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TECHNICAL REPORT

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ABSTRACT

The paper illustrates the use of continuum models in control design for stabilizing flexible structures. A 6-DOF anisotropic Timoshenko beam with discrete nodes where lumped masses or actuators are located provides a sufficiently rich model to be of interest for mathematical theory as well as practical application. We develop concepts and tools to help answer engineering questions without having to resort to ad hoc heuristic ("physical") arguments or faith. In this sense the paper is more mathematically oriented than engineering papers and vice versa at the same time. For instance we make precise time-domain solutions using the theory of semigroups of operators rather than formal "inverse Laplace transforms." We show that the modes arise as eigenvalues of the generator of the semigroup, which are then related to the eigenvalues of the stiffness operator. With the feedback control, the modes are no longer orthogonal and the question naturally arises as to whether there is still a modal expansion. Here we prove that the eigenfunctions yield a biorthogonal Riesz basis and indicate the corresponding expansion. We prove mathematically that the number of eigenvalues is nonfinite, based on the theory of zeros of entire functions. We make precise the notion of asymptotic modes and indicate how to calculate them. Although limited by space, we do consider the root locus problem and show for instance that the damping at first increases as the control gain increases but starts to decrease at a critical value, and goes to zero as the gain increases without bound. The undamped oscillatory modes remain oscillatory and the rigid-body modes go over into deadbeat modes.

The Timoshenko model dynamics are translated into a canonical wave equation in a Hilbert space. The solution is shown to require the use of an "energy" norm which is no more than the total energy: potential plus kinetic. We show that, under an appropriate extension of the notion of controllability, rate feedback with a collocated sensor can stabilize the structure in the sense that all modes are damped and the energy decays to zero. An example, non-numeric, is worked out in some detail illustrating the concepts and theory developed.

1. INTRODUCTION

The purpose of the paper is to illustrate the use of continuum models in control design for flexible structures: to provide the tools necessary to address relevant engineering issues. This is admittedly hazardous on two counts: on the one hand the complicated 3D geometry of realistic structures makes it almost impossible to use continuum models; while on the other hand the mathematics of continuum models is, for the most part, of mathematical interest only, and reduction to engineering practice is seldom undertaken. The alternative, universally the rule now, is to stay with the finite-dimensional "Finite Element" models. However the latter has the drawback that any control design is limited to specific numerical values of system parameters, and the dimensions, can be prohibitively large.

The 6-DOF anisotropic 1-D Timoshenko beam model is a convenient compromise from both sides. As shown in [Noor and Anderson 1979, Noor and Russell 1986, Wang 1994, and Balakrishnan 1992] it is excellent for modeling lattice trusses. On the other hand the mathematical theory strikes a good balance between the trivial and the nontractable.

The purpose of the control is to enhance the stability of the system, and the main interest in the theory centers on the modes and the damping attainable — the eigenvalue problem. A purely formal Laplace transform analysis can yield an entire function whose zeros in the complex plane are the eigenvalues. But the main difficulty is in determining the time-domain solution and the nature of the stability. Here is where it becomes necessary to use the theory of semigroups of operators and associated techniques from abstract (functional) analysis. In particular the system with control is no longer self-adjoint — the mode-shapes (eigenfunctions) are no longer orthogonal — and the problem of modal expansion of the solution must needs draw substantially on non-self-adjoint operator theory. A standard reference for the latter is [Gohberg-Krein 1969].

While some background in Hilbert space and linear operator theory is assumed (and almost all that is needed is covered in [Achieser-Glassman 1966, Riesz-Nagy 1955, Balakrishnan 1981]) every effort has been made to make the treatment self-contained.

We begin in Section 2 by translating the basic beam equations into a canonical abstract "vibration" or "wave" equation in a Hilbert space. The main feature of the choice of state is the inclusion of the displacements at the discrete nodes (the "boundary" points) in it — as pioneered by the author for the Bernoulli beam model of SCOLE in [Balakrishnan 1991a]. Not unlike the FEM version, the "boundary" conditions for the elastic equations are so chosen (in particular making the stiffness operator self-adjoint and nonnegative definite) as to yield the correct form for the potential energy.

Section 3 deals with spectral analysis. We show that the undamped structure modes are the zeros of an entire function and that the mode shapes yield an orthogonal basis for the space. We characterize the rigid-body modes showing that they span a six-dimensional space. We develop the Green's function for the eigenvalue problem. We show the relation of the square root of the stiffness operator to the potential energy of the beam.

Section 4 treats the time domain solutions of the structure dynamics, including "weak" and "strong" solutions. We introduce the notion of the "energy" inner product and show how the theory of semigroups of operators applies and the relation of the resolvent to the familiar Laplace transform.

In Section 5 we show that, under an appropriate generalization of the controllability condition to infinite dimensions, rate feedback using a collocated sensor can stabilize the structure in the sense that all modes are damped and the elastic energy decays to zero. We show that the damping coefficient goes to zero as the mode number increases without bound.

Section 6 is devoted to calculating the asymptotic modes of the structure with rate feedback. We construct the 6×6 matrix which defines the mode shapes, the modes being the zeros of the determinant. We show that the latter is an entire function of exponential type. We show that there are deadbeat modes, equal in number exactly to the (dimension) number of rigid-body modes. The oscillatory modes of the undamped structure remain oscillatory regardless of the control gain. Because of space limitation we do not go into detail on the root locus problem. Making precise the notion of "asymptotic modes," we show that the asymptotic modes are the "clamped" modes where all nodes are clamped — no displacement is allowed.

Section 7 deals with modal expansion. Since with feedback control the modes are no longer orthogonal we have to use a "biorthogonal" system. We show that the eigenfunctions do provide a Riesz basis and develop a "modal" expansion.

Finally, in Section 8, we include a non-numerical example, albeit simple, to illustrate the concepts and theory developed in the paper.

2. THE ANISOTROPIC TIMOSHENKO BEAM: DYNAMICS AND STATE SPACE FORMULATION

We begin by describing the dynamics of a 6-DOF–1D Timoshenko beam articulated with lumped masses, and/or control actuators at a finite number of "nodes" distributed along the beam, including the end points where the lumped masses may also be "offset." This is a natural extension of the familiar 1DOF–1D Timoshenko beam models found in standard texts (e.g., [Timoshenko 1974], [Meirovich 1967]).

The model does not include the inherent damping since we are concerned only with the damping attainable with control — we refer to this as the "undamped" structure.

Next we choose an appropriate Hilbert space as the state space and define the mass-inertia operator, the stiffness operator and the control operator — extending the notions familiar in finite dimensions. In particular the definition of the stiffness operator is based on the (elastic) protential energy of the structure. With these definitions the partial differential equations translate into a "vibration" or "wave" equation in a Hilbert space which needs to be interpreted appropriately but provides the canonical model for the rest of the paper.

The anisotroic Timoshenko beam model adopted here appears to have been introduced by [Noor and Anderson 1979], and refined in [Noor and Russell 1986], to model the lattice truss strctures used for deployment in space. Later [Wang 1994] showed how such models could be derived starting from the general elastic solid equations using homogenization theory. Details of deriving the beam equations and the elastic and mass-inertia parameters therein illustrated by a specific example can be found in [Balakrishnan 1992], including a comparison with FEM for calculating the step response. To minimize complexity we only consider the case of a single flexible beam. An example of a multibeam structure modelled as interconnected Timoshenko beams is given in [Balakrishnan 1991b].

Let (x_1, x_2, x_3) denote the coordinates of a rectangular coordinate system, and let the beam axis be the x_1-axis. We shall use "s" to denote the position along the beam: $0 \leq s \leq L$, where L is the length of the beam. We consider the "uniform" case where the beam properties do not depend on s. Let s_i denote discrete points (referred to as "nodes") along the axis

$$s_i < s_{i+1}, \quad i = 2, \ldots, m-1$$

$$s_1 = 0; \quad s_m = L \ .$$

Between nodes, that is to say for $s_i < s < s_{i+1}$, we have the basic anisotropic Timoshenko beam equations governing the beam displacements, $u(\cdot), v(\cdot), w(\cdot)$:

$u(\cdot)$	axial displacement (or, x_1-component)
$v(\cdot)$	bending displacement in the x_1-x_2 plane (or, x_2-component)
$w(\cdot)$	bending displacement in the x_1-x_3 plane (or, x_3-component)

and the torsion angles $\phi_1(\cdot), \phi_2(\cdot), \phi_3(\cdot)$:

$\phi_1(\cdot)$	rotation angle about the x_1-axis
$\phi_2(\cdot)$	rotation angle about the x_2-axis
$\phi_3(\cdot)$	rotation angle about the x_3-axis

are given by

$$\left.\begin{array}{c} m_{11}\ddot{u} - c_{11}u'' - c_{14}v'' - c_{15}w'' - c_{15}\phi_2' + c_{14}\phi_3' = 0 \\ m_{22}\ddot{v} - c_{44}v'' - c_{14}u'' - c_{45}\phi_2' - c_{45}w'' = 0 \\ m_{33}\ddot{w} - c_{55}w'' - c_{15}u'' - c_{45}\phi_2' - c_{45}v'' = 0 \\ m_{44}\ddot{\phi}_1 - c_{66}\phi_1'' - c_{36}\phi_2'' - c_{26}\phi_3'' = 0 \\ m_{55}\ddot{\phi}_2 + m_{56}\ddot{\phi}_3 + c_{15}u' + c_{55}w' - c_{36}\phi_1'' + c_{55}\phi_2 - c_{33}\phi_2'' - c_{23}\phi_3'' - c_{45}\phi_3 = 0 \\ m_{66}\ddot{\phi}_3 + m_{56}\ddot{\phi}_2 - c_{14}u' + c_{44}v' - c_{26}\phi_1'' - c_{23}\phi_2'' + c_{44}\phi_3 - c_{22}\phi_3'' - c_{45}\phi_2 = 0 \end{array}\right\} \tag{2.1}$$

where the superdots denote time derivatives and the primes, the space derivatives (with respect to s). The matrices

$$C_1 = \begin{vmatrix} c_{11} & c_{14} & c_{15} \\ c_{14} & c_{44} & c_{45} \\ c_{15} & c_{45} & c_{55} \end{vmatrix}; \quad C_3 = \begin{vmatrix} c_{66} & c_{36} & c_{26} \\ c_{36} & c_{33} & c_{23} \\ c_{26} & c_{23} & c_{22} \end{vmatrix}$$

are both strictly positive definite. We shall also use the notation:

$$C_2 = \begin{vmatrix} 0 & -c_{15} & c_{14} \\ 0 & -c_{45} & c_{44} \\ 0 & -c_{55} & c_{45} \end{vmatrix}; \quad C_4 = \begin{vmatrix} 0 & 0 & 0 \\ 0 & c_{55} & -c_{45} \\ 0 & -c_{45} & c_{44} \end{vmatrix}.$$

With f denoting the 6×1 (column) vector:

$$f = \begin{vmatrix} u \\ v \\ w \\ \phi_1 \\ \phi_2 \\ \phi_3 \end{vmatrix}, \tag{2.2}$$

these equations can be conveniently rewritten in the vector form:

$$M_0 \ddot{f} - A_2 f'' + A_1 f' + A_0 f = 0, \quad s_i < s < s_{i+1} \tag{2.3}$$

where M_0 is the mass/inertia matrix

$$M_0 = \begin{vmatrix} m_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & m_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & m_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & m_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & m_{55} & m_{56} \\ 0 & 0 & 0 & 0 & m_{56} & m_{66} \end{vmatrix}$$

where M is also required to be strictly positive definite;

$$A_2 = \begin{vmatrix} C_1 & 0 \\ 0 & C_3 \end{vmatrix}$$

$$A_1 = \begin{vmatrix} 0 & C_2 \\ -C_2* & 0 \end{vmatrix}$$

$$A_0 = \begin{vmatrix} 0 & 0 \\ 0 & C_4 \end{vmatrix}$$

and where f stands for

$$f(t, s), \quad 0 < t, \ 0 < s < L.$$

Here the m_{ij} are the "mass inertia" coefficients and $\{c_{ij}\}$ the "flexibility" coefficients. They are constrained by the positive definite requirements of C_1 and C_3; in particular $c_{ii} > 0$, $m_{ii} > 0$ for all i.

The nodes $\{s_i\}$ are points where the slope — spatial derivative $f'(\cdot)$ — is discontinuous because of controllers or lumped masses located there. The displacements at the nodes thus have to be included as part of the definition of the state. Let $L_2(0, L)^6$ denote the L_2-space of 6×1 vector functions $f(\cdot)$ and let $\mathcal{H}$ denote the Hilbert Space

$$\mathcal{H} = L_2(0, L)^6 \times (R^6)^m$$

where we use the notation $(R^6)^m$ rather than R^{6m} to indicate 6×1 vectors replicated m times. For elements x in $\mathcal{H}$ we shall use the notation

$$x = \begin{vmatrix} f \\ b \end{vmatrix}$$

where $f \in L_2(0, L)^6$ and $b \in (R^6)^m$. To avoid possible confusion let us note explicitly that the inner product in $\mathcal{H}$ is given by

$$[x, y] = [f, g]_{L_2(0,L)^6} + \sum_1^m [b_i, c_i]_{R^6} \tag{2.4}$$

where

$$x = \begin{vmatrix} f \\ b \end{vmatrix}; \quad b = \begin{vmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{vmatrix}$$

$$y = \begin{vmatrix} g \\ c \end{vmatrix}; \quad c = \begin{vmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{vmatrix}.$$

(To avoid excessive notation we may often delete the signature under the inner products in the sequel if they are clear from the context.) The elements of $\mathcal{H}$ will be our "states." We begin with the stiffness operator since the mass operator will depend on the control masses, the stiffness operator defined as the "differential" operator A with domains, denoted $\mathcal{D}(A)$, in $\mathcal{H}$ given by:

$$\mathcal{D}(A) = \Bigg[x = \begin{vmatrix} f \\ b \end{vmatrix} \quad \text{where defining } f_i(s) = f(s),\ s_i < s < s_{i+1},\ i = 1,\ldots,m-1$$
$$f_i(\cdot), f_i'(\cdot) \text{ and } f_i''(\cdot) \in L_2(s_i, s_{i+1})^6$$
$$\lim_{s \uparrow s_{i+1}} f_i(s) = \lim_{s \downarrow s_{i+1}} f_{i+1}(s),\ i = 1,\ldots,m-1 \Bigg]$$

$$b = \begin{vmatrix} f(s_1) \\ \vdots \\ f(s_m) \end{vmatrix}.$$

In other words the functions $f(\cdot)$ in $\mathcal{D}(\mathcal{A})$ are "piecewise smooth"; continuous in $0 \leq s \leq L$, but the derivative can have a jump discontinuity at each node s_i. We refer to b as the "boundary value." The operator A is defined by:

$$Ax = y; \quad x = \begin{vmatrix} f \\ b \end{vmatrix}; \quad y = \begin{vmatrix} g \\ c \end{vmatrix} \tag{2.5}$$

where

$$g(s) = -A_2 f''(s) + A_1 f'(s) + A_0 f(s), \quad s_i < s < s_{i+1},\ i = 1,\ldots,m-1. \tag{2.6}$$

and c is defined by:

$$c = \begin{vmatrix} -L_1 f(0+) - A_2 f'(0+) \\ -A_2(f'(s_2+) - f'(s_2-)) \\ \vdots \\ -A_2(f'(s_{m-1}+) - f'(s_{m-1}-)) \\ L_1 f(L-) + A_2 f'(L-) \end{vmatrix}. \tag{2.7}$$

where

$$L_1 = \begin{vmatrix} 0 & -C_2 \\ 0 & 0 \end{vmatrix}.$$

We shall find it convenient to use the notation:

$$c = A_b f.$$

This definition is made so that we get the right expression for the potential energy — so that

$$\frac{[Ax, x]}{2} = \text{Potential Energy} \tag{2.8}$$

where the potential energy of the structure is defined by

$$\int_0^L \left[C_1 \left| \begin{array}{c} u' \\ v' - \phi_3 \\ w' + \phi_2 \end{array} \right|, \left| \begin{array}{c} u' \\ v' - \phi_3 \\ w' + \phi_2 \end{array} \right| \right] ds \; + \; \int_0^L \left[C_3 \left| \begin{array}{c} \phi_1' \\ \phi_2' \\ \phi_3' \end{array} \right|, \left| \begin{array}{c} \phi_1' \\ \phi_2' \\ \phi_3' \end{array} \right| \right] ds. \qquad (2.9)$$

Let us verify (2.8). Integration by parts yields:

$$\int_{s_i}^{s_i+1} [A_2 f''(s), f(s)]\; ds$$

$$= \; [A_2 f'(s_{i+1}-), f(s_{i+1}-)] \; - \; [A_2 f'(s_i+), f(s_i+)] \; - \; \int_{s_i}^{s_i+1} [A_2 f'(s), f'(s)]\; ds.$$

Hence

$$- \int_0^L [A_2 f''(s), f(s)] ds$$

$$= - \sum_{i=1}^{m-1} [A_2 f'(s_{i+1}-), f(s_{i+1}-)] \; + \; \sum_{i=1}^{m-1} [A_2 f'(s_i+), f(s_i+)] \; + \; \int_0^L [A_2 f'(s), f(s)] ds.$$

But

$$- \sum_{i=1}^{m-1} [A_2 f'(s_{i+1}-), f(s_{i+1}-)] \; = \; \sum_{i=2}^{m} [A_2 f'(s_i-), f(s_i-)].$$

And using

$$f(s_i-) = f(s_i+) = f(s_i)$$

(x being in $\mathcal{D}(A)$), we have that

$$- \sum_{i=0}^{m-1} [A_2 f'(s_{i+1}-), f(s_{i+1}-)] + \sum_{i=1}^{m-1} [A_2 f'(s_i+), f(s_i+)]$$

$$= \; [A_2 f'(L-), f(L-)] \; + \; \sum_{2}^{m-1} [A_2 (f'(s_i+) - f'(s_i-)), f(s_i)][A_2 f'(0+), f(0+)].$$

Also, noticing that

$$A_1 = L_1^* - L_1,$$

we can calculate that

$$\int_0^L [A_1 f'(s), f(s)] ds \; = \; -[L_1 f(L-), f(L-)] + [L_1 f(0+), f(0+)]$$

$$+ \int_0^L [L_1 f(s), f'(s)] ds + \int_0^L [L_1^* f'(s), f(s)] ds.$$

But

$$
\begin{aligned}
[A_b f, b] &= [(L_1 f(L-) + A_2 f'(L-)), f(L-)] \\
&\quad + \sum_{2}^{m-1} [A_2(f'(s_i-) - f'(s_i+)), f(s_i)] \\
&\quad -[L_1 f(0+) + A_2 f'(0+), f(0+)].
\end{aligned}
$$

Hence

$$
\begin{aligned}
[Ax, x] &= [A_b f, b] + \int_0^L [(A_0 f + A_1 f' - A_2 f''), f(s)]\, ds \\
&= \int_0^L [A_2 f'(s), f'(s)]\, ds + \int_0^L [L_1 f(s), f'(s)]\, ds \\
&\quad + \int_0^L [L_1^* f'(s), f(s)]\, ds + \int_0^L [A_0 f(s), f(s)]\, ds.
\end{aligned}
$$

Hence

$$
[Ax, x] = \int_0^L \left[H \begin{vmatrix} f' \\ f \end{vmatrix}, \begin{vmatrix} f' \\ f \end{vmatrix} \right] ds \tag{2.10}
$$

where

$$
H = \begin{vmatrix} C_1 & 0 & 0 & -C_2 \\ 0 & C_3 & 0 & 0 \\ 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{A_0}} \\ -C_{2*} & 0 & & \end{vmatrix}
$$

from which (2.9) readily follows, noting that

$$
C_2 = -C_1 D_3
$$

$$
D_3 = \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{vmatrix}
$$

$$
C_4 = D_3^* C_1 D_3
$$

or, equivalently:

$$
A_0 = L_1^* A_2^{-1} L_1.
$$

This technique of defining the "boundary conditions" on the "differential" operator appropriately to make it self-adjoint and nonnegative definite should be compared with the FEM method where (in the Hamiltonian) the potential energy is specified and the stiffness matrix derived therefrom.

Remark

By our definition, a node is a point where there is a controller or lumped mass. There may be neither at the end points $s = 0$, $s = L$, and if so, they are not included in the nodes, and additional conditions need to be imposed. Typically, either

$$f(0) = 0 \quad \text{(clamped at zero)} \tag{2.11}$$

or

$$L_1 f(0) \;+\; A_2 f'(0) \;=\; 0 \quad \text{(free at zero)} \tag{2.12}$$

and similarly at $s = L$.

