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Abstract

A general methodology was constructed to develop the fundamental solution for a crack embedded in an infinite non-homogeneous material in which the shear modulus varies exponentially with the y coordinate (thickness). The fundamental solution was used to generate a solution to fully interactive multiple crack problems for stress intensity factors and strain energy release rates. Parametric studies were conducted for two crack configurations. The model displayed sensitivity to crack distance, relative angular orientation, and to the coefficient of nonhomogeneity.

Introduction

One of the disadvantages of composites is the mismatch of the thermal expansion coefficients between its constituents. This mismatch produces residual stresses, which may initiate debonding, delamination, and micro-cracks. For example, application of ceramics as a thermal coating for a metal substrate often produce debonding at the interface after a small number of thermo-mechanical load cycles. In order to minimize the mismatch between the ceramics and metal a new technology was developed. This technology allows fully tailored processing of materials and interfacial zones with predetermined continuously varying mechanical properties, that are known as Functionally Graded Materials (FGM) (see Asish et. al., 1997 and Holt et. al., 1993).

FGM could be described, as two-phase particulate composites wherein the volume fraction of its constituents differs continuously in the thickness direction (see Niino and Maeda, 1990; Hirano and Yamada, 1988; Hirano et. al., 1988; and Kawasaki and Watanabe R., 1990). This implies that the composition profile could be tailored to give the appropriate thermomechanical properties. Their physical properties can be determined either experimentally or using higher order theory for FGMs developed by Aboudi, Pindera and Arnold (1997).

Delale and Erdogan (1983) solved the crack problem for a nonhomogeneous plane. The authors considered the plane elasticity problem, in which the material is isotropic, has a constant Poisson's ratio (ν), and the Young's modulus (E) is of an exponential form varying in the x-direction, namely,

$$E(x) = E_0 e^{\beta x} \tag{1}$$

where β is a non-homogeneity constant and E_0 the Young modulus of the homogeneous material. They found that Poisson's ratio did not have much effect on the resulting stress intensity factors. And that the strain-energy release rate at the crack embedded in the portion of the medium with higher stiffness is lower than that corresponding to the crack tip in the less stiff side of the material. Hence, the crack will grow in the direction of the less stiff material.

Also, Delale and Erdogan (1988) solved the collinear crack problem for two dissimilar homogeneous elastic half–planes bonded to a very thin nonhomogeneous layer. The elastic properties of the interfacial material varied continuously between those of the two semi-infinite planes. The Airy stress function was used in their formulation of the solution in which it was assumed that it is composed of two functions, one is associated with an infinite plane containing the crack on the x-axis, while the second is an uncracked strip. Their results showed that if the crack location approaches the less stiff material, the strain energy release rate increases.

It can be noticed that multiple oriented crack problems embedded in a non-homogeneous infinite plate have not yet been addressed. Thus the scope of this work will deal with the general solution to a single and multiple oriented cracks embedded in a nonhomogeneous infinite plate. It is assumed that the FGM has a constant Poisson's ratio and the shear modulus is of an exponential form. The solution is valid for both plane stress and plane strain.

General Problem Formulation

The solution of the mixed boundary value problems for stress intensity factors or strain energy release rates at a crack tip is obtained from the perturbation part of the problem, see Figure 1. Before any particular problem is addressed, the general strategy of solution is discussed in this section.

Assume that there are two states of stresses, one is associated with a local coordinate system (x_1-y_1) in an infinite plate, while the other is associated with boundaries of a finite plate defined in a structural coordinate system (x-y). The crack lies on the x_1-axis, which is at an angle θ from the x-axis. In the case of infinite plate problems only the first state of stress exists, but for the general problem the total stresses in the local coordinate system are expressed as:

$$\begin{aligned} \sigma^T_{y_1y_1}(x_1, y_1) &= \sigma^{(1)}_{y_1y_1}(x_1, y_1) + \sigma^{(2)}_{y_1y_1}(x_1, y_1) \\ \tau^T_{x_1y_1}(x_1, y_1) &= \tau^{(1)}_{x_1y_1}(x_1, y_1) + \tau^{(2)}_{x_1y_1}(x_1, y_1) \end{aligned} \tag{2}$$

where,

$$\begin{aligned}
\sigma^{(2)}_{y_1y_1}(x_1, y_1) &= \sin^2(\theta)\sigma_{xx}(x, y) + \cos^2(\theta)\sigma_{yy}(x, y) - 2\sin(\theta)\cos(\theta)\tau_{xy}(x, y) \\
\tau^{(2)}_{x_1y_1}(x_1, y_1) &= -\sin(\theta)\cos(\theta)\sigma_{xx}(x, y) + \sin(\theta)\cos(\theta)\sigma_{yy}(x, y) \\
&\quad + (\cos^2(\theta) - \sin^2(\theta))\tau_{xy}(x, y) \\
x &= x_1\cos(\theta) - y_1\sin(\theta) \\
y &= x_1\sin(\theta) + y_1\cos(\theta)
\end{aligned}$$

The stress boundary conditions obtained from the perturbation problem are:

$$\begin{aligned}
-p_1(x_1) &= \lim_{y_1\to 0}\sigma^T_{y_1y_1}(x_1, y_1) \\
-p_2(x_1) &= \lim_{y_1\to 0}\tau^T_{x_1y_1}(x_1, y_1)
\end{aligned} \tag{3}$$

where $p_1(x_1)$ and $p_2(x_1)$ are the normal and shear tractions of the inner crack surfaces.

Upon substitution of (2) into (3) the boundary conditions becomes:

$$-p_1(x_1) = \lim_{y_1\to 0}\sigma^{(1)}_{y_1y_1}(x_1, y_1) + \lim_{y_1\to 0}\sigma^{(2)}_{y_1y_1}(x_1, y_1) \tag{4}$$

$$-p_2(x_1) = \lim_{y_1\to 0}\tau^{(1)}_{x_1y_1}(x_1, y_1) + \lim_{y_1\to 0}\tau^{(2)}_{x_1y_1}(x_1, y_1) \tag{5}$$

It is noticed that the principal part will be produced from the first part of (4) and (5). The most general form of the stresses are expressed as:

$$\begin{aligned}
\sigma^{(1)}_{y_1y_1}(x_1, y_1) &= \frac{1}{2\pi}\int_a^b K^{(1)}_{1j}(x_1, y_1, t)f_j(t)dt \\
\tau^{(1)}_{x_1y_1}(x_1, y_1) &= \frac{1}{2\pi}\int_a^b K^{(1)}_{2j}(x_1, y_1, t)f_j(t)dt \\
\sigma^{(2)}_{y_1y_1}(x_1, y_1) &= \frac{1}{2\pi}\int_a^b K^{(2)}_{1j}(x_1, y_1, t)f_j(t)dt \\
\tau^{(2)}_{x_1y_1}(x_1, y_1) &= \frac{1}{2\pi}\int_a^b K^{(2)}_{2j}(x_1, y_1, t)f_j(t)dt \qquad \text{for} \qquad a < (x_1, t) < b
\end{aligned} \tag{6}$$

where f_j are so called auxiliary functions defined as derivatives with respect to x_1 of displacement jumps along the crack. The kernels are expressed as:

$$K_{ij}^{(1)}(x_1,y_1,t)=\int_{-\infty}^{\infty}X_{ij}^{(1)}(\alpha,y_1)e^{\alpha i(x_1-t)}d\alpha$$
$$K_{ij}^{(2)}(x_1,y_1,t)=\int_{-\infty}^{\infty}X_{ij}^{(2)}(\alpha,x_1\cos\theta-y_1\sin\theta)e^{\alpha i(x_1\sin\theta+y_1\cos\theta-t)}d\alpha \tag{7}$$

The expressions of $X_{ij}^{(1)}$ and $X_{ij}^{(2)}$ depend on the stress and displacement continuity of the problem. If $X_{ij}^{(1)}$ do not vanish as $|\alpha|$ approaches infinity then an asymptotic analysis is done to separate the singular part from the regular. Consequently equation (7) can be integrated numerically.

