IN-39

401504

VIBRATING SYSTEMS WITH SINGULAR MASS-INERTIA MATRICES†

A. V. Balakrishnan*
Flight Systems Research Center
UCLA

VIBRATING SYSTEMS WITH
SINGULAR MASS-INERTIA MATRICES

A. V. Balakrishnan*
Flight Systems Research Center
UCLA

Abstract

Vibrating systems with singular mass-inertia matrices arise in recent continuum models of Smart Structures (beams with PZT strips) in assessing the damping attainable with rate feedback. While they do not quite yield "distributed" controls, we show that they can provide a fixed nonzero lower bound for the damping coefficient at all mode frequencies. The mathematical machinery for modelling the motion involves the theory of Semigroups of Operators. We consider a Timoshenko model for torsion only — a "smart string," where the damping coefficient turns out to be a constant at all frequencies. We also observe that the damping increases initially with the feedback gain but decreases to zero eventually as the gain increases without limit.

1. Introduction

We consider a class of vibrating systems with singular mass-inertia matrices. This would appear to be a characteristic feature of continuum models of smart structures [1-4] — beams with self-straining material. In this paper in order to keep the exposition simple and illustrate the nature of the results, we limit ourselves to a Timoshenko model for torsional motion only — a "smart string" — with rate feedback. We show that the damping coefficient is a constant at all frequencies. Also the damping increases as the gain increases initially but decreases to zero eventually as the gain increases without limit. The description of the motion dynamics involves the use of theory of Semigroups of Operators.

2. Timoshenko Model: Torsion with Rate Feedback

We consider a simplified Timoshenko "smart" beam confined to torsion mode only — clamped at one end and self-straining material along the entire length with rate feedback. The continuum model dynamics are then described by (see [4]):

$$\left.\begin{array}{rcl} \rho\ddot{\theta}(t,s) - c\theta''(t,s) &=& 0, \quad 0 < s < L;\ 0 < t \\ \theta(t,L) &=& 0 \\ c\theta'(t,0) + \alpha\dot{\theta}(t,0) &=& 0 \end{array}\right\} \tag{2.1}$$

Here $\theta(t,s)$ is the torsion angle and ρ the inertia parameter and c the stiffness parameter; and $\alpha > 0$ is the feedback gain. The dots indicate derivative with respect to time and the primes the derivative with respect to s, the position along the beam, L being the beam length. We may consider this as a "singular" version of the "regular" problem where we do have a nonzero "tip-inertia" — with the boundary equations

$$m\ddot{\theta}(t,0) + c\theta'(t,0) + \alpha\dot{\theta}(t,0) = 0$$

considered for example in the original version of the SCOLE [5]. We have

$$m = 0$$

in the present case. To underscore the "singular" feature, let us consider first how the problem is treated for the regular case:

$$m \neq 0.$$

As in [5], we begin with the "state" space:

$$\mathcal{H} = L_2[0,L] \times E^1$$

1

including in other words the boundary value in the state. We use the notation:

$$x = \left| \begin{array}{c} f(\cdot) \\ b \end{array} \right|, \quad f(\cdot) \in L_2[0, L]; \ b \in E^1.$$

We define next the stiffness operator A by

$$\mathcal{D}(A) = \left[x = \left| \begin{array}{c} f(\cdot) \\ b \end{array} \right|, \ f'(\cdot), f''(\cdot) \in L_2[0, L]; \ b = f(0); \ f(L) = 0 \right]$$

and

$$Ax = \left| \begin{array}{c} -cf''(\cdot) \\ -cf'(0) \end{array} \right|.$$

Thus defined A is self-adjoint and nonnegative definite and

$$[Ax, x] = c \int_0^L |f'(s)|^2 \, ds = \text{Potential Energy}.$$

We can therefore define the positive square root, $\sqrt{A}$, and we note that

$$\mathcal{D}\left(\sqrt{A}\right) = \left[x = \left| \begin{array}{c} f \\ f(0) \end{array} \right|, \ f'(\cdot) \in L_2[0, L]; \ f(L) = 0 \right].$$