Mass/Inertia Operator

To obtain the mass/inertia operator we have to specify the control mass/ inertia and end masses, possibly offset (modeling antennas). Let us begin with the interior controllers, force (reaction jets, proof-mass actuators) and moment (cmg's). Then we have for $i \neq 1$ or m

$$M_{b,i}\ddot{f}(t, s_i) + A_2(f'(t, s_i-) - f'(t, s_i+)) + U_i(t) \;=\; 0 \tag{2.13}$$

where

$$M_{b,i} \;=\; \left|\begin{array}{cccccc} m_i & 0 & 0 & 0 & 0 & 0 \\ 0 & m_i & 0 & 0 & 0 & 0 \\ 0 & 0 & m_i & 0 & 0 & 0 \\ 0 & 0 & 0 & & & \\ 0 & 0 & 0 & & I_i & \\ 0 & 0 & 0 & & & \end{array}\right|$$

and U_i is the 6×1 vector of force and moment controls at $s = s_i$; m_i is the control mass and I_i the corresponding moment of inertia (matrix). For the ends, allowing for offset end masses m_0 at $s = 0$ and m_L at $s = s_L$ and I_0 and I_c the moments of inertia of antennas and controller, each about the center of gravity, and r_0 and r_L the 3×1 position vectors of the centers of gravity respectively of the end masses we have at $s = 0$:

$$M_{b,0}\ddot{f}(t, 0+) - L_1 f(t, 0+) - A_2 f'(t, 0+) + U_0(t) \;=\; 0 \tag{2.14}$$

where

$$r_0 = \begin{vmatrix} 0 \\ r_{0,2} \\ r_{0,3} \end{vmatrix}$$

and

$$M_{b,0} = \left|\begin{array}{ccc|ccc} m_0+m_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & m_0+m_1 & 0 & m_0 r_{0,2} & 0 & 0 \\ 0 & 0 & m_0+m_1 & m_0 r_{0,3} & 0 & 0 \\ \hline 0 & m_0 r_{0,2} & m_0 r_{0,3} & & & \\ 0 & 0 & 0 & & \hat{I}_0 & \\ 0 & 0 & 0 & & & \end{array}\right|$$

$$\hat{I}_0 = I_0 + I_c + (m_0+m_1)\begin{vmatrix} r_{0,2}^2 + r_{0,2}^2 & 0 & 0 \\ 0 & r_{0,2}^2 & -r_{0,2}r_{0,3} \\ 0 & -r_{0,2}r_{0,3} & r_{0,3}^2 \end{vmatrix}$$

and where m_1 is the controller mass; and finally U_0 is the control vector (forces and moments).

Similarly for the end $s = L$, we have

$$M_{b,L}\ddot{f}(t, L-) + L_1 f(t, L-) + A_2 f'(t, L-) + U_L(t) = 0 \tag{2.15}$$

where

$$r_L = \begin{vmatrix} L \\ r_{L,2} \\ r_{L,3} \end{vmatrix}$$

$$M_{b,L} = \left|\begin{array}{ccc|ccc} m_L+m_m & 0 & 0 & 0 & 0 & 0 \\ 0 & m_L+m_m & 0 & m_L r_{L,2} & 0 & 0 \\ 0 & 0 & m_L+m_m & m_L r_{L,3} & 0 & 0 \\ \hline 0 & m_L r_{L,2} & m_L r_{L,3} & & & \\ 0 & 0 & 0 & & \hat{I}_L & \\ 0 & 0 & 0 & & & \end{array}\right|$$

$$\hat{I}_L = I_L + I_c + (m_L+m_m)\begin{vmatrix} r_{L,2}^2 + r_{L,2}^2 & 0 & 0 \\ 0 & r_{L,2}^2 & -r_{L,2}r_{L,3} \\ 0 & -r_{L,2}r_{L,3} & r_{L,3}^2 \end{vmatrix}$$

and m_m controller mass and U_L is the control vector of forces and moments. Again, $(m_L + m_m)$ is the total moving mass.

Let M_b denote the composite matrix of all the control and end masses/ inertia:

$$M_b = \begin{vmatrix} M_{b,0} & & & & \\ & M_{b,2} & & & \\ & & \ddots & & \\ & & & M_{b,m-1} & \\ & & & & M_{b,L} \end{vmatrix}.$$

The mass-inertia operator, denoted M, is then defined by

$$Mx = y; \quad x \begin{vmatrix} f \\ b \end{vmatrix}; \quad y = \begin{vmatrix} M_0 f \\ M_b b \end{vmatrix} \tag{2.16}$$

and is a linear bounded self-adjoint nonnegative definite opeator $\mathcal{H}$ onto $\mathcal{H}$ with a linear bound inverse:

$$M^{-1}x = \begin{vmatrix} M_0^{-1} b \\ M_b^{-1} b \end{vmatrix}$$

and we note that

$$M_b^{-1} b = \begin{vmatrix} M_{b,0}^{-1} b_1 \\ \vdots \\ M_{b,i}^{-1} b_i \\ \vdots \\ M_{b,L}^{-1} b_m \end{vmatrix}.$$

Also

$$\sqrt{M}\, x = \begin{vmatrix} \sqrt{M_0}\, f \\ \sqrt{M_b}\, b \end{vmatrix}$$

$$\sqrt{M_b}\, b = \begin{vmatrix} \sqrt{M_{b,0}}\, b \\ \vdots \\ \sqrt{M_{b,i}}\, b_i \\ \vdots \\ \sqrt{M_{b,L}}\, b_m \end{vmatrix}.$$

Control Operator

The control operator maps the control inputs into $\mathcal{H}$. We denote it by B. We note that it is possible that not all nodes may have controllers. Let the number of controllers — force, moment or proof-mass — be denoted m_c,

$$m_c < 6^m.$$

Then we can regard any control u as an $m_c \times 1$ column vector. We define

$$B_u u = b$$

where

$$b = \begin{vmatrix} B_1 u \\ \vdots \\ B_m u \end{vmatrix},$$

where each B_i is a $6 \times m_c$ matrix such that $B_i^* B_i$ is nonsingular and further

$$\left[B_i^* b,\ B_j^* b\right] = 0, \quad i \neq j$$

or

$$B_i B_j^* = 0, \quad i \neq j.$$

In particular it follows

$$B_u B_u^* b = \begin{vmatrix} B_1 B_1^* b_1 \\ \vdots \\ B_m B_m^* b_m \end{vmatrix}.$$

We shall use the notation

$$B_i B_i^* = D_i$$

where the D_i, which are 6×6 each, may be taken to be diagonal with entries 1 or 0. Finally

$$Bu = \begin{vmatrix} 0 \\ B_u u \end{vmatrix}$$

$$BB^* \begin{vmatrix} f \\ b \end{vmatrix} = \begin{vmatrix} 0 \\ B_u B_u^* b \end{vmatrix}.$$

Note that if

$$B^* x = 0, \quad x = \begin{vmatrix} f \\ b \end{vmatrix}$$

we must have:

$$B_u^* b = 0$$

or b must be such that:

$$B_i^* b_i = 0, \quad i = 1, \dots, m,$$

or

$$D_i b_i = 0, \quad i = 1, \dots, m.$$

Another equally useful representation for BB^* can be obtained in the following way. Let $\{e_i\}$, $i = 1, \dots, m_c$, denote the unit coordinate vectors in $\mathcal{R}^{m_c}$. Then let

$$B_{(i)} = Be_i.$$

Then

$$[B_{(i)}, B_{(j)}] = 0, \quad i \neq j$$

and

$$BB^* = \sum_1^{m_c} B_{(i)} B_{(i)}^*$$

where the $B_{(i)}$ are orthogonal, and each $B_{(i)}$ is nonzero.

With these definitions we can assemble the canonical "state space" version of the dynamic equations (2.1), (2.13), (2.14), (2.15):

$$M\ddot{x}(t) + Ax(t) + Bu(t) = 0, \quad t > 0 \tag{2.17}$$

where, formally, we have taken:

$$\ddot{x}(t) = \left| \begin{array}{c} \ddot{f}(t, \cdot) \\ \ddot{b}(t, \cdot) \end{array} \right|.$$

We recognize (2.17) as "an abstract wave equation in a Hilbert space"; the precise relationship between the original space-time dynamic equations and this abstract version will be clarified later, in stages.

3. SPECTRAL ANALYSIS: MODES/MODAL EXPANSION

Before we proceed to "solve" (2.17) we need to examine the spectrum of the stiffness operator A. The eigenvalues of A (with respect to the mass operator M)

are the modes of the undamped structure. A crucial result is that the modes are the zeros of an entire function — the determinant of the $m \times m$ "condensed dynamic stiffness matrix." The corresponding "mode shapes," the eigenfunctions, yield an M-orthogonal basis for the Hilbert space. We also characterize the "rigid body" modes, the eigenfunctions corresponding to the zero eigenvalue. We also obtain the "Green's function" for the eigenvalue problem. Finally we define the square root of the stiffness operator and indicate its relation to the potential energy.

Elementary Properties of the Stiffness Operator

Let us begin cataloging the elementary yet crucial properties of the stiffness operator A. We have already seen that it is self-adjoint:

$$[Ax, y] = [x, Ay] \quad \text{for } x, y \text{ in } \mathcal{D}(A)$$

and nonnegative definite

$$[Ax, x] \geq 0, \quad \text{for } x \in \mathcal{D}(A).$$

The next property is that the domain of A is dense in $\mathcal{H}$. In other words, while not every element x in $\mathcal{H}$ is in the domain of A as the definition of A clearly shows, we can find elements in the domain of A that approximate x as closely as needed. More precisely, we can find a sequence $\{x_n\}$ in $\mathcal{D}(A)$ such that

$$||x - x_n|| \to 0.$$

This is a feature of "differential" operators; but since in our case we also have "boundary values" to contend with, we shall present a formal argument. Thus let

$$x = \begin{vmatrix} f \\ b \end{vmatrix}.$$

Let

$$f_i(s) = f(s), \quad s_i \leq s \leq s_{i+1}.$$

It should be noted that b is arbitrary and is not necessarily the vector of "boundary" values of $f(\cdot)$ at $s = s_i$; the latter need not of course be even defined. But for each i, we can find a sequence of functions $\{f_{i,n}(\cdot)\}$ such that $f_{i,n}(s)$ is "infinitely" differentiable in $s_i < s < s_{i+1}$, and

$$\int_{s_i}^{s_{i+1}} ||f_i(s) - f_{i,n}(s)||^2 \, ds \to 0 \quad \text{as } n \to \infty$$

and

$$f_{i,n}(s_i) = b_i; \quad f_{i,n}(s_{i+1}) = b_{i+1}.$$

Defining

$$f_n(s) = f_{i,n}(s), \quad s_i < s < s_{i+1},$$

$$f_n(s_i) = f_{i,n}(s),$$

we see that $f_n(\cdot)$ is continuous and

$$x_n = \begin{vmatrix} f_n \\ b \end{vmatrix}$$

belongs to the domain of A, and

$$||x - x_n||^2 \to 0 \quad \text{as } n \to \infty$$

as required.

Eigenvalues and Eigenfunctions

If we proceed formally and take the Laplace transform of (2.17), setting

$$\phi \;=\; \int_0^\infty e^{-\lambda t} x(t) \, dt, \quad \operatorname{Re} \lambda > 0,$$

we obtain for fixed λ:

$$\lambda^2 M\phi + A\phi - \psi \;=\; 0 \tag{3.1}$$

where ψ is an element of $\mathcal{H}$. The solution of this equation plays an important role in the theory. We now formulate this more precisely as: find ϕ in $\mathcal{D}(A)$ such that

$$\lambda^2 M\phi + A\phi \;=\; \psi, \quad \psi \in \mathcal{H} \tag{3.2}$$

for given λ. We see that in order for (3.2) to have a unique solution the "homogeneous" equation

$$\lambda^2 M\phi + A\phi \;=\; 0 \tag{3.3}$$

can only have the zero solution. Rewriting this equation as

$$A\phi = -\lambda^2 M\phi$$

we may consider more generally

$$A\phi = \gamma M\phi, \quad \phi \neq 0. \tag{3.4}$$

Here γ is called an eigenvalue of A with respect to the mass matrix M, and ϕ a corresponding eigenfunction. From

$$0 \leq [A\phi, \phi] = \gamma[M\phi, \phi]$$

it follows that

$$\gamma \geq 0.$$

If $\gamma_1,\ \gamma_2$ are two distinct eigenvalues with corresponding eigenvectors $\phi_1,\ \phi_2$ respectively, we have

$$[A\phi_1,\ \phi_2] = \gamma_1[M\phi_1,\ \phi_2] = [\phi_1,\ A\phi_2] = \gamma_2[M\phi_1,\ \phi_2]$$

and hence we must have:

$$[M\phi_1,\ \phi_2] = 0$$

or, eigenfunctions corresponding to distinct eigenvalues must be M-orthogonal.

We shall show now that the set of eigenvalues is nonfinite, countable and can be taken as

$$\left\{\omega_k^2\right\}, \quad \omega_k \leq \omega_{k+1}, \quad k = 0, 1, 2, \ldots$$

and

$$\omega_k \to \infty \quad \text{as } k \to \infty$$

and

$$A\phi_k = \omega_k^2 M\phi_k.$$

First let us consider the eigenvalue zero.

Rigid Body Modes

Because we are considering the case where both $s = 0$ and $s = L$ are nodes, we shall show that zero is an eigenvalue. An eigenvector corresponding to the zero eigenvalue is called a rigid body mode. This is because if

$$A\phi = 0$$

we have

$$[A\phi, \phi] = 0$$

or, from (2.9) the associated potential energy is zero. Let

$$\phi = \begin{vmatrix} f \\ b \end{vmatrix},$$

where b is the boundary vector, ϕ being in $\mathcal{D}(A)$; if the potential energy is zero, we have from (2.9) that $f(\cdot)$ must be of the form:

$$f(s) \;=\; \begin{vmatrix} u(s) = u(0) \\ v(s) = v(0) + s\phi_3(0) \\ w(s) = w(0) - s\phi_2(0) \\ \phi_1(s) = \phi_1(0) \\ \phi_2(s) = \phi_2(0) \\ \phi_3(s) = \phi_3(0) \end{vmatrix}. \tag{3.5}$$

The derivative $f'(s)$ is continuous in $0 \le s \le L$. Note that the "free-free" boundary conditions (2.12) are satisfied at both ends, and $f(\cdot)$ is the same regardless of the number and location of the interior nodes. Also the dimension of the eigenfunction space is six. In our notation we shall set

$$\omega_0 = 0.$$

Nonzero Eigenvalues

Let us now consider the general case in the form

$$\lambda^2 M\phi + A\phi \;=\; 0.$$

Let

$$\phi = \begin{vmatrix} f \\ b \end{vmatrix}.$$

Then

$$\lambda^2 M_0 f + g \;=\; 0 \tag{3.6}$$

where $g(\cdot)$ is defined by (2.6), and

$$\lambda^2 M_b b + A_b f \;=\; 0. \tag{3.7}$$

To solve (3.3), let

$$\mathcal{A}(\lambda) \;=\; \begin{vmatrix} 0 & I_6 \\ A_2^{-1}(A_0 + \lambda^2 M_0) & A_2^{-1} A_1 \end{vmatrix}. \tag{3.8}$$

Then

$$f(s) \;=\; e^{\mathcal{A}(\lambda)(s-s_i)} \begin{vmatrix} f(s_i) \\ f'(s_i+) \end{vmatrix}, \quad s_i \le s \le s_{i+1},\; i = 1, 2, \ldots, m-1. \tag{3.9}$$

Let

$$a \;=\; \begin{vmatrix} f(s_1) \\ \delta f'(s_1) \\ \delta f'(s_2) \\ \vdots \\ \delta f'(s_{m-1}) \end{vmatrix}$$

where

$$\delta f'(s_i) \;=\; f'(s_i+) \;-\; f'(s_i-)$$

and we set

$$f'(0-) = 0.$$

Then

$$\begin{vmatrix} f(s) \\ f'(s) \end{vmatrix} \;=\; e^{\mathcal{A}(\lambda)s} \begin{vmatrix} f(0) \\ 0 \end{vmatrix} \;+\; \sum_{j=1}^{i} e^{\mathcal{A}(\lambda)(s-s_j)} \begin{vmatrix} 0 \\ \delta f'(s_j) \end{vmatrix},$$

$$s_i < s < s_{i+1},\; i = 1, \ldots, m-1. \tag{3.10}$$

Hence we can write

$$b = L(\lambda)\, a$$

where $L(\lambda)$ is a $6m \times 6m$ block lower-triangular matrix:

$$\begin{aligned} L(\lambda) &= \{\ell_{i,j}(\lambda)\} \\ \ell_{i,j}(\lambda) &= 0, \quad j > i \\ \ell_{11}(\lambda) &= I_6. \end{aligned}$$

For $i > 1$

$$\begin{aligned} \ell_{i,1}(\lambda) &= P_{11}(\lambda, s_i) \\ \ell_{i,2}(\lambda) &= P_{12}(\lambda, s_i) \\ \ell_{i,3}(\lambda) &= P_{12}(\lambda,\ s_i - s_2) \\ &\vdots \\ \ell_{i,i-1}(\lambda) &= P_{12}(\lambda,\ s_i - s_{i-2}) \\ \ell_{i,i}(\lambda) &= P_{12}(\lambda,\ s_i - s_{i-1}) \end{aligned}$$

where

$$e^{A(\lambda)s} = \begin{vmatrix} P_{11}(\lambda, s) & P_{12}(\lambda, s) \\ P_{21}(\lambda, s) & P_{22}(\lambda, s) \end{vmatrix},$$

$$P_{21}(\lambda, s) = \frac{\partial}{\partial s} P_{11}(\lambda, s), \quad P_{22}(\lambda, s) = \frac{\partial}{\partial s} P_{12}(\lambda, s).$$

Note that

$$\det L(\lambda) = \det P_{12}(\lambda, s_2) \cdots \det P_{12}(\lambda,\ s_i - s_{i-1}) \cdots \det P_{12}(\lambda,\ L - s_{m-1}). \quad (3.11)$$

Also, we can express $A_b f$ in terms of a, using (3.10). Thus we can write

$$A_b f = K(\lambda) a$$

where $K(\lambda)$ is also a $6m \times 6m$ (block-lower-triangular) matrix and hence (3.3) yields:

$$\left(\lambda^2 M_b L(\lambda) + K(\lambda)\right) a = 0. \quad (3.12)$$

Hence the eigenvalues are the roots of

$$\det\left(\lambda^2 M_b L(\lambda) + K(\lambda)\right) = 0. \quad (3.13)$$

The left side defines an entire function of the complex number λ and can have at most a countable number of zeros which must grow without bound in magnitude, if nonfinite in number. The corresponding eigenfunction is determined by (3.10). Also (3.12) shows that the dimension of each eigenfunction space is finite, not more than $6m$.

We can now deduce that the set of eigenvalues which is countable is not finite. For if it were finite, the number of eigenfunctions would be finite. But the eigenfunctions must be complete in the M-inner product. If not, there must be a function ψ which is M-orthogonal to all the eigenfunctions. The class of such functions form a Hilbert space and A must map a dense domain of this space into itself. But A being self-adjoint, we can now recall the fundamental result (see, for example [Riesz-Nagy 1955]) that is must have at least one eigenvalue, which leads to a contradiction. But the dimension of $\mathcal{H}_2$ being nonfinite, we see that the set of eigenvalues cannot be finite. In particular we see that the eigenfunctions are complete, and M-orthogonal. We can therefore make them an M-orthonormal basis for $\mathcal{H}$. For each eigenvalue ω_k^2, let P_k denote the projection (operator) corresponding to the eigenfunction space:

$$A(P_k\phi) = \omega_k^2(P_k\phi).$$

Since the dimension of each eigenfunction space is finite it is convenient to continue to use $\{\phi_k\}$ to denote the M-orthonormalized basis in each eigenfunction space, "counting each eigenfunction as many times as the dimension," as is customary. Then we can write

$$x = \sum_0^\infty \alpha_k \phi_k \tag{3.14}$$

where

$$\alpha_k = [x, M\phi_k]$$

and

$$\sum_0^\infty |\alpha_k|^2 = [Mx, x].$$

Hence

$$M^{-1}x = \sum_0^\infty [x, \phi_k]\phi_k$$

or,

$$x = \sum_0^\infty [x, \phi_k]M\phi_k. \tag{3.14a}$$

Using (3.14) and (3.14a) we have

$$[x,x] = \sum_0^\infty [x,\phi_k][M\phi_k,x]$$

conversely

$$x = \sum_1^\infty \alpha_k\phi_k \in \mathcal{H}$$

if and only if

$$\sum_1^\infty |\alpha_k|^2 < \infty$$

and

$$[x,Mx] = \sum_1^\infty |\alpha_k|^2.$$

Since

$$A\phi_k = \omega_k^2 M\phi_k$$

we have that

$$A\sum_1^N [x,M\phi_k]\phi_k = \sum_1^N [x,M\phi_k]\omega_k^2 M\phi_k$$

it follows that

$$x \in \mathcal{D}(A)$$

if and only if

$$\sum_1^\infty |[x,M\phi_k]|^2\omega_k^4 < \infty.$$

Also for x, y in $\mathcal{H}$:

$$[x,My] = \sum_0^\infty [x,M\phi_k][M\phi_k,y] \tag{3.15}$$

$$[x,y] = \sum_0^\infty [x,\phi_k][M\phi_k,y]. \tag{3.15a}$$

What we are exploiting here is the fact that the sequences $\{\phi_k\}$, $\{M\phi_k\}$ are "biorthogonal" and complete. We shall return to this concept later in Section 6. We refer to (3.14) as a "modal" expansion. Expanding on (3.14), let

$$x = \begin{vmatrix} f \\ b \end{vmatrix}, \quad \phi_k = \begin{vmatrix} f_k \\ b_k \end{vmatrix}.$$

Then (3.14) yields:

$$b \;=\; \sum_0^{\infty} [x, M\phi_k] b_k.$$

Taking

$$x = \begin{vmatrix} 0 \\ b \end{vmatrix},$$

we have

$$b \;=\; \sum_0^{\infty} [b,\; M_b b_k]\, b_k.$$

Hence

$$\sum |[b,\; M_b b_k]|^2 \;<\; \infty$$

and hence it follows that

$$\sum_0^{\infty} ||M_b b_k||^2 \;<\; \infty$$

and hence that

$$\sum_0^{\infty} ||b_k||^2 \;<\; \infty.$$

In particular therefore

$$b_k \to 0 \quad \text{as } k \to \infty.$$

Returning now to (3.10) we have

$$f(s) \;=\; P_{11}(\lambda, s)\, f(0) \;+\; \sum_{j=1}^{i} P_{12}(\lambda,\; s - s_j)\, \delta f'(s_j), \quad s_i < s < s_{i+1} \tag{3.16}$$

which we can express as a linear transformation:

$$f = \mathcal{L}(\lambda)\, a, \tag{3.17}$$

where $\mathcal{L}(\lambda)$ maps $(R^6)^m$ into $L_2(0, L)^6$. Let

$$f_1 \;=\; \mathcal{L}(\lambda)\, a_1, \quad a_1 \in (R^6)^m$$

$$f_2 \;=\; \mathcal{L}(\lambda)\, a_2, \quad a_2 \in (R^6)^m.$$

Then

$$\psi_1 \;=\; \begin{vmatrix} \mathcal{L}(\lambda)\, a_1 \\ L(\lambda)\, a_1 \end{vmatrix}, \qquad \psi_2 \;=\; \begin{vmatrix} \mathcal{L}(\lambda)\, a_2 \\ L(\lambda)\, a_2 \end{vmatrix}$$

are elements in $\mathcal{D}(A)$ and

$$\begin{aligned}[A\psi_1, \psi_2] &= [A_b f_1, b_2] - \lambda^2[M_0 f_1, f_2] \\ &= [K(\lambda)a_1, L(\lambda)a_2] - \lambda^2[M_0 f_1, f_2] \\ &= [\psi_1, A\psi_2] \\ &= [L(\lambda)a_1, K(\lambda)a_2] - \overline{\lambda}^2[M_0 f_1, f_2].\end{aligned}$$