As $|\alpha|$ approaches infinity equation (7) becomes:

$$K_{ij}^{(1)}(x_1,y_1,t)=\int_{-\infty}^{\infty}X_{ij}^{(1)}(\alpha)e^{-|\alpha|y_1+\alpha i(x_1-t)}d\alpha$$
$$K_{ij}^{(2)}(x_1,y_1,t)=\int_{-\infty}^{\infty}X_{ij}^{(2)}(\alpha)e^{-|\alpha|(x_1\sin\theta+y_1\cos\theta)+\alpha i(x_1\cos\theta-y_1\sin\theta-t)}d\alpha \tag{8}$$

Substituting equation (8) into (6), the following is obtained:

$$\sigma_{y_1y_1}^{(1)}(x_1,y_1)=\frac{1}{2\pi}\int_{-\infty}^{\infty}X_{1j}^{(1)}(\alpha)e^{-|\alpha|y_1+i\alpha(x_1-t)}d\alpha\int_a^b f_j(t)dt$$
$$\tau_{x_1y_1}^{(1)}(x_1,y_1)=\frac{1}{2\pi}\int_{-\infty}^{\infty}X_{2j}^{(1)}(\alpha)e^{-|\alpha|y_1+i\alpha(x_1-t)}d\alpha\int_a^b f_j(t)dt \tag{9}$$

$$\sigma_{y_1y_1}^{(2)}(x_1,y_1)=\frac{1}{2\pi}\int_{-\infty}^{\infty}X_{1j}^{(2)}(\alpha)e^{-|\alpha|(x_1\sin\theta+y_1\cos\theta)+i\alpha(x_1\cos\theta-y_1\sin\theta-t)}d\alpha\int_a^b f_j(t)dt$$
$$\tau_{x_1y_1}^{(2)}(x_1,y_1)=\frac{1}{2\pi}\int_{-\infty}^{\infty}X_{2j}^{(2)}(\alpha)e^{-|\alpha|(x_1\sin\theta+y_1\cos\theta)+i\alpha(x_1\cos\theta-y_1\sin\theta-t)}d\alpha\int_a^b f_j(t)dt \tag{10}$$

Further simplifications can be achieved by taking the limit of equation (9) so that the first terms in equations (4) and (5) can be determined:

$$\lim_{y_1 \to 0}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X_{ij}^{(1)}(\alpha) e^{-|\alpha| y_1 + i\alpha(x_1 - t)} d\alpha \int_a^b f_j(t) dt\right] \tag{11}$$

Assume that $X_{ij}^{(1)}$ have the following asymptotes:

$$\begin{aligned} asy1 &= c_{ij}^{(1)} + id_{ij}^{(1)} \cdots\cdots \alpha \to +\infty \\ asy2 &= c_{ij}^{(1)} - id_{ij}^{(1)} \cdots\cdots \alpha \to -\infty \end{aligned} \tag{12}$$

Splitting equation (12) at α=0 into two parts and making the change of variable for the part from -∞ to 0 by letting α=-β, adding and subtracting (12) from (11) and taking the limit as $y_1 \to 0$, the following is obtained:

$$\begin{aligned} &\frac{1}{2\pi}\int_0^{\infty}\{[X_{ij}^{(1)}(\alpha) + X_{ijc}^{(1)}(\alpha) - 2c_{ij}^{(1)}]\cos(\alpha(t - x_1)) + \\ &\quad [X_{ijc}^{(1)}(\alpha) - X_{ij}^{(1)}(\alpha) + 2id_{ij}^{(1)}] i \sin(\alpha(t - x_1))\} d\alpha + \lim_{y_1 \to 0}\left[\frac{1}{2\pi} SIP^{(1)}\right] \end{aligned} \tag{13}$$

where, $X_{ijc}^{(1)}(\alpha)$ is the complex conjugate of $X_{ij}^{(1)}(\alpha)$ and the term denoted by $SIP^{(1)}$ is:

$$SIP^{(1)} = \int_0^{\infty} 2e^{-\alpha y_1}\{c_{ij}^{(1)} \cos(\alpha(t - x_1)) + d_{ij}^{(1)} \sin(\alpha(t - x_1))\} d\alpha \tag{14}$$

The following integral identities can be used to evaluate equation (14) (Abramowitz and Stegun, 1964):

$$\begin{aligned} \int_0^{\infty} e^{-\alpha n} \cos(m\alpha) d\alpha &= \frac{n}{n^2 + m^2} \\ \int_0^{\infty} e^{-\alpha n} \sin(m\alpha) d\alpha &= \frac{m}{n^2 + m^2} \end{aligned} \tag{15}$$

Hence, it can be shown that (14) becomes,

$$2\frac{c_{ij}^{(1)} y_1}{(t - x_1)^2 + y_1^2} + 2\frac{d_{ij}^{(1)}(t - x_1)}{(t - x_1)^2 + y_1^2} \tag{16}$$

where upon taking the limit of (16) and substituting the result into (13) the following expression is obtained:

$$\frac{1}{2\pi}\int_0^{\infty}\{[X_{ij}^{(1)} + X_{ijc}^{(1)} - 2c_{ij}^{(1)}]\cos(\alpha(t - x_1)) + [X_{ijc}^{(1)} - X_{ij}^{(1)} + 2id_{ij}^{(1)}]i\sin(\alpha(t - x_1))\}d\alpha + \frac{1}{\pi}\frac{d_{ij}^{(1)}}{t - x_1} \quad (17)$$

In some problems the integrand $X_{ij}^{(1)}(\alpha)$ does not converge rapidly to zero, consequently U (where $X_{ij}^{(1)}$ is close to zero) is large, thus for computational efficiency an additional term of the asymptote is taken which is of the order (α^{-1}), namely:

$$asy11 = \frac{e_{ij}^{(1)}}{\alpha} + i\frac{g_{ij}^{(1)}}{\alpha}\cdots\cdots\alpha \to +\infty \quad (18)$$

Equation (17) remains the same if the asymptote (18) is subtracted and added at the same time. It becomes:

$$\begin{aligned}&\frac{1}{2\pi}\int_0^{\infty}\{[X_{ij}^{(1)}(\alpha) + X_{ijc}^{(1)}(\alpha) - 2(c_{ij}^{(1)} + \frac{e_{ij}^{(1)}}{\alpha})]\cos(\alpha(t - x_1)) + \\ &[X_{ijc}^{(1)}(\alpha) - X_{ij}^{(1)}(\alpha) + 2i(d_{ij}^{(1)} + \frac{g_{ij}^{(1)}}{\alpha})]i\sin(\alpha(t - x_1))\}d\alpha + \frac{1}{\pi}\frac{d_{ij}^{(1)}}{t - x_1} \\ &+ \frac{1}{2\pi}\int_0^{\infty}\frac{2e_{ij}^{(1)}\cos(\alpha(t - x_1))}{\alpha}d\alpha - \frac{1}{2\pi}\int_0^{\infty}\frac{2g_{ij}^{(1)}\sin(\alpha(t - x_1))}{\alpha}d\alpha\end{aligned} \quad (19)$$

