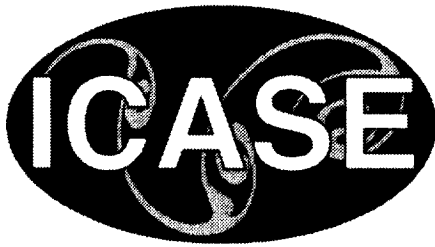


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## **Eigenmode Analysis of Boundary Conditions for the One-dimensional Preconditioned Euler Equations**

*David L. Darmofal*  
*Massachusetts Institute of Technology, Cambridge, Massachusetts*

*Institute for Computer Applications in Science and Engineering*  
*NASA Langley Research Center*  
*Hampton, VA*

*Operated by Universities Space Research Association*



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# EIGENMODE ANALYSIS OF BOUNDARY CONDITIONS FOR THE ONE-DIMENSIONAL PRECONDITIONED EULER EQUATIONS

DAVID L. DARMOFAL\*

**Abstract.** An analysis of the effect of local preconditioning on boundary conditions for the subsonic, one-dimensional Euler equations is presented. Decay rates for the eigenmodes of the initial boundary value problem are determined for different boundary conditions. Riemann invariant boundary conditions based on the unpreconditioned Euler equations are shown to be reflective with preconditioning, and, at low Mach numbers, disturbances do not decay. Other boundary conditions are investigated which are non-reflective with preconditioning and numerical results are presented confirming the analysis.

**Subject classification.** Applied & Numerical Mathematics

**Key words.** local preconditioning, boundary conditions, Euler equations

**1. Introduction.** Local preconditioning has been successfully utilized to accelerate the convergence to a steady-state for Euler and Navier-Stokes simulations[15, 18, 2, 19, 17, 7, 11, 4, 3]. Local preconditioning is introduced into a time-dependent problem as,

$$\mathbf{u}_t + \mathbf{P}(\mathbf{u})\mathbf{r}(\mathbf{u}) = 0,$$

where  $\mathbf{u}$  is the state vector of length  $m$ ,  $\mathbf{r}$  is the residual vector of length  $m$ , and  $\mathbf{P}$  is the  $m \times m$  preconditioning matrix. Since preconditioning effectively alters the time-dependent properties of the governing partial differential equation, modifications of the numerical discretization can be required. For example, upwind methods for inviscid problems must be based on the characteristics of the preconditioned equations instead of the unpreconditioned equations[18]. Similarly, the behavior of boundary conditions in conjunction with preconditioning will also be altered. While the effect of preconditioning on boundary conditions is known[9, 6, 16], to date, no quantitative analysis has been performed.

The purpose of this paper is to analyze the effect of preconditioning on several different boundary conditions commonly used in numerical simulations. Specifically, we consider the one-dimensional, preconditioned Euler equations linearized about a steady, uniform, subsonic mean state. The work is an extension of the analysis of Giles[5] for the one-dimensional, unpreconditioned Euler equations. As discussed by Giles, the exact eigenmodes and eigenfrequencies for this initial boundary value problem can be analytically determined. From these, we find the exponential decay rates for perturbations under different sets of boundary conditions. Finally, we demonstrate the validity of the analysis through numerical results.

**2. Analysis.** We first present the general analysis of the initial boundary value problem following the work of Giles[5]. The linear, preconditioned Euler equations are given by,

$$(2.1) \quad \begin{pmatrix} \tilde{\rho} \\ \tilde{q} \\ \tilde{p} \end{pmatrix}_T + \mathbf{P} \begin{pmatrix} \bar{q} & \bar{\rho} & 0 \\ 0 & \bar{q} & \bar{\rho}^{-1} \\ 0 & \bar{\rho}\bar{c}^2 & \bar{q} \end{pmatrix} \begin{pmatrix} \tilde{\rho} \\ \tilde{q} \\ \tilde{p} \end{pmatrix}_X = 0,$$

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\*Massachusetts Institute of Technology, Cambridge, MA 02139 (email: darmofal@mit.edu). This research was supported by the National Aeronautics and Space Administration under NASA Contract No. NAS1-97046 while the author was in residence at the Institute for Computer Applications in Science and Engineering (ICASE), NASA Langley Research Center, Hampton, VA 23681-2199.

where  $\tilde{\rho}$ ,  $\tilde{q}$ ,  $\tilde{p}$  are the perturbation density, velocity, and pressure, and  $\bar{\rho}$ ,  $\bar{q}$ ,  $\bar{c}$  are the mean density, velocity, and speed of sound. The speed of sound is related to the pressure and density through  $\bar{c}^2 = \gamma\bar{p}/\bar{\rho}$ . The subsonic inflow is located at  $X = 0$  and the outflow is at  $X = 1$ .