Hence for λ such that λ^2 is real

$$L(\lambda)^* K(\lambda)$$

is self-adjoint and is nonnegative definite if λ is real, since

$$0 \leq [A\psi_1, \psi_1].$$

Also $L(\lambda)$ is an entire function of the complex variable λ, and

$$\det |L(\lambda)| = 0$$

(equivalently

$$\det P_{12}(\lambda;\, s_i - s_{i-1}) = 0$$

for some $i \geq 2$, from (3.11)) for at most a countable set of λ. And, omitting this set (which are recognized as "clamped" modes of the structure, where every node is clamped so that the displacement is zero), we have

$$a = L(\lambda)^{-1} b$$

and

$$\left(\lambda^2 M_b L(\lambda) + K(\lambda)\right) a = \left(\lambda^2 M_b + K(\lambda)L(\lambda)^{-1}\right) b.$$

Hence we can express (3.12) as:

$$\left(\lambda^2 M_b + K(\lambda)L(\lambda)^{-1}\right) b = 0.$$

Following [Wittrick-Williams 1971], we shall call

$$\lambda^2 M_b + K(\lambda)L(\lambda)^{-1}$$

the "condensed dynamic stiffness matrix." Let

$$T(\lambda) = K(\lambda)L(\lambda)^{-1}.$$

Then we have:

$$\begin{aligned}
[A\phi_1, \phi_2] &= [A_b f_1, b_2] - \lambda^2[M_0 f_1, f_2] \\
&= \left[K(\lambda)L(\lambda)^{-1}b_1, b_2\right] - \lambda^2[M_0 f_1, f_2] \\
&= [\phi_1, A\phi_2] \\
&= \left[b_1, K(\lambda)L(\lambda)^{-1}b_2\right] - \overline{\lambda}^2[M_0 f_1, f_2].
\end{aligned}$$

It follows that for λ such that λ^2 is real, $T(\lambda)$ is self-adjoint; and if λ is real, also nonnegative definite. Also, if

$$K(\lambda)a = 0, \quad a \neq 0,$$

taking

$$f = \mathcal{L}(\lambda)a, \quad b = L(\lambda)a, \quad \phi = \begin{vmatrix} f \\ b \end{vmatrix}$$

we have

$$A_b f = 0$$

and hence

$$\begin{aligned}
[A\phi, \phi] &= [A_b f, b] - \lambda^2[M_0 f, f] \\
&= -\lambda^2[M_0 f, f]
\end{aligned}$$

which is impossible unless λ is pure imaginary. Zero is an eigenvalue of $K(0)$. By the "zeros of the dynamic stiffness matrix" we mean the roots of

$$\det(\lambda^2 M_b + T(\lambda)) = 0$$

which are of course the same as those of

$$\det\left(\lambda^2 M_b L(\lambda) + K(\lambda)\right) = 0.$$

The poles of the condensed dynamic stiffness matrix are the zeros of

$$\det L(\lambda),$$

or the "clamped" modes of the structure. The dimension of the eigenfunction space for $\lambda \neq 0$ is equal to 1 if the condensed dynamic stiffness matrix has distinct eigenvalues.

Green's Function

We are now ready to solve the non-homogeneous equation:

$$(\lambda^2 M + A)\phi = \psi. \tag{3.18}$$

In fact for λ such that

$$\det\left(\lambda^2 M_b + T(\lambda)\right) \neq 0$$

or

$$\lambda^2 \neq -\omega_k^2,$$

we can calculate the solution in two ways. First we can obtain a "modal" solution using the modal expansion (3.14). For this purpose we note first that:

$$(\lambda^2 M + A)^{-1} M\phi_k = \frac{\phi_k}{\lambda^2 + \omega_k^2}$$

since

$$(\lambda^2 M + A)\phi_k = \left(\lambda^2 + \omega_k^2\right) M\phi_k.$$

Hence we obtain:

$$(\lambda^2 M + A)^{-1} x = \sum_0^\infty \frac{[x, \phi_k]\phi_k}{\lambda^2 + \omega_k^2}$$

and

$$\left[(\lambda^2 M + A)^{-1} x,\ y\right] = \sum_0^\infty \frac{[x, \phi_k][\phi_k, y]}{\lambda^2 + \omega_k^2}.$$

For λ^2 positive we see that

$$(\lambda^2 M + A)^{-1}$$

is self-adjoint and nonnegative definite,

$$\left[(\lambda^2 M + A)^{-1}\phi_k,\ M\phi_k\right] = \frac{1}{\lambda^2 + \omega_k^2}$$

so that

$$\sum_0^\infty \left[(\lambda^2 M + A)^{-1}\phi_k,\ M\phi_k\right] \;=\; \sum_0^\infty \frac{1}{\lambda^2+\omega_k^2}\,. \tag{3.19}$$

We shall show that the right side is finite. For this purpose we need to derive the Green's function for the eigenvalue problem (3.18).

For this purpose let

$$\phi \;=\; \begin{vmatrix} f \\ b \end{vmatrix}; \qquad \psi \;=\; \begin{vmatrix} g \\ c \end{vmatrix}.$$

Then (3.18) becomes

$$\lambda^2 M_0 f + h \;=\; g$$

where $h(\cdot)$ is defined by (2.6), and

$$\lambda^2 M_b b + A_b f \;=\; c. \tag{3.20}$$

With $\mathcal{A}(\lambda)$, as before we see that we can solve (2.38) as

$$\begin{vmatrix} f(s) \\ f'(s) \end{vmatrix} \;=\; e^{\mathcal{A}(\lambda)s}\begin{vmatrix} f(0) \\ 0 \end{vmatrix} \;+\; \sum_{j=1}^{i} e^{\mathcal{A}(\lambda)(s-s_j)}\begin{vmatrix} 0 \\ \delta f'(s_j) \end{vmatrix} \;+\; \int_0^s e^{\mathcal{A}(\lambda)(s-\sigma)}\begin{vmatrix} g(\sigma) \\ 0 \end{vmatrix}\, d\sigma. \tag{3.21}$$

Or, with $a, \mathcal{L}(\lambda)$ as before, we have:

$$f \;=\; \mathcal{L}(\lambda)a \;+\; \mathcal{N}(\lambda)g$$

where the operator $\mathcal{N}(\lambda)$ mapping $L_2[0,L]^6$ into itself is defined by:

$$h = \mathcal{N}(\lambda)g; \quad h(s) \;=\; \int_0^s P_{11}(\lambda; s-\sigma)\, g(\sigma)\, d\sigma, \quad 0 < s < L.$$

Also, we have

$$b \;=\; L(\lambda)a + N(\lambda)g$$

where

$$N(\lambda)g \;=\; \begin{vmatrix} h(0) \\ \vdots \\ h(L) \end{vmatrix}$$

and

$$A_b f \;=\; K(\lambda)a + M(\lambda)g$$

where

$$M(\lambda)g \;=\; A_b(\mathcal{N}(\lambda)g).$$

Hence (3.20) becomes

$$\left(\lambda^2 M_b L(\lambda) \;+\; K(\lambda)\right) a \;=\; c \;-\; N(\lambda)g \;-\; M(\lambda)g$$

and hence for

$$\lambda^2 + \omega_k^2 \neq 0,$$

we have:

$$a \;=\; \left(\lambda^2 M_b L(\lambda) + K(\lambda)\right)^{-1} (c - N(\lambda)g - M(\lambda)g).$$

Hence finally

$$f \;=\; \mathcal{L}(\lambda)a(\lambda) \;+\; \mathcal{N}(\lambda)g \tag{3.22}$$

$$b \;=\; L(\lambda)a(\lambda) \;=\; \left(\lambda^2 M_b + T(\lambda))\right)^{-1} (c - N(\lambda)g - M(\lambda)g) \tag{3.23}$$

where

$$a(\lambda) \;=\; \left(\lambda^2 M_b L(\lambda) \;+\; K(\lambda)\right)^{-1} (c - N(\lambda)g - M(\lambda)g).$$

From (3.22) we see that we can write

$$f(s) \;=\; \int_0^L G(\lambda, s, \sigma)\, g(\sigma)\; d\sigma, \quad 0 < s < L$$

$$b \;=\; \left(\lambda^2 M_b + T(\lambda)\right)^{-1} c \;+\; \int_0^L G_b(\lambda, \sigma)\, g(\sigma)\; d\sigma \;=\; N(\lambda)f$$

where the kernel $G(\lambda, s, \sigma)$ is continuous in $0 \leq s, \sigma \leq L$ and $G_b(\lambda, \sigma)$ is continuous in $0 \leq \sigma \leq L$. It follows from this

$$(\lambda^2 M \;+\; A)^{-1}$$

is nuclear (see [Balakrishnan 1981] for the definition) and in particular (3.19) is finite and hence it follows also that

$$\sum_1^\infty \frac{1}{\omega_k^2} \;<\; \infty.$$

Of particular interest to us is the solution of (3.18) when ψ is of the form

$$\psi = Bu$$

and (3.22), (3.23) simplify to:

$$f = \mathcal{L}(\lambda)\left(\lambda^2 M_b L(\lambda) + K(\lambda)\right)^{-1} B_u u \tag{3.24}$$

$$b = \left(\lambda^2 M_b + T(\lambda)\right)^{-1} B_u u. \tag{3.25}$$

Hence

$$B^*(\lambda^2 M + A)^{-1} B = B_u^* \left(\lambda^2 M_b + T(\lambda)\right)^{-1} B_u. \tag{3.26}$$

The Square Root of the Stiffness Operator

Finally we note that we can define $\sqrt{A}$ as a self-adjoint nonnegative operator. This is treated in standard texts — e.g., [Riesz-Nagy 1955]. It is known that

$$D\left(\sqrt{A}\,\right) \supset D(A).$$

Unfortunately we cannot use the expansion (3.14), since we cannot evaluate

$$\sqrt{A}\,\phi_k \quad \text{or} \quad \sqrt{A}\,M\phi_k.$$

However we do know that

$$\begin{aligned} \left[\sqrt{A}\,\phi_k,\ \sqrt{A}\,\phi_j\right] &= [A\phi_k,\ \phi_j] \\ &= 0, \qquad k \neq j \\ &= \omega_k^2, \qquad k = j. \end{aligned}$$

Using this we have

$$\left[\sqrt{A}\,x,\ \sqrt{A}\,x\right] = \sum_1^\infty \omega_k^2 |[x, M\phi_k]|^2 \tag{3.27}$$

which if $x \in \mathcal{D}(A)$, is

$$= [Ax, x] = 2(\text{Potential Energy}).$$

Hence we see that if

$$x \in \mathcal{D}\left(\sqrt{A}\right)$$

then we can define

$$\text{(Potential Energy)} = \frac{||\sqrt{A}\, x||^2}{2} \, .$$

Thus we can extend the definition of Potential Energy for all x in $\mathcal{D}\left(\sqrt{A}\right)$, which is larger than $\mathcal{D}(A)$.

Since the stiffness operator A has nothing to do with the mass operator M, we may consider the case

$$M = \text{Identity}$$

in which case, our eigenfunctions become

$$A\tilde{\phi}_k = \tilde{\omega}_k^2 \tilde{\phi}_k.$$

Then in terms of these eigenfunctions we can define

$$\mathcal{D}\left(\sqrt{A}\right) = \left[x \,\middle|\, \sum_1^\infty \tilde{\omega}_k^2 \left| \left[x, \tilde{\phi}_k \right] \right|^2 < \infty \right] \tag{3.28}$$

and

$$\sqrt{A}\, x = \sum_1^\infty \tilde{\omega}_k \left[x, \tilde{\phi}_k \right] \tilde{\phi}_k$$

$$[Ax, x] = \sum_1^\infty \tilde{\omega}_k^2 \left| \left[x, \tilde{\phi}_k \right] \right|^2$$

which is then also the domain on which the Potential Energy can be defined. However we need a characterization similar to that for A, which in particular does not invoke eigenfunction. For more on this see [Balakrishnan 1990].

4. TIME-DOMAIN ANALYSIS

We have tools enough to consider the "time domain" solution of the dynamic equations: (2.14) and in turn (2.1). Unlike the finite-dimensional case, we can have more than one kind of solution depending on the interpretation of the equation (2.14) and the properties demanded of the solution. For the most satisfactory form of solution, we need to introduce the notion of the "energy norm" space in which the norm is determined in terms of the total system energy — kinetic-plus-potential — and the theory of semigroups of operators applies.

We begin with the weaker notion first.

Weak Solution/Modal Solution

With respect to (2.14) we have to specify first how the time-derivative therein is to be defined and whether the solution $x(t),\ 0 < t$, which will depend only on the initial conditions at $t = 0$ and the input $u(\cdot)$, is required to be in the domain of A. Whatever the definition, we must have that for every ϕ in $\mathcal{D}(A)$:

$$[M\ddot{x}(t),\ \phi] + [Ax(t), \phi] + [Bu(t), \phi] \ = \ 0.$$

We can rewrite this as

$$\frac{d^2}{dt^2}[x(t), M\phi] + [x(t), A\phi] + [Bu(t), \phi] = 0. \tag{4.1}$$

As for the input $u(\cdot)$, we assume that

$$\int_0^T |u(t)|^2\, dt \ < \ \infty, \qquad \text{for every } T, \quad 0 < T < \infty\,.$$

That is, $u(\cdot) \in L_2[0,T]^{m_c}$ for every $T < \infty$. Note that in (4.1) we only need the "weak derivative" (see [Balakrishnan 1981] for definition) and further we have circumvented the requirement that $x(t) \in \mathcal{D}(A)$. By a "weak solution" of (2.14) we mean a function $x(t),\ t \geq 0$ such that for every ϕ in $\mathcal{D}(A)$:

$$[x(t), M\phi]$$

and its derivative are absolutely continuous in $t > 0$, satisfies (4.1) and the initial conditions:

$$[x(t), M\phi] \to [x_1, M\phi] \quad \text{ as } t \to 0+$$
$$\frac{d}{dt}[x(t), M\phi] \to [x_2, M\phi] \quad \text{ as } t \to 0+$$

where x_1, x_2 are given elements in $\mathcal{H}$.

Even weaker (and easiest to construct) is the notion of a "modal solution" where we only require that (4.1) hold for every eigenfunction ϕ_k. In this case (4.1) becomes

$$\frac{d^2}{dt^2}[x(t), M\phi_k] \ + \ \omega_k^2[x(t), M\phi_k] \ + \ [u(t), B^*\phi_k] \ = \ 0. \tag{4.2}$$

Let

$$a_k(t) = [x(t), M\phi_k]$$

$$u_k(t) = [u(t), B^*\phi_k].$$

Then for each k, we have the ordinary differential equation

$$\ddot{a}_k(t) + \omega_k^2 a_k(t) + u_k(t) = 0, \quad \text{a.e. } t \geq 0 \tag{4.3}$$

with the initial conditions:

$$a_k(0) = [x_1, M\phi_k]$$

$$\dot{a}_k(0) = [x_2, M\phi_k]$$

which we can solve, to yield, for $\omega_k \neq 0$:

$$a_k(t) = \frac{\dot{a}_k(0)\sin\omega_k t}{\omega_k^2} + a_k(0)\cos\omega_k t + \int_0^t \frac{1}{\omega_k}\sin\omega_k(t-s)\, u_k(s)\, ds \tag{4.4}$$

and for $\omega_k = 0$:

$$a_k(t) = a_k(0) + t\dot{a}_k(0) + \int_0^t (t-s)u_k(s)\, ds. \tag{4.5}$$

It is easy to verify that

$$\sum_0^\infty |a_k(t)|^2 < \infty, \quad 0 < t < \infty.$$

Hence we can define

$$x(t) = \sum_0^\infty a_k(t)\phi_k. \tag{4.6}$$

Thus defined, we do have

$$a_k(t) = [x(t), M\phi_k]$$

and $x(\cdot)$ satisfies (4.2). Hence we have a "modal" solution. However $x(\cdot)$ need not qualify as a weak solution in general and additional restrictions will need to be placed on the initial conditions — on x_1, x_2. Thus for

$$u(t) = 0, \quad t > 0$$

we would need to require that

$$x_1 \in \mathcal{D}(A)$$

so that

$$\sum_1^\infty a_k(0)^2\omega_k^4 \;<\; \infty\,.$$

In this case

$$\sum_1^\infty |\dot{a}_k(t)|^2 \;+\; \sum_1^\infty |\ddot{a}_k(t)|^2 \;<\; \infty$$

and hence for ϕ in $\mathcal{D}(A)$

$$\begin{aligned}
\frac{d^2}{dt^2}\,[x(t), M\phi] &= \sum_0^\infty \ddot{a}_k(t)[\phi_k, M\phi] \\
&= \sum_0^\infty a_k(t)\left[(-\omega_k^2)M\phi_k,\ \phi\right] \\
&= -\sum_0^\infty a_k(t)[\phi_k, A\phi] \\
&= -[x(t),\ A\phi]
\end{aligned}$$

as required.

A better technique, avoiding these special considerations, is to go over to "state space" solution.

State Space Solution: Need for Energy Norm

To proceed further with (2.14), let us cast it in "state-space" form. Thus we let

$$Y(t) = \begin{vmatrix} x(t) \\ \dot{x}(t) \end{vmatrix}$$

and, formally, (2.14) goes over into

$$\dot{Y}(t) = \mathcal{A}Y(t) + \mathcal{B}u(t) \tag{4.7}$$

where

$$Y(t) \in \mathcal{H} \times \mathcal{H},$$

$$\mathcal{A} = \begin{vmatrix} 0 & I \\ -M^{-1}A & 0 \end{vmatrix}$$

$$\mathcal{B} = \begin{vmatrix} 0 \\ -M^{-1}B \end{vmatrix}$$

and

$$\mathcal{D}(\mathcal{A}) \;=\; \left[Y = \begin{vmatrix} y_1 \\ y_2 \end{vmatrix}, \quad \begin{matrix} y_1 \in \mathcal{D}(A) \\ y_2 \in \mathcal{H} \end{matrix} \right].$$

We interpret (4.7) in the weak sense, satisfying also the initial condition:

$$||Y(t) - Y|| \to 0 \quad \text{as } t \to 0+\,.$$

In addition we require that for each t, $Y(t)$ is continuous with respect to Y:

$$||Y(t)|| \to 0 \quad \text{as } ||Y|| \to 0.$$

It is shown in [Balakrishnan-Triggiani 1993] that this is impossible unless we change the space $\mathcal{H} \times \mathcal{H}$ — change the product-norm, which is physically meaningless, to the "energy" norm. The total energy

$$\begin{aligned} &= \text{Potential (Elastic)Energy} + \text{Kinetic Energy} \\ &= \frac{[Ay_1, y_1]}{2} + \frac{[My_2, y_2]}{2} \end{aligned}$$

where

$$Y \;=\; \begin{vmatrix} y_1 \\ y_2 \end{vmatrix}, \quad y_1 \in \mathcal{D}(A).$$

As we have seen, we can extend the definition of Potential Energy to $\mathcal{D}\left(\sqrt{A}\right)$. Also the total energy vanishes for

$$Y \;=\; \begin{vmatrix} y_1 \\ 0 \end{vmatrix}, \quad y_1 \in \text{Nullspace of } A.$$

Let $\mathcal{H}_1$ denote the M-orthogonal complement of the null space of A:

$$\mathcal{H}_1 \;=\; [x \mid [Mx, \phi] = 0, \; \phi \in \text{Nullspace of } A].$$

Then

$$\phi_k \in \mathcal{H}_1 \quad \text{for every } k\,, \; \omega_k \neq 0.$$

In other words we consider $\mathcal{H}$ under the M-inner product,

$$[x, y]_M \;=\; [Mx, y]$$

which is equivalent to the original inner product since M has a bounded inverse. We now define the energy space $\mathcal{H}_E$ by

$$\mathcal{H}_E = \left(\mathcal{D}\left(\sqrt{A}\right) \cap \mathcal{H}_1\right) \times \mathcal{H}$$

with inner product defined by

$$[Y, Z]_E = \left[\sqrt{A}\, y_1, \ \sqrt{A}\, z_1\right] + [M y_2, z_2]$$

where

$$Y = \left| \begin{array}{c} y_1 \\ y_2 \end{array} \right|; \quad Z = \left| \begin{array}{c} z_1 \\ z_2 \end{array} \right|.$$

Note that

$$[Y, Y]_E = 2[\text{Total Energy}]$$

and hence the name "energy norm" space. We shall show now that $\mathcal{H}_E$ is actually complete. Let

$$Y_n = \left| \begin{array}{c} x_n \\ z_n \end{array} \right|$$

be a Cauchy sequence in $\mathcal{H}_E$. Then

$$||Y_n - Y_m||_E^2 = \left\|\sqrt{A}(x_n - x_m)\right\|^2 + \left\|\sqrt{M}(z_n - z_m)\right\|^2.$$

Since $\sqrt{M}$ has a bounded inverse,

$$z_n \to z \quad \text{in } \mathcal{H}.$$

Now

$$x_n \in \mathcal{H}_1,$$

and $\sqrt{A}$ restricted to $\mathcal{H}_1$ has a bounded inverse. Hence

$$x_n = \left(\sqrt{A}\right)^{-1} \sqrt{A}\, x_n$$

converges, and denoting the limit by x, we have

$$x = \left(\sqrt{A}\right)^{-1} (\lim \sqrt{A}\, x_n)$$

and

$$x \in \mathcal{D}\left(\sqrt{A}\right) \cap \mathcal{H}_1.$$

Hence $\mathcal{H}_E$ is complete.