Define the mass-inertia operator M by

$$Mx = \left| \begin{array}{c} \rho f(\cdot) \\ mb \end{array} \right|$$

and note that M is nonsingular, m being nonzero. Define the control operator B on E^1 into $\mathcal{H}$ by

$$Bu = \left| \begin{array}{c} 0 \\ u \end{array} \right|.$$

Then the equations (2.1) go over into the abstract version:

$$M\ddot{x}(t) + Ax(t) + \alpha BB^*\dot{x}(t) = 0. \tag{2.2}$$

Note that this corresponds to the "collocated" case, with rate feedback. To proceed further we need to construct the energy space

$$\mathcal{H}_E = \mathcal{D}\left(\sqrt{A}\right) \times \mathcal{H}$$

in which the inner product is the energy inner product:

$$Y = \left| \begin{array}{c} x_1 \\ x_2 \end{array} \right|; \qquad Z = \left| \begin{array}{c} z_1 \\ z_2 \end{array} \right|$$

$$[Y, Z]_E = \left[\sqrt{A}\, x_1, \ \sqrt{A}\, z_1 \right] + [M x_2, \ z_2] \tag{2.3}$$

$$\sim \ \text{Potential Energy} + \text{Kinetic Energy} .$$

Defining

$$\mathcal{A} = \left| \begin{array}{cc} 0 & I \\ -M^{-1}A & -\alpha M^{-1} B B^* \end{array} \right| \tag{2.4}$$

$$\mathcal{D}(\mathcal{A}) = \left[Y = \left| \begin{array}{c} \theta_1(\cdot) \\ \theta_1(0) \\ \theta_2(\cdot) \\ \theta_2(0) \end{array} \right|, \quad \begin{array}{l} y_1 = \left| \begin{array}{c} \theta_1(\cdot) \\ \theta_1(0) \end{array} \right| \in \mathcal{D}(A) \\[2em] y_2 = \left| \begin{array}{c} \theta_2(\cdot) \\ \theta_2(0) \end{array} \right| \in \mathcal{D}\left(\sqrt{A} \right) \end{array} \right]$$

$$\mathcal{A}Y = \left| \begin{array}{c} \theta_2(\cdot) \\ \theta_2(0) \\ \frac{c}{\rho}\theta_1''(\cdot) \\ \frac{c}{m}\theta_1'(0) - \frac{\alpha}{m}\theta_2(0) \end{array} \right| \tag{2.5}$$

we see that

$$\text{Re}\,[\mathcal{A}Y, \ Y]_E = -\alpha \|B^* y_2\|^2 = -\alpha |\theta_2(0)|^2$$

and hence $\mathcal{A}$ is dissipative and generates a contraction c_0-semigroup:

$$\|Y(t)\|_E \leq \|Y(0)\|_E .$$

We can show that

$$\|Y(t)\|_E \to 0 \quad \text{as } t \to \infty$$

but the semigroup is *not* exponentially damped. The resolvent $\mathcal{R}(\lambda, \mathcal{A})$ is compact. The eigenalvues of $\mathcal{A}$ are the roots of

$$D(\lambda) = (\alpha + m\lambda)\,\text{Sinh}\,\lambda\nu L + c\nu\,\text{Cosh}\,\lambda\nu L = 0; \quad \nu^2 = \frac{\rho}{c} . \tag{2.6}$$

We note that $D(\lambda)$ is an entire function of order one and completely regular growth [6] and hence has a countable number of eigenvalues. Denoting them by $\{\lambda_k\}$ we have

$$\lambda_k = -|\sigma_k| \pm i\omega_k; \quad |\omega_k| \uparrow \infty$$

3

and [5,6]:

$$\sum_1^\infty |\sigma_k| = \frac{\alpha}{m}$$

$$|\sigma_k| \to 0, \quad \text{not monotonic.}$$

Thus the damping due to feedback goes to zero as mode frequency increases. The damping increases as α increases and as m decreases.

Singular Case

With this background, let us now get back to our problem where

$$m = 0.$$

We see that (2.5) no longer is defined, since m is zero. M is singular and hence we can no longer use (2.3) as an inner product. Hence we need to change the definition of the energy space $\mathcal{H}_E$. We note that the total energy is in this case given by

$$c \int_0^L |\theta'(t,s)|^2 \, ds + \rho \int_0^L |\dot\theta(t,s)|^2 \, ds.$$

Hence we define now

$$\mathcal{H}_E = \mathcal{D}\left(\sqrt{A}\right) \times L_2[0,L]$$

$$Y = \begin{vmatrix} f_1(\cdot) \\ f_1(0) \\ f_2(\cdot) \end{vmatrix}, \qquad
\begin{aligned} y_1 &= \begin{vmatrix} f_1(\cdot) \\ f_1(0) \end{vmatrix} \in \mathcal{D}\left(\sqrt{A}\right) \\ y_2 &= f_2(\cdot) \in L_2[0,L] \end{aligned}$$

$$[Y,Y]_E = \left[\sqrt{A}\, y_1, \sqrt{A}\, y_1\right] + \rho[y_2, y_2].$$

Define $\mathcal{A}$ by:

$$\mathcal{D}(\mathcal{A}) = \begin{vmatrix} Y = \begin{vmatrix} f_1(\cdot) \\ f_1(0) \\ f_2(\cdot) \end{vmatrix}, & \begin{aligned} y_1 &= \begin{vmatrix} f_1(\cdot) \\ f_1(0) \end{vmatrix} \in \mathcal{D}(A) \\ &\begin{vmatrix} f_2(\cdot) \\ \frac{c}{\alpha} f_1'(0) \end{vmatrix} \in \mathcal{D}\left(\sqrt{A}\right) \end{aligned} \end{vmatrix}$$

$$\mathcal{A}Y = \begin{vmatrix} f_2(\cdot) \\ \frac{c}{\alpha} f_1'(0) \\ \frac{c}{\rho} f_1''(\cdot) \end{vmatrix}.$$

4

We can then verify that

$$\text{Re}\,[\mathcal{A}Y,\,Y] \;=\; \frac{-c^2}{\alpha}\,|f_1'(0)|^2 \;\leq\; 0.$$

Hence $\mathcal{A}$ is dissipative, generates a c_0-semigroup which is a contraction, and (2.1) goes over into:

$$\dot{Y}(t) \;=\; \mathcal{A}Y(t)$$

and

$$\|Y(t)\|_E \;\leq\; \|Y(0)\|_E.$$

The resolvent $\mathcal{R}(\lambda,\,A)$ is again compact. The eigenvalues are the roots of

$$D(\lambda) \;=\; c\nu\,\text{Cosh}\,\lambda\nu L \;+\; \alpha\,\text{Sinh}\,\lambda\nu L \;=\; 0 \tag{2.7}$$

where we note that $D(\lambda)$ is an entire function of order one and of completely regular growth [6] with a countable number of eigenvalues $\{\lambda_k\}$:

$$\lambda_k \;=\; -|\sigma_k| \pm i\omega_k, \qquad |\omega_k| \uparrow \infty.$$

In fact $|\sigma_k|$ is given by

$$|\sigma_k| \;=\; \frac{1}{2L\nu}\,\text{Log}\,\left|\frac{\alpha + c\nu}{\alpha - c\nu}\right| \;=\; |\sigma| \tag{2.8}$$

and

$$\begin{aligned}
\lambda_k \;&=\; \frac{-|\sigma|}{L\nu} \pm \frac{i(2k+1)}{L\nu}\,\frac{\pi}{2}, \qquad \alpha < c\nu \\[2mm]
&=\; \frac{-|\sigma|}{L\nu} \pm \frac{ik\pi}{L\nu}, \qquad \alpha > c\nu.
\end{aligned}$$

The eigenfunctions are

$$\theta_k \;=\; A_k\,\text{Sinh}\,\lambda_k\nu(L-s), \qquad 0 < s < L$$

$$\Phi_k \;=\; A_k \left|\begin{array}{c} \theta_k(\cdot) \\ \theta_k(0) \\ \lambda_k\theta_k(\cdot) \end{array}\right|$$

$$\mathcal{A}\Phi_k \;=\; \lambda_k\Phi_k$$

$$\Psi_k \;=\; A_k \left|\begin{array}{c} \overline{\theta_k}(\cdot) \\ \overline{\theta_k}(0) \\ -\overline{\lambda_k}\,\overline{\theta_k}(\cdot) \end{array}\right|$$

5

$$\mathcal{A}^* \Psi_k = \overline{\lambda_k} \, \Psi_k.$$

We note that $\{\Phi_k, \Psi_k\}$ yield a Riesz basis for *all* values of α, and in turn we have a bi-orthogonal modal expansion for any Y in $\mathcal{H}_E$:

$$Y = \sum_k \frac{[Y, \Psi_k]}{[\Phi_k, \Psi_k]} \, \Phi_k.$$

The semigroup generated by $\mathcal{A}$ is exponentially damped:

$$\|Y(t)\|_E \leq \|Y(0)\|_E \, e^{-|\sigma|t}.$$

Finally we note the dependence of the damping coefficient on the gain α. We have that the damping increases as α increases from zero to $c\nu$. At $\alpha = c\nu$, we have

$$|\sigma| = \infty$$

and for $\alpha > c\nu$, $|\sigma|$ decreases as α increases. The function is plotted in Figure 1. The apparent paradox of decreasing damping can be explained by noting that as $\alpha \to \infty$, we are in the neighborhood of $\alpha = \infty$ and here the modes are determined by the roots of

$$\text{Sinh}\,\lambda\nu L = 0$$

and for large α we may write:

$$\text{Sinh}\,\lambda\nu L + \frac{c\nu}{\alpha}\,\text{Cosh}\,\lambda\nu L = 0$$

and consider $\frac{1}{\alpha}$ in place of α. In practice, the attainable gain is such that

$$\frac{\alpha}{c\nu} \ll 1.$$

These results extend to the more general anisotropic Timoshenko models considered in [7] — the damping coefficients are no longer constant but tend to a nonzero limit as the mode frequency increases. Details of this analysis will appear elsewhere.
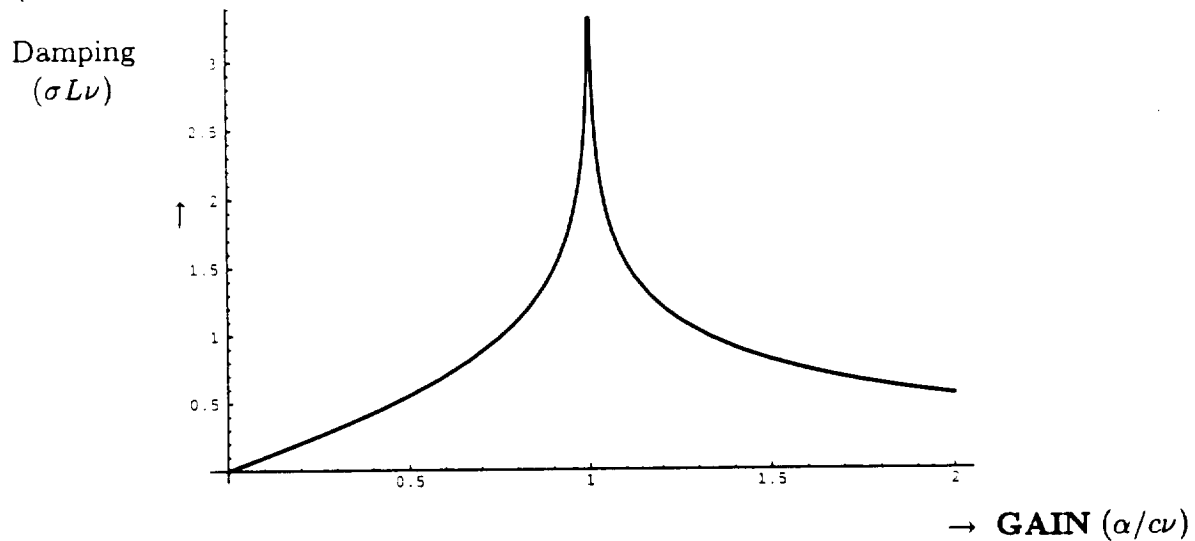
Damping vs. Gain

Damping
($\sigma L\nu$)

$\rightarrow$ **GAIN** ($\alpha/c\nu$)

Figure 1

References

1.   T. Bailey and J.E. Hubbard, Jr. "Distributed Piezoelectric-Polymer Active Vibration Control of a Cantilever Beam," *J. Guidance*, vol. 8, no. 5 (Sept-Oct 1985), pp. 605–611.

2.   H.S. Tzou and M. Gadre. "Theoretical Analysis of a Multi-layered Thin Shell Coupled with Piezoelectric Shell Actuators for Distributed Vibration Controls," *Journal of Sound and Vibration*, vol. 132, no. 3 (1989), pp. 433–450.

3.   S. Im and S.N. Atluri. "Effects of a Piezo-Actuator on a Finitely Deformed Beam Subjected to a General Loading," *AIAA Journal*, vol. 27, no. 12 (Dec 1989), pp. 1801–1807.

4.   C.-K. Lee, W.-W. Chiang and T.C. O'Sullivan. "Piezoelectric Modal Sensor/Acutator Pairs for Critical Active Damping Vibration Control," *J. Acoust. Soc. Am.*, vol. 90, no. 1 (July 1991), pp. 374-384.

5.   A.V. Balakrishnan. "Compensator Design for Stability Enhancement with Collocated Controllers," *IEEE Trans. on Automatic Control*, vol. 36, no. 9 (Sept 1991), pp. 994–1007.

6.   B.Ja. Levin. *Distribution of Zeros of Entire Functions*. Revised edition. Translations of Mathematical Monographs, Vol. 5. Providence, Rhode Island: American Mathematical Society, 1980.

7.   A.V. Balakrishnan. "Dynamics and Control of Articulated Anisotropic Timoshenko Beams." In: *Dynamics and Control of Distributed Systems*. Edited by H.S. Tzou and L.A. Bergman. Cambridge University Press, 1996.