Next, we define the following non-dimensionalizations to simplify the analysis,

$$(2.2) \quad \rho \equiv \frac{\tilde{\rho}}{\bar{\rho}}, \quad q \equiv \frac{\tilde{q}}{\bar{c}}, \quad p \equiv \frac{\tilde{p}}{\bar{\rho}\bar{c}^2}, \quad x \equiv \frac{X}{L}, \quad t \equiv \frac{T}{L/\bar{c}}.$$

The non-dimensional version of Equation (2.1) is,

$$(2.3) \quad \mathbf{u}_t + \mathbf{PA}\mathbf{u}_x = 0,$$

where,

$$\mathbf{u} = \begin{pmatrix} \rho \\ q \\ p \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} M & 1 & 0 \\ 0 & M & 1 \\ 0 & 1 & M \end{pmatrix},$$

and the mean flow Mach number is  $M = \bar{q}/\bar{c}$ . The boundary conditions for subsonic flow require two inflow quantities and one outflow quantity to be specified. The inflow boundary conditions can be expressed as,

$$(2.4) \quad \mathbf{C}_{in}\mathbf{u}(0, t) = 0,$$

where  $\mathbf{C}_{in}$  is a  $2 \times 3$  matrix dependent on the specific choice of inflow conditions. Similarly, the single outflow boundary condition can be expressed as,

$$(2.5) \quad \mathbf{C}_{out}\mathbf{u}(1, t) = 0,$$

where  $\mathbf{C}_{out}$  is a  $1 \times 3$  matrix dependent on the specific choice of outflow conditions.

Equations (2.3), (2.4), and (2.5) represent the initial boundary value problem which we wish to study. An eigenmode of the initial boundary value problem is given by

$$(2.6) \quad \mathbf{u} = e^{-i\omega t} \sum_{i=1}^3 \alpha_i \mathbf{r}_i e^{i\omega x/\lambda_i}.$$

$\mathbf{r}_i$  and  $\lambda_i$  are the right eigenvectors and eigenvalues, respectively, of the matrix  $\mathbf{PA}$ , i.e.,

$$(\mathbf{PA} - \lambda_i \mathbf{I}) \mathbf{r}_i = 0.$$

In the following developments, we assume that the eigenvalues have been ordered such that the two forward-moving characteristics are  $i = 1, 2$  (i.e.  $\lambda_{1,2} > 0$ ) and the backward-moving characteristic is  $i = 3$  (i.e.  $\lambda_3 < 0$ ). The constants  $\alpha_i$  are the strengths of each eigenmode. Given an eigenmode of the initial boundary value problem as in Equation (2.6), the strength  $\alpha_i$  can be determined by multiplying  $\mathbf{u}$  by the left eigenvector,  $\mathbf{l}_i$ , where left and right eigenvectors are related by,

$$(2.7) \quad \begin{pmatrix} \mathbf{l}_1 & \mathbf{l}_2 & \mathbf{l}_3 \end{pmatrix}^T = \begin{pmatrix} \mathbf{r}_1 & \mathbf{r}_2 & \mathbf{r}_3 \end{pmatrix}^{-1}.$$

Specifically, since Equation (2.7) implies that  $\mathbf{l}_i^T \mathbf{r}_j = \delta_{ij}$ , we may find,

$$\mathbf{l}_i^T \mathbf{u} = \mathbf{l}_i^T e^{-i\omega t} \sum_{j=1}^3 \alpha_j \mathbf{r}_j e^{i\omega x/\lambda_j} = \alpha_i e^{i\omega(x/\lambda_i - t)}.$$