Let $\phi_{0,k}$, $k = 1, \ldots, 6$, denote an M-orthonormal basis for the null space of A:

$$[M\phi_{0,k},\ \phi_{0,j}] \ = \ \delta_j^k.$$

Define the projection operator (self-adjoint in the M-inner product) by

$$\mathcal{P}_0 x \ = \ \sum_1^6 [x,\ M\phi_{0,k}]\,\phi_{0,k}$$

and let

$$\mathcal{P}_1 \ = \ I - \mathcal{P}_0$$

so that

$$[\mathcal{P}_1 x, M\phi] \ = \ 0, \qquad \text{if } \mathcal{P}_0\phi = \phi.$$

Then

$$\mathcal{P}_1\mathcal{H} = \mathcal{H}_1.$$

The operator $\mathcal{A}$ is defined then as follows:

$$\mathcal{D}(\mathcal{A}) \ = \ \left[y = \begin{vmatrix} x_1 \\ x_2 \end{vmatrix},\ \begin{array}{l} x_1 \in \mathcal{D}(A) \cap \mathcal{H}_1 \\ x_2 \in \mathcal{D}\left(\sqrt{A}\right) \end{array} \right] \tag{4.8}$$

$$\mathcal{A} \ = \ \begin{vmatrix} 0 & \mathcal{P}_1 \\ -M^{-1}A & 0 \end{vmatrix}$$

$$\mathcal{A}Y \ = \ \begin{vmatrix} \mathcal{P}_1 x_2 \\ -M^{-1}Ax_1 \end{vmatrix};$$

thus defined $\mathcal{A}$ is closed and has a dense domain.

Let $\mathcal{A}^*$ denote the adjoint. Then

$$\mathcal{A}^* \ = \ \begin{vmatrix} 0 & -\mathcal{P}_1 \\ M^{-1}A & 0 \end{vmatrix} \ = \ -\mathcal{A}.$$

To prove this, let

$$Y = \begin{vmatrix} y_1 \\ y_2 \end{vmatrix}; \quad Z = \begin{vmatrix} z_1 \\ z_2 \end{vmatrix}; \quad Y, Z \in \mathcal{D}(\mathcal{A}).$$

Then

$$
\begin{aligned}
[\mathcal{A}Y, Z]_E &= \left[\sqrt{A}\, y_2,\ \sqrt{A}\, z_1\right] - [Ay_1, z_2] \\
&= [M y_2,\ M^{-1}Az_1] - \left[\sqrt{A}\, y_1, \sqrt{A}\, z_2\right] \\
&= [Y,\ \mathcal{A}^* z]_E \\
&= -[Y, \mathcal{A}z]_E.
\end{aligned}
$$

In what follows we shall omit the subscript E in inner products and norms where elements of $\mathcal{H}_E$ are involved. In particular, for Y in the domain of $\mathcal{A}$:

$$[\mathcal{A}Y, Y] + [Y, \mathcal{A}Y] = \operatorname{Re}[\mathcal{A}Y, Y] = 0.$$

Hence (see [Balakrishnan 1981]), $\mathcal{A}$ generates a strongly continuous semigroup $S(t)$, $t \geq 0$: actually a group:

$$S(t)^* = (S(t))^{-1} = S(-t).$$

For Y in the domain of $\mathcal{A}$, $S(t)Y$ is also in the domain of $\mathcal{A}$, and

$$\frac{d}{dt} S(t)Y = \mathcal{A}S(t)Y.$$

Hence the equation

$$\dot{Y}(t) = \mathcal{A}Y(t)$$

with the derivative interpreted in the strong sense:

$$\left\| \dot{Y}(t) - \frac{Y(t+\Delta) - Y(t)}{\Delta} \right\| \to 0 \quad \text{as } \Delta \to 0$$

has the unique solution:

$$Y(t) = S(t)Y(0)$$

such that

$$||Y(t) - Y(0)|| \to 0.$$

Note that for Y in $\mathcal{D}(\mathcal{A})$, we can exploit strong differentiability to yield:

$$\frac{d}{dt}[S(t)Y, S(t)Y] = [\mathcal{A}S(t)Y, S(t)Y] + [S(t)Y, \mathcal{A}S(t)Y] = 0. \tag{4.9}$$

Hence

$$||S(t)Y||^2 = ||Y||^2.$$

The domain of $\mathcal{A}$ being dense, we have that

$$||S(t)Y|| = ||Y||, \quad Y \in \mathcal{H}_E.$$

In particular, the energy stays constant in time.

More generally, for any $Y(0)$ in $\mathcal{H}_E$,

$$\dot{Y}(t) = \mathcal{A}Y(t) + \mathcal{B}u(t)$$

interpreted in the weak sense

$$\frac{d}{dt}[Y(t), Y] = [Y(t), \mathcal{A}^*Y] + [\mathcal{B}u(t), Y]$$

for every Y in the domain of $\mathcal{A}^*$, has the unique solution (see [Balakrishnan 1981]):

$$Y(t) = S(t)Y(0) + \int_0^t S(t-\sigma)\mathcal{B}u(\sigma)\, d\sigma, \quad t \geq 0$$

for

$$\int_0^t ||u(\sigma)||^2\, d\sigma < \infty \quad \text{for every } t > 0.$$

Remark

We can finally relate (2.14) to (2.1). Requiring the initial condition vector Y to be in $\mathcal{D}(\mathcal{A})$ means in particualr that we can define the necessary partial derivatives in (2.1) and thus have a solution in the " ordinary" or pointwise sense.

Spectral Properties of $\mathcal{A}$

The spectral properties of $\mathcal{A}$ are readily deduced from the spectral properties of A. Thus the equation

$$\lambda X - \mathcal{A}X = Y$$

$$X = \begin{vmatrix} x_1 \\ x_2 \end{vmatrix}; \quad Y =, \begin{vmatrix} y_1 \\ y_2 \end{vmatrix}$$

yields for X:

$$\lambda x_1 - \mathcal{P}_1 x_2 = y_1,$$

$$\lambda x_2 + M^{-1} A x_1 = y_2.$$

Or,

$$\left.\begin{aligned} \lambda^2 M x_1 + A x_1 &= M(y_2 + \lambda y_1 - \lambda \mathcal{P}_0 x_2) \\ \mathcal{P}_1 x_2 &= \lambda x_1 - y_1 \end{aligned}\right\}. \tag{4.10}$$

The eigenvalues of $\mathcal{A}$ are thus given by

$$\left.\begin{aligned} \lambda^2 M x_1 + A x_1 &= \lambda \mathcal{P}_0 x_2 \\ \mathcal{P}_1 x_2 &= \lambda x_1 \end{aligned}\right\}. \tag{4.11}$$

Zero is an eigenvalue of $\mathcal{A}$, since zero is an eigenvalue of A. In fact

$$\mathcal{A}\begin{vmatrix} 0 \\ \phi \end{vmatrix} = \begin{vmatrix} \mathcal{P}_1\phi \\ 0 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \end{vmatrix}, \quad \text{if } A\phi = 0.$$

Thus the dimension of the null space of $\mathcal{A}$ is 6.

For nonzero λ, we note that

$$\Phi_k = \begin{vmatrix} \phi_k \\ i\omega_k \phi_k \end{vmatrix}$$

is an eigenvector with eigenvalue $(i\omega_k)$. For

$$[M\phi_k, \mathcal{P}_0 x] = 0, \quad \text{or} \quad \phi_k \in \mathcal{D}(A) \cap \mathcal{H}_1$$

$$\mathcal{P}_0 \phi_k = 0; \quad -\omega_k^2 M\phi_k + A\phi_k = 0,$$

and hence (4.11) is satisfied. The dimension of the eigenfunction space is equal to the dimension of the eigenfunction space of A corresponding to ω_k^2. We shall for simplicity take this dimension to be equal to one. Hence the eigenvalues of $\mathcal{A}$ may be enumerated as:

$$\omega_0 = 0$$

$$\{\pm i|\omega_k|\}, \quad \omega_k \neq 0, \quad k > 0; \; \omega_{k+1} > \omega_k$$

$$\mathcal{A}\Phi_k = i|\omega_k|\Phi_k, \quad \omega_k \neq 0$$

$$\mathcal{A}\overline{\Phi}_k = -i|\omega_k|\,\overline{\Phi}_k, \quad \omega_k \neq 0, \quad \text{since } \phi_k = \overline{\phi}_k$$

where

$$\Phi_k = \begin{vmatrix} \phi_k \\ i|\omega_k|\phi_k \end{vmatrix}$$

$$\Phi_{0,i}, \quad i = 1, \ldots, 6 = \begin{vmatrix} 0 \\ \phi_{0,i} \end{vmatrix}$$

where

$$\overline{\Phi}_{0,i} = \Phi_{0,i}$$

$$[M\phi_{0,i}, \phi_{0,j}] = \delta^i_j.$$

Note that

$$\mathcal{A}^*\Phi_k = -i\omega_k\Phi_k$$

$$\mathcal{A}^*\overline{\Phi}_k = i|\omega_k|\,\overline{\Phi}_k$$

$$\mathcal{A}^*\Phi_{0,i} = 0.$$

These eigenfunctions are orthogonal:

$$\begin{aligned} \left[\Phi_k, \overline{\Phi}_j\right] &= 0, \\ [\Phi_k, \Phi_j] &= 0, \quad k \neq j \\ [\Phi_k, \Phi_k] &= 2\omega_k^2 \, [M\phi_k, \phi_k]. \end{aligned}$$

To orthonormalize the Φ_k, we need only to take

$$[M\phi_k, \phi_k] = \frac{1}{2\omega_k^2} \tag{4.12}$$

which we shall assume in what follows.

Let us show that the $\{\Phi_k, \overline{\Phi}_k\}$ are complete. Suppose for some Ψ in $\mathcal{H}_E$

$$[\Phi_k, \psi] = [\overline{\Phi}_k, \psi] = 0 \quad \text{for every } k.$$

Writing

$$\Psi = \begin{vmatrix} \psi_1 \\ \psi_2 \end{vmatrix}$$

we have

$$0 = [\Phi_k, \Psi] = \omega_k^2[M\phi_k, \psi_1] + i\omega_k[M\phi_k, \psi_2],$$

$$0 = \left[\overline{\Phi}_k, \Psi\right] = \omega_k^2[M\phi_k, \psi_1] - i\omega_k[M\phi_k, \psi_2]$$

$$0 = [\Phi_{0,k}, \Psi] = 0$$

from which it follows that

$$[M\phi_k, \psi_2] = 0 \quad \text{for every } k$$

and from the completeness of $\{\phi_k\}$ we have that

$$\psi_2 = 0.$$

We also have that

$$[M\phi_k, \psi_1] = 0, \quad \omega_k \neq 0$$

and by definition ψ_1 is in $\mathcal{H}_1$. Hence it follows that

$$\psi_1 = 0.$$

As a consequence we have the modal expansion:

$$Y = \sum_1^\infty [Y, \Phi_k]\Phi_k + \sum_1^\infty \left[Y, \overline{\Phi}_k\right]\overline{\Phi}_k + \sum_1^6 [Y, \Phi_{0,k}]\Phi_{0,k}. \tag{4.13}$$

Since

$$S(t)\Phi_k = e^{i\omega_k t}\Phi_k; \quad S(t)\overline{\Phi}_k = e^{-i\omega_k t}\overline{\Phi}_k; \quad S(t)\Phi_{0,k} = e^{i\omega_k t}\Phi_{0,k}$$

we have also

$$S(t)Y = \sum_1^\infty [Y, \Phi_k]e^{i\omega_k t}\Phi_k + \sum_1^\infty \left[Y, \overline{\Phi}_k\right] e^{-i\omega_k t}\overline{\Phi}_k$$
$$+ \sum_1^6 [Y, \Phi_{0,k}]\,\Phi_{0,k}. \tag{4.14}$$

Resolvent of $\mathcal{A}$

For

$$\lambda \neq \pm i|\omega_k|, \quad \lambda \neq 0,$$

we shall now show that

$$\lambda I - \mathcal{A}$$

has a bounded inverse. To calculate this inverse we go back to (4.8), where we let

$$\mathcal{P}_0 x_2 = z.$$

Then

$$\lambda^2 M x_1 + A x_1 = M y_2 + \lambda M y_1 - \lambda M z.$$

For $\lambda \neq \pm i|\omega_k|$, $\lambda \neq 0$, we can define

$$w = (\lambda^2 M + A)^{-1}(M y_2 + \lambda M y_1).$$

Then

$$\lambda^2 M(x_1 - w) + A(x_1 - w) = -\lambda M z.$$

Hence

$$\lambda^2 M(x_1 - \mathcal{P}_1 w) + A(x_1 - \mathcal{P}_1 w) = \lambda^2 M \mathcal{P}_0 w - \lambda M z = 0$$

if we let

$$z = \lambda \mathcal{P}_0 w$$

or,

$$x_2 = \lambda \mathcal{P}_1 w - y_1 + \lambda \mathcal{P}_0 w = \lambda w - y_1.$$

Hence we obtain

$$\begin{vmatrix} x_1 \\ x_2 \end{vmatrix} = \begin{vmatrix} \mathcal{P}_1 w \\ \lambda w - y_1 \end{vmatrix} = (\lambda I - \mathcal{A})^{-1} \begin{vmatrix} y_1 \\ y_2 \end{vmatrix} \tag{4.15}$$

where

$$w = (\lambda^2 M + A)^{-1}(M y_2 + \lambda y_1)$$

and

$$w \in \mathcal{D}(A).$$

We use the notation:

$$R(\lambda, \mathcal{A}) = (\lambda I - \mathcal{A})^{-1}$$

and refer to the left side as the "resolvent of $\mathcal{A}$." We have also the modal representation:

$$R(\lambda, \mathcal{A})Y \;=\; \sum_1^\infty \frac{[Y, \Phi_k]}{\lambda - i\omega_k}\,\Phi_k \;+\; \sum_1^\infty \frac{[Y, \overline{\Phi}_k]}{\lambda + i\omega_k}\,\overline{\Phi}_k \;+\; \sum_1^6 \frac{[Y, \Phi_{0,k}]}{\lambda}\,\Phi_{0,k}. \tag{4.16}$$

It is indeed the Laplace Transform:

$$R(\lambda, \mathcal{A})Y \;=\; \int_0^\infty e^{-\lambda t} S(t)Y\,dt, \quad \operatorname{Re}\lambda > 0, \tag{4.17}$$

but of course is defined and analytic in λ except for poles at the eigenvalues as follows from (4.13) and (4.14). Note in particular that

$$\mathcal{B}^*\mathcal{R}(\lambda, \mathcal{A})Y \;=\; -B^*(\lambda w - y_1)$$

and hence it follows that

$$\mathcal{B}^*\mathcal{R}(\lambda, \mathcal{A})\mathcal{B} \;=\; \lambda B^*(\lambda^2 M + A)^{-1} B \;=\; \lambda B_u^*\left(\lambda^2 M_b + T(\lambda)\right)^{-1} B_u. \tag{4.18}$$

5. CONTROLLABILITY AND STABILIZABILITY

In this section we show that under a controllability condition, rate feedback using a collocated sensor can stabilize the system. All modes will decay even though the damping coefficient will decrease with mode frequency. Whatever the initial conditions, the elastic energy will eventually dissipate to zero. (This is known as "strong" stability.) In our model we neglect any inherent damping in the structure. However the controller is robust in the sense it will not destabilize any mode — it will only increase the damping. We have always stability enhancement, in other words.

The main results are known — see [Balakrishnan 1981] and the references therein. The presentation here tries as far as possible to be self-contained, using the specific features of the problem at hand rather than merely quoting general results.

In reference to our dynamic equation:

$$\dot{Y}(t) = \mathcal{A}Y(t) + \mathcal{B}u(t) \tag{5.1}$$

we do not include a damping operator; $\mathcal{A}$ is not "stable." In fact

$$||S(t)Y||_E = ||Y||_E.$$

The elastic energy is not dissipated in the absence of control. In finite dimensional theory, the system is "exponentially" stabilizable, if $(\mathcal{A}, \mathcal{B})$ is "exactly" controllable — by that we mean given any Y_1, Y_2 we can find a control $u(\cdot)$ such that

$$Y_2 = S(t)Y_1 + \int_0^t S(t-\sigma)\mathcal{B}u(\sigma)\, d\sigma$$

for some $t \geq 0$. This is impossible in our case because (range of) $\mathcal{B}$ is finite-dimensional. (See [Balakrishnan 1981], for a proof.) The next best thing we can do is to require just "controllability." Thus we say that $(\mathcal{A}, \mathcal{B})$ is "controllable" if

$$\bigcup_{t\geq 0} \text{(range of } S(t)\mathcal{B})$$

is dense in $\mathcal{H}_E$. This is equivalent to saying that

$$\bigcup_{t\geq 0} \left(\int_0^t S(t-\sigma)\mathcal{B}u(\sigma)\, d\sigma;\ u(\cdot) \in L_2(0,t) \right)$$

is dense in $\mathcal{H}_E$. In other words we require that the states "reachable" from the zero state are dense in $\mathcal{H}_E$.

Theorem 5.1

$\mathcal{A} \sim \mathcal{B}$ is controllable in $\mathcal{H}_E$ if and only if

$$B^*\phi \neq 0 \tag{5.2}$$

for any eigenfunction ϕ, defined by:

$$A\phi = \omega^2 M\phi, \quad \phi \neq 0.$$

Equivalently

$$\mathcal{B}^*\Phi \neq 0 \tag{5.3}$$

for any mode Φ:

$$\mathcal{A}\Phi = i\omega\Phi, \quad \Phi \neq 0.$$

Proof

It is convenient to use a modal expansion for any Y as:

$$Y \;=\; \sum_{1}^{\infty} P_k Y \;+\; \sum_{1}^{\infty} P_{-k} Y \;+\; P_0 Y$$

where P_k is the Projector onto the eigenfunction space corresponding to the eigenvalue $i\omega_k$, and P_{-k} corresponding to $-i\omega_k$, and P_0 onto the null space of $\mathcal{A}$. Then

$$S(t)Y \;=\; \sum_{1}^{\infty} e^{i\omega_k t} P_k Y \;+\; \sum_{1}^{\infty} e^{-i\omega_k t} P_{-k} Y \;+\; P_0 Y. \tag{5.4}$$

If the set

$$\bigcup_{t \geq 0} S(t)\mathcal{B}u$$

where u ranges over all of $\mathcal{R}^{m_c}$ is *not* dense in $\mathcal{H}_E$, we can find a nonzero element Y in $\mathcal{H}_E$ such that

$$[S(t)\mathcal{B}u, Y] \;=\; 0, \quad t \geq 0, \; u \in \mathcal{R}^{m_c}.$$

Hence using (5.4) we must have

$$\sum_{1}^{\infty} e^{i\omega_k t} \mathcal{B}[u,\; \mathcal{B}^* P_k Y] \;+\; \sum_{1}^{\infty} e^{-i\omega_k t} [u,\; \mathcal{B}^* P_{-k} Y] \;+\; [u,\; \mathcal{B}^* P_0 Y] \;=\; 0, \quad t \geq 0.$$

But the left side is an almost periodic function in t, and can vanish identically if and only if for every k:

$$0 \;=\; [u,\; \mathcal{B}^* P_k y] \;=\; [u,\; \mathcal{B}^* P_{-k} Y] \;=\; [u, \mathcal{B}^* P_0 Y].$$

Since Y is not zero, there must be at least one k such that

$$P_k Y \neq 0$$

or

$$P_k Y = \Phi_k,$$

where

$$\mathcal{A}\Phi_k \;=\; i\omega_k \Phi_k, \quad \Phi_k \neq 0$$

$$[u,\; \mathcal{B}^* \Phi_k] \doteq 0.$$

But u being arbitrary, we must have

$$\mathcal{B}^* \Phi_k = 0$$

which is a contradiction. Since Φ_k must be of the form:

$$\Phi_k = \left| \begin{array}{c} \phi_k \\ i\omega_k \phi_k \end{array} \right|$$

it follows that

$$B^*\phi_k = 0$$

which is again impossible by assumption.

Next, suppose $\mathcal{A} \sim \mathcal{B}$ is controllable. Suppose

$$B^*\phi = 0$$

for some mode ϕ

$$A\phi = \omega^2 M\phi, \quad \phi \neq 0.$$

Then for

$$\Phi = \left| \begin{array}{c} \phi \\ i\omega\phi \end{array} \right|$$

$$[S(t)\mathcal{B}u, \Phi] = e^{i\omega t}[\mathcal{B}u, \Phi] = e^{i\omega t}(i\omega)[u, B^*\phi] = 0.$$

Hence Φ is orthogonal to

$$S(t)\mathcal{B}u$$

for every t and $u \in \mathcal{R}^{m_c}$. Hence $(\mathcal{A} \sim \mathcal{B})$ is not controllable — contradicting the hypothesis.

We shall show that $(\mathcal{A} - \mathcal{B})$ is controllable for our system in Section 4 where we study the eigenvalue problem.

Corollary

Suppose $(\mathcal{A} - \mathcal{B})$ is controllable. Then the number of controls (the dimension of the control space) must be at least 6. More generally, the dimension of the control space must be at least equal to the largest eigenfunction-space dimension.

