a linear integral equation for the wind velocity (magnitude). In this paper we show that the normalized moments of the scintillation spectral density yield a similar relationship and calculate the corresponding path weighting functions. The spectral moment of order two is of particular importance because of its relation to the zero-crossing rate (measurable real time) given by Rice's theorem [18]. We have thus a potential real time alternate technique for estimating wind velocity.

We also consider a random wind model in which the wind velocity is assumed to be a Gaussian 2D random field with given mean and covariance matrix. We calculate the spectral density as well as the spectral moments of order two and the corresponding path weighting function.

We begin with a brief review in Section 2. The necessary propagation theory is well described in references [1,2,3] and is reviewed further in [4]. We use the Markov turbulence model leading to the random parabolic Tatarkskii-Klyatskin propagation partial differential equation. We use the generally accepted approximation to the solution due to Rytov, and calculate the slow-varying component of the logarithm of the amplitude assuming the Taylor hypothesis to account for the wind. Section 3 contains the spectral density and spectral moment calculations for the deterministic case and Section 4 for the random wind model.

2. Review of Propagation Theory

The starting point of all propagation theory (as in [1,2,3]) is the Helmholtz equation for the scalar electric field over R^3 for each component E of the (complex) electric field (in the usual notation):

$$\nabla^2 E + k^2 n^2 E = 0, \quad n = c^2 \mu \varepsilon$$

where k is the wave number:

$$k = \frac{2\pi}{\text{wavelength}}$$

and $n(\cdot)$ is the refractive index. Next we choose a "propagation" direction: say, X-axis, and write

$$r = ix + jy + kz$$

$$\rho = jy + kz$$

where i, j, k are orthonormal unit vectors in 3-space. Then we let

$$E(x,\rho) = v(x,\rho)e^{ikx}$$

and substituting in the Helmholtz equation we obtain:

$$\frac{\partial^2 v}{\partial x^2} + 2ik\frac{\partial v}{\partial x} + k^2(n^2 - 1)v + \Delta v = 0$$

where

$$\Delta v = \left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) v.$$

Now we can write:

$$\frac{\partial^2 v}{\partial x^2} + 2ik\frac{\partial v}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x} + 2ikv\right).$$

We assume that v is "slow-varying" in x, so that

$$\frac{\partial v}{\partial x} + 2ikv \approx 2ikv,$$

or that the $\frac{\partial^2 v}{\partial x^2}$ term can be neglected. Then we have the "parabolic" approximation [1,3,4,17]:

$$2ik\frac{\partial v}{\partial x} + \Delta v + k^2(n^2 - 1)v = 0 \tag{2.1}$$

which we may treat as an "initial value" problem in $x \geq 0$.

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If $n = 1$, (2.1) simplifies to:

$$2ik \frac{\partial v}{\partial x} + \Delta v = 0 \qquad (2.2)$$

or

$$\frac{\partial v}{\partial x} = \frac{-1}{2ik} \Delta v = \frac{i}{2k} \Delta v, \quad x > 0. \qquad (2.3)$$

We can treat this as in [4] as an abstract Cauchy problem for $v(x, \cdot)$ over the Hilbert space $\mathcal{H} = L^2(\mathbf{R^2})$:

$$\frac{dv}{dx} = Av$$

where

$$A = \frac{i}{2k} \Delta$$

is the infinitesimal generator with

$$\mathcal{D}(A) = \mathcal{H}^2.$$

The solution to the initial value problem is then given by

$$v(x, \cdot) = S(x)v(0, \cdot)$$

where $S(\cdot)$ is the semigroup generated by A. As in [4], we can use Fourier transforms with $\mathcal{F}(\cdot)$ denoting Fourier transform for each x:

$$\mathcal{F}(v(x); \lambda) = \int_{R^2} e^{2\pi i [\lambda, \rho]} v(x, \rho)|d\rho|, \quad \lambda \in R^2.$$

By slight abuse of notation, we shall write ρ^2 for $|\rho|^2$; λ^2 for $|\lambda|^2$. Since

$$\mathcal{F}(\Delta v; \lambda) = -4\pi^2 \lambda^2 \mathcal{F}(v; \lambda)$$

we have

$$\mathcal{F}(v(x); \lambda) = \mathcal{F}(S(x)v(0, \cdot)) = e^{-4\pi^2 \lambda^2 (ix/2k)} \mathcal{F}(v(0); \lambda).$$

Hence

$$v(x, \rho) = \int_{R^2} G(x, \rho - \rho')v(0, \rho')|d\rho'|$$

where the Fourier transform of $G(x, \rho)$, for each $x \geq 0$ is given by:

$$e^{-4\pi^2 \lambda^2 (ix/2k)}. \qquad (2.4)$$

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Hence

$$G(x,\rho) \;=\; \frac{1}{2\pi\left(\frac{ix}{2k}\right)} \exp\frac{-1}{2}\frac{k}{ix}\rho^2 \;=\; \frac{k}{2\pi ix}\exp\frac{1}{2}\frac{ik\rho^2}{x}$$

and

$$v(x,\rho) \;=\; \frac{k}{2\pi ix}\int_{R^2}\exp\frac{1}{2}\frac{ik|\rho-\rho'|^2}{x}\,v(0,\rho')\,|d\rho'|. \tag{2.5}$$

For the special case where $v(0,\cdot)$ is a δ-function at the origin in R^2, we obtain

$$E(x,\rho) \;=\; \frac{ke^{(ikx+(ki\rho^2)/2x)}}{2\pi ix}\;.$$

Since constant multipliers do not matter, we may take

$$E(x,\rho) \;=\; \frac{1}{4\pi x}e^{ik(x+(\rho^2/2x))} \tag{2.6}$$

which is the result obtained by Ishimaru [2, p. 377] although by somewhat different arguments.

Beam Wave

For the beam wave case, the initial condition is (as in [2]) given by:

$$E(0,\rho) \;=\; \exp\frac{-\rho^2\alpha k}{2}\;,$$

where

$$\alpha = \left(\frac{\lambda}{\pi W^2} + \frac{i}{R_0}\right);\qquad W = \text{beam size}\,;\; R_0 = \text{focal length}\,.$$

Ishimaru [2] goes through a similar procedure as in deriving (2.6) to obtain $E(x,\cdot)$ but here we again get this directly from our parabolic approximation (2.1). Thus

$$E[x,\rho] \;=\; e^{ikx}v(x,\rho)$$

and writing $v(x) = v(x,\cdot)$,

$$v(x) = S(x)v(0); \quad v(0,\cdot) = E(0,\cdot).$$

Using Fourier transforms to evaluate (2.5), we have

$$\mathcal{F}(v(x); \lambda) = \left(\frac{2\pi}{\alpha k}\right) \exp -4\pi^2 \left(\frac{\lambda^2}{2\alpha k}\right) \exp -4\pi^2 \lambda^2 \left(\frac{ix}{2k}\right)$$

$$= \left(\frac{2\pi}{\alpha k}\right) \exp -4\pi^2 \lambda^2 \left(\frac{\sigma^2 + \frac{1}{\alpha k}}{2}\right)$$

$$= \left(\frac{2\pi}{\alpha k}\right) \exp -4\pi^2 \lambda^2 \left(\frac{\frac{ix}{k} + \frac{1}{\alpha k}}{2}\right).$$

Hence

$$v(x, \rho) = \frac{1}{1 + ix\alpha} \exp \frac{-1}{2} \left(\frac{\alpha k \rho^2}{1 + ix\alpha}\right)$$

and

$$E(x, \rho) = \left(\frac{e^{ikx}}{1 + ix\alpha}\right) \exp \frac{-1}{2} \left(\frac{\alpha k \rho^2}{1 + ix\alpha}\right) \tag{2.7}$$

as in [2, p. 381].

From (2.7) it follows as in [2, p. 381] that the beam size at x is given by

$$W \sqrt{\left(1 - \frac{x}{R_0}\right)^2 + \frac{4x^2}{k^2 W^4}}.$$

Turbulence Equation

Consider now the case $n \neq 1$, we have, rewriting (2.1)

$$\frac{\partial v}{\partial x} + \frac{\Delta v}{2ik} = \frac{-k^2(n^2 - 1)v}{2ik}$$

$$\frac{\partial v}{\partial x} = \frac{i}{2k}\Delta v + \frac{ik}{2}(n^2 - 1)v.$$

If we take

$$n = 1 + \tilde{n},$$

then

$$n^2 - 1 = (1 + \tilde{n})^2 - 1 = \tilde{n}^2 + 2\tilde{n}$$

omitting the $\tilde{n}^2$ as small compared to $2\tilde{n}$, we have:

$$n^2 - 1 = 2\tilde{n}$$

or,

$$\frac{\partial v}{\partial x} = \frac{i}{2k}\Delta v + ik\tilde{n}v \qquad (2.8)$$

where $\tilde{n}v$ denotes the function $\tilde{n}(x,\rho)v(x,\rho)$.