To evaluate the last two terms of equation (19) the following identities are used (Abramowitz and Stegun, 1964):

$$\begin{aligned}&\int_0^{\infty}\frac{1}{\alpha}\sin(\alpha(t - x_1))d\alpha = \frac{\pi}{2}\frac{(t - x_1)}{|t - x_1|} \\ &\int_U^{\infty}\frac{1}{\alpha}\cos(\alpha(t - x_1))d\alpha = -Ci(U(t - x_1)) \\ &= -(C_0 + \log|U(t - x_1)| + \int_0^{|U(t - x_1)|}\frac{\cos\beta - 1}{\beta}d\beta)\end{aligned} \quad (20)$$

where, C_0 is the Euler constant. The following expression replaces equation (19),

$$\begin{aligned}
&\frac{1}{2\pi}\int_0^{\infty}\{[X_{ijc}^{(1)}(\alpha)-X_{ij}^{(1)}(\alpha)+2i(d_{ij}^{(1)}+\frac{g_{ij}^{(1)}}{\alpha})]i\sin(\alpha(t-x_1))\}d\alpha \\
&+\frac{1}{2\pi}\int_0^{U}\{[X_{ij}^{(1)}(\alpha)+X_{ijc}^{(1)}(\alpha)-2c_{ij}^{(1)}]\cos(\alpha(t-x_1))\}d\alpha \\
&+\frac{1}{2\pi}\int_U^{\infty}\{[X_{ij}^{(1)}(\alpha)+X_{ijc}^{(1)}(\alpha)-2(c_{ij}^{(1)}+\frac{e_{ij}^{(1)}}{\alpha})]\cos(\alpha(t-x_1))\}d\alpha \\
&+\frac{1}{\pi}\frac{d_{ij}^{(1)}}{t-x_1}-\frac{g_{ij}^{(1)}}{2}\frac{(t-x_1)}{|t-x_1|}-\frac{e_{ij}^{(1)}}{\pi}Ci(U(t-x_1))
\end{aligned} \tag{21}$$

The first part of (4) and (5) can be expressed as:

$$\begin{aligned}
\Rightarrow \lim_{y_1\to 0}\sigma_{y_1y_1}^{(1)}(x_1,y_1) = &\frac{1}{\pi}\int_a^b \frac{d_{1j}^{(1)}f_j(t)}{t-x_1}dt+\frac{1}{2\pi}\int_a^b k_{1j}^{(1)}(x_1,t)f_j(t)dt \\
&+\int_a^b [\frac{g_{1j}^{(1)}}{2}\frac{(t-x_1)}{|t-x_1|}-\frac{e_{1j}^{(1)}}{\pi}Ci(U(t-x_1))]f_j(t)dt
\end{aligned} \tag{22}$$

$$\begin{aligned}
\Rightarrow \lim_{y_1\to 0}\tau_{x_1y_1}^{(1)}(x_1,y_1) = &\frac{1}{\pi}\int_a^b \frac{d_{2j}^{(1)}f_j(t)}{t-x_1}dt+\frac{1}{2\pi}\int_a^b k_{2j}^{(1)}(x_1,t)f_j(t)dt \\
&+\int_a^b [\frac{g_{2j}^{(1)}}{2}\frac{(t-x_1)}{|t-x_1|}-\frac{e_{2j}^{(1)}}{\pi}Ci(U(t-x_1))]f_j(t)dt
\end{aligned} \tag{23}$$

where,

$$\begin{aligned}
k_{ij}^{(1)}(x_1,t)=&\int_0^{\infty}\{[X_{ijc}^{(1)}(\alpha)-X_{ij}^{(1)}(\alpha)+2i(d_{ij}^{(1)}+\frac{g_{ij}^{(1)}}{\alpha})]i\sin(\alpha(t-x_1))\}d\alpha \\
&+\int_0^{U}\{[X_{ij}^{(1)}(\alpha)+X_{ijc}^{(1)}(\alpha)-2c_{ij}^{(1)}]\cos(\alpha(t-x_1))\}d\alpha \\
&+\int_U^{\infty}\{[X_{ij}^{(1)}(\alpha)+X_{ijc}^{(1)}(\alpha)-2(c_{ij}^{(1)}+\frac{e_{ij}^{(1)}}{\alpha})]\cos(\alpha(t-x_1))\}d\alpha
\end{aligned} \tag{24}$$

A similar procedure is applied procedure to the second terms in equations (4) and (5), expressed in the form shown in (10). The terms denoted by $X_{ij}^{(2)}(\alpha)$ have the following asymptotes:

$$\begin{aligned}
asy3 &= (a_{ij}^{(2)}+ib_{ij}^{(2)})\alpha+c_{ij}^{(2)}+id_{ij}^{(2)}\cdots\cdots\alpha\to+\infty \\
asy4 &= (-a_{ij}^{(2)}+ib_{ij}^{(2)})\alpha+c_{ij}^{(2)}-id_{ij}^{(2)}\cdots\cdots\alpha\to-\infty
\end{aligned} \tag{25}$$

Simplifications can be made with the use of the integral identities (Abramowitz and Stegun, 1964):

$$\begin{aligned}
\int_0^\infty \alpha e^{-\alpha n} \cos(m\alpha)d\alpha &= \frac{n^2 - m^2}{(n^2 + m^2)^2} \\
\int_0^\infty \alpha e^{-\alpha n} \sin(m\alpha)d\alpha &= \frac{2nm}{(n^2 + m^2)^2}
\end{aligned} \tag{26}$$

Hence, the second part of equations (4) and (5) becomes:

$$\begin{aligned}
\Rightarrow \lim_{y_1 \to 0} \sigma^{(2)}_{x_1 x_1}(x_1, y_1) = \frac{1}{\pi} \int_a^b [&\frac{a^{(2)}_{1j}(x_1^2 \sin^2\theta - \cos^2\theta (t - x_1)^2)}{(x_1^2 \sin^2\theta + \cos^2\theta (t - x_1)^2)^2} \\
&+ \frac{2b^{(2)}_{1j} x_1 (t - x_1) \sin\theta \cos\theta}{(x_1^2 \sin^2\theta + \cos^2\theta (t - x_1)^2)^2} \\
&+ \frac{c^{(2)}_{1j} x_1 \sin\theta + 2d^{(2)}_{1j} \cos\theta (t - x_1)}{(x_1^2 \sin^2\theta + \cos^2\theta (t - x_1)^2)^2}] f_j(t)dt \\
&+ \frac{1}{2\pi} \int_a^b k^{(2)}_{1j}(x_1, t) f_j(t)dt
\end{aligned} \tag{27}$$

$$\begin{aligned}
\Rightarrow \lim_{y_1 \to 0} \tau^{(2)}_{x_1 y_1}(x_1, y_1) = \frac{1}{\pi} \int_a^b [&\frac{a^{(2)}_{2j}(x_1^2 \sin^2\theta - \cos^2\theta (t - x_1)^2)}{(x_1^2 \sin^2\theta + \cos^2\theta (t - x_1)^2)^2} \\
&+ \frac{2b^{(2)}_{2j} x_1 (t - x_1) \sin\theta \cos\theta}{(x_1^2 \sin^2\theta + \cos^2\theta (t - x_1)^2)^2} \\
&+ \frac{c^{(2)}_{2j} x_1 \sin\theta + 2d^{(2)}_{2j} \cos\theta (t - x_1)}{(x_1^2 \sin^2\theta + \cos^2\theta (t - x_1)^2)^2}] f_j(t)dt \\
&+ \frac{1}{2\pi} \int_a^b k^{(2)}_{2j}(x_1, t) f_j(t)dt
\end{aligned} \tag{28}$$

where,

$$\begin{aligned}
k^{(2)}_{ij}(x_1, t) = &\int_0^\infty \{[X^{(2)}_{ijc}(\alpha) - X^{(2)}_{ij}(\alpha) + 2i(\alpha b^{(2)}_{ij} + d^{(2)}_{ij}) e^{-\alpha x_1 \sin\theta}] i \sin(\alpha(t - x_1))\} d\alpha \\
&+ \int_0^\infty \{[X^{(2)}_{ij}(\alpha) + X^{(2)}_{ijc}(\alpha) - 2(\alpha a^{(2)}_{ij} + c^{(2)}_{ij}) e^{-\alpha x_1 \sin\theta}] \cos(\alpha(t - x_1))\} d\alpha
\end{aligned} \tag{29}$$

It should be noted that (27) and (28) do not contain any singularity and the asymptotic expansion is applied only for computational efficiency.