Proof

Let $\phi_{0,i}$, $i = 1, \ldots, 6$, be a basis for the null space of A. Suppose $B^*\phi_{0,i}$ are linearly dependent. Then

$$\sum_1^6 a_k B^* \phi_{0,k} = 0, \qquad \text{not all } a_k = 0$$

or

$$B^* \left(\sum_1^6 a_k \phi_{0,k} \right) = 0.$$

But

$$\sum_1^6 a_k \phi_{0,k}$$

is a nonzero eigenfunction function of A corresponding to the eigenvalue zero and controllability is thus violated.

Next we shall prove the fundamental relationship of controllability to stabilizabilty.

Theorem 5.2

Suppose $(\mathcal{A}, \mathcal{B})$ is controllable. Then the feedback control

$$u(t) = -\alpha \mathcal{B}^* Y(t), \quad \alpha > 0 \tag{5.5}$$

is such that

$$||Y(t)|| \to 0 \quad \text{as } t \to \infty$$

for every $Y(0)$.

Proof

The closed-loop system dynamic is now

$$\left.\begin{array}{c} \dot{Y}(t) = (\mathcal{A} - \alpha \mathcal{B}\mathcal{B}^*) Y(t), \\ Y(0) \quad \text{given.} \end{array}\right\} \tag{5.6}$$

Let

$$Y(t) = \left| \begin{array}{c} x_1(t) \\ x_2(t) \end{array} \right|.$$

Then we have

$$\dot{x}_1(t) \;=\; \mathcal{P}_1 x_2(t) \tag{5.7}$$

$$\dot{x}_2(t) \;=\; -M^{-1}Ax_1(t) \;-\; \alpha M^{-1}BB^*x_2(t). \tag{5.8}$$

To relate this to (2.17), let the initial conditions for the latter be given as

$$\begin{vmatrix} x(0) \\ \dot{x}(0) \end{vmatrix}.$$

For (3.6) let

$$Y(0) \;=\; \begin{vmatrix} x_1(0) \\ x_2(0) \end{vmatrix}$$

where

$$x_1(0) \;=\; \mathcal{P}_1 x(0)$$

$$x_2(0) \;=\; \dot{x}(0).$$

Define

$$x(t) \;=\; x_1(t) \;+\; \mathcal{P}_0 x(0) \;+\; \int_0^t \mathcal{P}_0 x_2(\sigma)\, d\sigma.$$

Then

$$\dot{x}(t) \;=\; \dot{x}_1(t) + \mathcal{P}_0 x_2(t)$$

and (5.7), (5.8) yield

$$\dot{x}(t) \;=\; x_2(t)$$

$$\begin{aligned} \dot{x}_2(t) \;&=\; \mathcal{P}_1\dot{x}_2(t) + \mathcal{P}_0\dot{x}_2(t) \\ &=\; \ddot{x}(t) \\ &=\; -M^{-1}Ax(t) \;-\; \alpha M^{-1}BB^*\dot{x}(t) \end{aligned}$$

or,

$$M\ddot{x}(t) + Ax(t) + \alpha BB^*\dot{x}(t) \;=\; 0 \tag{5.9}$$

so that the control $u(t)$ is now

$$u(t) \;=\; \alpha B^*\dot{x}(t) \;=\; -\alpha \mathcal{B}^* Y(t). \tag{5.10}$$

In other words we have rate feedback using a collocated sensor — a feedback principle for stabilization that is age-old, but still requires proof in our infinite-dimensional context.

First we note that for Y in $\mathcal{D}(\mathcal{A})$:

$$\begin{aligned} \mathrm{Re}\,[(\mathcal{A} - \alpha\mathcal{B}\mathcal{B}^*)Y,\, Y] &= -\alpha||\mathcal{B}^*Y||^2 \\ &= \mathrm{Re}\,[(\mathcal{A}^* - \alpha\mathcal{B}\mathcal{B}^*)Y,\, Y]. \end{aligned}$$

Hence, see [Balakrishnan 1981],

$$(\mathcal{A} - \alpha\mathcal{B}\mathcal{B}^*)$$

generates a dissipative strongly continuous semigroup. Denoting the latter by

$$S_\alpha(t), \quad t \geq 0$$

we note that

$$||S_\alpha(t)|| \leq 1.$$

Eigenvalues

Let us consider next the eigenvalues:

$$(\mathcal{A} - \alpha\mathcal{B}\mathcal{B}^*)Y = \lambda Y \tag{5.11}$$

or, then

$$(\mathrm{Re}\,\lambda)[Y, Y] = -\alpha||\mathcal{B}^*Y||^2.$$

Here

$$||\mathcal{B}^*Y|| \quad \text{cannot be zero}$$

for if

$$\mathcal{B}^*Y = 0$$

$$(A - \alpha\mathcal{B}\mathcal{B}^*)Y = \mathcal{A}Y = \lambda Y$$

and we violate the controllability condition. Hence it follows that

$$\mathrm{Re}\,\lambda = \frac{-\alpha||\mathcal{B}^*Y||^2}{[Y, Y]} < 0.$$

Hence every eigenvalue has a strictly negative real part. Hence rewriting (5.11) as

$$(\lambda I - \mathcal{A} + \alpha\mathcal{B}\mathcal{B}^*)Y \;=\; 0$$

and multiplying on the left by $\mathcal{R}(\lambda, \mathcal{A})$, we have

$$Y \,+\, \alpha\mathcal{R}(\lambda, \mathcal{A})\,\mathcal{B}\mathcal{B}^*Y \;=\; 0.$$

Hence

$$\mathcal{B}^*Y \,+\, \alpha\mathcal{B}^*\mathcal{R}(\lambda, \mathcal{A})\,\mathcal{B}\mathcal{B}^*Y \;=\; 0$$

or,

$$(I + \alpha\mathcal{B}^*\mathcal{R}(\lambda, \mathcal{A})\mathcal{B})\mathcal{B}^*Y \;=\; 0$$

where inside the parentheses on the left side is an $m_c \times m_c$ matrix. Hence the eigenvalues are the roots of

$$D(\lambda, \alpha) = 0 \tag{5.12}$$

where

$$D(\lambda, \alpha) \;=\; \det(I + \alpha\mathcal{B}^*\mathcal{R}(\lambda, A)\mathcal{B}). \tag{5.13}$$

But using (4.18), we have

$$D(\lambda, \alpha) \;=\; \det\Big[I \,+\, \alpha\lambda B_u^*(\lambda^2 M_b + T(\lambda))^{-1}B_u\Big]. \tag{5.14}$$

Let $\{\lambda_k\}$ denote the eigenvalues, where we know that λ_k must have the form:

$$\lambda_k \;=\; -|\sigma_k| \,+\, i\nu_k, \quad \sigma_k, \nu_k \text{ real.}$$

The corresponding eigenfunctions, denote them Y_k, are then given by

$$Y_k \;=\; \mathcal{R}(\lambda_k, \mathcal{A})\,\mathcal{B}u(\lambda_k) \tag{5.15}$$

where

$$(I + \alpha\mathcal{M}(\lambda_k))u(\lambda_k) \;=\; 0, \quad ||u(\lambda_k)|| = 1$$

where

$$\mathcal{M}(\lambda) \;=\; \mathcal{B}^*\mathcal{R}(\lambda, \mathcal{A})\mathcal{B}.$$

We note that the dimension of the eigenfunction space is the dimension of the eigenvector space corresponding to the eigenvalue zero of the $m_c \times m_c$ matrix:

$$I + \alpha\mathcal{M}(\lambda_k))$$

and is thus less than m_c. It is equal to one, if we assume that the matrix has distinct eigenvalues, which we shall, for simplicity, in what follows.

At this point we leave open whether the sequence $\{\lambda_k\}$ is finite or not. (We shall eventually see that it is not.)

The eigenfunctions $\{Y_k\}$ it must be noted are *not* orthogonal. We shall normalize them so that

$$||Y_k|| = 1.$$

Lemma 5.1

$$\sum_1^{\infty}(-\sigma_k) \;\leq\; \alpha \mathrm{Tr}\, \mathcal{B}\mathcal{B}^* \;=\; \alpha \mathrm{Tr}\, B_u^* B_u. \tag{5.16}$$

Proof

We follow essentially [Gohberg-Krein 1969, p. 101]. Let us orthogonalize $\{Y_k\}$ following the Gram-Schmidt procedure. Then

$$Z_k \;=\; Y_k - \sum_{j=1}^{k-1} a_{kj} Y_j, \quad [Z_k\, Z_j] \;=\; 0, \quad k \neq j.$$

Recall that the $\{Y_k\}$, $k = 1, \ldots, n$, cannot be linearly dependent for any n, since the $\{\lambda_k\}$ are distinct. Hence

$$||Z_k|| \neq 0.$$

Now

$$(\mathcal{A} - \alpha\mathcal{B}\mathcal{B}^*)Z_k \;=\; \lambda_k Y_k - \sum_{j=1}^{k-1} a_{kj} \lambda_j Y_j\,.$$

Hence

$$[(\mathcal{A} - \alpha\mathcal{B}\mathcal{B}^*)Z_k\,,\, Z_k] \;=\; \lambda_k[Y_k, Z_k] \;=\; \lambda_k[Z_k, Z_k].$$

Hence

$$\mathrm{Re}\,\lambda_k \;=\; \frac{-\alpha[\mathcal{B}\mathcal{B}^* Z_k,\, Z_k]}{[Z_k, Z_k]}\,.$$

Now $\mathcal{B}\mathcal{B}^*$ is nuclear, $\mathcal{B}$ being finite-dimensional. Since

$$\left\{ \frac{Z_k}{\sqrt{[Z_k, Z_k]}} \right\}$$

is now an orthonormal sequence, we have:

$$\sum_1^\infty \frac{[\mathcal{B}\mathcal{B}^* Z_k\,,\, Z_k]}{[Z_k, Z_k]} \leq \operatorname{Tr} \mathcal{B}\mathcal{B}^*.$$

We note that

$$\operatorname{Tr} \mathcal{B}\mathcal{B}^* = \operatorname{Tr} \mathcal{B}^*\mathcal{B} = \operatorname{Tr} B_u^* B_u.$$

Hence (3.9) follows. We have an obvious corollary:

Corollary

If the sequence $\{\lambda_k\}$ is not finite, then

$$|\sigma_k| \to 0 \qquad \text{as } k \to \infty$$

$$|\nu_k| \to \infty \qquad \text{as } k \to \infty.$$

Proof

The first part of the statement follows from (3.16) and the second part from the fact

$$\det(I + \alpha \mathcal{M}(\lambda))$$

is an analytic function for Re λ negative, so that no subsequence of the sequence $\{\lambda_k\}$ can have a finite limit point. Hence

$$|\lambda_k| \to \infty \qquad \text{as } k \to \infty.$$

This will also follow from the fact that the resolvent

$$\mathcal{R}(\lambda,\ \mathcal{A} - \alpha\mathcal{B}\mathcal{B}^*), \quad \lambda \neq \lambda_k$$

is compact, as we shall see presently. Nevertheless, we still have to prove that the sequence $\{\lambda_k\}$ is *not* finite, which we shall in Section 7.

Now

$$S_\alpha(t)Y_k \;=\; e^{-|\sigma_k|t}e^{i\mu_k t}Y_k, \quad t \geq 0$$

and hence each mode is damped with damping coefficient σ_k. However, the number of modes is not finite, and the damping coefficient eventually goes to zero, so that we cannot guarantee a finite gain margin. In particular the fact that each mode decays is not enough to prove (5.6).

For this purpose we can invoke a general result due to [Benchimol 1978]. We are assuming that $\mathcal{A} - \mathcal{B}$ is controllable and we have seen that the resolvent of $\mathcal{A}$ is compact, and that

$$\mathcal{A} + \mathcal{A}^* = 0.$$

Hence by a theorem of Benchimol — [Benchimol 1978] — it follows that the semigroup $S_\alpha(\cdot)$ is "strongly stable":

$$||S_\alpha(t)Y|| \to 0 \quad \text{as } t \to \infty.$$

Since

$$Y(t) \;=\; S_\alpha(t)Y(0),$$

the theorem is proved. We shall give an independent and self-contained proof in Section 7.

Corollary

The solution of (5.9) is such that the total energy — elastic plus kinetic:

$$||\sqrt{A}\, x(t)||^2 \,+\, [M\dot{x}(t),\, \dot{x}(t)]$$

is monotonic nonincreasing as t increases and decays to zero as t increases without bound. Moreover

$$\mathcal{P}_0(x(t)) \;\to\; \mathcal{P}_0\left(x(0) - \dot{x}(0)\right), \quad \text{as } t \to \infty\,. \tag{5.17}$$

Proof

We have only to note that

$$Y(t) = \begin{vmatrix} \mathcal{P}_1 x(t) \\ \dot{x}(t) \end{vmatrix}$$

and

$$||Y(t)||^2 = ||\sqrt{A}\, x(t)||^2 + [M\dot{x}(t),\ \dot{x}(t)].$$

Thus the elastic energy decays to zero. The rigid-body component is given by

$$\mathcal{P}_0 x(t) = \int_0^t \mathcal{P}_0 \dot{x}(\sigma)\, d\sigma + \mathcal{P}_0 x(0)$$

where we so far only know that

$$||\mathcal{P}_0 \dot{x}(t)|| \to 0 \quad \text{as } t \to \infty.$$

If the initial state is such that

$$\begin{vmatrix} \mathcal{P}_1 x(0) \\ \dot{x}(0) \end{vmatrix} = Y_k$$

is an eigenfunction, then

$$Y(t) = e^{\lambda_k t} Y_k$$

so that

$$\mathcal{P}_0 \dot{x}(t) = \lambda_k e^{\lambda_k t} \mathcal{P}_0 \dot{x}(0)$$

and hence

$$\begin{aligned} \mathcal{P}_0(x(t)) &= \mathcal{P}_0(x(0)) + e^{\lambda_k t}\mathcal{P}_0(\dot{x}(0)) - \mathcal{P}_0(\dot{x}(0)) \\ &\to \mathcal{P}_0(x(0)) - \mathcal{P}_0(\dot{x}(0)) \quad \text{as } t \to \infty. \end{aligned}$$

We shall show in Section 7 that this holds generally, using the modal expansion.

Resolvent

Let us see how the resolvent of $(\mathcal{A} - \alpha\mathcal{B}\mathcal{B}^*)$ can be expressed in terms of the resolvent of $\mathcal{A}$. We have

$$\mathcal{R}(\lambda,\ \mathcal{A} - \alpha\mathcal{B}\mathcal{B}^*)X = Y$$

or

$$(\lambda I - \mathcal{A} + \alpha \mathcal{B}\mathcal{B}^*)Y \;=\; X.$$

For $\lambda \neq \lambda_k$ and $\lambda \neq i\omega_k$, 0, we can multiply on the left by $\mathcal{R}(\lambda, \mathcal{A})$ and obtain

$$Y \;+\; \alpha \mathcal{R}(\lambda, \mathcal{A})\, \mathcal{B}\mathcal{B}^* Y \;=\; \mathcal{R}(\lambda, \mathcal{A})X.$$

Hence

$$\mathcal{B}^* Y \;+\; \alpha \mathcal{B}^* \mathcal{R}(\lambda, \mathcal{A})\, \mathcal{B}\mathcal{B}^* Y \;=\; \mathcal{B}^* \mathcal{R}(\lambda, \mathcal{A})X$$

$$(I + \alpha \mathcal{M}(\lambda))\, \mathcal{B}^* Y \;=\; \mathcal{B}^* \mathcal{R}(\lambda, \mathcal{A})X$$

and the matrix on the left side being nonsingular,

$$\mathcal{B}^* Y \;=\; (I + \alpha \mathcal{M}(\lambda))^{-1} \mathcal{B}^* R(\lambda, A)X. \tag{5.18}$$

Hence it follows that

$$\mathcal{R}(\lambda,\; \mathcal{A} - \alpha \mathcal{B}\mathcal{B}^*) \;=\; \mathcal{R}(\lambda, \mathcal{A}) \;-\; \alpha \mathcal{R}(\lambda, \mathcal{A})\, \mathcal{B}(I + \alpha \mathcal{M}(\lambda))^{-1} \mathcal{B}^* \mathcal{R}(\lambda, \mathcal{A}). \tag{5.19}$$

It follows in particular that the resolvent of $(\mathcal{A} - \alpha \mathcal{B}\mathcal{B}^*)$ has all the properties of $\mathcal{R}(\lambda, \mathcal{A})$ such as being compact, Hilbert-Schmidt, etc., being a perturbation of the latter by a finite-dimensional operator.

6. ASYMPTOTIC MODES

In this section we examine in more detail the modes, both open-loop system — the undamped structure, and closed-loop — with rate feedback, in particular obtaining asymptotic estimates.

As we have seen in Section 2, the open-loop mode frequencies are the roots of

$$\det[-\omega^2 M_b + T(i\omega)] \;=\; 0 \tag{6.1}$$

where $T(\lambda)$ is self-adjoint, and $T(i\omega)$ being a function of ω^2. The mode shape is determined by

$$\mathcal{L}(i\omega_k)\, L(i\omega_k)^{-1} b(i\omega_k) \tag{6.2}$$

where

$$\left(-\omega_k^2 M_b + T(i\omega_k)\right) b(i\omega_k) \;=\; 0. \tag{6.3}$$

Our first step is to show how the dimension of the matrix that determines the eigenvalues can be reduced. In (6.1) this dimension is $6m \times 6m$. We shall show that it can be reduced to 6×6, regardless of how large m is.

We shall consider actually the closed loop system, with the corresponding eigenvalue problem:

$$(\mathcal{A} - \alpha \mathcal{B}\mathcal{B}^*)Y \;=\; \lambda Y$$

proceeding in a slightly different way than before. Let

$$Y \;=\; \begin{vmatrix} y_1 \\ y_2 \end{vmatrix}.$$

Then we have

$$\lambda y_1 \;=\; \mathcal{P}_1 y_2, \quad \lambda \neq 0$$

$$\lambda y_2 + M^{-1} A y_2 + \alpha M^{-1} B B^* y_2 \;=\; 0.$$

Hence

$$\lambda y_2 \;+\; \tfrac{1}{\lambda} M^{-1} A y_2 \;+\; \alpha M^{-1} B B^* y_2 \;=\; 0$$

or, we need to find y_2 satisfying

$$\lambda^2 M y_2 \;+\; A y_2 \;+\; \alpha B B^* y_2 \;=\; 0 \tag{6.4}$$

and then

$$y_1 \;=\; \frac{\mathcal{P}_1 y_2}{\lambda}. \tag{6.5}$$

We shall call y_2 the mode shape even though it is not purged of rigid-body modes, as (6.5) is. The advantage in going to (6.4) is that we get the undamped mode frequencies by setting $\alpha = 0$, which we cannot do with (5.13).

To proceed with (6.4), let

$$y_2 \;=\; \begin{vmatrix} f \\ b \end{vmatrix}.$$

We have

$$-A^2 f''(s) + A_1 f'(s) + A_0 f(s) + \lambda^2 M_0 f = 0, \quad s_i < s < s_{i+1} \tag{6.6}$$

$$\lambda^2 M_b b + \alpha\lambda B_u B_u^* b + A_b f = 0. \tag{6.7}$$

As we have seen in Section 2:

$$B_u B_u^* b = Db$$

where D is the "diagonal" in the sense that

$$Db = \left| \begin{matrix} D_1 b_1 \\ \vdots \\ D_m b_m \end{matrix} \right|, \quad D_i,\ 6 \times 6, \text{ self-adjoint, nonnegative definite, diagonal.}$$

$$\left| \begin{matrix} f(s) \\ f'(s) \end{matrix} \right| = e^{\mathcal{A}(\lambda)(s-s_i)} \left| \begin{matrix} f(s_i) \\ f'(s_i+) \end{matrix} \right|, \quad s_i < s < s_{i+1} \tag{6.8}$$

where

$$\mathcal{A}(\lambda) = \left| \begin{matrix} 0 & I \\ A_2^{-1}(A_0 + \lambda^2 M_0) & A_2^{-1} A_1 \end{matrix} \right|.$$

The boundary conditions (2.7) relate $f'(s_i+)$ to $f(s_i)$ and $f'(s_i-)$:

$$f'(0+) = A_2^{-1}\left(-L_1 + \alpha\lambda D_1 + \lambda^2 M_{b,0}\right) f(0)$$

$$f'(s_i+) = f'(s_i-) + A_2^{-1}\left(\alpha\lambda D_i + \lambda^2 M_{b,i}\right) f(s_i)$$

$$2 \le i \le m-1$$

and for $i = m$:

$$f'(L) = -A_2^{-1}\left(L_1 + \lambda\alpha D_m + \lambda^2 M_{b,L}\right) f(L). \tag{6.9}$$

Hence it follows that we can calculate $f(L)$ and $f'(L)$ in terms of $f(0)$ and then invoke (6.9) to obtain

$$h(\lambda, \alpha) f(0) = 0$$

where

$$
\begin{aligned}
h(\lambda;\alpha) \;=\; & \left| \begin{array}{cc} A_2^{-1}(L_1+\alpha\lambda D_m+\lambda^2 M_{b,L}) & I \end{array} \right| \\
& \cdot\, e^{\mathcal{A}(\lambda)(L-s_{m-1})} \left| \begin{array}{cc} I & 0 \\ A_2^{-1}(\alpha\lambda D_{m-1}+\lambda^2 M_{b,m-1}) & I \end{array} \right| \\
& \cdots\, e^{\mathcal{A}(\lambda)(s_{i+1}-s_i)} \left| \begin{array}{cc} I & 0 \\ A_2^{-1}(\alpha\lambda D_i+\lambda^2 M_{b,i}) & I \end{array} \right| \\
& \cdots\, e^{\mathcal{A}(\lambda)s_2} \left| \begin{array}{c} I \\ A_2^{-1}(-L_1+\alpha\lambda D_1+\lambda^2 M_{b,0}) \end{array} \right|
\end{aligned}
\tag{6.10}
$$

where I is the 6×6 Identity matrix.