The eigenfrequency  $\omega$  and eigenmode strengths  $\alpha_i$  are determined by the boundary conditions. For the inflow boundary, substitution of Equation (2.6) into Equation (2.4) leads to,

$$(2.8) \quad \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = 0,$$

where

$$(2.9) \quad \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix} = \mathbf{C}_{in} \begin{pmatrix} \mathbf{r}_1 & \mathbf{r}_2 & \mathbf{r}_3 \end{pmatrix}.$$

As described by Giles[5], a necessary condition for the well-posedness of the initial boundary value problem is that the incoming characteristics,  $\alpha_1$  and  $\alpha_2$ , can be determined as functions of the outgoing characteristic,  $\alpha_3$ . This requires that the  $2 \times 2$  matrix,

$$\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

is non-singular. Also, the boundary condition at the inflow will be non-reflecting if the outgoing characteristic does not cause a perturbation in the incoming characteristics. In other words, a necessary condition for non-reflecting inflow conditions is that  $b_{13} = b_{23} = 0$ .

For the outflow boundary, substitution of Equation (2.6) into Equation (2.5) leads to,

$$(2.10) \quad \begin{pmatrix} b_{31} & b_{32} & b_{33} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = 0,$$

where

$$(2.11) \quad \begin{pmatrix} b_{31} & b_{32} & b_{33} \end{pmatrix} = \mathbf{C}_{out} \begin{pmatrix} e^{i(\omega/\lambda_1)}\mathbf{r}_1 & e^{i(\omega/\lambda_2)}\mathbf{r}_2 & e^{i(\omega/\lambda_3)}\mathbf{r}_3 \end{pmatrix}.$$

In this case, well-posedness of the initial boundary value problem requires that incoming characteristic,  $\alpha_3$ , can be determined as a function of the outgoing characteristics,  $\alpha_1$  and  $\alpha_2$ . Thus,  $b_{33}$  must be non-zero. Also, the boundary condition at the outflow will be non-reflecting if  $b_{31} = b_{32} = 0$ .

The inflow and outflow boundary conditions in Equations (2.8) and (2.10) can be combined as,

$$(2.12) \quad \mathbf{B}(\omega) \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = 0.$$

In order for a non-trivial solution of the initial boundary value problem to exist, a non-zero vector,  $(\alpha_1, \alpha_2, \alpha_3)^T$ , must exist which satisfies Equation (2.12). This is possible for values of  $\omega$  for which,

$$\det \mathbf{B}(\omega) = 0.$$

Separating the eigenfrequency into its real and imaginary parts,  $\omega = \omega_r + i\omega_i$ , the amplitude of the eigenmodes grows as  $\exp(\omega_i t)$ . Thus, in order for the eigenmodes to decay,  $\omega_i < 0$  for all possible values of  $\omega$ .

Finally, we note that the steady-state problem is well-posed if and only if  $\det \mathbf{B}(0)$  is non-zero.[5].

**3. Examples.** In this analysis, we will consider the one-dimensional version of the van Leer-Lee-Roe preconditioner[18] which is discussed by Lee[8]. For this preconditioner, the resultant  $\mathbf{PA}$  matrix is,

$$\mathbf{PA} = \begin{pmatrix} M & 0 & -2M \\ 0 & M & 2 \\ 0 & 0 & -M \end{pmatrix}.$$

The specific form of  $\mathbf{P}$  is described in the Appendix. The eigenvalues of  $\mathbf{PA}$  are  $\lambda_{1,2,3} = M, M, -M$ , and the eigenvectors are,

$$(3.1) \quad \begin{pmatrix} \mathbf{r}_1 & \mathbf{r}_2 & \mathbf{r}_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & M \\ 0 & 1 & -1 \\ 0 & 0 & M \end{pmatrix}, \quad \begin{pmatrix} \mathbf{l}_1^T \\ \mathbf{l}_2^T \\ \mathbf{l}_3^T \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1/M \\ 0 & 0 & 1/M \end{pmatrix}.$$