Exact solution of (2.8) not being possible, we invoke the two well-known approximation techniques [1,2,3]: the Born approximation and the Rytov approximation.

Born Approximation

Rewriting (2.8) as an integral equation:

$$v(x) = S(x)v(0) + ik\int_0^x S(x-\sigma)(\tilde{n}(\sigma)v(\sigma))\,d\sigma$$

and expressing the solution in a Volterra series expansion (see [4]), and truncating the series at the linear term yields the Born approximation [1,2,3]:

$$v(x) = S(x)v(0) + ik\int_0^x S(x-\sigma)[\tilde{n}(\sigma)S(\sigma)v(0)]\,d\sigma$$

or,

$$v(x,\rho) = E_0(x,\rho) + ik\int_0^x\int_{R^2} G(x-\sigma,\rho-\rho'),\tilde{n}(\sigma,\rho')E_0(\sigma,\rho')\,|d\rho'|, \quad \rho\in R^2.$$

Using the next approximation

$$\log(1+z) = z \quad \text{for } |z| \text{ small}\,,$$

we can reduce this further to yield

$$\chi(x,\rho) = \log|v(x,\rho)| = \log|E_0(x,\rho)| + \operatorname{Re}\psi_B(x,\rho)$$

where

$$\psi_B(x,\rho) = \frac{ik}{E_0(x,\rho)}\int_0^x\int_{R^2} G(x-\sigma,\rho-\rho')\tilde{n}(\sigma,\rho')E_0(\sigma,\rho')\,|d\rho'|. \qquad (2.9)$$

We now specialize to two cases: the plane wave and the beam wave.

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Case 1. Plane Wave

$$E_0(x, \rho) = e^{ikx}$$

and (2.9) yields

$$\psi_B(x, \rho) = \frac{k^2}{2\pi} \int_0^x \frac{e^{ik(\sigma-x)}}{(x-\sigma)} \, d\sigma \int_{R^2} \exp \frac{1}{2} \frac{ik|\rho - \rho'|^2}{(\sigma - x)} \, \tilde{n}(\sigma, \rho') \, |d\rho'|. \qquad (2.13a)$$

Case 2. Beam Wave

Here

$$E_0(x, \rho) = \frac{e^{ikx}}{1 + i\alpha x} \exp \frac{-1}{2} \frac{\alpha k \rho^2}{1 + i\alpha x} \, .$$

Hence (2.9) yields

$$\psi_B(x, \rho) = ik(1 + i\alpha x)e^{-ikx} \left(\exp \frac{1}{2} \frac{\alpha k \rho^2}{1 + i\alpha x} \right) \cdot \int_0^x \frac{d\sigma}{x - \sigma}$$

$$\cdot \frac{k}{2\pi i} \int_{R^2} \left(\exp \frac{1}{2} \frac{ik|\rho - \rho'|^2}{x - \sigma} \right) \tilde{n}(\sigma, \rho') \frac{e^{ik\sigma}}{1 + i\sigma\alpha} \exp \frac{-1}{2} \frac{\alpha k \rho^2}{1 + i\alpha x} \, |d\rho'|$$

which, as in (2.14), can be expressed

$$\frac{k^2}{2\pi} \int_0^x \frac{e^{8k(\sigma-x)}}{(x-\sigma)\gamma(\sigma)} \, d\sigma \int_{R^2} \exp \frac{ik}{2} \frac{|\rho' - \gamma(\rho)\rho|^2}{\gamma(\sigma)(x-\sigma)} \, \tilde{n}(\sigma, \rho') \, |d\rho'|. \qquad (2.14a)$$

Next we consider the Rytov approximation [1,2,3].

Rytov Approximation

We begin by expressing the electric field as:

$$E = E_0 e^{\psi}; \quad \psi(0) = 0; \quad E(0) = E_0(0)$$

in the Helmholtz equation, where E_0 is the solution of (2.1) for $n = 0$, or equivalently of (2.3). Then

$$(\nabla^2 + n^2 k^2)(E_0 e^{\psi}) = 0$$

yields

$$(\nabla^2 + k^2)(E_0\psi) = k^2(n^2 - 1)E_0. \tag{2.10}$$

We now let

$$E_0\psi = ve^{ikx}.$$

Then

$$(\nabla^2 + k^2)ve^{ikx} = \left(\frac{\partial^2 v}{\partial x^2} + \Delta v + 2ik\frac{\partial v}{\partial x} - k^2 v\right)e^{ikx}$$

where we invoke "the parabolic approximation" and neglect the $\frac{\partial^2 v}{\partial x^2}$ term. Then (2.10) yields

$$2ik\frac{\partial v}{\partial x} + \Delta v = -k^2(n^2 - 1)(E_0 e^{-ikx})$$

or

$$\frac{\partial v}{\partial x} = \frac{i\Delta v}{2k} + ik\tilde{n}E_0 e^{-ikx} \tag{2.11}$$

where

$$v(0) = E_0(0)\psi(0) = 0$$

since

$$\psi(0) = 0.$$

Hence, solving (2.11) we have the so-called "Rytov First Iteration Solution," [2],

$$\psi(x, \rho) = \frac{e^{ikx}ik}{E_0(x, \rho)}\int_0^x S(x - \sigma)(\tilde{n}(\sigma, \cdot)E_0(\sigma, \cdot))e^{-ik\sigma}\,d\sigma \tag{2.12}$$

where

$$\tilde{n}(\sigma, \cdot)E_0(\sigma, \cdot)$$

represents the function

$$\tilde{n}(\sigma, \rho)E_0(\sigma, \rho), \quad \rho \in R^2.$$

We specialize next to two cases.

Case 1. Plane Wave

Here we take

$$E_0(x) = e^{ikx}$$

and hence

$$\tilde{n}(\sigma, \rho)E_0(\sigma, \rho) = \tilde{n}(\sigma, \rho)e^{ik\sigma}$$

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and (2.12) yields

$$\psi(x) = ik \int_0^x S(x - \sigma)(\tilde{n}(\sigma, \cdot)) \, d\sigma$$

or,

$$\psi(x, \rho) = (ik) \int_0^x d\sigma \int_{R^2} \frac{k}{2\pi i(x - \sigma)} \exp \frac{1}{2} \frac{ik|\rho - \rho'|^2}{x - \sigma} \tilde{n}(\sigma, \rho') \, |d\rho'|$$

$$= \frac{k^2}{2\pi} \int_0^x \frac{1}{x - \sigma} \int_{R^2} \frac{1}{x - \sigma} \exp \frac{1}{2} \frac{ik|\rho - \rho'|^2}{x - \sigma} \tilde{n}(\sigma, \rho') \, |d\rho'|. \qquad (2.13)$$

This agrees with [2, (17–27a)] but we obtain it by the parabolic approximation.

Case 2. Beam Wave

For the beam wave case

$$E(0, \rho) = E_0(0, \rho) = \exp \frac{-\rho^2 \alpha k}{2}$$

and as we have seen (cf. 2.7))

$$E_0(x, \rho) = \frac{e^{ikx}}{1 + ix\alpha} \exp \frac{-1}{2} \frac{\alpha k \rho^2}{1 + ix\alpha} \ .$$

Hence

$$\psi(x, \rho) = (ik) \left(\frac{1 + ix\alpha}{e^{ikx}} \exp \frac{1}{2} \frac{\alpha k \rho^2}{1 + ix\alpha} \right)$$

$$\frac{k}{2\pi i} \int_0^x \frac{d\sigma}{x - \sigma} \int_{R^2} \left(\exp \frac{1}{2} \frac{ik|\rho - \rho'|^2}{x - \sigma} \right) \tilde{n}(\sigma, \rho') \frac{1}{1 + i\sigma\alpha} \exp \frac{-1}{2} \frac{\alpha k \rho'^2}{1 + i\sigma\alpha} |d\rho'|$$

which as Ishimaru [2] has shown can be simplified to

$$\psi(x, \rho) = \int_0^x d\sigma \frac{k^2}{2\pi \gamma(\sigma)(x - \sigma)} \int_{R^2} \exp \frac{ik}{2} \frac{|\rho' - \gamma(\sigma)\rho|^2}{\gamma(\sigma)(x - \sigma)} \tilde{n}(\sigma, \rho') \, |d\rho'| \qquad (2.14)$$

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where

$$\gamma(\sigma) = \frac{1 + i\alpha\sigma}{1 + i\alpha x} = \gamma_R(\sigma) + i\gamma_I(\sigma)$$

$$\gamma_R(\sigma) = \frac{\left(1 - \frac{\sigma}{R_0}\right)\left(1 - \frac{x}{R_0}\right) + \alpha_R^2 x\sigma}{\left(1 - \frac{x}{R_0}\right)^2 + \alpha_R^2 x^2}$$

$$\gamma_I(\sigma) = \frac{\alpha_R(\sigma - x)}{\left(1 - \frac{x}{R_0}\right)^2 + \alpha_R^2 x^2} \leq 0$$

$$\alpha_R = \frac{2}{kW^2}.$$

As is known (see [2, p. 349]) the Rytov approximation is generally accepted as superior to the Born approximation, and so we shall use only the Rytov solution in the rest of the paper.