Let us use the notation

$$\Delta_i \;=\; s_{i+1}-s_i, \quad i=1,\ldots,m-1$$

so that

$$
\begin{aligned}
\Delta_1 &= s_2-s_1 \;=\; s_2 \\
\Delta_{m-1} &= L-s_{m-1}.
\end{aligned}
$$

Let

$$t_i(\lambda,\alpha) \;=\; A_2^{-1}\left(\lambda^2 M_{b,i}+\alpha\lambda D_i\right), \quad 1\le i\le m \tag{6.11}$$

$$T_i(\lambda,\alpha) \;=\; \left| \begin{array}{cc} I & 0 \\ t_i(\lambda,\alpha) & I \end{array} \right|, \quad i=2,3,\ldots,m-1 \tag{6.12}$$

$$
\begin{aligned}
T_1(\lambda;\alpha) &= \left| \begin{array}{cc} A_2^{-1}L_1+t_m(\lambda,\alpha) & I \end{array} \right| \\
T_m(\lambda;\alpha) &= \left| \begin{array}{c} I \\ -A_2^{-1}L_1+t_1(\lambda,\alpha) \end{array} \right|.
\end{aligned}
$$

Then we can write $h(\lambda,\alpha)$ as:

$$h(\lambda,\alpha) \;=\; T_m(\lambda;\alpha)\,Q(\lambda,\alpha)T_1(\lambda;\alpha) \tag{6.13}$$

where

$$Q(\lambda,\alpha) \;=\; e^{\mathcal{A}(\lambda)\Delta_{m-1}}T_{m-1}(\lambda;\alpha)\,\cdots\, e^{\mathcal{A}(\lambda)\Delta_2}T_2(\lambda,\alpha)\,\cdot\, e^{\mathcal{A}(\lambda)\Delta_1}.$$

We note that the $t_i(\lambda;\alpha)$ as well as $T_i(\lambda;\alpha)$ are polynomials in λ while

$$e^{\mathcal{A}(\lambda)\Delta_i}$$

involve transcendental functions of λ.

The coefficients of the terms of the highest degree in λ as well as α are contained in the term

$$A_2^{-1}(\alpha\lambda D_m + \lambda^2 M_{b,L})P_{12}(\lambda;\Delta_{m-1}) \cdot A_2^{-1}(\alpha\lambda D_{m-1} + \lambda^2 M_{b,m-1}) \cdot \cdots$$
$$\cdot\; P_{12}(\lambda;\Delta_2)\, A_2^{-1}(\alpha\lambda D_2 + \lambda^2 M_{b,2}) \;\cdot\; P_{12}(\lambda;\Delta_1)\, A_2^{-1}(\alpha\lambda D_1 + \lambda^2 M_{b,0}). \tag{6.14}$$

In particular, the term containing highest powers of λ that occurs is:

$$\lambda^{2m} A_2^{-1} M_{b,L}\, P_{12}(\lambda;\Delta_{m-1})\, A_2^{-1} M_{b,m-1} \;\cdots\; P_{12}(\lambda;\Delta_1)\, A_2^{-1} M_{b,0}. \tag{6.15}$$

The term containing the highest powers of α that occurs is

$$\alpha^m \left(\lambda^m A_2^{-1} D_m P_{12}(\lambda;\Delta_{m-1}) A_2^{-1} D_{m-1} \;\cdots\; P_{12}(\lambda;\Delta_1) A_2^{-1} D_1\right). \tag{6.16}$$

From (4.13) we have that

$$||h(\lambda;\alpha)|| \;\leq\; ||T_1(\lambda;\alpha)|| \;\cdots\; ||T_m(\lambda;\alpha)|| \;\cdot\; ||e^{\mathcal{A}(\lambda)\Delta_1}|| \;\cdots\; ||e^{\mathcal{A}(\lambda)\Delta_{m-1}}|| \tag{6.17}$$

where $||\cdot||$ denote matrix norm.

The corresponding mode shape function $f(\cdot)$ (corresponding to y_2) is given by:

$$f(s) \;=\; |\, I \quad 0\,|\, e^{\mathcal{A}(\lambda)(s-s_i)} T_i(\lambda;\alpha) \;\cdot\; e^{\mathcal{A}(\lambda)\Delta_{i-1}} T_{i-1}(\lambda;\alpha)$$
$$\cdots\; e^{\mathcal{A}(\lambda)\Delta_i} \left| \begin{array}{c} I \\ A_2^{-1}(-L_1 + \lambda^2 M_{b,0} + \lambda\alpha D) \end{array} \right| f(0),$$
$$s_i \leq s \leq s_{i+1}. \tag{6.18}$$

Let

$$d(\lambda;\alpha) \;=\; \det h(\lambda;\alpha)$$

Then $d(\lambda;\alpha)$ is an entire function of λ and the eigenvalues $\{\lambda_k\}$ of $(\mathcal{A} - \alpha\mathcal{B}\mathcal{B}^*)$ are the nonzero roots of

$$d(\lambda;\alpha) \;=\; 0.$$

Behavior at $\lambda = 0$

Let us first consider

$$\lambda = 0.$$

We have:

$$\begin{aligned} h(0,\alpha) &= h(0,0) = \left| \begin{array}{cc} A_2^{-1}L_1 & I \end{array} \right| e^{\mathcal{A}(0)L} \left| \begin{array}{c} I \\ A_2^{-1}L_1 \end{array} \right| \\ &= 0. \end{aligned}$$

Moreover we have the power series expansion about zero:

$$h(\lambda;\alpha) = \lambda h'(0,\alpha) + \frac{\lambda^2}{2} h''(0,\alpha) + \text{ terms of higher order in } \lambda$$

and correspondingly

$$d(\lambda;\alpha) = \lambda^6 \alpha^6 d_6(\alpha) + \text{ terms of higher order in } \lambda \tag{6.19}$$

where

$$d_6(0) \neq 0.$$

Thus $d(\lambda;\alpha)$ for nonzero α has a zero of order 6 at $\lambda = 0$, while $d(\lambda, 0)$ has a zero of order 12 at $\lambda = 0$.

Relation of $d(\lambda;\alpha)$ to $D(\lambda;\alpha)$

Let us examine next the relation of $d(\lambda;\alpha)$ to $D(\lambda,\alpha)$, the latter defined in (5.13). Now

$$\begin{aligned} \mathcal{M}(\lambda) &= \mathcal{B}^*\mathcal{R}(\lambda,\mathcal{A})\mathcal{B} \\ &= \sum_1^\infty \frac{\mathcal{B}^* P_k \mathcal{B}}{\lambda - i\omega_k} + \sum_1^\infty \frac{\mathcal{B}^* P_{-k} \mathcal{B}}{\lambda + i\omega_k} + \frac{\mathcal{B}^* P_0 \mathcal{B}}{\lambda} \end{aligned} \tag{6.20}$$

with P_k as in Section 3 (see also (4.16)) and hence as λ goes to zero

$$\begin{aligned} D(\lambda,\alpha) &\sim \det\left(I + \frac{\alpha \mathcal{B}^* P_0 \mathcal{B}}{\lambda}\right) \\ &\sim \left(\frac{\alpha}{\lambda}\right)^{m_c} \det\left[\frac{\lambda}{\alpha} + \mathcal{B}^* P_0 \mathcal{B}\right]. \end{aligned}$$

Now the range space of $\mathcal{B}^* P_0 \mathcal{B}$ is of dimension 6 and hence the null space is of dimension

$$m_c - 6$$

(which is nonnegative by virtue of the controllability assumption!). Hence

$$\det \left[\frac{\lambda}{\alpha} + \mathcal{B}^* P_0 \mathcal{B} \right] = \left(\frac{\lambda}{\alpha} \right)^{m_c - 6} \cdot \text{(nonzero constant)}.$$

Hence it follows that

$$D(\lambda; \alpha) \sim \left(\frac{\alpha}{\lambda} \right)^6 \cdot \text{(nonzero constant)}, \quad \text{as } \lambda \to 0.$$

In a similar way we see from (6.20) that the nonzero poles of $D(\lambda, \alpha)$ are the zeros of $d(\lambda, 0)$ to the same order. The dimension of the eigenfunction space of $\mathcal{A}$ for $\lambda \neq 0$ may be taken to be unity, since we see from (6.18) that the dimension is equal to the dimension of the eigenvector space of $h(\lambda, 0)$ corresponding to the zero eigenvalue, and if $h(\lambda, 0)$ has distinct eigenvalues, the dimension is equal to one. We can make a similar statement for the eigenfunction spaces corresponding to the eigenvalues with nonzero imaginary parts of

$$\mathcal{A} - \alpha \mathcal{B}\mathcal{B}^*.$$

Hence it follows that

$$d(\lambda; 0)\, D(\lambda; \alpha)$$

is an entire function with zeros coinciding with that of $d(\lambda; \alpha)$. Hence we know that we must have:

$$d(\lambda; \alpha) = e^{q(\lambda)} d(\lambda; 0)\, D(\lambda; \alpha) \tag{6.21}$$

where $q(\lambda)$ is an entire function.

Order of $d(\lambda; \alpha)$

Next we shall show that the order of the entire function $d(\lambda; \alpha)$ is less than or equal to one. Let

$$m(r; \alpha) = \max_{|\lambda| = r} |d(\lambda, \alpha)|.$$

Now, the determinant of a matrix being the product of the eigenvalues, we have

$$\begin{aligned} |d(\lambda;\alpha)| &\leq \ (\text{spectral radius of } h(\lambda;\alpha)\,)^6 \\ &\leq \ ||h(\lambda;\alpha)||^6. \end{aligned}$$

From (6.17) we see that

$$\log ||h(\lambda;\alpha)|| \ \leq \ \sum_{i=1}^{m} \log ||T_i(\lambda;\alpha)|| \ + \ \sum_{i=1}^{m-1} \log ||e^{\mathcal{A}(\lambda)\Delta_i}||.$$

Since the $T_i(\lambda;\alpha)$ are polynomials in λ, we need only to consider the order of

$$\left\| e^{\mathcal{A}(\lambda)\Delta} \right\|.$$

For this purpose, we proceed to evaluate the eigenvalues of $\mathcal{A}(\lambda)$.

An eigenvector of $\mathcal{A}(\lambda)$, corresponding to the eigenvalue $\gamma(\lambda)$, must be of the form

$$\left| \begin{array}{c} y(\lambda) \\ \gamma(\lambda)y(\lambda) \end{array} \right|$$

where

$$\gamma(\lambda)^2 y(\lambda) \ = \ A_2^{-1}(\lambda^2 M_0 + A_0)y(\lambda) \ + \ \gamma(\lambda)A_2^{-1}A_1 y(\lambda). \tag{6.22}$$

Hence

$$\gamma(\lambda)^2 A_2\, y(\lambda) \ = \ \lambda^2 M_0 y(\lambda) \ + \ A_0 y(\lambda) \ + \ \gamma(\lambda)A_1 y(\lambda). \tag{6.23}$$

Let

$$\begin{aligned} a_2(\lambda) \ &= \ [A_2 y(\lambda),\, y(\lambda)]; \quad a_1(\lambda) \ = \ [A_1 y(\lambda),\, y(\lambda)] \\ m(\lambda) \ &= \ \lambda^2 m_0(\lambda) \ + \ [A_0 y(\lambda),\, y(\lambda)] \\ m_0(\lambda) \ &= \ [M_0 y(\lambda),\, y(\lambda)]. \end{aligned}$$

Then solving the quadratic equation

$$\gamma(\lambda)^2 a_2(\lambda) \ = \ m(\lambda) \ + \ \gamma(\lambda)a_1(\lambda)$$

we have

$$\gamma(\lambda) \ = \ \frac{1}{2a_2(\lambda)}\left(+a_1(\lambda) \ \pm \ \sqrt{a_1(\lambda)^2 + 4m(\lambda)a_2(\lambda)}\right). \tag{6.24}$$

From (6.22) we have

$$\left(\frac{\gamma(\lambda)}{\lambda}\right)^2 y(\lambda) \;=\; \left(A_2^{-1}M_0 \;+\; \frac{1}{\lambda^2}A_0 \;+\; \left(\frac{\gamma(\lambda)}{\lambda}\right)\frac{1}{\lambda}A_1\right)y(\lambda).$$

Since

$$a_2(\lambda) \;\geq\; (\text{smallest eigenvalue of } A_2)||y(\lambda)||^2$$

and

$$|a_1(\lambda)| \quad \text{is bounded,}$$

it follows from (6.24) that

$$\left\|\frac{\gamma(\lambda)}{\lambda}\right\| \quad \text{is bounded.}$$

Hence

$$\left\|\left(A_2^{-1}M_0 \;+\; \frac{1}{\lambda^2}A_0 \;+\; \frac{\gamma(\lambda)}{\lambda}\frac{1}{\lambda}A_1\right) \;-\; A_2^{-1}M_0\right\| \;=\; O\left(\frac{1}{|\lambda|}\right) \qquad \text{as } \lambda \to \infty.$$

Hence "normalizing" the eigenvectors so that

$$||y(\lambda)|| = 1,$$

every sequence $\{y(\lambda_n)\}$ has a subsequence which converges to one of the eigenvectors of

$$A_2^{-1}M_0$$

as $|\lambda_n| \;\to\; \infty$. We assume (for simplicity) that the eigenvalues of this matrix are distinct, which are of course strictly positive. Let μ_i, $i \;=\; 1,\ldots,6$ denote these eigenvalues and $\mathbf{e}_i$ the corresponding eigenvector of unit norm. Then for $|\lambda|$ large enough we can arrange so that the eigenvalues of $\mathcal{A}(\lambda)$ are

$$\gamma_i^+(\lambda), \gamma_i^-(\lambda), \quad i = 1,\ldots,6$$

$$\left|\left(\frac{\gamma_i^{\pm}(\lambda)}{\lambda}\right)^2 \;-\; \mu_i\right| \;\to\; 0 \quad \text{as } |\lambda| \to \infty$$

with corresponding eigenvectors:

$$\left|\begin{array}{c} y_i(\lambda) \\ \gamma_i^+(\lambda) y_i(\lambda) \end{array}\right|, \qquad \left|\begin{array}{c} y_i(\lambda) \\ \gamma_i^-(\lambda) y_i(\lambda) \end{array}\right|$$

$$||y_i(\lambda)|| \;=\; 1, \quad i = 1, \ldots, 6, \quad y_i(\lambda) \to \mathbf{e}_i.$$

In particular

$$[A_1 y_i(\lambda),\, y_i(\lambda)] \to [A_1 \mathbf{e}_i,\, \mathbf{e}_i].$$

But the $\mathbf{e}_i$ being real-valued and

$$A_1 = -L_1 + L_1^*,$$

we have that

$$[A_1 y_i(\lambda),\, y_i(\lambda)] \to 0.$$

And as a result, in (6.24)

$$\gamma^{\pm}(\lambda) \;-\; \frac{\pm 1}{2a_2(\lambda)} \sqrt{a_1(\lambda)^2 + 4m(\lambda)a_2(\lambda)} \;\to\; 0 \quad \text{as } |\lambda| \to \infty$$

and hence

$$\gamma_i^{\pm}(\lambda) \;-\; \pm\lambda\sqrt{\mu_i} \;\to\; 0 \quad \text{as } |\lambda| \to \infty. \tag{6.25}$$

Hence we have

$$\left|\frac{e^{\gamma_i^{\pm}(\lambda)\Delta}}{e^{\pm\lambda\sqrt{\mu_i}\Delta}}\right| \;\to\; 1 \quad \text{as } |\lambda| \to \infty.$$

Hence

$$\text{spectral radius of } e^{\mathcal{A}(\lambda)\Delta} \;=\; \max_k \left|e^{\pm\lambda\sqrt{\mu_k}\,\Delta}\right|,$$

for $|\lambda|$ sufficiently large. Again, since $y_i(\lambda) \to \mathbf{e}_i$ we can find a constant M such that

$$\left\|e^{\mathcal{A}(\lambda)\Delta}\right\| \;\leq\; M \max_k \left|e^{\pm\lambda\sqrt{\mu_k}\,\Delta}\right|, \quad 0 < \Delta, \tag{6.26}$$

for all $|\lambda|$ sufficiently large. Hence it follows that

$$\max_{|\lambda|=r} \left\|e^{\mathcal{A}(\lambda)\Delta}\right\| \;\leq\; e^{r\Delta \max \sqrt{\mu_k}}$$

and hence

$$\max_{|\lambda|=r} \sum_{i=1}^{m-1} \log ||e^{\mathcal{A}(\lambda)\Delta_i}|| \;\sim\; r\,(\max \sqrt{\mu_k}\,)\, L.$$

Hence we have that $d(\lambda;\alpha)$ is of order less than or equal to one — or is of "exponential type" in the terminology of [Levin 1980].

Since

$$|D(\lambda,\alpha)| \to 1 \quad \text{as } |\lambda| \to \infty,$$

we note that

$$d(\lambda,0)\,D(\lambda,\alpha)$$

is also of exponential type and hence [Levin 1980, p. 24] we can sharpen (6.21) to:

$$d(\lambda,\alpha) \;=\; e^{p_1(\lambda)} d(\lambda,0) D(\lambda,\alpha) \tag{6.27}$$

where $p_1(\lambda)$ is a polynomial of degree one at most.

Deadbeat Modes

An eigenvalue which is real is often referred to as a "deadbeat" mode. They occur in closed loop only if there are rigid-body modes (zero eigenvalues in the open loop). In fact we have:

Theorem 6.1

For each $\alpha > 0$, the number of deadbeat modes is equal to the number of (linearly independent) rigid-body modes (= dimension of the null space of A).

Proof

Let Π_k denote the M-orthogonal projection operator projecting $\mathcal{H}$ into the eigenfunction space corresponding to the eigenvalue ω_k^2 of A. Then we have

$$H(\lambda,\alpha) \;=\; I \;+\; \frac{\alpha B^*\Pi_0 B}{\lambda} \;+\; \alpha \sum_1^\infty \frac{\lambda B^*\Pi_k B}{\lambda^2 + \omega_k^2}\,. \tag{6.28}$$

We are only ineterested in $\lambda < 0$. Hence we can write

$$|\lambda|\,H(\lambda,\alpha) \;=\; |\lambda| \;-\; \alpha B^*\Pi_0 B \;-\; \alpha \sum_1^\infty \frac{\lambda^2}{\lambda^2 + \omega_k^2}\, B^*\Pi_k B \tag{6.29}$$

where we note

$$\sum_1^\infty \frac{\lambda^2}{\lambda^2 + \omega_k^2}\, B^*\Pi_k B \;\to\; 0 \quad \text{as } |\lambda| \to 0.$$

Hence

$$|\lambda|\, H(\lambda,\alpha) \;\rightarrow\; -\alpha B^{*}\Pi_0 B \quad \text{as } |\lambda| \rightarrow 0.$$

Let

$$\gamma_k(\lambda,\alpha), \quad k = 1,\ldots,m_c$$

denote the eigenvalues of $H(\lambda,\alpha)$. Then

$$|\lambda|\,\gamma_k(\lambda,\alpha) \;\rightarrow\; (-\alpha)\,\cdot\,\text{eigenvalues of } B^{*}\Pi_0 B, \quad \text{as } |\lambda| \rightarrow 0.$$

Now $B^{*}\Pi_0 B$ has exactly 6 nonzero eigenvalues. For, since B is one-to-one, the range space of $B^{*}\Pi_0 B$ is the same

$$\{B^{*}\phi\}, \quad \phi\varepsilon \text{ null space of } A$$

and the latter, as we have seen, has dimension 6, by controllability. Hence

$$|\lambda|\gamma_k(\lambda,\alpha) \;\rightarrow\; -|\gamma_k|, \quad k = 1,\ldots,6, \quad \text{as } |\lambda| \rightarrow 0.$$

Hence

$$\gamma_k(\lambda,\alpha) \;\rightarrow\; -\infty, \quad k = 1,\ldots,6, \quad \text{as } |\lambda| \rightarrow 0.$$

On the other hand

$$H(\lambda,\alpha) \;\rightarrow\; I \quad \text{as } |\lambda| \rightarrow \infty$$

and hence

$$\gamma_k(\lambda,\alpha) \;>\; 0 \quad \text{for } |\lambda| > \lambda_0.$$

Hence it follows that

$$\gamma_k(\lambda,\alpha) \;=\; 0 \quad \text{for some } \lambda, \;\; -|\lambda_0| < \lambda < 0.$$

Hence it follows that

$$D(\lambda_k,\alpha) \;=\; 0, \quad \lambda_k < 0,\; k = 1,\ldots,6.$$

Or, we have exactly 6 deadbeat modes.