Some physical interpretation can be given to the preconditioned eigenmodes by considering the strengths of each mode in dimensional form, i.e.,

$$(3.2) \quad \mathbf{l}_1^T \mathbf{u} \propto \tilde{p} - \tilde{c}^2 \tilde{\rho},$$

$$(3.3) \quad \mathbf{l}_2^T \mathbf{u} \propto \tilde{p} - \tilde{\rho} \tilde{q} \tilde{q},$$

$$(3.4) \quad \mathbf{l}_3^T \mathbf{u} \propto \tilde{p}.$$

The strength of the first eigenmode is proportional to the linearized change in entropy and, under this preconditioning, is unchanged from the Euler equations without preconditioning. The second eigenmode is proportional to the linearized streamwise momentum,  $\tilde{p} + \tilde{\rho} \tilde{q} \tilde{q}$ . Finally, the third eigenmode is directly proportional to the pressure. Thus, with preconditioning, the upstream-running wave is exactly a pressure wave. As we will show in Section 3.2, this will have a significant impact on the behavior of boundary conditions for which a downstream pressure is specified.

**3.1. Euler Riemann boundary conditions.** We first consider the use of boundary conditions based on the Riemann invariants of the Euler equations. Without preconditioning, Giles[5] showed that these boundary conditions are non-reflecting with  $\omega_i = -\infty$ . In other words, all perturbations are eliminated in the finite time required for them to propagate out of the domain. Specifically, the Riemann invariant boundary conditions are,

$$X = 0, \quad \left\{ \begin{array}{l} p'/\rho'^\gamma = \bar{p}/\bar{\rho}^\gamma \\ q' + \frac{2}{\gamma-1}c' = \bar{q} + \frac{2}{\gamma-1}\bar{c} \end{array} \right. \quad X = L, \quad q' - \frac{2}{\gamma-1}c' = \bar{q} - \frac{2}{\gamma-1}\bar{c},$$

where the primed quantities are the sum of the mean and perturbation states, i.e.  $p' = \bar{p} + \tilde{p}$ . Linearization and non-dimensionalization of these boundary conditions gives,

$$\mathbf{C}_{in} = \begin{pmatrix} -1 & 0 & 1 \\ -1 & \gamma-1 & \gamma \end{pmatrix},$$

$$\mathbf{C}_{out} = \begin{pmatrix} 1 & \gamma-1 & -\gamma \end{pmatrix}.$$

From  $\mathbf{C}_{in}$  and  $\mathbf{C}_{out}$ , we find the matrix  $\mathbf{B}$ ,

$$\mathbf{B} = \begin{pmatrix} -1 & 0 & 0 \\ -1 & \gamma-1 & (\gamma-1)(M-1) \\ e^{i\omega/M} & (\gamma-1)e^{i\omega/M} & -(\gamma-1)(M+1)e^{-i\omega/M} \end{pmatrix}.$$

The determinant of  $\mathbf{B}$  is,

$$\det \mathbf{B} = -(\gamma - 1)^2 \left[ (M - 1)e^{i\omega/M} + (M + 1)e^{-i\omega/M} \right],$$

and the eigenfrequencies which result in a zero determinant are,

$$\begin{aligned} \omega_r &= \pi M n, & \text{for integer } n, \\ \omega_i &= -\frac{M}{2} \log \frac{1 + M}{1 - M}. \end{aligned}$$

In contrast to the unpreconditioned Euler equations, the Riemann invariant boundary conditions are reflective for the preconditioned Euler equations since the value of  $\omega_i$  is not  $-\infty$ . Within a computational simulation, perhaps the best measure of the decay rate is actually the decay per time step. Assuming the time step is given by a CFL condition of the form  $\Delta t = \nu \Delta x / \lambda_{\max}$  where  $\nu$  is a constant dependent on the temporal integration,  $\Delta x$  is the cell size, and  $\lambda_{\max}$  is the maximum eigenvalue, then  $\omega \Delta t \propto \omega_i / \lambda_{\max}$ . Since  $\lambda_{\max} = M$  for this preconditioned system,

$$\omega_i / \lambda_{\max} = -\frac{1}{2} \log \frac{1 + M}{1 - M}.$$

In particular, we note that as  $M \rightarrow 0$ ,  $\omega_i / \lambda_{\max} \rightarrow 0$ . Thus, at low Mach numbers, disturbances will not decay indicating that the use of Riemann invariant boundary conditions based on the Euler equations is likely to impede convergence to a steady state.