Markov Turbulence Model

To proceed further we need to specify the stationary covariance function of the refractive index field $\tilde{n}(\cdot)$ in the propagation equation in (2.8). Here we follow the Tatarskii-Klyatskin theory [1] and invoke the "Markov turbulence model" — that it is "delta-correlated" along the propagation direction. Denoting the covariance function by $R_{\tilde{n}}(\sigma, \rho)$ we have:

$$R_{\tilde{n}}(\sigma, \rho) = C_n^2 \delta(\sigma) R(\rho) \tag{2.15}$$

where C_n^2 in the usual notation denotes the turbulence intensity. To determine the function $R(\cdot)$ we again follow [1,2] and require that

$$\int_{-\infty}^{\infty} \tilde{R}_n(\sigma, \rho) \, d\sigma = \int_{-\infty}^{\infty} R_3(\sigma, \rho) \, d\sigma$$

where $R_3(\sigma, \rho)$ is the covariance corresponding to an isotropic random field, with the Kolmogorov spectral density. This yields

$$R(\rho) = \int_{R^2} e^{2\pi i [\lambda, \rho]} Q(|\lambda|) \, |d\lambda|$$

where

$$Q(\lambda) = \frac{1}{\left(\frac{1}{L^2} + 4\pi^2\lambda^2\right)^{11/6}} \exp{-4\pi^2 \ell_0^2 \lambda^2}. \tag{2.16}$$

It is customary to allow the turbulence intensity C_n^2 to depend on the propagation direction so that $\tilde{n}(\cdot)$ is now a "locally stationary process." However usually little is known about this dependence. We shall avoid this uncertainty by restricting consideration, if necessary, to propagation path intervals small enough so that it may be assumed to be constant. Moreover the quantities of interest to us will be "normalized" as in [9] so that we do not need to know the precise value either. In what follows we shall omit this as a multiplicative constant.

3. Response to Cross Wind

We are now ready to study the response to wind. Let $v(\sigma)$, $0 < \sigma < x$, denote the "crosswind," i.e., projection of the wind velocity in the plane normal to the beam axis. Invoking the Taylor hypothesis as in [1,2,3,8] we replace $\tilde{n}(\sigma, \rho')$ in (2.14) by

$$\tilde{n}(\sigma, \ \rho' - v(\sigma)t)$$

so that the response is now a function of time t as well. Thus (2.14) becomes:

$$\psi(x, \rho, t) \ = \ \int_0^x \frac{k^2}{2\pi} \frac{d\sigma}{\gamma(\sigma)(x - \sigma)} \int_{R^2} \exp \frac{ik}{2} \frac{|\rho' - \gamma(\sigma)\rho|^2}{\gamma(\sigma)(x - \sigma)} \, \tilde{n}(\sigma, \rho' - v(\sigma)t) \, |d\rho'|, \quad (3.1)$$

$$\rho \in R^2.$$

Log Amplitude Scintillation

The log-amplitude scintillation is given by:

$$\chi(x, \rho, t) \ = \ \log |E(x, \rho, t)| \ = \ \log \left| E_0(x) e^{\psi(x,\rho,t)} \right|$$

$$= \ \log |E_0(x)| \ + \ \operatorname{Re} \psi(x, \rho, t). \quad (3.2)$$

We are only interested in the part due to the wind:

$$\operatorname{Re} \psi(x, \rho, t)$$

which we shall continue to denote

$$\chi(x, \rho, t).$$

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Thus we have for the log amplitude scintillation due to wind:

$$\chi(x, \rho, t) = \int_0^x d\sigma \int_{R^2} h_R(x - \sigma, \sigma, \rho', \rho)\tilde{n}(\sigma, \rho' - v(\sigma)t) \, |d\rho'| \qquad (3.3)$$

where

$$h_R = \mathrm{Re}\left(\frac{k^2}{2\pi} \frac{1}{\gamma(\sigma)(x - \sigma)} \exp \frac{ik}{2} \frac{|\rho' - \gamma(\sigma)\rho|^2}{\gamma(\sigma)(x - \sigma)}\right). \qquad (3.4)$$

This is our basic "scintillation response" space-time field, which we shall use to deduce the spectrum and covariance functions.

Scintillation Spectrum

From (3.3), we see that for fixed x, and each $\rho \in R^2$, $\chi(x, \rho, t)$ describes a temporally stationary Gaussian random process in $-\infty < t < \infty$. We proceed now to calculate the corresponding spectral density. For this purpose we first calculate the covariance function.

$$R_\chi(\rho_1, \rho_2, t) = E[\chi(x, \rho_1, t + s) \, \chi(x, \rho_2, s)]$$

$$= \int \int \int_0^x d\sigma \; h_R(x - \sigma, \sigma, \rho', \rho_1) h_R(x - \sigma, \sigma, \rho'', \rho_2)$$

$$\cdot R(\rho' - \rho'' - v(\sigma)t)$$

$$= \int_{R^2} \int_0^x d\sigma \left(\int_{R^2} h_R(x - \sigma, \sigma, \rho', \rho_1)e^{2\pi i[\lambda, \rho']} \, d\rho'\right)$$

$$\cdot \left(\int_{R^2} h_R(x - \sigma, \sigma, \rho'', \rho_2)e^{-2\pi i[\lambda, \rho'']} \, d\rho''\right) e^{-2\pi i[\lambda, v(\sigma)]t} Q(\lambda) \, |d\lambda| \quad (3.5)$$

$$h_R(x - \sigma, \sigma, \rho', \rho_1) = \frac{h(x - \sigma, \sigma\rho', \rho_1) + \overline{h(x - \sigma, \sigma, \rho', \rho_1)}}{2}$$

where

$$h(x - \sigma, \sigma, \rho', \rho_1) = \frac{k^2}{2\pi} \frac{1}{\gamma(\sigma)(x - \sigma)} \exp \frac{ik}{2} \frac{[\rho' - \gamma(\sigma)\rho_1, \rho' - \gamma(\sigma)\rho_1]}{\gamma(\sigma)(x - \sigma)} .$$

Now for $\mathrm{Re}\,\mu > 0$:

$$\frac{1}{2\pi\mu} \int_{R^2} e^{2\pi i[\lambda, \rho']} \exp \frac{-1}{2} \frac{[\rho' - m, \rho' - m]}{\mu} \, |d\rho'| = \exp(-2\pi^2 \lambda^2 \mu + 2\pi i[\lambda, m]). \quad (3.6)$$

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We note that:

$$\operatorname{Re}\left[i\gamma(\sigma)\right] = -\gamma_I(\sigma) > 0, \quad 0 < \sigma < x.$$

Hence

$$\int_{R^2} e^{2\pi i[\lambda,\rho']} h(x - \sigma, \sigma, \sigma', \rho_1) \, |d\rho'|$$

$$= ik \exp\frac{-4\pi^2\lambda^2 i\gamma(\sigma)(x-\sigma)}{2k} \exp 2\pi i[\lambda,\rho_1]\overline{\gamma(\sigma)}$$

$$\int_{R^2} e^{2\pi i[\lambda,\rho']} \overline{h(x - \sigma, \sigma, \rho', \rho_1)} \, |d\rho'|$$

$$= -ik \exp\frac{4\pi^2\lambda^2 i\overline{\gamma(\sigma)}(x-\sigma)}{2k} \exp 2\pi i[\lambda,\rho_1]\gamma(\sigma)$$

$$\int_{R^2} e^{-2\pi i[\lambda,\rho'']} h_R(x - \sigma, \sigma, \rho'', \rho_2) \, |d\rho''|$$

$$= \frac{1}{2}\left[ik \exp\frac{-4\pi^2\lambda^2 i\gamma(\sigma)(x-\sigma)}{2k} \exp -2\pi i[\lambda,\rho_2]\overline{\gamma(\sigma)} \right.$$

$$\left. - ik \exp\frac{4\pi^2\lambda^2 i\overline{\gamma(\sigma)}(x-\sigma)}{2k} \exp -2\pi i[\lambda,\rho_2]\gamma(\sigma) \right].$$