Remark

For $\alpha = 0$, the eigenvalues are the zeros of $d(\lambda, 0)$. We want to consider now the limiting case $\alpha = \infty$. For this purpose we consider now

$$\left(\frac{I}{\alpha} + \mathcal{M}(\lambda)\right)$$

and note that the zeros of

$$\det\left(\frac{I}{\alpha} + \mathcal{M}(\lambda)\right) \tag{6.30}$$

are the same as those of $D(\lambda, \alpha)$. However the form (6.30) allows us to consider the case for large α, or $\alpha =$ infinity. The matrix

$$\frac{I}{\alpha} + \mathcal{M}(\lambda) \;\rightarrow\; \mathcal{M}(\lambda) \quad \text{as } \alpha \rightarrow \infty \tag{6.31}$$

for each $\lambda \neq i\omega_k$, and hence (6.30)

$$\rightarrow\; m(\lambda) \;=\; \det \mathcal{M}(\lambda).$$

Hence we define the eigenvalues corresponding to $\alpha = +\infty$ as the roots of

$$m(\lambda) = 0. \tag{6.32}$$

We shall show now that the roots are pure-imaginary. For suppose

$$\mathcal{M}(\lambda)u = 0.$$

Then

$$[\mathcal{R}(\lambda)\mathcal{B}u,\, \mathcal{B}u] \;=\; 0.$$

Let

$$\mathcal{R}(\lambda)\mathcal{B}u = Y.$$

Then

$$[\mathcal{R}(\lambda)\mathcal{B}u,\, \mathcal{B}u] \;=\; [Y,\, \lambda I - \mathcal{A}Y] \;=\; \overline{\lambda}\,[Y, Y] \;-\; [Y, \mathcal{A}Y].$$

Hence

$$\operatorname{Re}\,[\mathcal{A}(\lambda)\mathcal{B}u,\, \mathcal{B}u] \;=\; \operatorname{Re}\left(\overline{\lambda}\right)[Y, Y]$$

$$(\mathcal{B}^*\mathcal{R}(\lambda)\mathcal{B} \;\text{ is a "positive-real" matrix})$$

and hence

$$\mathrm{Re}\left(\overline{\lambda}\right) = 0$$

or λ is pure-imaginary.

Let us examine the eigenvalue further. We have:

$$\lambda B_u^* \left(\lambda^2 M_b + T(\lambda)\right)^{-1} B_u u = 0.$$

We assume that no zero of $m(\lambda)$ is a zero of $L(\lambda)$. In that case we can write

$$\lambda B_u^* L(\lambda) \left(\lambda^2 M_b L(\lambda) + K(\lambda)\right)^{-1} B_u u = 0.$$

Since

$$T(\lambda)^* = T(\lambda)$$

$$\begin{aligned} \left(L(\lambda)\left(\lambda^2 M_b L(\lambda) + K(\lambda)\right)^{-1}\right)^* &= \left(\lambda^2 M_b L(\lambda)^* + K(\lambda)^*\right)^{-1} L(\lambda)^* \\ &= L(\lambda)\left(\lambda^2 M_b L(\lambda) + K(\lambda)\right)^{-1}. \end{aligned}$$

Hence

$$\lambda B_u^* \left(\lambda^2 M_b L(\lambda)^* + K(\lambda)^*\right)^{-1} L(\lambda)^* B_u u = 0.$$

In the special case where

$$m_c = 6m, \quad B_u = \text{Identity}$$

we have that the eigenvalues corresponding to $\alpha = \infty$ are the "clamped" modes. This is not true in general, as in fact the example in Section 8 shows.

The mode "shape" associated with these eigenvalues is given by

$$\phi = \begin{vmatrix} \mathcal{L}(\lambda)\,a(\lambda) \\ L(\lambda)\,a(\lambda) \end{vmatrix}$$

where

$$a(\lambda) = \left(\lambda^2 M_b + K(\lambda)\right)^{-1} B_u u.$$

Then letting

$$f = \mathcal{L}(\lambda)\,a(\lambda)$$

we see that

$$\lambda^2 M_0 f + g \;=\; 0, \tag{6.33}$$

where g is defined by (2.6), and

$$B_u^* L(\lambda)\, \mathcal{a}(\lambda) \;=\; 0, \tag{6.34}$$

or, these are modes in which the control nodes are clamped. We shall characterize these modes more precisely below.

Root Locus

From Theorem 6.1, it follows that oscillatory modes of the undamped structure remain oscillatory for *all* values of α, however large. The behavior of the set of eigenvalues is such that as α increases from zero they migrate from the imaginary axis to the left half-plane and then back to the imaginary axis. It is possible to define the eigenvalues each as a function of α and show that the real part decreases first and then at a critical value of α starts to increase as α increases, going to zero as α increases to infinity. We can also show that the critical value increases as the mode number increases. The loci describe differential arcs in the complex plane. Thus let

$$\lambda_k(0) \;=\; i\omega_k$$

where $i\omega_k$ is a zero of $d(\lambda, 0)$. Then we define $\lambda_k(\alpha)$ using the derivatives at $\alpha = 0$. Thus

$$\left.\frac{d\lambda_k}{d\alpha}\right|_{\alpha=0} \;=\; \frac{-d_\alpha(i\omega_k, 0)}{d_\lambda(i\omega_k, 0)} \tag{6.35}$$

where the subscripts denote partial derivatives, and calculate similarly higher order derivatives using the identity:

$$d(\lambda_k(\alpha),\, \alpha) \;=\; 0.$$

We can show that it is real and negative, and in particular leading to an approximation for σ_k, the real part, via the Newton formula:

$$\sigma_k \;\sim\; -\alpha\, \frac{d_\alpha(i\omega_k, 0)}{d_\lambda(i\omega_k, 0)} \quad \text{for small } \alpha. \tag{6.36}$$

Owing to space limitations we must stop here; and refer to the example in Section 8 for more.

Asymptotic Modes

The modes are the roots of the equation:

$$d(\lambda; \alpha) = 0.$$

Our interest is not evaluating the roots — which in a given case will be a problem in numerical analysis — but rather in their asymptotic behavior as the mode number increases without bound.

Let $\{\lambda_k\}$ denote a sequence of modes, where we note that

$$|\lambda_k| \to \infty \quad \text{as } k \to \infty.$$

We shall say that the sequence $\{\tilde{\lambda}_k\}$ is asymptotically equivalent if the sequence

$$|\tilde{\lambda}_k - \lambda_k|$$

is bounded. In our case we shall show that it actually goes to zero. We call $\{\tilde{\lambda}_k\}$ "asymptotic" modes. Note that the "percent error"

$$\frac{|\tilde{\lambda}_k - \lambda_k|}{\lambda_k} \to 0 \quad \text{as } k \to \infty.$$

Since all mode determination is approximate only, this is clearly the best we can do.

The zeros of $d(\lambda, \alpha)$ for each $\alpha \geq 0$ are confined to the strip

$$-|\sigma| \leq \operatorname{Re} \lambda \leq 0, \quad \sigma = \sup |\sigma_i| < \infty \tag{6.37}$$

and from (6.26) we have the important result that in this strip

$$\left\| e^{A(\lambda)\Delta_i} \right\|, \quad i = 1, \ldots, m-1$$

is bounded. Now we can express $h(\lambda; \alpha)$ as:

$$h(\lambda, \alpha) = \sum_0^{2m} \lambda^k h_k(\lambda; \alpha) \tag{6.38}$$

where the coefficient matrices $h_k(\cdot,\cdot)$ are bounded in the strip (6.37). The coefficient of λ^{2m} is given by (from (6.14)):

$$h_{2m}(\lambda;\alpha) = A_2^{-1}M_{b,L}P_{12}(\lambda,\Delta_{m-1}) \cdot \cdots \cdot P_{12}(\lambda,\Delta_1)A_2^{-1}M_{b,0} \tag{6.39}$$

and does not depend on α. The zeros of $d(\lambda,\alpha)$ are those of

$$\det\left(\frac{h(\lambda,\alpha)}{\lambda^{2m}}\right)$$

and hence are "asymptotically" those of

$$\det h_{2m}(\lambda;\alpha)$$

as $|\lambda| \to \infty$. Since A_2 and $M_{b,i}$ are nonsingular we have that

$$\det h_{2m}(\lambda;\alpha) = \prod_{1}^{m-1} \det P_{12}(\lambda;\Delta_i). \tag{6.40}$$

Next we shall show that:

Lemma 6.1

$$\det P_{12}(\lambda;\Delta) = \prod_{k=1}^{6}\left(\frac{\sinh\Delta\gamma_k(\lambda)}{\Delta\gamma_k(\lambda)}\right) \tag{6.41}$$

where

$$|\gamma_k(\lambda) - \lambda\sqrt{\mu_k}| \to 0, \quad |\lambda| \to \infty.$$

Proof

Using

$$\begin{vmatrix} y_k(\lambda) \\ \gamma_k(\lambda)y_k(\lambda)\end{vmatrix} - \begin{vmatrix} y_k(\lambda) \\ -\gamma_k(\lambda)y_k(\lambda)\end{vmatrix} = \begin{vmatrix} 0 \\ 2\gamma_k(\lambda)y_k(\lambda)\end{vmatrix}$$

we have

$$\begin{aligned} P_{12}(\lambda;\Delta)\, y_k(\lambda) &= |\, I \quad 0\,|\, e^{A(\lambda)\Delta} \begin{vmatrix} 0 \\ y_k(\lambda)\end{vmatrix} \\ &= \frac{\sinh\gamma_k(\lambda)\Delta}{\gamma_k(\lambda)\Delta}\, y_k(\lambda). \end{aligned}$$

Hence

$$\det P_{12}(\lambda;\Delta) \;=\; \prod_{k=1}^{6} \frac{\sinh \gamma_k(\lambda)\Delta}{\gamma_k(\lambda)\Delta}$$

where by (6.25),

$$|\gamma_k(\lambda) \;-\; \lambda\sqrt{\mu_k}\,| \;\to\; 0, \qquad \text{as } |\lambda| \to \infty$$

as required.

Next, let

$$\mathcal{A}_\infty(\lambda) \;=\; \left| \begin{array}{cc} 0 & I \\ \lambda^2 A_2^{-1} M_0 & 0 \end{array} \right|.$$

Let

$$A_s \;=\; \sqrt{A_2^{-1} M_0}$$

where the eigenvalues of A_s are

$$\sqrt{\mu_k}\,, \qquad k = 1, \ldots, 6.$$

Then

$$e^{\mathcal{A}_\infty(\lambda)\Delta} \;=\; \left| \begin{array}{cc} \cosh(\lambda A_s \Delta) & (\lambda A_s \Delta)^{-1} \sinh(\lambda A_s \Delta) \\ (\lambda A_s \Delta) \sinh(\lambda A_s \Delta) & \cosh(\lambda A_s \Delta) \end{array} \right|. \tag{6.42}$$

Now

$$||P_{12}(\lambda;\Delta)||$$

is bounded in the strip (6.37) and hence it follows that for every k

$$\left\| \Big(P_{12}(\lambda;\Delta) \;-\; (\lambda A_s\Delta)^{-1} \sinh(\lambda A_s \Delta) \Big)\, \mathbf{e}_k \right\| \;\to\; 0$$

and hence

$$\left\| P_{12}(\lambda;\Delta) \;-\; (\lambda A_s\Delta)^{-1} \sinh(\lambda A_s \Delta) \right\| \;\to\; 0$$

as $|\lambda| \to \infty$ in the strip (6.37).

The zeros of

$$\frac{\sinh \gamma_k(\lambda)\Delta}{\gamma_k(\lambda)\Delta}$$

are given by

$$\gamma_k(\lambda_n)\Delta \;=\; in\pi.$$

Let

$$\lambda_n' \sqrt{\mu_k}\, \Delta \;=\; in\pi$$

being one sequence of zeros of

$$\det\Big((\lambda A_s \Delta)^{-1} \sinh(\lambda A_s \Delta)\Big)$$

corresponding to the eigenvalue

$$\lambda_n' \sqrt{\mu_k} \quad \text{and eigenvector } \mathbf{e}_k\,.$$

Then

$$|(\gamma_k(\lambda_n) \;-\; \lambda_n \sqrt{\mu_k}\,)\,\Delta| \;=\; |(\lambda_n - \lambda_n')\sqrt{\mu_k}\,\Delta| \;\to\; 0, \qquad \text{as } n \to \infty$$

and hence

$$|\lambda_n - \lambda_n'| \to 0, \qquad \text{as } n \to \infty.$$

We say in this case $\{\lambda_n\}$ is asymptotically equivalent to $\{\lambda_n'\}$ — or that the asymptotic zeros of $\det P_{12}(\lambda;\Delta)$ are given by

$$\lambda_{n,k} \;=\; \frac{\pm in\pi}{\sqrt{\mu_k}\,\Delta}\,, \qquad k = 1,\ldots,6. \tag{6.43}$$

Thus the asymptotic zeros of (6.40) are given by

$$\lambda_{n,k,j} \;=\; \frac{\pm in\pi}{\sqrt{\mu_k}\,\Delta_j}\,, \qquad k = 1,\ldots,6, \;\; j = 1,\ldots,m-1. \tag{6.44}$$

Let us turn now to the asymptotic zeros of $d(\lambda;\alpha)$. Let

$$Q_{21}(\lambda;\Delta) \;=\; \lambda P_{21}(\lambda;\Delta); \quad Q_{12}(\lambda;\Delta) \;=\; \lambda P_{12}(\lambda;\Delta)$$

and

$$h(\lambda;\alpha) \;=\; \lambda^m \left(q_m(\lambda;\alpha) \;+\; \frac{r(\lambda;\alpha)}{\lambda}\right) \tag{6.45}$$

where, using (6.14),

$$\begin{aligned} q_m(\lambda;\alpha) \;=\; & A_2^{-1}\left(M_{b,L} \;+\; \frac{\alpha}{\lambda} D_m\right) Q_{12}(\lambda;\Delta_{m-1}) \\ & \cdots\; Q_{12}(\lambda;\Delta_1)\, A_2^{-1}\left(M_{b,0} \;+\; \frac{\alpha}{\lambda} D_1\right) \end{aligned} \tag{6.46}$$

and

$$||r(\lambda;\alpha)||$$

is bounded in the strip (6.37). Hence we can write

$$d(\lambda;\alpha) \;=\; \det q_m(\lambda;\alpha) \;+\; \sum_1^6 \frac{1}{\lambda^k}\, d_k(\lambda;\alpha)$$

where

$$d_k(\lambda;\alpha) \quad \text{are bounded in the strip.} \tag{6.47}$$

For large $|\lambda|$ we can use the approximation (6.42) and hence

$$\det q_m(\lambda;\alpha) \;=\; \prod_{k=1}^{m} \det\left[A_2^{-1}\left(M_{b,L} \;+\; \frac{\alpha}{\lambda} D_k\right)\right] \cdot \prod_{k=1}^{m-1} \det\left[(A_s\Delta_k)^{-1} \sinh \lambda A_s \Delta_k\right]$$

which for all $|\lambda|$ sufficiently large can be expressed:

$$(\text{constant}) \;\prod_{k=1}^{m-1}\left(\prod_{i=1}^{6} \frac{\sinh \gamma_i \lambda \Delta_k}{\sqrt{\mu_i}\,\Delta_k}\right) \;+\; \frac{1}{\lambda}\left(d_1(\lambda;\alpha) \;+\; \sum_2^6 \frac{1}{\lambda_k}\, d_k(\lambda;\alpha)\right). \tag{6.48}$$

Given $\varepsilon > 0$, we can make $|\lambda|$ large enough so that the second term is less than ε. Hence taking

$$\gamma_k(\lambda)\Delta_k \;=\; in\pi \;+\; \theta,$$

where

$$|\sinh\theta| < \varepsilon$$

or, approximately

$$\lambda \;=\; \frac{in\pi + \theta}{\Delta_k\sqrt{\mu_k}}\,.$$

We see that there is a value of θ such that (6.48) is zero. Since we know that the real part of the eigenvalue must be negative, we have that

$$\operatorname{Re}\theta < 0.$$

We see however that the zeros are again asymptotically the same as that given by (6.35). In summary asymptotically the zeros of $d(\lambda,\alpha)$ for any α are given by the zeros of

$$\det\left(\prod_{k=1}^{m-1} (\Delta_i A_s)^{-1} \sinh \lambda A_s \Delta_i\right).$$

Mode Shape

The (unpurged!) asymptotic mode shapes are determined by the eigenvector y:

$$h_{2m}(\lambda;\alpha)y = 0$$

where y can be determined as follows. Fix j, and let

$$\lambda_n = \frac{in\pi}{\Delta_{m-1}\sqrt{\mu_j}}$$

and assuming for simplicity that the Δ_i are distinct, we see that $P_{12}(\lambda_n;\Delta_i)$ are nonsingular for i not equal to $m-1$, and hence we can take

$$y = \left(A_2^{-1}M_{b,m-2}P_{12}(\lambda_n;\Delta_{m-2}) \cdots P_{12}(\lambda_n;\Delta_1)A_2^{-1}M_{b,0}\right)^{-1}\mathbf{e}_j$$

so that

$$P_{12}(\lambda_n;\Delta_{m-1})\mathbf{e}_j = 0.$$

We can repeat this procedure for $\Delta_{m-2},\ldots,\Delta_1$. The corresponding mode shape is then determined by (6.18) where $f(0)$ is now denoted y.

7. MODAL EXPANSION

For the undamped structure ($\alpha = 0$), we have seen that we have a "modal expansion" in terms of the eigenfunctions of $\mathcal{A}$ given by (4.16). The eigenfunctions are orthogonal and complete. The question arises as to what extent this property holds in the closed-loop case — for the eigenfunctions of

$$\mathcal{A} - \alpha\mathcal{B}\mathcal{B}^*, \quad \alpha > 0.$$

We have seen that these functions are not orthogonal and hence we need to examine what happens to the modal expansion.

We have seen (cf. (5.15)) that the eigenfunctions are of the form

$$Y_k = \mathcal{R}(\lambda_k,\mathcal{A})\mathcal{B}^*u(\lambda_k) \tag{7.1}$$

where $\{\lambda_k\}$ are the eigenvalues. For simplicity we shall assume that the dimension of the eigenfunction space is unity. Let Z_k denote the eigenfunction of

$$(\mathcal{A} - \alpha\mathcal{B}\mathcal{B}^*)^* \;=\; \mathcal{A}^* - \alpha\mathcal{B}\mathcal{B}^*$$

corresponding to the eigenvalue $\overline{\lambda}_k$. Then the pertinent properties of $\mathcal{A}^*$ being similar to those of $\mathcal{A}$, we have:

$$\begin{aligned} Z_k &= \mathcal{R}\left(\overline{\lambda}_k, \mathcal{A}^*\right)\mathcal{B}^* v\left(\overline{\lambda}_k\right), \quad v(\cdot) \in E^{m_c} \\ &= -\mathcal{R}\left(-\overline{\lambda}_k, \mathcal{A}\right)\mathcal{B}^* v\left(\overline{\lambda}_k\right). \end{aligned} \tag{7.2}$$

The main feature of these eigenfunctions is that they are "biorthogonal"

$$\begin{aligned} [Y_k, Z_j] &= 0, \quad k \neq j \\ [Y_k, Z_k] &\neq 0. \end{aligned}$$

Riesz Basis

Recall now that a sequence $\{\Psi_k\}$ of elements in a Hilbert space $\mathcal{H}$ is called a "basis" if every element Y in the space can be expressed as

$$Y = \sum_1^\infty a_k \Psi_k$$

where

$$\sum_1^\infty |a_k| \, ||\psi_k|| < \infty$$

and

$$0 = \sum_1^\infty a_k \psi_k$$

implies

$$a_k = 0 \quad \text{for every } k.$$

A biorthogonal sequence $\{\Phi_k, \psi_k\}$ is called a Riesz basis if there is a linear bounded operator T on $\mathcal{H}$ into $\mathcal{H}$ with bounded inverse such that $\{T\phi_k\}$ is an orthonormal basis. This implies in particular that we have the expansion:

$$Y = \sum_1^\infty [Y, \psi_k]\Phi_k = \sum_1^\infty [Y, \Phi_i]\Psi_k \tag{7.3}$$

where

$$\sum_1^\infty |[Y, \psi_k]|^2 < \infty; \quad \sum_1^\infty |[Y, \Phi_k]|^2 < \infty$$

$$T\phi_k = T^{*-1}\psi_k.$$

Also: For Y, Z in $\mathcal{H}$

$$[Y, Z] = \sum_1^\infty [Y, \psi_k][\phi_k, Z]. \tag{7.4}$$

The main result in this section is that $\{Y_k, Z_k\}$ upon "normalization" so that

$$[Y_k, Z_k] = 1$$

("normalizing" $u(\lambda_k), v\left(\overline{\lambda}_k\right)$ appropriately, which we shall assume from now on) form a Riesz basis. This will follow from [Balakrishnan 1996] upon verifying the conditions

i)

$$\left|\lambda_k + \overline{\lambda}_j\right| \geq \delta > 0, \quad \text{for} \quad k \neq j$$

ii)

$$\sum_1^\infty |\sigma_k| < \infty$$

iii) (algebraic) multiplicity of each eigenvalue is equal to unity.

The condition (i) follows readily from the asymptotic estimate (5.35). Condition (ii) is verified in Section 5, (5.16). Condition (iii) is automatic since we are assuming the dimension of the eigenfunction space for each $\alpha > 0$ is unity.