Fundamental Solution

The formulated equations in the previous section will be used to solve the problem of radial multiple cracks in an infinite isotropic FGM as depicted in Figure 2. Since dealing with an infinite plate only equations (22), (23) and (24) will be used. But before doing so, the fundamental solution of a single crack is required. Konda and Erdogan (1994) solved the single crack problem using Navier equations. In this work the same problem will be solved using Airy stress function and the shear modulus μ will vary exponentially with the global y-axis.

The shear modulus is defined as follows:

$$\begin{aligned} \mu_2(\mathrm{y}) &= \mu_0 e^{\gamma y} \\ \mu_2(x_1, y_1) &= \mu_0 e^{\beta x_1 + \delta y_1} \\ \delta &= \gamma \cos(\theta) \\ \beta &= \gamma \sin(\theta) \end{aligned} \tag{30}$$

where γ, δ, and β are real constants and represent the coefficients of nonhomogeneity.

The Airy stress function $U(x_1,y_1)$ are defined as,

$$\begin{aligned} \sigma_{x_1x_1}(x_1, y_1) &= \frac{\partial^2 U}{\partial y_1^2} \\ \sigma_{y_1y_1}(x_1, y_1) &= \frac{\partial^2 U}{\partial x_1^2} \\ \tau_{x_1y_1}(x_1, y_1) &= -\frac{\partial^2 U}{\partial x_1 \partial y_1} \end{aligned} \tag{31}$$

The stresses and strains are related through:

$$\begin{aligned} \varepsilon_{x_1x_1}(x_1, y_1) &= \frac{\partial u_1}{\partial x_1} = \frac{1}{8\mu(x_1, y_1)}\left[(\kappa+1)\sigma_{x_1x_1} + (\kappa-3)\sigma_{y_1y_1}\right] \\ \varepsilon_{y_1y_1}(x_1, y_1) &= \frac{\partial v_1}{\partial y_1} = \frac{1}{8\mu(x_1, y_1)}\left[(\kappa-3)\sigma_{x_1x_1} + (\kappa+1)\sigma_{y_1y_1}\right] \\ \varepsilon_{x_1y_1}(x_1, y_1) &= \frac{1}{2}\left(\frac{\partial u_1}{\partial y_1} + \frac{\partial v_1}{\partial x_1}\right) = \frac{1}{2\mu(x_1, y_1)}\tau_{x_1y_1} \end{aligned} \tag{32}$$

where κ is defined as,

$$\kappa = 3 - 4\nu \cdots\cdots \text{for plane strain}$$
$$\kappa = \frac{3-\nu}{1+\nu} \cdots\cdots \text{for plane stress} \tag{33}$$

and the compatibility equation is defined as:

$$\frac{\partial^2 \varepsilon_{x_1x_1}}{\partial x_1^2} + \frac{\partial^2 \varepsilon_{y_1y_1}}{\partial y_1^2} - 2\frac{\partial^2 \varepsilon_{x_1y_1}}{\partial x_1 \partial y_1} = 0 \tag{34}$$

From the compatibility equation the following fourth order governing equation for $U(x_1,y_1)$ is obtained,

$$\begin{aligned}
&\nabla^4 U(x_1,y_1) - 2\delta\frac{\partial}{\partial y_1}(\nabla^2 U(x_1,y_1)) - 2\beta\frac{\partial}{\partial x_1}(\nabla^2 U(x_1,y_1)) + \\
&\frac{8\beta\delta}{1+\kappa}\frac{\partial^2 U(x_1,y_1)}{\partial x_1 \partial y_1} + \beta^2\left(\frac{\kappa-3}{1+\kappa}\frac{\partial^2 U(x_1,y_1)}{\partial y_1^2} + \frac{\partial^2 U(x_1,y_1)}{\partial x_1^2}\right) + \\
&\delta^2\left(\frac{\partial^2 U(x_1,y_1)}{\partial y_1^2} + \frac{\kappa-3}{1+\kappa}\frac{\partial^2 U(x_1,y_1)}{\partial x_1^2}\right) = 0
\end{aligned} \tag{35}$$

where

$$\nabla = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial y_1^2}\right) \tag{36}$$

Defining the Fourier transform as follows,

$$V(\alpha, y_1) = \int_{-\infty}^{\infty} U(x_1,y_1)e^{-i\alpha x_1}dx_1 \tag{37}$$

and applying equation (37) to (35), the following characteristic equation is obtained,

$$\begin{aligned}
&n^4 - 2\delta n^3 + \left(\beta^2\frac{\kappa_2 - 3}{\kappa_2 + 1} + \delta^2 - 2\alpha(i\beta + \alpha)\right)n^2 + \alpha\delta\left(\frac{8\beta}{\kappa_2 + 1}i + 2\alpha\right)n \\
&\quad + \alpha^2\left(\alpha^2 + \delta^2\frac{3-\kappa_2}{\kappa_2 + 1} + \beta(2i\alpha - \beta)\right) = 0
\end{aligned} \tag{38}$$

the roots of which are:

$$
\begin{aligned}
n_1 &= \frac{1}{2}\left(\delta + \beta\sqrt{\frac{3-\kappa}{\kappa_2+1}}\right) - \frac{1}{2}\sqrt{\left(\delta + \beta\sqrt{\frac{3-\kappa}{\kappa+1}}\right)^2 + 4(\alpha^2 + i\alpha\left(\beta - \delta\sqrt{\frac{3-\kappa}{\kappa+1}}\right)} \\
n_2 &= \frac{1}{2}\left(\delta - \beta\sqrt{\frac{3-\kappa}{\kappa+1}}\right) - \frac{1}{2}\sqrt{\left(\delta - \beta\sqrt{\frac{3-\kappa}{\kappa+1}}\right)^2 + 4(\alpha^2 + i\alpha\left(\beta + \delta\sqrt{\frac{3-\kappa}{\kappa+1}}\right)} \\
n_3 &= \frac{1}{2}\left(\delta + \beta\sqrt{\frac{3-\kappa}{\kappa+1}}\right) + \frac{1}{2}\sqrt{\left(\delta + \beta\sqrt{\frac{3-\kappa}{\kappa+1}}\right)^2 + 4(\alpha^2 + i\alpha\left(\beta - \delta\sqrt{\frac{3-\kappa}{\kappa+1}}\right)} \\
n_4 &= \frac{1}{2}\left(\delta - \beta\sqrt{\frac{3-\kappa}{\kappa+1}}\right) + \frac{1}{2}\sqrt{\left(\delta - \beta\sqrt{\frac{3-\kappa}{\kappa+1}}\right)^2 + 4(\alpha^2 + i\alpha\left(\beta + \delta\sqrt{\frac{3-\kappa}{\kappa+1}}\right)}
\end{aligned}
\tag{39}
$$