Hence a little calculation leads to

$$R_\chi(\rho_1,\rho_2,t) = \int_0^x d\sigma \int_{R^2} e^{-2\pi i[\lambda, v(\sigma)]t} Q_\chi(\lambda,\sigma,\rho_1,\rho_2) Q(\lambda) \, |d\lambda| \qquad (3.7)$$

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where

$$Q_\chi(\lambda, \sigma, \rho_1, \rho_2) = \frac{k^2}{4} \left(\exp 4\pi^2 \lambda^2 \gamma_I(\sigma) \frac{x - \sigma}{k} \right) \exp 2\pi i[\lambda, \rho_1 - \rho_2]\gamma_R(\sigma)$$

$$\cdot \Big[2 \cosh 2\pi[\lambda, \rho_1 + \rho_2]\gamma_I(\sigma)$$

$$- e^{4\pi^2 i \lambda^2 \gamma_R(\sigma)\frac{x-\sigma}{k}} e^{-2\pi[\lambda, \rho_1 - \rho_2]\gamma_I(\sigma)}$$

$$- e^{-4\pi^2 i \lambda^2 \gamma_R(\sigma)\frac{x-\sigma}{k}} e^{2\pi[\lambda, \rho_1 - \rho_2]\gamma_I(\sigma)} \Big] ,$$

$$= \frac{k^2}{4} \exp\left(4\pi^2 \lambda^2 \gamma_I(\sigma) \frac{x - \sigma}{k} \right) \exp(2\pi i[\lambda, \rho_1 - \rho_2]\gamma_R(\sigma))$$

$$\cdot \Big[2 \cosh\left(2\pi[\lambda, \rho_1 + \rho_2]\gamma_I(\sigma) \right)$$

$$- 2 \cosh\left(4\pi^2 i \lambda^2 \frac{\gamma_R(x - \sigma)}{k} - 2\pi[\lambda, \rho_1 - \rho_2]\gamma_I(\sigma) \right) \Big] ,$$

$$0 \le \sigma \le x. \tag{3.8}$$

From (3.7) we see that the scintillation intensity at ρ:

$$R_\chi(\rho, \rho, 0) = E[(\chi(x, \rho, t))^2]$$

does *not* depend on the wind velocity. Defining

$$Q_\chi(\lambda, \sigma, \rho) = Q_\chi(\lambda, \sigma, \rho, \rho)$$

we have:

$$Q_\chi(\lambda, \sigma, \rho) = \frac{k^2}{4} \left(\exp 4\pi^2 \lambda^2 \gamma_I(\sigma) \frac{x - \sigma}{k} \right)$$

$$\cdot \left[2 \cosh 4\pi[\lambda, \rho]\gamma_I(\sigma) - 2 \cos 4\pi^2 \lambda^2 \gamma_R(\sigma) \frac{x - \sigma}{k} \right]. \tag{3.9}$$

The cross-spectral density $P(\rho_1, \rho_2, f)$ is defined by the Fourier transform:

$$P(\rho_1, \rho_2, f) = \int_{-\infty}^{\infty} e^{-2\pi i f t} R_\chi(\rho_1, \rho_2, t) \, dt.$$

Thus the spectral density matrix of the 2×1 scintillation process at any two points ρ_1, ρ_2:

$$\left| \begin{array}{c} \operatorname{Re} \psi(x, \rho_1, t) \\ \operatorname{Re} \psi(x, \rho_2, t) \end{array} \right| , \quad \rho_1, \rho_2 \in R^2,$$

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for fixed x is given by

$$\begin{vmatrix} P(\rho_1, \rho_1, f) & P(\rho_1, \rho_2, f) \\ P(\rho_2, \rho_1, f) & P(\rho_2, \rho_2, f) \end{vmatrix}.$$

We shall now proceed to calculate this matrix function, which is the main contribution of this paper. First we shall specialize to the plane wave case to compare with known results.

Spectral Density: Plane Wave Case

As we have indicated, the plane wave case is obtained by setting

$$\gamma_I(\sigma) = 0; \quad \gamma_R(\sigma) = \gamma(\sigma) = 1.$$

Substituting into (3.8) we have:

$$Q_\chi(\lambda, \sigma, \rho_1, \rho_2) = \frac{k^2}{2}\left[1 - \cos 4\pi^2\lambda^2\frac{x-\sigma}{k}\right]\exp 2\pi i[\lambda, \rho_1 - \rho_2] \qquad (3.10)$$

and in turn we have:

$$\begin{aligned} R_\chi(\rho_1, \rho_2, t) &= \int_0^x d\sigma \int_{R^2} e^{-2\pi i[\lambda, v(\sigma)]t} \frac{k^2}{2} e^{2\pi i[\lambda, \rho_1 - \rho_2]} \\ &\qquad \cdot \left[1 - \cos 4\pi^2\lambda^2\frac{(x-\sigma)}{k}\right] Q(\lambda) |d\lambda| \\ &= \int_0^x d\sigma \int_0^\infty (2\pi)\left(\frac{k^2}{2}\right)\lambda J_0\left(2\pi\lambda|v(\sigma)t + (\rho_2 - \rho_1)|\right) \\ &\qquad \cdot \left[1 - \cos 4\pi^2\lambda^2\frac{(x-\sigma)}{k}\right] Q(\lambda)\, d\lambda. \qquad (3.11) \end{aligned}$$

This depends only on the difference in position $(\rho_1 - \rho_2)$ and

$$\chi(x, \rho, t)$$

is thus both spatially and temporally stationary. This function is noticeably different from that for the case of the spherical wave derived by Lawrence *et al.* in [9].

Specializing to $\rho_1 = \rho_2 = \rho$, we have:

$$R_\chi(\rho, \rho, t) = \int_0^x d\sigma \; e^{2\pi i[\lambda, v(\sigma)]t} \int_{R^2} \frac{k^2}{2} \left[1 - \cos 4\pi^2 \lambda^2 \frac{x - \sigma}{k} \right] Q(\lambda) \; |d\lambda|$$

$$= \int_0^x d\sigma \; (2\pi) \left(\frac{k^2}{2} \right) \int_0^\infty \lambda J_0(2\pi \lambda |v(\sigma)t + \rho_1 - \rho_2|)$$

$$\cdot \left[1 - \cos \left(\frac{4\pi^2 \lambda^2 (x - \sigma)}{k} \right) \right] Q(\lambda) \; d\lambda. \qquad (3.12)$$

Note that the covariance function does not depend on ρ; and depends only on the magnitude of the wind velocity $|v(\sigma)|$.

We shall denote corresponding spectral density $P(f, \rho)$ by $P(f)$:

$$R_\chi(\rho, \rho, t) = \int_{-\infty}^\infty e^{2\pi i f t} P(f) \; df.$$

Then

$$P(f) = \int_0^x \frac{1}{|v(\sigma)|} P_1 \left(\sigma, \frac{|f|}{|v(\sigma)|} \right) \; d\sigma, \qquad (3.13)$$

where for $f > 0$,

$$P_1(\sigma, f) = \frac{k^2}{2} \int_f^\infty \frac{\lambda}{\sqrt{\lambda^2 - f^2}} \left(1 - \cos 4\pi^2 \lambda^2 \frac{x - \sigma}{k} \right) Q(\lambda) \; d\lambda. \qquad (3.14)$$