**3.2. Entropy, stagnation enthalpy at inflow; pressure at outflow.** Another common set of boundary conditions for subsonic, internal flows is the specification of entropy and stagnation enthalpy at the inflow and pressure at the outflow. Specifically,

$$X = 0, \quad \left\{ \begin{array}{l} p' / \rho'^{\gamma} = \bar{p} / \bar{\rho}^{\gamma} \\ \frac{1}{2} q'^2 + \frac{\gamma}{\gamma-1} \frac{p'}{\rho'} = \frac{1}{2} \bar{q}^2 + \frac{\gamma}{\gamma-1} \frac{\bar{p}}{\bar{\rho}} \end{array} \right. \quad X = L, \quad p' = \bar{p}.$$

For these boundary conditions,  $\mathbf{C}_{in}$  and  $\mathbf{C}_{out}$  are,

$$\mathbf{C}_{in} = \begin{pmatrix} -1 & 0 & 1 \\ -1 & (\gamma - 1)M & \gamma \end{pmatrix},$$

$$\mathbf{C}_{out} = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}.$$

From  $\mathbf{C}_{in}$  and  $\mathbf{C}_{out}$ , we find the matrix  $\mathbf{B}$ ,

$$\mathbf{B} = \begin{pmatrix} -1 & 0 & 0 \\ -1 & (\gamma - 1)M & 0 \\ 0 & 0 & M e^{-i\omega/M} \end{pmatrix}.$$

We note that  $b_{13} = b_{23} = b_{31} = b_{32} = 0$  for these boundary conditions. Thus, outgoing waves do not generate incoming waves at either the inflow or outflow boundaries. At the inflow, specification of the entropy guarantees that  $b_{12} = b_{13} = 0$  since the change in entropy is proportional to the strength of the first eigenmode as shown in Equation (3.2). Specification of the stagnation enthalpy is also a non-reflecting inflow condition since the change in stagnation enthalpy,  $\tilde{H}$ , can be written as a linear combination of the first and second eigenmodes,  $\tilde{H} \propto -\mathbf{l}_1^T \mathbf{u} + (\gamma - 1)M \mathbf{l}_2^T \mathbf{u}$ . Finally, as observed from Equation (3.4), the

pressure change is proportional to the change in the third (upstream-running) eigenmode, thus, specification of the pressure at the outflow is non-reflective.

Since the boundary conditions at inflow and outflow are individually non-reflective, the entire system will have infinite, perturbation decay rates. Specifically, the determinant of  $\mathbf{B}$  is,

$$\det \mathbf{B} = -(\gamma - 1)M^2 e^{-i\omega/M},$$

and the eigenfrequencies which result in a zero determinant are,

$$\begin{aligned} \omega_r &= 2\pi M n, & \text{for integer } n, \\ \omega_i &= -\infty. \end{aligned}$$

Since  $\omega_i = -\infty$ , disturbances are eliminated via propagation in finite time. An interesting aspect of this result is that these boundary conditions are actually reflective for the unpreconditioned Euler equations (see Giles[5]).

**3.3. Velocity, temperature at inflow; pressure at outflow.** The final set of boundary conditions we consider is setting velocity and temperature at the inflow and pressure at the outflow. These conditions are fairly common in low speed viscous flow applications. Specifically,

$$X = 0, \quad \begin{cases} p'/\rho' = \bar{p}/\bar{\rho} \\ q' = \bar{q} \end{cases} \quad X = L, \quad p' = \bar{p}.$$

For these boundary conditions,  $\mathbf{C}_{in}$  and  $\mathbf{C}_{out}$  are,

$$\begin{aligned} \mathbf{C}_{in} &= \begin{pmatrix} -1 & 0 & \gamma \\ 0 & 1 & 0 \end{pmatrix}, \\ \mathbf{C}_{out} &= \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