The solution to the ODE becomes:

$$V(\alpha, y_1) = B_1(\alpha)e^{n_1 y_1} + B_2(\alpha)e^{n_2 y_1} + B_3(\alpha)e^{n_3 y_1} + B_4(\alpha)e^{n_4 y_1} \tag{40}$$

so that,

$$U(x_1, y_1) = \frac{1}{2\pi}\int_{-\infty}^{\infty}[B_1(\alpha)e^{n_1 y_1} + B_2(\alpha)e^{n_2 y_1} + B_3(\alpha)e^{n_3 y_1} + B_4(\alpha)e^{n_4 y_1}]e^{ix_1\alpha}d\alpha \tag{41}$$

Bounded form of equation (41) can be obtained upon examination of the roots of the characteristic equations. The real part of n_1 and n_2 are negative while that of n_3 and n_4 are positive as α approaches infinity. Hence U_2 is defined for positive and negative y_1 as,

$$
\begin{aligned}
U_2(x_1, y_1) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}(B_1 e^{n_1 y_1} + B_2 e^{n_2 y_1})e^{ix_1\alpha}d\alpha; \cdots\cdots\cdots y_1 > 0 \\
U_2(x_1, y_1) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}(B_3 e^{n_3 y_1} + B_4 e^{n_4 y_1})e^{ix_1\alpha}d\alpha; \cdots\cdots\cdots y_1 < 0
\end{aligned}
\tag{42}
$$

Normal and shear stresses must be continuos at y_1=0. The continuity conditions can be represented by equation (31) as:

$$
\begin{aligned}
\frac{\partial U(x_1, 0^+)}{\partial y_1} &= \frac{\partial U(x_1, 0^-)}{\partial y_1} \\
U(x_1, 0^+) &= U(x_1, 0^-)
\end{aligned}
\tag{43}
$$

where, 0^+ is for $y_1>0$ and 0^- is for $y_1<0$. Using conditions (43) we can eliminate B_3 and B_4 :

$$
\begin{aligned}
B_3 &= \frac{n_4 - n_1}{n_4 - n_3} B_1 + \frac{n_4 - n_2}{n_4 - n_3} B_2 \\
B_4 &= \frac{n_1 - n_3}{n_4 - n_3} B_1 + \frac{n_2 - n_3}{n_4 - n_3} B_2
\end{aligned}
\tag{44}
$$

The remaining two unknowns can be expressed in terms of the auxiliary functions:

$$
\begin{aligned}
f_1(x_1) &= \frac{\partial}{\partial x_1}[u_1(x_1,0^+) - u_1(x_1,0^-)] \\
f_2(x_1) &= \frac{\partial}{\partial x_1}[v_1(x_1,0^+) - v_1(x_1,0^-)]
\end{aligned}
\tag{45}
$$

The final expressions of the stresses are obtained using (31) and Hooke's law,

$$
\begin{aligned}
\sigma_{x_1x_1}(x_1, y_1^+) = \frac{1}{2\pi}\int_{-\infty}^{\infty}[n_1^2 \frac{F_1(\alpha)h_{22} - F_2(\alpha)h_{12}}{h_{11}h_{22} - h_{12}h_{21}} e^{n_1y_1} \\
+ n_2^2 \frac{-F_1(\alpha)h_{21} + F_2(\alpha)h_{11}}{h_{11}h_{22} - h_{12}h_{21}} e^{n_2y_1}]e^{ix_1\alpha}d\alpha
\end{aligned}
\tag{46}
$$

$$
\begin{aligned}
\sigma_{y_1y_1}(x_1, y_1^+) = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\alpha^2[\frac{F_1(\alpha)h_{22} - F_2(\alpha)h_{12}}{h_{11}h_{22} - h_{12}h_{21}} e^{n_1y_1} \\
+ \frac{-F_1(\alpha)h_{21} + F_2(\alpha)h_{11}}{h_{11}h_{22} - h_{12}h_{21}} e^{n_2y_1}]e^{ix_1\alpha}d\alpha
\end{aligned}
\tag{47}
$$

$$
\begin{aligned}
\tau_{x_1y_1}(x_1, y_1^+) = -\frac{i}{2\pi}\int_{-\infty}^{\infty}\alpha[n_1 \frac{F_1(\alpha)h_{22} - F_2(\alpha)h_{12}}{h_{11}h_{22} - h_{12}h_{21}} e^{n_1y_1} \\
+ n_2 \frac{-F_1(\alpha)h_{21} + F_2(\alpha)h_{11}}{h_{11}h_{22} - h_{12}h_{21}} e^{n_2y_1}]e^{ix_1\alpha}d\alpha
\end{aligned}
\tag{48}
$$

where,

$$
\begin{aligned}
h_{11} &= \frac{\kappa+1}{8\mu_0}(n_1-n_3)(n_1-n_4) \\
h_{12} &= \frac{\kappa+1}{8\mu_0}(n_2-n_3)(n_2-n_4) \\
h_{21} &= \frac{i\alpha-\beta}{8\mu_0}(\alpha^2(1+\kappa)-\delta^2(\kappa_2-3))[\frac{(n_1-n_3)(n_1-n_4)}{(\delta-n_1)(\delta-n_3)(\delta-n_4)}] \\
h_{22} &= \frac{i\alpha-\beta}{8\mu_0}(\alpha^2(1+\kappa)-\delta^2(\kappa_2-3))[\frac{(n_2-n_3)(n_2-n_4)}{(\delta-n_2)(\delta-n_3)(\delta-n_4)}] \\
F_j(\alpha) &= \int_a^b f_j(t)e^{(\beta-i\alpha)t}dt \cdots\cdots \mathrm{j}=1,2.
\end{aligned}
\tag{49}
$$

The singular integral equations can be solved for the auxiliary functions using the boundary conditions:

$$
\begin{aligned}
-p_1(x_1) &= \lim_{y_1\to 0}\sigma_{y_1y_1}(x_1,y_1)\cdots\cdots \mathrm{a} < \mathrm{x}_1 < \mathrm{b} \\
-p_2(x_1) &= \lim_{y_1\to 0}\tau_{x_1y_1}(x_1,y_1)\cdots\cdots \mathrm{a} < \mathrm{x}_1 < \mathrm{b}
\end{aligned}
\tag{50}
$$

Equation (47) and (48) are rearranged as follows,

$$
\begin{aligned}
\sigma_{y_1y_1}(x_1,y_1^+) = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\alpha^2[&\frac{h_{22}e^{n_1y_1}-h_{21}e^{n_2y_1}}{h_{11}h_{22}-h_{12}h_{21}}F_1(\alpha) \\
&+\frac{h_{11}e^{n_2y_1}-h_{12}e^{n_1y_1}}{h_{11}h_{22}-h_{12}h_{21}}F_2(\alpha)]e^{ix_1\alpha}d\alpha
\end{aligned}
\tag{51}
$$

$$
\begin{aligned}
\tau_{x_1y_1}(x_1,y_1^+) = -\frac{i}{2\pi}\int_{-\infty}^{\infty}\alpha[&\frac{n_1h_{22}e^{n_1y_1}-n_2h_{21}e^{n_2y_1}}{h_{11}h_{22}-h_{12}h_{21}}F_1(\alpha) \\
&+\frac{n_2h_{11}e^{n_2y_1}-n_1h_{12}e^{n_1y_1}}{h_{11}h_{22}-h_{12}h_{21}}F_2(\alpha)]e^{ix_1\alpha}d\alpha
\end{aligned}
\tag{52}
$$