If the velocity is a constant:

$$|v(\sigma)| = v$$

the spectral density formula (3.14) simplifies, since we can perform the integration with respect to σ. In fact

$$P(f) = \frac{1}{v} P_1 \left(\frac{f}{v} \right)$$

where

$$P_1(f) = \frac{k^2 x}{2} \int_f^\infty \frac{\lambda Q(\lambda)}{\sqrt{\lambda^2 - f^2}} \left(1 - \frac{\sin 4\pi^2 \lambda^2 \left(\frac{x}{k} \right)}{4\pi^2 \lambda^2 \left(\frac{x}{k} \right)} \right) \; d\lambda, \qquad (3.15)$$

agreeing with the result obtained by Ishimaru [2], who also obtains therefrom the estimate:

$$P_1(f) \sim f^{-8/3} \quad \text{for large } f.$$

Cross-Spectral Density: Plane Wave

Let us next consider the cross-spectral density. We have:

$$P(f, \rho_1, \rho_2) = \int_0^x P(\sigma, f, \rho_1, \rho_2) \, d\sigma$$

where

$$P(\sigma, f, \rho_1, \rho_2) = \int_{-\infty}^{\infty} e^{-2\pi i f t} R_\chi(\sigma, \rho_1, \rho_2, t) \, dt$$

is the "distributed" spectral density, where

$$R_\chi(\sigma, \rho_1, \rho_2, t) = \frac{2\pi k^2}{2} \int_0^\infty \lambda J_0(2\pi \lambda |v(\sigma)t + \rho|)$$

$$\cdot \left(1 - \cos 4\pi^2 \lambda^2 \frac{x - \sigma}{k} \right) Q(\lambda) \, d\lambda$$

where

$$\rho = \rho_1 - \rho_2.$$

By the Bessel function summation formula:

$$
\begin{aligned}
J_0(2\pi \lambda |v(\sigma)t + \rho|) &= J_0(2\pi \lambda \, |v(\sigma)| \, t) J_0(2\pi \lambda |\rho|) \\
&\quad + 2 \sum_1^\infty J_m(2\pi \lambda \, |v(\sigma)| \, t) J_m(2\pi \lambda |\rho|) \cos(m\pi + \phi(\sigma)) \quad t > 0 \\
&= J_0(2\pi \lambda \, |v(\sigma)| \, t) J_0(2\pi \lambda |\rho|) \\
&\quad + 2 \sum_1^\infty J_m(2\pi \lambda \, |v(\sigma)| \, t) J_m(2\pi \lambda |\rho|) \cos m\phi(\sigma),
\end{aligned}
$$

where the angle $\phi(\sigma)$ is defined by

$$[v(\sigma), \rho] = |v(\sigma)| \, |\rho| \cos \phi(\sigma).$$

We see that the cross-spectral density depends now on the angle $\phi(\sigma)$, although we can write

$$P(\sigma, f, \rho_1, \rho_2) = \frac{1}{|v(\sigma))|} P_1 \left(\sigma, \frac{f}{|v(\sigma)|}, \rho_1, \rho_2 \right)$$

where the subscript "1" again stands for the case $|v(\sigma)| = 1$, and we note that

$$\int_{-\infty}^{\infty} J_0(2\pi\lambda\,|t+\rho|)e^{-2\pi ift}\,dt \;=\; \int_{-\infty}^{\infty} J_0(2\pi\lambda t)e^{-2\pi ift}\,dt\; J_0(2\pi\lambda|\rho|)$$

$$+\,4i\sum_{1}^{\infty}\int_{0}^{\infty}\cos 2\pi ft\, J_{2m}(2\pi\lambda t)\,dt\; J_{2m}(2\pi\lambda|\rho|)$$

$$-\,4i\sum_{1}^{\infty}\int_{0}^{\infty}\sin 2\pi ft\, J_{2m-1}(2\pi\lambda t)\,dt\; J_{2m-1}(2\pi\lambda|\rho|).$$

Now by [6, p. 405], for λ, $f > 0$:

$$\int_{0}^{\infty}\cos 2\pi ft\, J_{2m}(2\pi\lambda t)\,dt \;=\; 0, \qquad\qquad\qquad \lambda < f$$

$$=\; \frac{\cos\left(2m\arcsin\left(\frac{f}{\lambda}\right)\right)}{2\pi\sqrt{\lambda^2 - f^2}}, \qquad \lambda > f \quad (3.16)$$

$$\int_{0}^{\infty}\sin 2\pi ft\, J_{2m-1}(2\pi\lambda t)\,dt \;=\; 0, \qquad\qquad\qquad \lambda < f$$

$$=\; \frac{\sin\left((2m-1)\arcsin\left(\frac{f}{\lambda}\right)\right)}{2\pi\sqrt{\lambda^2 - f^2}}, \quad \lambda > f. \quad (3.17)$$

Hence

$$P_1(\sigma, f, \rho_1, \rho_2) \;=\; \int_{f}^{\infty}\frac{2}{\sqrt{\lambda^2 - f^2}}J_0(2\pi\lambda|\rho|)\lambda Q_2(\lambda,\sigma)\,d\lambda$$

$$+\,4i\sum_{1}^{\infty}\cos(2m\phi(\sigma))\int_{f}^{\infty}\frac{\cos\left(2m\arcsin\left(\frac{f}{\lambda}\right)\right)}{\sqrt{\lambda^2 - f^2}}$$

$$\cdot\; J_{2m}(2\pi\lambda|\rho|)\lambda Q_2(\lambda,\sigma)\,d\lambda$$

$$-\,4i\sum_{1}^{\infty}\cos((2m-1)\phi(\sigma))\int_{f}^{\infty}\frac{\sin\left((2m-1)\arcsin\left(\frac{f}{\lambda}\right)\right)}{\sqrt{\lambda^2 - f^2}}$$

$$\cdot\; J_{2m-1}(2\pi\lambda|\rho|)\lambda Q_2(\lambda,\sigma)\,d\lambda \qquad (3.18)$$

where

$$Q_2(\lambda,\sigma) \;=\; \frac{k^2}{2}\left[1 - \cos\left(4\pi^2\lambda^2\,\frac{x-\sigma}{k}\right)\right]Q(\lambda).$$

This result would appear to be new with this paper. We note that

$$P_1(\sigma, f, \rho_2, \rho_1) \;=\; \overline{P_1(\sigma, f, \rho_1, \rho_2)}\,. \qquad (3.19)$$

Spectral Density: General Beam Wave Case

Let us now consider the general beam wave case. We shall calculate the spectral density of the scintillation at the point ρ. Let

$$R(\rho, t) = R_\chi(\rho, \rho, t).$$

Then

$$R(\rho, t) = \int_0^x d\sigma \; R(\sigma, \rho, t) \; d\sigma$$

where

$$R(\sigma, \rho, t) = \int_{R^2} e^{-2\pi i [\lambda, \, v(\sigma)] t} Q_\chi(\lambda, \sigma, \rho) Q(\lambda) \; |d\lambda|. \tag{3.20}$$

Since

$$Q_\chi(\lambda, \sigma, \rho) > 0,$$

we see that $R(\sigma, \rho, t)$ is a covariance function in t, $-\infty < t < \infty$. Also

$$R(\rho, 0) = \int_0^x d\sigma \int_{R^2} Q_\chi(\lambda, \sigma, \rho) Q(\lambda) \; |d\lambda| \tag{3.21}$$

and as a function of ρ, depends only on $|\rho|$, and is a minimum at $\rho = 0$, since

$$\cosh 4\pi^2 [\lambda, \rho] \gamma_I(\sigma) \geq 1.$$

As noted earlier (3.21) does *not* depend on the velocity $v(\cdot)$. Let $\phi(\sigma)$ denote the angle between the vector ρ and $v(\sigma)$, so that:

$$[v(\sigma), \rho] = |v(\sigma)| \, |\rho| \cos \phi(\sigma).$$

Then

$$\begin{aligned}
R(\sigma, \rho, t) = \int_0^\infty \int_0^{2\pi} & e^{-2\pi i |\lambda| |v(\sigma)| t \cos\theta} \lambda Q_1(\lambda, \sigma) \\
& \cdot \Big[\cosh(4\pi^2 |\lambda| \, |\rho| \cos(\theta - \phi(\sigma)) \gamma_I(\sigma)) \\
& \qquad - \cos\left(4\pi^2 \lambda^2 \gamma_R(\sigma) \frac{x - \sigma}{k} \right) \Big] \; d\lambda \; d\theta
\end{aligned} \tag{3.22}$$

where we have used the notation for short:

$$Q_1(\lambda, \sigma) = \frac{k^2}{2} \, Q(\lambda) \exp\left(4\pi^2 \lambda^2 \gamma_I(\sigma) \frac{x - \sigma}{k} \right). \tag{3.23}$$

Now we recall the relation [6, p. 358]

$$\frac{1}{2\pi} \int_0^{2\pi} e^{ia\cos\theta} e^{b\cos(\theta-\phi)}\, d\theta \;=\; J_0(a)J_0\left(\frac{b}{i}\right) \;+\; 2\sum_1^\infty J_k(a)J_k\left(\frac{b}{i}\right)\cos k\phi. \qquad (3.24)$$