From  $\mathbf{C}_{in}$  and  $\mathbf{C}_{out}$ , we find the matrix  $\mathbf{B}$ ,

$$\mathbf{B} = \begin{pmatrix} -1 & 0 & (\gamma - 1)M \\ 0 & 1 & -1 \\ 0 & 0 & M e^{-i\omega/M} \end{pmatrix}.$$

Unlike the boundary conditions in Section 3.2, the specification of velocity and temperature at the inflow is a reflective condition and create reflections of the outgoing characteristic wave. This is evident from the non-zero values of  $b_{13}$  and  $b_{23}$ . Specifically, at the inflow,  $\alpha_1 = (\gamma - 1)M\alpha_3$  and  $\alpha_2 = \alpha_3$ . However, since  $b_{31} = b_{32} = 0$ , the outflow boundary condition does not create any perturbations in the incoming characteristics. Thus, the reflected waves from the inflow conditions would be emitted at the outflow boundary without further reflection and all disturbances would be eliminated in finite time. Specifically, the determinant of  $\mathbf{B}$  is,

$$\det \mathbf{B} = -M e^{-i\omega/M},$$

and the eigenfrequencies which result in a zero determinant are,

$$\begin{aligned} \omega_r &= 2\pi M n, & \text{for integer } n, \\ \omega_i &= -\infty. \end{aligned}$$



While perturbations again have infinite decay rates for these boundary conditions, in practice, the reflective inflow condition may slow convergence somewhat compared to the non-reflecting boundary conditions discussed in Section 3.2. This convergence slowdown is observed in the numerical results of the following section.

**4. Numerical Results.** To illustrate the effect of different boundary conditions on numerical convergence as well as check the accuracy of the analysis, we simulate the two-dimensional flow in a duct with a straight upper wall and a bump on the lower wall between  $0 \leq x \leq 1$  described by  $y = .042 \sin^2(\pi x)$ . The domain is 5 unit lengths long and 2 lengths high. The grid is structured with clustering toward the wall boundary.

Specifically, we use the algorithm described by Darmofal and Siu[3] which employs the semi-coarsening technique of Mulder[12, 13] in conjunction with a multi-stage, block Jacobi relaxation[10, 1]. The discretization is a 2nd order upwind scheme with a Roe approximate Riemann solver[14]. The calculations are performed on a grid of  $32 \times 16$  cells. A three level, V-cycle is utilized with 2 pre and post smooths. All calculations are initialized to uniform flow.

A form of Turkel's preconditioner is employed which is smoothly turned off for Mach numbers above 0.5. While this preconditioner is different than the one-dimensional van Leer-Lee-Roe preconditioner for which the analysis was performed, we expect the low Mach number behavior of these preconditioners to be similar. The major differences between the analysis and the numerical results will occur for higher subsonic Mach numbers where the preconditioning is turned off in the numerical simulations.

We have implemented the boundary conditions described above by constructing a boundary face state vector and calculating the boundary flux directly from this state vector. For example, for the enthalpy, entropy, pressure boundary conditions at an inflow, entropy, enthalpy, and the tangential velocity are prescribed from the exterior and the pressure is extrapolated from the interior. At an outflow, we reverse the procedure and specify pressure from the exterior and extrapolate entropy, enthalpy, and tangential velocity from the interior. Note, regardless of the specific boundary conditions, we always use the tangential velocity as the additional variable for the two-dimensional boundary implementation.

The number of cycles required to converge the solution six orders of magnitude from the initial residual are given in Table 4.1. As can be clearly seen, the results are in good agreement with the analysis at low Mach numbers. In particular, the Riemann boundary conditions are unstable at low Mach numbers. The entropy, enthalpy, pressure (HSP) boundary conditions perform best while the velocity, temperature, pressure boundary (QTP) conditions are about 75% more expensive. This would indicate that the reflective nature of the inflow for the QTP boundary conditions does indeed slow the convergence.