As α goes to infinity $n_1 \cong n_2$ so X_{ij} can be expressed as defined in equation (9):

$$
\begin{aligned}
X_{11} &= -\alpha^2[\frac{h_{22}-h_{21}}{h_{11}h_{22}-h_{12}h_{21}}] \quad & X_{12} &= -\alpha^2[\frac{h_{11}-h_{12}}{h_{11}h_{22}-h_{12}h_{21}}] \\
X_{21} &= -i\alpha[\frac{n_1h_{22}-n_2h_{21}}{h_{11}h_{22}-h_{12}h_{21}}] \quad & X_{22} &= -i\alpha[\frac{n_2h_{11}-n_1h_{12}}{h_{11}h_{22}-h_{12}h_{21}}]
\end{aligned}
\tag{53}
$$

the asymptotes of X_{ij} are found as:

$$\begin{aligned}
&X_{11}(+\infty) = \frac{\delta}{\alpha(1+\kappa)} \quad X_{11}(-\infty) = \frac{\delta}{-\alpha(1+\kappa)} \\
&X_{12}(+\infty) = \frac{2i}{(1+\kappa)} + \frac{\beta}{\alpha(1+\kappa)} \quad X_{12}(-\infty) = \frac{-2i}{(1+\kappa)} + \frac{\beta}{-\alpha(1+\kappa)} \\
&X_{21}(+\infty) = \frac{2i}{(1+\kappa)} + \frac{\beta}{\alpha(1+\kappa)} \quad X_{21}(-\infty) = \frac{-2i}{(1+\kappa)} + \frac{\beta}{-\alpha(1+\kappa)} \\
&X_{22}(+\infty) = \frac{-\delta}{\alpha(1+\kappa)} \quad X_{22}(-\infty) = \frac{-\delta}{-\alpha(1+\kappa)}
\end{aligned} \tag{54}$$

so that from (54) it is concluded that $d_{ij}^1, e_{ij}^1, c_{ij}^1$ and g_{ij}^1 described earlier in equations (12) and (18) become,

$$\begin{aligned}
&d_{12}^1 = d_{21}^1 = \frac{2}{1+\kappa} \\
&e_{11}^1 = -e_{22}^1 = \frac{\delta}{1+\kappa} \\
&e_{12}^1 = e_{21}^1 = \frac{\beta}{1+\kappa} \\
&c_{11}^1 = c_{12}^1 = c_{21}^1 = c_{22}^1 = d_{11}^1 = d_{22}^1 = g_{11}^1 = g_{12}^1 = g_{21}^1 = g_{22}^1 = 0
\end{aligned} \tag{55}$$

Now substituting (53) and (55) into (22) and (23) the final singular integral equations are reached,

$$\begin{aligned}
-\frac{\kappa+1}{2\mu(x_1,0)} p_1(x_1) = &\frac{1}{\pi}\int_a^b \frac{f_2(t)}{t-x_1}dt + \frac{1}{\pi}\int_a^b k_{12}(x_1,t) f_2(t)dt \\
&-\frac{1}{\pi}\int_a^b \frac{\beta}{2} Ci(U(t-x_1))] f_2(t)dt \\
&+\frac{1}{\pi}\int_a^b k_{11}(x_1,t) f_1(t)dt - \frac{1}{\pi}\int_a^b \frac{\delta}{2} Ci(U(t-x_1))] f_1(t)dt
\end{aligned} \tag{56}$$

$$\begin{aligned}
-\frac{\kappa+1}{2\mu(x_1,0)} p_2(x_1) = &\frac{1}{\pi}\int_a^b \frac{f_1(t)}{t-x_1}dt + \frac{1}{\pi}\int_a^b k_{21}(x_1,t) f_1(t)dt \\
&-\frac{1}{\pi}\int_a^b \frac{\beta}{2} Ci(U(t-x_1))] f_1(t)dt \\
&+\frac{1}{\pi}\int_a^b k_{22}(x_1,t) f_2(t)dt + \frac{1}{\pi}\int_a^b \frac{\delta}{2} Ci(U(t-x_1))] f_2(t)dt
\end{aligned} \tag{57}$$

where,

$$k_{11}(x_1,t) = \frac{\kappa+1}{4}\{\int_0^\infty\{[X_{11c} - X_{11}]i\sin(\alpha(t-x_1))\}d\alpha$$
$$+\int_0^U\{[X_{11} + X_{11c}]\cos(\alpha(t-x_1))\}d\alpha \tag{58}$$
$$+\int_U^\infty\{[X_{11} + X_{11c} - 2\frac{\delta}{\alpha(1+\kappa)}]\cos(\alpha(t-x_1))\}d\alpha\}$$

$$k_{12}(x_1,t) = \frac{\kappa+1}{4}\{\int_0^\infty\{[X_{12c} - X_{12} + \frac{4i}{1+\kappa}]i\sin(\alpha(t-x_1))\}d\alpha$$
$$+\int_0^U\{[X_{12} + X_{12c}]\cos(\alpha(t-x_1))\}d\alpha \tag{59}$$
$$+\int_U^\infty\{[X_{11} + X_{11c} - 2\frac{\beta}{\alpha(1+\kappa)}]\cos(\alpha(t-x_1))\}d\alpha\}$$

$$k_{21}(x_1,t) = \frac{\kappa+1}{4}\{\int_0^\infty\{[X_{21c} - X_{21} + \frac{4i}{1+\kappa}]i\sin(\alpha(t-x_1))\}d\alpha$$
$$+\int_0^U\{[X_{21} + X_{21c}]\cos(\alpha(t-x_1))\}d\alpha \tag{60}$$
$$+\int_U^\infty\{[X_{21} + X_{21c} - 2\frac{\beta}{\alpha(1+\kappa)}]\cos(\alpha(t-x_1))\}d\alpha\}$$

$$k_{22}(x_1,t) = \frac{\kappa+1}{4}\{\int_0^\infty\{[X_{22c} - X_{22}]i\sin(\alpha(t-x_1))\}d\alpha$$
$$+\int_0^U\{[X_{22} + X_{22c}]\cos(\alpha(t-x_1))\}d\alpha \tag{61}$$
$$+\int_U^\infty\{[X_{22} + X_{22c} + 2\frac{\delta}{\alpha(1+\kappa)}]\cos(\alpha(t-x_1))\}d\alpha\}$$

The definitions for the stress intensity factors (SIF) and the strain energy release rate (SERR) can be found in Konda and Erdogan (1994). Applying the Lobatto-Chebyshev collocation integration technique, as in Binienda and Arnold (1995), to the system of singular integral equations (56) and (57), the normalized mode I SIF were produced and compared to that of Konda and Erdogan (1994) as summarized in Table I.