Letting

$$a = 2\pi|\lambda|\,|\boldsymbol{v}(\sigma)|\,t$$
$$b = 4\pi|\lambda|\,|\rho|\,\gamma_I(\sigma)$$

we have, using (3.24), that

$$\int_0^{2\pi} e^{-2\pi i|\lambda|\,|\boldsymbol{v}(\sigma)|t\cos\theta}\cosh\left(4\pi^2|\lambda|\,|\rho|\gamma_I(\sigma)\cos(\theta-\phi(\sigma))\right)\,d\theta$$

$$= \; 2\pi\left[J_0(a)I_0(b) \;+\; 2\sum_1^\infty (-1)^k J_{2k}(a)I_{2k}(b)\cos 2k\phi(\sigma)\right] \qquad (3.25)$$

$$= \; 2\pi J_0(a) \qquad\qquad\qquad \text{for } \rho = 0. \qquad (3.26)$$

Also

$$R(\sigma,0,t) \;=\; 4\pi\int_0^\infty J_0(2\pi|\lambda|\,|\boldsymbol{v}(\sigma)|t)\lambda Q_1(\lambda,\sigma)$$

$$\cdot \left(1 - \cos\left(4\pi^2\lambda^2\gamma_R(\sigma)\frac{x-\sigma}{k}\right)\right)\,d\sigma. \qquad (3.27)$$

Since

$$Q_\chi(\lambda,\sigma,\rho) - Q_\chi(\lambda,\sigma,0) \;\geq\; 0,$$

we see that

$$R(\sigma,\rho,t) - R(\sigma,0,t) \;=\; \int_{R^2} e^{-2\pi i[\lambda,\,\boldsymbol{v}(\sigma)]t}\left(Q_\chi(\lambda,\sigma,\rho) - Q_\chi(\lambda,\sigma,0)\right)\,|d\lambda|$$

is a covariance function in t, $-\infty < t < \infty$; and further, using (3.25), (3.26),

$$= \; 2\pi\int_0^\infty \lambda Q_1(\lambda,\sigma)\left[J_0(a)(I_0(b)-1) \;+\; 2\sum_1^\infty (-1)^k J_{2k}(a)I_{2k}(b)\cos 2k\phi\right]\,d\lambda. \qquad (3.28)$$

Here we may decompose $R(\sigma,\rho,t)$ as:

$$R(\sigma,\rho,t) \;=\; R(\sigma,0,t) + \tilde{R}(\sigma,\rho,t)$$

where $\tilde{R}(\sigma,\rho,t)$ is given by (3.28). Correspondingly we decompose the spectral density

$$P(f,\rho) = \int_0^x P(\sigma,f,\rho)\,d\sigma, \quad -\infty < f < \infty,$$

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setting

$$P(\sigma, f, \rho) = P(\sigma, f, 0) + \tilde{P}(\sigma, f, \rho)$$

$$P(\sigma, f, 0) = 2 \int_0^\infty R(\sigma, 0, t) \cos 2\pi f t \, dt$$

$$\tilde{P}(\sigma, f, \rho) = 2 \int_0^\infty \tilde{R}(\sigma, \rho, t) \cos 2\pi f t \, dt.$$

Now

$$P(\sigma, f, 0) = \frac{1}{|\boldsymbol{v}(\sigma)|} P_1\left(\sigma, \frac{f}{|\boldsymbol{v}(\sigma)|}, 0\right) \qquad (3.29)$$

where the subscript "1" corresponds to the special case

$$|\boldsymbol{v}(\sigma)| = 1.$$

Using (3.16) we have for $0 \le f$:

$$P_1(\sigma, f, 0) = \int_f^\infty \frac{\lambda Q_1(\lambda, \sigma)}{\sqrt{\lambda^2 - f^2}} \left(1 - \cos\left(4\pi^2\lambda^2\gamma_R(\sigma)\frac{x-\sigma}{k}\right)\right) \, d\lambda.$$

Similarly we have

$$\tilde{P}(\sigma, f, \rho) = \frac{1}{|\boldsymbol{v}(\sigma)|} \tilde{P}_1\left(\sigma, \frac{f}{|\boldsymbol{v}(\sigma)|}, \rho\right)$$

where

$$\tilde{P}_1(\sigma, f, \rho) = \int_f^\infty \frac{\lambda Q_1(\lambda, \sigma)}{\sqrt{\lambda^2 - f^2}}$$

$$\cdot \left[(I_0(b) - 1) + 2\sum_1^\infty (-1)^k \cos\left(2k \arcsin\left(\frac{f}{\lambda}\right)\right) I_{2k}(b) \cos 2k\phi(\sigma)\right] \, d\lambda$$

where

$$b = 4\pi^2\lambda|\rho|\gamma_I(\sigma).$$

Hence finally we have:

$$P(\sigma, f, \rho) = \frac{1}{|\boldsymbol{v}(\sigma)|} P_1\left(\sigma, \frac{f}{|\boldsymbol{v}(\sigma)|}, \rho\right)$$

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and for $f > 0$:

$$P_1(\sigma, f, \rho) = \int_f^\infty \frac{\lambda Q_1(\lambda, \sigma)}{\sqrt{\lambda^2 - f^2}}$$

$$\cdot \left(I_0(b) + 2\sum_1^\infty (-1)^k \cos\left(2k \arcsin\left(\frac{f}{\lambda}\right)\right) I_{2k}(b) \cos 2k\phi(\sigma) \right.$$

$$\left. - \cos\left(4\pi^2 \lambda^2 \gamma_R(\sigma) \frac{x - \sigma}{k}\right)\right) d\lambda. \qquad (3.30)$$

For $\rho = 0$:

$$P_1(f, 0) = \int_f^\infty \frac{k^2}{2} \frac{\lambda Q(\lambda)}{\sqrt{\lambda^2 - f^2}}$$

$$\cdot \int_0^x \left[1 - \cos\left(\frac{4\pi^2 \lambda^2 \gamma_R(\sigma)(x - \sigma)}{k}\right)\right] \exp\left(\frac{4\pi^2 \lambda^2}{k} \gamma_I(\sigma)(x - \sigma)\right) d\sigma \quad (3.31)$$

If we specialize to the plane wave case,

$$P_1(\sigma, f, \rho) = P_1(\sigma, f, 0)$$

and in particular (3.31) simplifies of course to (3.15). Put another way, (3.31) is the generalization of (3.15) to the beam wave case.

Spectral Moments/Path Weighting Function

As we have noted, the primary impetus in studying scintillation was in "remote sensing," or estimating, the wind velocity. Thus Lee and Harp [8] noted that the time derivative at time zero of the space time cross-covariance function could be expressed as a linear function of the wind velocity. They were working with the spherical wave model but we can see this as well from (3.11) derived for the plane wave. Thus we can calculate that

$$\frac{d}{dt} R_\chi(\rho_1, \rho_2, t)\Big|_{t=0} = \int_0^x |v(\sigma)| W(\sigma, \rho) d\sigma \qquad (3.11a)$$

where

$$W(\sigma, \rho) = \int_0^\infty 4\pi^2 \left(\frac{k^2}{2}\right) \lambda^2 J_1(2\pi\lambda|\rho|) \left(1 - \cos 4\pi^2 \lambda^2 \frac{(x - \sigma)}{k}\right) Q(\lambda) d\lambda$$

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and note that it does not depend on the wind velocity. Thus it serves as a "path weighting function."

One disadvantage of this technique involves taking the slope of the cross correlation function which is an off-line operation, and further the path-weighting function is equal to zero when $\rho = 0$.

We shall show that it is possible instead to work with the spectral moments.

Given any spectral density $P(\cdot)$ (of any real-valued process), we may define the "spectral moment of order n," by

$$\overline{f^n} = \frac{\int_0^\infty f^n P(f) \, df}{\int_0^\infty P(f) \, df} \, , \tag{3.32}$$

yielding a measure of the spectral spread. We can express even-order spectral moments in terms of derivatives of $R(\cdot)$, the corresponding covariance function. Thus for $n = 2$, we have

$$4\pi^2 \overline{f^2} = \frac{-R''(0)}{R(0)} \, . \tag{3.33}$$

On the other hand we should note that all odd derivatives of $R(\cdot)$ vanish at the origin.