We can now proceed to exploit the modal expansion (7.3). First we note that if λ_k is an eigenvalue so is $\overline{\lambda}_k$ if

$$\operatorname{Im} \lambda_k \neq 0$$

and there are exactly six real-valued λ_k. Let

$$(\mathcal{A} - \alpha\mathcal{B}\mathcal{B}^*)Y_k = \lambda_k Y_k.$$

Then

$$(\mathcal{A} - \alpha\mathcal{B}\mathcal{B}^*)\overline{Y}_k = \overline{\lambda}_k \overline{Y}_k$$

$$(\mathcal{A} - \alpha\mathcal{B}\mathcal{B}^*)^* \overline{Z}_k = \lambda_k \overline{Z}_k.$$

where $\{\lambda_k\}$ are the eigenvalues. For simplicity we shall assume that the dimension of the eigenfunction space is unity. Let Z_k denote the eigenfunction of

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$$\operatorname{Im} \lambda_k \neq 0$$

and there are exactly six real-valued λ_k. Let

$$(\mathcal{A} - \alpha\mathcal{B}\mathcal{B}^*)Y_k = \lambda_k Y_k.$$

Then

$$(\mathcal{A} - \alpha\mathcal{B}\mathcal{B}^*)\overline{Y}_k = \overline{\lambda}_k \overline{Y}_k$$

$$(\mathcal{A} - \alpha\mathcal{B}\mathcal{B}^*)^* \overline{Z}_k = \lambda_k \overline{Z}_k.$$

Hence numbering so that $\lambda_1, \ldots, \lambda_6$ are real, and the λ_k are in increasing order in $|\lambda_k|$, we can express the modal expansion as:

$$Y = \sum_1^6 [Y, Z_k] Y_k + \sum_7^\infty \left([Y, Z_k] Y_k + \left[Y, \overline{Z}_k\right] - Y_k \right). \tag{7.5}$$

Correspondingly the solution of the closed-loop system:

$$\dot{Y}(t) = (\mathcal{A} - \alpha \mathcal{B}\mathcal{B}^*) Y(t); \quad Y(0) = Y$$

can be expressed:

$$Y(t) = \sum_1^6 [Y, Z_k] e^{-|\sigma_k| t} Y_k + \sum_7^\infty \left([Y, Z_k] e^{\lambda_k t} Y_k + \left[Y, \overline{Z}_k\right] e^{\overline{\lambda}_k t} \overline{Y}_k \right). \tag{7.6}$$

We can easily deduce strong stability of the semigroup $S_\alpha(\cdot)$ from (7.6), or equivalently from the fact that the eigenfunctions $\{Y_k\}$ are complete in $\mathcal{H}_E$, exploiting the dissipativity:

$$||S_\alpha(t)|| \leq 1.$$

If Y is real-valued, we note that

$$\overline{[Y, Z_k]} = \left[Y, \overline{Z}_k\right]$$

and hence

$$[Y, Z_k] e^{\lambda_k t} Y_k + \left[Y, \overline{Z}_k\right] e^{\overline{\lambda}_k t} \overline{Y}_k$$

can be expressed

$$= e^{-|\sigma_k| t} \left[[Y, Z_k] Y_k + \left[Y, \overline{Z}_k\right] \overline{Y}_k \right] \cos \omega_k t + e^{-|\sigma_k| t} \left[[Y, Z_k] Y_k - \left[Y, \overline{Z}_k\right] \overline{Y}_k \right] \sin \omega_k t. \tag{7.7}$$

Since we know that we must have:

$$Y_k = \begin{vmatrix} y_k \\ \lambda_k y_k \end{vmatrix}; \quad Z_k = \begin{vmatrix} z_k \\ \overline{\lambda}_k z_k \end{vmatrix}$$

we can proceed to develop expansions for $x_1(t)$, $x_2(t)$ where

$$Y(t) = \begin{vmatrix} x_1(t) \\ x_2(t) \end{vmatrix}$$

going back to (5.7), (5.8). We omit the details. From (7.6) we can readily deduce (5.17).

8. ILLUSTRATIVE EXAMPLE

To illustrate the foregoing theory and concepts, we consider now an example — simplified in the extreme to reduce notational complexity and wholly non-numeric to avoid computer calculation. Thus we consider beam torsion about a single axis with a control at one end and a lumped mass at the other, and no interior nodes.

Retaining the nomenclature of Section 2 as much as possible but using $\theta(t,s)$ in place of $\phi(t,s)$, the dynamics can be described by:

$$m_{44}\ddot{\theta} - c_{66}\theta'' = 0; \quad 0 < s < L; \; 0 < t \tag{8.1}$$

yielding in the notation of Section 2:

$$M_0 = m_{44}; \quad A_2 = c_{66}.$$

The abstract version becomes:

$$\mathcal{H} = L_2(0,L) \times E^2$$

$$x = \begin{vmatrix} f \\ b \end{vmatrix}, \quad f(\cdot) \in L_2(0,L), \; b \in E^2, \quad b = \begin{vmatrix} b_0 \\ b_L \end{vmatrix}.$$

The stiffness operator A is then given by:

$$\text{Domain of } A = \left[x = \begin{vmatrix} f \\ b \end{vmatrix}, \quad f, f', f'' \in L_2(0,L); \quad b = \begin{vmatrix} f(0) \\ f(L) \end{vmatrix} \right];$$

$$Ax = y; \quad y = \begin{vmatrix} g \\ c \end{vmatrix}$$

$$g(s) = -c_{66}g''(s), \quad 0 < s < L,$$

$$c = \begin{vmatrix} -c_{66}f'(0) \\ c_{66}f'(L) \end{vmatrix}.$$

Thus defined,

$$
\begin{aligned}
[Ax, x] &= -c_{66} \int_0^L f''(s) \overline{f(s)}\, ds \; + \; c_{66} f(L) \overline{f''(L)} \; - \; c_{66} f(0) \overline{f'(0)} \\
&= c_{66} \int_0^L |f'(s)|^2 \, ds
\end{aligned}
\tag{8.2}
$$

yielding the potential energy, as required. There is a rigid-body mode:

$$Ax = 0,$$

where x is of the form

$$x = \begin{vmatrix} f \\ a \\ a \end{vmatrix} \tag{8.3}$$

where

$$f(s) = a, \quad 0 \leq s \leq L.$$

Placing the control at $s = 0$, the control operator B is given by

$$Bu = \begin{vmatrix} 0 \\ B_u u \end{vmatrix}; \quad B_u u = \begin{vmatrix} u \\ 0 \end{vmatrix}.$$

For x in $\mathcal{D}(A)$,

$$x = \begin{vmatrix} f \\ b \end{vmatrix},$$

we see that

$$B^* x = f(0).$$

Since there is a control at one end, we see that all modes are controllable and that $(A \sim B)$ is controllable.

Finally, the mass operator M is given by

$$Mx = y; \quad y = \begin{vmatrix} m_{44} \\ M_b b \end{vmatrix}; \quad M_b = \begin{vmatrix} m_0 & 0 \\ 0 & m_L \end{vmatrix}.$$

Correspondingly we have the boundary equations:

$$\left.\begin{aligned} m_0 \ddot{\theta}(t, 0) \; - \; c_{66} \theta'(t, 0) \; + \; u(t) \; &= \; 0 \\ m_L \ddot{\theta}(t, L) \; + \; c_{66} \theta'(t, L) \; &= \; 0 \end{aligned}\right\}. \tag{8.1a}$$

The space $\mathcal{H}_1$ (M-orthogonal to the null space of A) consists of elements of the form

$$x = \begin{vmatrix} f(\cdot) \\ b_0 \\ b_L \end{vmatrix}$$

where

$$m_{44} \int_0^L f(s)\,ds + m_0 b_0 + m_L b_L = 0. \tag{8.4}$$

The domain of $\sqrt{A}$, by [Balakrishnan 1990] is characterized by elements of the form

$$x = \begin{vmatrix} f(\cdot) \\ f(0) \\ f(L) \end{vmatrix}$$

where $f(\cdot)$ is absolutely continuous and $f'(\cdot) \in L_2(0, L)$ and in particular

$$\|\sqrt{A}\, x\|^2$$

is the potential energy given by (8.2). Thus

$$\mathcal{D}\left(\sqrt{A}\right) \cap \mathcal{H}_1 = \left[\begin{vmatrix} f \\ f(0) \\ f(L) \end{vmatrix}, \quad \begin{array}{c} f \text{ is absolutely continuous with } f' \\ \text{in } L_2(0, L), \text{ and (8.4) holds} \end{array} \right].$$

Also

$$\mathcal{H}_E = \left(\mathcal{D}\left(\sqrt{A}\right) \cap \mathcal{H}_1\right) \times \mathcal{H}$$

with energy inner product, as in Section 2.

The feedback control is:

$$u(t) = \alpha B^* \dot{x}(t) = \alpha \dot{\theta}(t, 0), \quad \alpha > 0.$$

Closed-Loop Modes

We proceed directly to characterize the closed-loop modes. In the notation of Section 6:

$$\mathcal{A}(\lambda) = \begin{vmatrix} 0 & 1 \\ \lambda^2 \nu^2 & 0 \end{vmatrix},$$

where

$$\nu^2 = \frac{m_{44}}{c_{66}}$$

yielding

$$e^{\mathcal{A}(\lambda)s} = \begin{vmatrix} \cosh \lambda\nu s & \frac{\sinh \lambda\nu s}{\lambda\nu} \\ \lambda\nu \sinh \lambda\nu s & \cosh \lambda\nu s \end{vmatrix}$$

and

$$\begin{aligned} h(\lambda, \alpha) &= \begin{vmatrix} \frac{\lambda^2 m_L}{c_{66}} & 1 \end{vmatrix} e^{\mathcal{A}(\lambda)L} \begin{vmatrix} 1 \\ \frac{\alpha\lambda + \lambda^2 m_0}{c_{66}} \end{vmatrix} \\ &= \lambda \left(a_1(\lambda) \sinh \lambda\nu L \; + \; a_2(\lambda) \cosh \lambda\nu L\right) \end{aligned} \tag{8.5}$$

where

$$\begin{aligned} a_1(\lambda) &= \frac{1}{\nu c_{66}^2}\left(\nu^2 c_{66}^2 + \lambda\alpha m_L + \lambda^2 m_0 m_L\right) \\ a_2(\lambda) &= \frac{\alpha + \lambda(m_L + m_0)}{c_{66}} . \end{aligned}$$

We see that

$$d(\lambda; \alpha) = h(\lambda; \alpha)$$

is an entire function of order one. It has a zero of order one at $\lambda = 0$, for nonzero α, and of order two for $\alpha = 0$. It is of "completely regular growth" in the terminology of [Levin 1980]:

$$\lim_{r \to \infty} \frac{\log |d(re^{i\theta}, \alpha)|}{r} = \nu L |\cos\theta|$$

and hence [Levin 1980, p. 169]:

$$\lim_{r \to 0} \frac{N(r)}{r} = \frac{1}{2\pi}\int_0^{2\pi} \nu L |\cos\theta| \, d\theta > 0$$

where $N(r)$ is the number of zeros in the circle of radius r. Hence the number of zeros is not finite. For large $|\lambda|$:

$$d(\lambda; \alpha) \sim \lambda^3 m_0 m_L \sinh \lambda\nu L$$

and hence the asymptotic modes are the roots of

$$\sinh \lambda\nu L = 0,$$

or

$$\lambda_n = \frac{\pm in\pi}{\nu L} \tag{8.6}$$

for all $\alpha \geq 0$. Here however we can make a more exact calculation. Thus the eigenvalues $\{\lambda_k\}$ are the roots of

$$\tanh \lambda\nu L + b(\lambda;\alpha) = 0 \tag{8.7}$$

where

$$b(\lambda;\alpha) = \frac{m_{44}}{\nu} \frac{\alpha + \lambda(m_L + m_0)}{m_{44}c_{66} + \lambda\alpha m_L + \lambda^2 m_0 m_L} \tag{8.8}$$

and

$$|b(\lambda;\alpha)| = O\left(\frac{1}{|\lambda|}\right) \quad \text{as } |\lambda| \to \infty.$$

We can rewrite (8.4) as

$$\lambda\nu L + \tanh^{-1} b(\lambda;\alpha) = 0$$

where

$$b(\lambda;\alpha) = \frac{a_2(\lambda)}{a_1(\lambda)}$$

and since

$$\tanh x = \tanh(x \pm 2in\pi), \quad n \text{ integer}$$

we have

$$\lambda\nu L = \pm in\pi + \tfrac{1}{2}\log\frac{1+b(\lambda;\alpha)}{1-b(\lambda;\alpha)} = 0, \tag{8.7a}$$

using the principal value of $\log x$, real when x is positive. For $|\lambda|$ large, (8.7a) becomes

$$\lambda\nu L = \pm in\pi + \frac{m_{44}}{\nu}\cdot\left(\frac{m_L + m_0}{m_L m_0}\right)\frac{1}{\lambda} \tag{8.9}$$

yielding a slightly better approximation than (8.6), for large n. For nonzero α, we can see that (8.4) has exactly one real root, approximately:

$$\lambda = \frac{-\alpha}{m_{44}L + m_L + m_0}. \tag{8.10}$$

Clamped Modes

We can calculate that

$$L(\lambda) = \begin{vmatrix} 1 & 0 \\ \cosh \lambda\nu L & \frac{\sinh \lambda\nu L}{\lambda\nu} \end{vmatrix}$$

and the clamped modes are the zeros of $\sinh \lambda\nu L$ or,

$$\lambda_k = \frac{\pm ik\pi}{\nu L}, \quad k = 1, 2, \ldots . \tag{8.11}$$

Dynamic Stiffness Matrix

We can calculate that

$$K(\lambda) = c_{66} \begin{vmatrix} 0 & -1 \\ \lambda\nu \sinh \lambda\nu L & \cosh \lambda\nu L \end{vmatrix}$$

and hence that

$$T(\lambda) = \frac{\lambda\nu c_{66}}{\sinh \lambda\nu L} \begin{vmatrix} \cosh \lambda\nu L & -1 \\ -1 & \cosh \lambda\nu L \end{vmatrix}$$

which is clearly nonnegative definite for λ real, and nonsingular except for $\lambda = 0$. Also the inverse of the dynamic stiffness matrix:

$$(\lambda^2 m_b + T(\lambda))^{-1}$$

$$= \frac{1}{\lambda h(\lambda;0)} \begin{vmatrix} \lambda^2 m_L \sinh \lambda\nu L + c_{66}\lambda\nu \cosh \lambda\nu L & -\lambda\nu c_{66} \\ -\lambda\nu c_{66} & \lambda^2 m_0 \sinh \lambda\nu L + c_{66}\lambda\nu \cosh \lambda\nu L \end{vmatrix}.$$

Hence

$$\begin{aligned} \lambda B_u^* \left(\lambda^2 M_b + T(\lambda)\right)^{-1} Bu &= \frac{1}{h(\lambda;0)} \left(\lambda^2 m_L \sinh \lambda\nu L \; + \; c_{66}\lambda\nu \cosh \lambda\nu L\right) \\ &= m(\lambda). \end{aligned} \tag{8.12}$$

Hence we can verify that for this example

$$h(\lambda;0) = h(\lambda,0)\, D(\lambda;\alpha).$$

In other words in (6.27)

$$p_1(\lambda) = 0$$

which we may conjecture holds in general.

Root Locus

Beginning first with the limiting eigenvalues as α goes to infinity, given by the roots of

$$m(\lambda) = 0;$$

we have from (8.9)

$$\tanh \lambda\nu L \; + \; \frac{\nu c_{66}}{\lambda m_L} \; = \; 0 \tag{8.13}$$

or directly from (8.7) by taking the limit as α goes to infinity in (8.8). The roots are of course pure imaginary:

$$\lambda_n = i\beta_n, \quad \beta_n \text{ real},$$

$$\beta_n \nu L \; = \; \pm in\pi \; + \; i\delta_n, \quad |\delta_n| < \pi.$$

These are the modes which satisfy

$$m_L\ddot{\theta}(t,L) \; + \; c_{66}\theta'(t,L) \; = \; 0; \quad \theta(t,0) = 0$$

$$m_{44}\ddot{\theta}(t,s) \; - \; c_{66}\theta''(t,s) \; = \; 0, \quad 0 < s < L.$$

These are *not* the clamped modes, although they are, asymptotically.

Since we are only interested in the nonzero eigenvalues, let

$$F(\lambda;\alpha) \; = \; (\nu^2 c_{66}^2 + \lambda\alpha m_L + \lambda^2 m_0 m_L)\sinh\lambda\nu L \; + \; \nu c_{66}(\alpha \; + \; \lambda(m_L + m_0))\cosh\lambda\nu L$$

whose zeros are the nonzero eigenvalues. Let $\{i\omega_k\}$ denote the zeros for $\alpha = 0$. Fix k. Now

$$F(\lambda(\alpha);\, \alpha) \; = \; 0 \tag{8.14}$$

defines an implicit function $\lambda_k(\alpha)$, with

$$\lambda_k(0) = i\omega_k$$

and we define all derivatives at $\alpha = 0$ using (8.14). In particular

$$\left.\frac{d\lambda_k(\alpha)}{d\alpha}\right|_{\alpha=0} \; = \; \left.\frac{-F_\alpha(\lambda_k(\alpha);\, \alpha)}{F_\lambda(\lambda_k(\alpha);\, \alpha)}\right|_{\alpha=0}$$

where the subscripts denote partial derivatives, and the main point is that it is real, and it is negative. In particular this shows that the real part is decreasing. Since we know that the real part goes to zero as α goes to infinity, we see that there is a value of α at which its derivative must change sign.

Closed-Loop Mode Shapes

Following Section 7, the (unpurged) closed-loop mode shape corresponding to the eigenvalue λ_k is given by

$$f_k(s) \;=\; |\,1 \quad 0\,|\, e^{\mathcal{A}(\lambda_k)s} \left| \begin{matrix} 1 \\ \frac{\alpha\lambda_k + \lambda_k^2 m_0}{c_{66}} \end{matrix} \right| \;=\; \left(\cosh \lambda_k \nu s \;+\; \frac{(\alpha + \lambda_k m_0)}{\nu c_{66}} \sinh \lambda_k \nu s\right) f(0).$$

Since arbitrary multiplicative constants can be used, we may define the mode shape as:

$$f_k(s) \;=\; A_k \sinh(\lambda_k \nu s + \theta_k), \quad 0 \le s \le L$$

where

$$\tanh \theta_k \;=\; \frac{\nu c_{66}}{\alpha + \lambda_k m_0}\,.$$

For $\lambda_k \neq 0$, the purged version would be in the notation of Section 7,

$$Y_k \;=\; \left| \begin{matrix} \phi_k \\ \lambda_k \phi_k \end{matrix} \right|; \qquad Z_k \;=\; \left| \begin{matrix} \bar{\phi}_k \\ \bar{\lambda}_k \phi_k \end{matrix} \right|$$

where (new notation, not to be confused with Section 2):

$$\phi_k \;=\; \left| \begin{matrix} \tilde{f}_k(\cdot) \\ \tilde{f}_k(0) \\ \tilde{f}_k(L) \end{matrix} \right|$$

$$\tilde{f}_k(s) \;=\; f_k(s) \;-\; \left(m_{44} \int_0^L f_k(s)\, ds \;+\; m_0 f_k(0) \;+\; m_L f_k(L)\right).$$

The constant A_k can be determined to normalize the biorthogonal system as in Section 7 and thus obtain a Riesz basis for $\mathcal{H}_E$.

Limiting Case: $m_L = \infty$

We illustrate finally how to handle the case when one end is clamped. We set $m_L = +\infty$. This results in the boundary condition

$$\theta(t, L) = 0$$

replacing the condition at L in (8.1a). We may take

$$\mathcal{H} = L_2(0, L) \times E^1$$

and

$$Ax = y; \quad y = \begin{vmatrix} -c_{66} f''(\cdot) \\ -c_{66} f'(0) \end{vmatrix}, \quad x = \begin{vmatrix} f(\cdot) \\ f(0) \end{vmatrix}.$$

There are no rigid body modes and the eigenvalues are roots of

$$(\alpha + \lambda m_0) \sinh \lambda \nu L \; + \; \nu c_{66} \cosh \lambda \nu L \; = \; 0 \tag{8.15}$$

or

$$\lambda_n \nu L \; = \; \pm i n \pi \; - \; \tanh^{-1} b(\lambda_n)$$

$$b(\lambda) \; = \; \frac{\nu c_{66}}{\alpha + \lambda m_0} \, .$$

The eigenfunctions are

$$Y_k = \begin{vmatrix} \phi_k \\ \lambda_k \phi_k \end{vmatrix}; \qquad \phi_k = \begin{vmatrix} f_k(\cdot) \\ f_k(0) \end{vmatrix}$$

$$f_k(s) \; = \; A_k \sinh \lambda_k \nu (L - s).$$

The root-locus problem becomes much simpler than before. Again, we omit the details. For $\alpha = \infty$, the modes are the zeros of $\sinh \lambda \nu L$ or

$$\lambda_n \; = \; \frac{\pm i n \pi}{\nu L} \, .$$

From (8.15) we see that

$$\frac{d\lambda_n}{d\alpha} \; = \; \frac{-1}{1 - \left(\frac{\nu c_{66}}{\alpha + \lambda_n m_0}\right)^2}$$

and is real negative at $\alpha = 0$ and goes to (-1) as α goes to infinity. For large n we have the approximation:

$$\frac{d\lambda_n}{d\alpha} \sim -(\nu c_{66})^2 \frac{1}{(\alpha + \lambda_n m_0)^2}$$

and

$$\mathrm{Re}\,\frac{d\lambda_n}{d\alpha} = 0$$

implies

$$\alpha_{\text{crit}} = m_0\left(|\mathrm{Re}\,\lambda_n| + |\mathrm{Im}\,\lambda_n|\right)$$

and shows that the critical value of α increases with mode number.

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NOMENCLATURE

A	stiffness operator
B	control operator
s	position along beam
(x_1, x_2, x_3)	rectangular coordinates
u	axial displacement (x_1-component)
v	displacement (x_2-component)
w	displacement (x_3-component)
ϕ_1	torsion angle about x_1-axis
ϕ_2	torsion angle about x_2-axis
ϕ_3	torsion angle about x_3-axis
M	mass/inertia operator
M_0	mass/inertia matrix
M_b	composite matrix of mass/inertia at nodes
$\mathcal{H}$	Hilbert space
$\mathcal{H}_1$	space M-orthogonal to null space of A
$L_2(0, L)^6$	L_2-space of 6×1 vector functions over $(0, L)$
R^6, E^6	Euclidean 6-space
m_c	number of control inputs
$[\ ,\ ]$	inner product
$\mathcal{D}(A)$	domain of operator A
I_0, I_c, I_L	moments of inertia
$u(t)$	control input
λ_k	eigenvalues
σ_k	Real part of λ_k
ω_k	angular mode frequencies
Tr	trace
$\mathrm{Re}\, z$	real part of z
$\mathrm{Im}\, z$	imaginary part of z
$\det M$	determinant of M
$\|z\|$	absolute value of z
$\|\|F\|\|$	norm of vector F; operator norm of matrix F
$\overline{z}$	conjugate of z
A^*	adjoint of A
SCOLE	Spacecraft COntrol Laboratory Experiment