Multiple Crack Formulation

To formulate the multiple crack problem the total stresses of the system needs to be determined. The cracks are located along their local x_i axes, which are related by the following coordinate transformation:

$$
\begin{aligned}
x_i &= x_{i+1}\cos(\theta_{i+1}-\theta_i) - y_{i+1}\sin(\theta_{i+1}-\theta_i) \\
y_i &= x_{i+1}\sin(\theta_{i+1}-\theta_i) + y_{i+1}\cos(\theta_{i+1}-\theta_i) \\
x_{i+1} &= x_i\cos(\theta_{i+1}-\theta_i) + y_i\sin(\theta_{i+1}-\theta_i) \\
y_{i+1} &= -x_i\sin(\theta_{i+1}-\theta_i) + y_i\cos(\theta_{i+1}-\theta_i) \\
&\theta_{i+1} > \theta_i
\end{aligned}
\tag{62}
$$

and the stresses are related through the Cauchy stress transformation tensor:

$$
\begin{aligned}
&\begin{Bmatrix} \sigma_{x_i x_i} \\ \sigma_{y_i y_i} \\ \tau_{x_i y_i} \end{Bmatrix} = \begin{bmatrix} m^2 & n^2 & -2mn \\ n^2 & m^2 & 2mn \\ mn & -mn & m^2-n^2 \end{bmatrix} \begin{Bmatrix} \sigma_{x_{i+1}x_{i+1}} \\ \sigma_{y_{i+1}y_{i+1}} \\ \tau_{x_{i+1}y_{i+1}} \end{Bmatrix} \\
&\begin{Bmatrix} \sigma_{x_{i+1}x_{i+1}} \\ \sigma_{y_{i+1}y_{i+1}} \\ \tau_{x_{i+1}y_{i+1}} \end{Bmatrix} = \begin{bmatrix} m^2 & n^2 & 2mn \\ n^2 & m^2 & -2mn \\ -mn & mn & m^2-n^2 \end{bmatrix} \begin{Bmatrix} \sigma_{x_i x_i} \\ \sigma_{y_i y_i} \\ \tau_{x_i y_i} \end{Bmatrix} \\
&m = \cos(\theta_{i+1}-\theta_i);\, n = \sin(\theta_{i+1}-\theta_i)
\end{aligned}
\tag{63}
$$

The material constants are:

$$
\begin{aligned}
\beta_i &= \gamma\sin(\theta_i) \\
\delta_i &= \gamma\cos(\theta_i)
\end{aligned}
\tag{64}
$$

The stresses for each coordinate system are expressed as:

$$
\begin{aligned}
\sigma_{x_i x_i}(x_i, y_i^+) = \frac{1}{2\pi}\int_{-\infty}^{\infty} [& \frac{\overset{(i)}{n}{}_1^2\,\overset{(i)}{h}_{22}\,e^{\overset{(i)}{n}_1 y_i} - \overset{(i)}{n}{}_2^2\,\overset{(i)}{h}_{21}\,e^{\overset{(i)}{n}_2 y_i}}{\overset{(i)}{h}_{11}\overset{(i)}{h}_{22} - \overset{(i)}{h}_{12}\overset{(i)}{h}_{21}} \overset{(i)}{F}_1(\alpha) \\
& + \frac{\overset{(i)}{n}{}_2^2\,\overset{(i)}{h}_{11}\,e^{\overset{(i)}{n}_2 y_i} - \overset{(i)}{n}{}_1^2\,\overset{(i)}{h}_{12}\,e^{\overset{(i)}{n}_1 y_i}}{\overset{(i)}{h}_{11}\overset{(i)}{h}_{22} - \overset{(i)}{h}_{12}\overset{(i)}{h}_{21}} \overset{(i)}{F}_2(\alpha)] e^{ix_i\alpha}\,d\alpha
\end{aligned}
\tag{65}
$$

$$
\begin{aligned}
\sigma_{y_i y_i}(x_i, y_i^+) = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\alpha^2\Bigg[&\frac{\overset{(i)}{h}_{22}\, e^{\overset{(i)}{n}_1 y_i} - \overset{(i)}{h}_{21}\, e^{\overset{(i)}{n}_2 y_i}}{\overset{(i)}{h}_{11}\overset{(i)}{h}_{22} - \overset{(i)}{h}_{12}\overset{(i)}{h}_{21}}\overset{(i)}{F}_1(\alpha) \\
&+ \frac{\overset{(i)}{h}_{11}\, e^{\overset{(i)}{n}_2 y_i} - \overset{(i)}{h}_{12}\, e^{\overset{(i)}{n}_1 y_i}}{\overset{(i)}{h}_{11}\overset{(i)}{h}_{22} - \overset{(i)}{h}_{12}\overset{(i)}{h}_{21}}\overset{(i)}{F}_2(\alpha)\Bigg]e^{i x_i \alpha}d\alpha
\end{aligned}
\tag{66}
$$

$$
\begin{aligned}
\tau_{x_i y_i}(x_i, y_i^+) = -\frac{i}{2\pi}\int_{-\infty}^{\infty}\alpha\Bigg[&\frac{\overset{(i)}{n}_1\overset{(i)}{h}_{22}\, e^{\overset{(i)}{n}_1 y_i} - \overset{(i)}{n}_2\overset{(i)}{h}_{21}\, e^{\overset{(i)}{n}_2 y_i}}{\overset{(i)}{h}_{11}\overset{(i)}{h}_{22} - \overset{(i)}{h}_{12}\overset{(i)}{h}_{21}}\overset{(i)}{F}_1(\alpha) \\
&+ \frac{\overset{(i)}{n}_2\overset{(i)}{h}_{11}\, e^{\overset{(i)}{n}_2 y_i} - \overset{(i)}{n}_1\overset{(i)}{h}_{12}\, e^{\overset{(i)}{n}_1 y_i}}{\overset{(i)}{h}_{11}\overset{(i)}{h}_{22} - \overset{(i)}{h}_{12}\overset{(i)}{h}_{21}}\overset{(i)}{F}_2(\alpha)\Bigg]e^{i x_i \alpha}d\alpha
\end{aligned}
\tag{67}
$$

$$
\begin{aligned}
\sigma_{x_i x_i}(x_i, y_i^-) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\Bigg[&\frac{\overset{(i)}{n}{}_3^2\overset{(i)}{N}_{22}\, e^{\overset{(i)}{n}_3 y_i} - \overset{(i)}{n}{}_4^2\overset{(i)}{N}_{21}\, e^{\overset{(i)}{n}_4 y_i}}{\overset{(i)}{N}_{11}\overset{(i)}{N}_{22} - \overset{(i)}{N}_{12}\overset{(i)}{N}_{21}}\overset{(i)}{F}_1(\alpha) \\
&+ \frac{\overset{(i)}{n}{}_4^2\overset{(i)}{N}_{11}\, e^{\overset{(i)}{n}_4 y_i} - \overset{(i)}{n}{}_3^2\overset{(i)}{N}_{12}\, e^{\overset{(i)}{n}_3 y_i}}{\overset{(i)}{N}_{11}\overset{(i)}{N}_{22} - \overset{(i)}{N}_{12}\overset{(i)}{N}_{21}}\overset{(i)}{F}_2(\alpha)\Bigg]e^{i x_i \alpha}d\alpha
\end{aligned}
\tag{68}
$$

$$
\begin{aligned}
\sigma_{y_i y_i}(x_i, y_i^-) = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\alpha^2\Bigg[&\frac{\overset{(i)}{N}_{22}\, e^{\overset{(i)}{n}_3 y_i} - \overset{(i)}{N}_{21}\, e^{\overset{(i)}{n}_4 y_i}}{\overset{(i)}{N}_{11}\overset{(i)}{N}_{22} - \overset{(i)}{N}_{12}\overset{(i)}{N}_{21}}\overset{(i)}{F}_1(\alpha) \\
&+ \frac{\overset{(i)}{N}_{11}\, e^{\overset{(i)}{n}_4 y_i} - \overset{(i)}{N}_{12}\, e^{\overset{(i)}{n}_3 y_i}}{\overset{(i)}{N}_{11}\overset{(i)}{N}_{22} - \overset{(i)}{N}_{12}\overset{(i)}{N}_{21}}\overset{(i)}{F}_2(\alpha)\Bigg]e^{i x_i \alpha}d\alpha
\end{aligned}
\tag{69}
$$