At the higher Mach numbers ( $M \geq 0.2$ ), the Riemann boundary condition cases begin to converge and the number of cycles decreases with increasing Mach. The QTP boundary conditions require an increasing number of cycles for increasing Mach number. Also, for the  $M_\infty = 0.5$  test case, the amount of work required to converge the HSP boundary conditions increased from 8 cycles to 10 cycles. These trends at higher Mach numbers are expected because the preconditioning which was implemented numerically was automatically phased out at  $M = 0.5$ . Thus, the behavior of the different boundary conditions will be described by the unpreconditioned Euler equations. In this case, the Riemann boundary conditions are non-reflective while the other boundary conditions are reflective.

**5. Remarks.** The present analysis of the preconditioned Euler equations shows the effect of preconditioning on boundary conditions. Boundary conditions based on the Riemann invariants of the Euler equations are found to be reflective in conjunction with preconditioning. The problem is most detrimental at low Mach

| Mach  | Riemann | HSP | QTP |
|-------|---------|-----|-----|
| 0.001 | UNS     | 8   | 13  |
| 0.01  | UNS     | 8   | 13  |
| 0.1   | UNS     | 8   | 13  |
| 0.2   | 20      | 8   | 14  |
| 0.3   | 14      | 8   | 15  |
| 0.4   | 11      | 8   | 18  |
| 0.5   | 9       | 10  | 20  |

TABLE 4.1

*Number of cycles required to drop residual six orders of magnitude for different Mach numbers and boundary conditions. Riemann: Euler Riemann invariant boundary conditions from Section 3.1. HSP: enthalpy, entropy, pressure boundary conditions from Section 3.2. QTP: velocity, temperature, pressure boundary conditions from Section 3.3. UNS: algorithm was unstable and aborted with infinite residual.*

numbers where the decay rate of perturbations approaches zero. Boundary conditions which specify entropy and stagnation enthalpy at an inflow and pressure at an outflow are found to be non-reflective with preconditioning. Numerical results were presented which are in good agreement with the analytic predictions.

Finally, an interesting possibility implied by this analysis would be to incorporate boundary condition considerations into the design of preconditioners. For example, given a set of physical boundary conditions which must be applied for a specific problem, a preconditioner could be designed such that these conditions are non-reflective.

**Appendix.** The one-dimensional van Leer-Lee-Roe preconditioner[8] is usually derived using the symmetrizing variables which in dimensional form are  $(\tilde{p}/\tilde{\rho}\tilde{c}, \tilde{q}, \tilde{p} - \tilde{c}^2\tilde{\rho})$ . Using Equation (2.2), the non-dimensional symmetrizing variables are,

$$\mathbf{v} = \begin{pmatrix} p & q & p - \rho \end{pmatrix}^T,$$

and are related to the  $\mathbf{u} = (\rho, q, p)^T$  variables through the transformation,  $\mathbf{v} = \mathbf{S}\mathbf{u}$ , where,

$$\mathbf{S} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

The preconditioned Euler equations in terms of  $\mathbf{v}$  are,

$$\mathbf{v}_t + \mathbf{P}_v \mathbf{A}_v \mathbf{v}_x = 0,$$

where,  $\mathbf{P}_v = \mathbf{S}\mathbf{P}\mathbf{S}^{-1}$ , and,

$$\mathbf{A}_v = \mathbf{S}\mathbf{A}\mathbf{S}^{-1} = \begin{pmatrix} M & 1 & 0 \\ 1 & M & 0 \\ 0 & 0 & M \end{pmatrix}.$$

Finally, the one-dimensional van Leer-Lee-Roe preconditioner is given by,

$$\mathbf{P}_v = \begin{pmatrix} \frac{M^2}{\beta^2} & -\frac{M}{\beta^2} & 0 \\ -\frac{M}{\beta^2} & 1 + \frac{1}{\beta^2} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and, for use with the  $(\rho, q, p)$  variables,

$$\mathbf{P} = \mathbf{S}^{-1} \mathbf{P}_v \mathbf{S} = \begin{pmatrix} 1 & -\frac{M}{\beta^2} & \frac{M^2}{\beta^2} - 1 \\ 0 & 1 + \frac{1}{\beta^2} & -\frac{M}{\beta^2} \\ 0 & -\frac{M}{\beta^2} & \frac{M^2}{\beta^2} \end{pmatrix},$$

where  $\beta^2 = 1 - M^2$ .

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