Here we shall calculate all the spectral moments directly from (3.30). Thus we have:

$$\overline{f^n(\rho)} = \frac{\int_0^x |v(\sigma)|^n a_n(\sigma, \rho) \, d\sigma}{\int_0^x a_0(\sigma, \rho) \, d\sigma} \tag{3.34}$$

where

$$a_n(\sigma, \rho) = \int_0^\infty f^n P_1(\sigma, f, \rho) \, df$$

$$= \left(\int_0^{\pi/2} (\sin \theta)^n \, d\theta \right)$$

$$\cdot \int_0^\infty \lambda^{n+1} Q_1(\lambda, \sigma) \left(I_0(b) - \cos \left(4\pi^2 \lambda^2 \gamma_R(\sigma) \frac{x - \sigma}{k} \right) \right) \, d\lambda$$

$$+ 2 \sum_1^\infty (-1)^k \left(\int_0^{\pi/2} (\sin \theta)^n \cos 2k\theta \, d\theta \right)$$

$$\cdot \int_0^\infty \lambda^{n+1} Q_1(\lambda, \sigma) \left(I_{2k}(b) \cos 2k\phi(\sigma) \right) \, d\lambda. \tag{3.35}$$

In particular:

$$a_0(\sigma, \rho) = \frac{\pi}{2} \int_0^\infty \lambda Q_1(\lambda, \sigma) \left(I_0(b) - \cos\left(4\pi^2 \lambda^2 \gamma_R(\sigma) \frac{x - \sigma}{k} \right) \right) d\lambda$$

$$a_1(\sigma, \rho) = \int_0^\infty \lambda^2 Q_1(\lambda, \sigma) \left(I_0(b) - \cos\left(4\pi^2 \lambda^2 \gamma_R(\sigma) \frac{x - \sigma}{k} \right) \right) d\lambda$$

$$a_2(\sigma, \rho) = \frac{\pi}{4} \int_0^\infty \lambda^3 Q_1(\lambda, \sigma) \left(I_0(b) + I_2(b) \cos 2\phi(\sigma) \right.$$
$$\left. - \cos\left(4\pi^2 \lambda^2 \gamma_R(\sigma) \frac{x - \sigma}{k} \right) \right) d\lambda.$$

We may express (3.32) as

$$\overline{f^n}(\rho) = \int_0^\infty |v(\sigma)|^n W_n(\sigma, \rho) \, d\sigma \qquad (3.36)$$

where $W_n(\sigma, \rho)$ is the "path weighting function," and is given by

$$W_n(\sigma, \rho) = \frac{a_n(\sigma, \rho)}{\int_0^x a_0(\sigma, \rho) \, d\sigma}. \qquad (3.37)$$

We note that for the general beam wave case the path weighting functions depend on the angle $\phi(\sigma)$ for nonzero ρ. However this dependence disappears when we specialize to the case of the plane wave:

$$a_n(\sigma, \rho) = \frac{k^2}{2} \left(\int_0^{\pi/2} (\sin\theta)^n \, d\theta \right) \int_0^\infty \lambda^{n+1} Q(\lambda) \left(1 - \cos 4\pi^2 \lambda^2 \frac{x - \sigma}{k} \right) d\lambda \qquad (3.38)$$

and in particular:

$$\int_0^x a_0(\sigma, \rho) \, d\sigma = \frac{\pi k^2}{4} \int_0^\infty \lambda Q(\lambda) \left(\int_0^x \left(1 - \cos 4\pi^2 \lambda^2 \frac{\sigma}{k} \right) d\sigma \right) d\lambda$$

$$= \frac{\pi k^2 x}{4} \int_0^\infty \lambda Q(\lambda) \left(1 - \frac{\sin 4\pi^2 \lambda^2 \frac{x}{k}}{4\pi^2 \lambda^2 \frac{x}{k}} \right) d\lambda.$$

Thus the path weighting functions do not depend on ρ, for the plane wave case.

Zero Crossing Rate

The spectral moment of order two is of particular importance because of Rice's theorem [18] relating it to the zero-crossing rate of the process:

$$\text{Average zero crossing rate} = 2\sqrt{\overline{f^2}}. \qquad (3.39)$$

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Thus yields, in our notation:

$$N_0(\rho)^2 = 4 \int_0^x |\boldsymbol{v}(\sigma)|^2 W_2(\sigma, \rho) \, d\sigma \qquad (3.40)$$

where $N_0(\rho)$ is the average zero-crossing rate at the position ρ. The zero-crossing rate can be readily measured on-line by analog instrumentation. Thus (3.40) offers a postential alternate technique for measuring wind velocity.

4. Random Wind Model

In this section we model the wind velocity vector field $\boldsymbol{v}(\sigma)$, $0 < \sigma < x$, as random — and more specifically, as Gaussian distributed with mean $m(\sigma)$ and nonsingular covariance matrix $R(\sigma)$. We seek to calculate the average spectral density:

$$\overline{P}(f, \rho) = E[P(f, \rho)].$$

Rather than use (3.30) we calculate

$$E[R(\rho, t)]$$

where by (3.20)

$$R(\rho, t) = \int_0^x d\sigma \int_{R^2} e^{-2\pi i [\lambda, \boldsymbol{v}(\sigma)] t} Q_\chi(\lambda, \sigma, \rho) Q(\lambda) \, |d\lambda|. \qquad (4.1)$$

Hence

$$E[R(\rho, t)] = \int_0^x d\sigma \int_{R^2} E\left[e^{-2\pi i [\lambda, \boldsymbol{v}(\sigma)] t}\right] Q_\chi(\lambda, \sigma, \rho) Q(\lambda) \, |d\lambda| \qquad (4.2)$$

and we see that only the first-order distributions of $v(\cdot)$ are involved. Now

$$E[e^{-2\pi i [\lambda, \boldsymbol{v}(\sigma)] t}] = \exp(-2\pi^2 [R(\sigma)\lambda, \lambda] t^2 + 2\pi i [\lambda, m(\sigma)] t) \qquad (4.3)$$

and hence

$$\int_{-\infty}^\infty e^{2\pi i f t} E[e^{-2\pi i [\lambda, \boldsymbol{v}(\sigma)] t}] \, dt$$

$$= \frac{1}{\sqrt{2\pi} \sqrt{[R(\sigma)\lambda, \lambda]}} \exp \frac{-1}{2} \frac{(f - [\lambda, m(\sigma)])^2}{[R(\sigma)\lambda, \lambda]} . \qquad (4.4)$$

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Hence

$$\overline{P}(f,\rho) = \int_0^x d\sigma \int_{R^2} \frac{1}{\sqrt{2\pi}\sqrt{[R(\sigma)\lambda,\lambda]}} Q_\chi(\lambda,\sigma,\rho)Q(\lambda)$$

$$\cdot \exp \frac{-1}{2} \frac{(f-[\lambda,m(\sigma)])^2}{[R(\sigma)\lambda,\lambda]} |d\lambda|. \tag{4.5}$$

Diagonal Covariance

While the mean $m(\sigma)$ can be arbitrary, the covariance matrix model may be reasonably simplified to be diagonal:

$$R(\sigma) = d^2(\sigma)I_{2\times 2}. \tag{4.6}$$

In this case we can derive a useful alternate form for (4.5). Thus

$$E[R(\rho,t)] = \int_0^x d\sigma \int_{R^2} Q_\chi(\lambda,\sigma,\rho)Q(\lambda)$$

$$\cdot \exp[2\pi i[\lambda,m(\sigma)]t - 2\pi^2\lambda^2 d^2(\sigma)t^2] |d\lambda|$$

$$= 2\pi \int_0^x d\sigma \int_0^\infty \lambda Q_1(\lambda,\sigma) \left(\exp -2\pi^2\lambda^2 d^2(\sigma)t^2\right)$$

$$\cdot \left[J_0(2\pi\lambda|m(\sigma)|\,|t|)I_0(4\pi\lambda|\rho|\gamma_I(\sigma)) \right.$$

$$+ 2\sum_1^\infty (-1)^k J_{2k}(2\pi\lambda|m(\sigma)|\,|t|)$$

$$\cdot I_{2k}(4\pi\lambda|\rho|\gamma_I(\sigma))\cos 2k\phi(\sigma)$$

$$\left. - \cos 4\pi^2\lambda^2\gamma_R(\sigma)\frac{x-\sigma}{k} \right] d\lambda, \tag{4.7}$$

where $\phi(\sigma)$ is the angle between ρ and $m(\sigma)$. Hence, for $|m(\sigma)| \neq 0$,

$$\int_{-\infty}^\infty e^{-2\pi i f t} \left(\exp -2\pi^2 d^2(\sigma)\lambda^2 t^2\right) J_{2k}(2\pi\lambda|m(\sigma)|\,|t|)\,dt$$

$$= \frac{1}{2\pi}\frac{1}{\lambda}\int_0^{\lambda|m(\sigma)|} \frac{1}{\sqrt{2\pi}d(\sigma)} \left(\exp\frac{-1}{2}\frac{(f-\nu)^2}{\lambda^2 d^2(\sigma)} + \exp\frac{-1}{2}\frac{(f+\nu)^2}{\lambda^2 d^2(\sigma)}\right)$$

$$\cdot \frac{\cos\left(2k\arcsin\left(\frac{\nu}{\lambda|m(\sigma)|}\right)\right)}{\sqrt{\lambda^2|m(\sigma)|^2 - \nu^2}}\,d\nu.$$

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Hence

$$
\overline{P}(f, \rho) = \int_0^x d\sigma \int_0^\infty d\lambda \cdot \lambda Q_1(\lambda, \sigma)
$$

$$
\cdot \left[I_0(4\pi\lambda|\rho|\gamma_I(\sigma)) + 2\sum_1^\infty (-1)^k I_{2k}(4\pi\lambda|\rho|\gamma_I(\sigma)) \right.
$$

$$
\cdot \cos\left(2k \arcsin\left(\frac{\nu}{\lambda|m(\sigma)|} \right) \right) \cos 2k\phi(\sigma)
$$

$$
\left. - \cos 4\pi^2 \lambda^2 \gamma_R(\sigma) \frac{x-\sigma}{k} \right]
$$

$$
\left(\int_0^{|m(\sigma)|\lambda} \frac{1}{\sqrt{2\pi}} \frac{1}{\lambda d(\sigma)} \left(\exp \frac{-1}{2} \frac{(f-\nu)^2}{\lambda^2 d^2(\sigma)} + \exp \frac{-1}{2} \frac{(f+\nu)^2}{\lambda^2 d^2(\sigma)} \right) \right.
$$

$$
\left. \cdot \frac{1}{\sqrt{\lambda^2|m(\sigma)|^2 - \nu^2}} \, d\nu \right). \qquad (4.8)
$$

We note that as $d(\sigma) \to 0$, the last factor in parentheses:

$$
\to \frac{1}{\sqrt{\lambda^2|m(\sigma)|^2 - f^2}} \quad \text{for} \quad f < \lambda|m(\sigma)|
$$

$$
= 0 \qquad \text{for} \quad f > \lambda|m(\sigma)|
$$

and thus (4.8) agrees with (3.30) for $d(\sigma) = 0$. We may interpret the deterministic case as mean wind of variance zero.

Case of Zero Mean Wind

On the other hand, one advantage of the random wind model is that we can consider the case where the mean wind is zero, unlike the deterministic case. Thus let

$$
m(\sigma) = 0, \quad 0 < \sigma < x
$$

and further consider a coaxial detector:

$$
\rho = 0.
$$

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Then (4.5) yields:

$$\overline{P}(f,0) = \int_0^x d\sigma \int_0^\infty \frac{\sqrt{2\pi}}{d(\sigma)} Q_1(\lambda,\sigma) \left(1 - \cos\left(4\pi^2\lambda^2\gamma_R(\sigma)\frac{x-\sigma}{k}\right)\right)$$

$$\cdot \exp\left(\frac{-1}{2}\frac{f^2}{\lambda^2 d^2(\sigma)}\right) d\lambda. \qquad (4.9)$$

This has no analog in the deterministic case. It is interesting to note that the spectral density is now a monotone decreasing function of the frequency. We can simplify further if we take wind-covariance to be constant:

$$d(\sigma) = d, \quad 0 < \sigma < x.$$

Specializing to the plane wave case, (4.9) simplifies to:

$$\overline{P}(f,0) = \frac{k^2 x}{2} \int_0^\infty \frac{\sqrt{2\pi}}{d} \exp\left(\frac{-1}{2}\frac{f^2}{\lambda^2 d^2}\right)\left(1 - \frac{\sin 4\pi^2\lambda^2\left(\frac{x}{k}\right)}{4\pi^2\lambda^2\left(\frac{x}{k}\right)}\right) Q(\lambda)\, d\lambda. \quad (4.10)$$

If further as in [9] we invoke the "inertial subrange approximation" for the Kolmogorov spectral density:

$$Q(\lambda) \sim \frac{1}{\left(\frac{1}{L^2} + 4\pi^2\lambda^2\right)^{11/6}} \sim \frac{1}{(2\pi\lambda)^{11/3}}$$

we can simplify (4.10) further to:

$$\overline{P}(f,0) \approx \frac{k^2 x}{2} \int_0^\infty \frac{\sqrt{2\pi}}{d} \frac{1}{(2\pi\lambda)^{11/3}} \exp\left(\frac{-f^2}{\lambda^2 d^2}\right) d\lambda$$

and evaluating the integral on the right yields:

$$\overline{P}(f,0) \approx \frac{\sqrt{2\pi}}{2d} k^2 x \frac{1}{(2\pi)^{11/3}} \left(\frac{f}{d}\right)^{-8/3} \left(\int_0^\infty \lambda^{-11/3}\exp\frac{-1}{2\lambda^2}\, d\lambda\right). \qquad (4.11)$$

Note the similarity to the estimate in the deterministic case (cf. (3.15)).

Spectral Moments/Path Weighting Functions

We shall consider only the most useful case: the spectral moment for $n = 2$ and the corresponding weighting function. Thus using (4.2) we have:

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$$2 \int_0^\infty f^2 \overline{P}(f, \rho) \, df$$

$$= \int_0^x d\sigma \int_{R^2} E([\lambda, v(\sigma)]^2) Q_\chi(\lambda, \sigma, \rho) Q(\lambda) \, |d\lambda|$$

$$= \int_0^x d\sigma \int_{R^2} \left(|[\lambda, m(\sigma)]|^2 + [R(\sigma)\lambda, \lambda] \right) Q_\chi(\lambda, \sigma, \rho) Q(\lambda) \, |d\lambda|.$$

For the term containing $m(\sigma)$ we have only to replace $v(\sigma)$ by $m(\sigma)$ in (3.33) for $n = 2$. For the second term, using (3.24), we have that

$$\int_{R^2} [R(\sigma)\lambda, \lambda] Q_\chi(\lambda, \sigma, \rho) Q(\lambda) \, |d\lambda|$$

$$= 2\pi d^2(\sigma) \int_0^\infty \lambda^3 Q_1(\lambda, \sigma) \left[I_0(4\pi\lambda|\rho|\gamma_I(\sigma)) - \cos\left(4\pi^2\lambda^2\gamma_R(\sigma)\frac{x-\sigma}{k}\right) \right].$$

Now

$$2 \int_0^\infty \overline{P}(f, \rho) \, df = E[R(\rho, 0)]$$

and by (4.7) yields:

$$E[R(\rho, 0)] = 2\pi \int_0^x d\sigma \int_0^\infty \lambda Q_1(\lambda, \sigma)$$

$$\cdot \left[I_0(b) - \cos\left(4\pi^2\lambda^2\gamma_R(\sigma)\frac{x-\sigma}{k}\right) \right] d\lambda.$$

Hence

$$\overline{f^2(\rho)} = \frac{\int_0^\infty f^2 \overline{P}(f, \rho) \, df}{\int_0^\infty \overline{P}(f, \rho) \, df}$$

can be expressed as

$$= \overline{f_m^2(\rho)} + \overline{f_0^2(\rho)}$$

where

$$\overline{f_m^2(\rho)} = \int_0^x |m(\sigma)|^2 W_2(\sigma, \rho) \, d\sigma$$

where $W_2(\sigma, \rho)$ is given by (3.36). And

$$\overline{f_0^2(\rho)} = \int_0^x (d^2(\sigma)) \, d\sigma \, W_2(\sigma, \rho) \, d\sigma.$$

In other words the spectral moment of order two:

$$\overline{f^2(\rho)}$$

for the random wind is the sum of two terms, the first corresponding to a deterministic wind velocity equal to the mean wind velocity and the second corresponding to a determinstic wind velocity equal to the "sigma" of the random wind:

$$d(\sigma) \;=\; \sqrt{d^2(\sigma)}\,.$$

The path weighting function of course does not depend on the statistics of the wind velocity. Note that the variance term represents an additive error term. This "additive" property does not extend to higher order spectral moments.

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