$$
\begin{aligned}
\tau_{x_i y_i}(x_i, y_i^-) = -\frac{i}{2\pi}\int_{-\infty}^{\infty}\alpha\Bigg[&\frac{\overset{(i)}{n}_3\overset{(i)}{N}_{22}\, e^{\overset{(i)}{n}_3 y_i} - \overset{(i)}{n}_4\overset{(i)}{N}_{21}\, e^{\overset{(i)}{n}_4 y_i}}{\overset{(i)}{N}_{11}\overset{(i)}{N}_{22} - \overset{(i)}{N}_{12}\overset{(i)}{N}_{21}}\overset{(i)}{F}_1(\alpha) \\
&+ \frac{\overset{(i)}{n}_4\overset{(i)}{N}_{11}\, e^{\overset{(i)}{n}_4 y_i} - \overset{(i)}{n}_3\overset{(i)}{N}_{12}\, e^{\overset{(i)}{n}_3 y_i}}{\overset{(i)}{N}_{11}\overset{(i)}{N}_{22} - \overset{(i)}{N}_{12}\overset{(i)}{N}_{21}}\overset{(i)}{F}_2(\alpha)\Bigg]e^{i x_i \alpha}d\alpha
\end{aligned}
\tag{70}
$$

where,

$$\overset{(i)}{N}_{11}=\frac{\kappa+1}{8\mu_0}(\overset{(i)}{n}_1-\overset{(i)}{n}_3)(\overset{(i)}{n}_3-\overset{(i)}{n}_2)$$

$$\overset{(i)}{N}_{12}=\frac{\kappa+1}{8\mu_0}(\overset{(i)}{n}_4-\overset{(i)}{n}_2)(\overset{(i)}{n}_1-\overset{(i)}{n}_4)$$

$$\overset{(i)}{N}_{21}=\frac{i\alpha-\beta}{8\mu_0}(\alpha^2(1+\kappa_2)-\delta_i^2(\kappa-3))[\frac{(\overset{(i)}{n}_1-\overset{(i)}{n}_3)(\overset{(i)}{n}_3-\overset{(i)}{n}_2)}{(\delta_i-\overset{(i)}{n}_1)(\delta_i-\overset{(i)}{n}_2)(\delta_i-\overset{(i)}{n}_3)}] \tag{71}$$

$$\overset{(i)}{N}_{22}=\frac{i\alpha-\beta}{8\mu_0}(\alpha^2(1+\kappa_2)-\delta_i^2(\kappa-3))[\frac{(\overset{(i)}{n}_4-\overset{(i)}{n}_2)(\overset{(i)}{n}_1-\overset{(i)}{n}_4)}{(\delta_i-\overset{(i)}{n}_2)(\delta_i-\overset{(i)}{n}_1)(\delta_i-\overset{(i)}{n}_4)}]$$

and h_{ij} are as defined in (49) for the i^{th} crack. Assume that there are n cracks present, then the stresses for the i^{th} crack could be expressed as:

$$\begin{aligned} {}_i\sigma^T_{y_iy_i}(x_i,y_i) &= \sigma^i_{y_iy_i}(x_i,y_i)+\sum_{j=i+1}^{n}\sigma^j_{y_jy_j}[x_j(x_i,y_i),y_j(x_i,y_i)] \\ {}_i\tau^T_{x_iy_i}(x_i,y_i) &= \tau^i_{x_iy_i}(x_i,y_i)+\sum_{j=i+1}^{n}\tau^j_{x_jy_j}[x_j(x_i,y_i),y_j(x_i,y_i)] \end{aligned} \tag{72}$$

where $\overset{(j)}{\sigma}_{y_iy_i}, \overset{(j)}{\tau}_{x_iy_i}$ are found using (63) and they are evaluated as in (9) and (10) respectively. One must be carefull when chossing the stress components for the positive y_i (y_i^+) and for the negative y_i (y_i^-). Thus the final singular integral equation could be expressesd for the i^{th} crack as:

$$
\begin{aligned}
{}_i(-\frac{\kappa+1}{2\mu(x_i,0)}\overset{(i)}{p_1}(x_i)) &= \frac{1}{\pi}\int_{a_i}^{b_i}\frac{\overset{(i)}{f_2}(t_i)}{t_i-x_i}dt_i+\frac{1}{\pi}\int_{a_i}^{b_i}\overset{(i)}{k_{12}}(x_i,t_i)\overset{(i)}{f_2}(t_i)\,dt_i \\
&-\frac{1}{\pi}\int_{a_i}^{b_i}\frac{\beta_i}{2}Ci(U_i(t_i-x_i))]\overset{(i)}{f_2}(t_i)\,dt_i \\
&+\frac{1}{\pi}\int_{a_i}^{b_i}\overset{(i)}{k_{11}}(x_i,t_i)\overset{(i)}{f_1}(t_i)\,dt_i-\frac{1}{\pi}\int_{a_i}^{b_i}\frac{\delta_i}{2}Ci(U(t_i-x_i))]\overset{(i)}{f_1}(t_i)\,dt_i \\
&+\sum_{j=i+1}^{n}\{\frac{1}{\pi}\int_{a_j}^{b_j}[\frac{a_{11}^{(j)}(x_i^2\sin^2\theta-(t_j-x_i\cos\theta)^2)}{(x_i^2\sin^2\theta+(t_j-x_i\cos\theta)^2)^2} \\
&+\frac{c_{11}^{(j)}x_i\sin\theta((t_j-x_i\cos\theta)^2+x_i^2\sin^2\theta)}{(x_i^2\sin^2\theta+(t_j-x_i\cos\theta)^2)^2} \\
&+\frac{2b_{11}^{(j)}x_i\sin\theta(t_j-x_i\cos\theta)}{(x_i^2\sin^2\theta+\cos^2\theta(t-x_i)^2)^2} \\
&+\frac{(t_j-x_i\cos\theta)d_{11}^{(j)}(x_i^2\sin^2\theta+(t_j-x_i\cos\theta)^2)}{(x_i^2\sin^2\theta+(t_j-x_i\cos\theta)^2)^2}]\overset{(j)}{f_1}(t_j)\,dt_j \\
&+\frac{1}{\pi}\int_{a_j}^{b_j}\overset{(j)}{k_{11}}(x_i,t_j)\overset{(j)}{f_1}(t_j)\,dt_j \\
&+\frac{1}{\pi}\int_{a_j}^{b_j}[\frac{a_{12}^{(j)}(x_i^2\sin^2\theta-(t_j-x_i\cos\theta)^2)}{(x_i^2\sin^2\theta+(t_j-x_i\cos\theta)^2)^2} \\
&+\frac{c_{12}^{(j)}x_i\sin\theta((t_j-x_i\cos\theta)^2+x_i^2\sin^2\theta)}{(x_i^2\sin^2\theta+(t_j-x_i\cos\theta)^2)^2} \\
&+\frac{2b_{12}^{(j)}x_i\sin\theta(t_j-x_i\cos\theta)}{(x_i^2\sin^2\theta+\cos^2\theta(t-x_i)^2)^2} \\
&+\frac{d_{12}^{(j)}(t_j-x_i\cos\theta)(x_i^2\sin^2\theta+(t_j-x_i\cos\theta)^2)}{(x_i^2\sin^2\theta+(t_j-x_i\cos\theta)^2)^2}]\overset{(j)}{f_2}(t_j)\,dt_j \\
&+\frac{1}{\pi}\int_{a_j}^{b_j}\overset{(j)}{k_{12}}(x_i,t_j)\overset{(j)}{f_2}(t_j)\,dt_j\} \qquad (73)
\end{aligned}
$$

$$
{}_i(-\frac{\kappa+1}{2\mu(x_i,0)}\overset{(i)}{p_2}(x_i)) = \frac{1}{\pi}\int_{a_i}^{b_i}\frac{\overset{(i)}{f_1}(t_i)}{t_i-x_i}dt_i + \frac{1}{\pi}\int_{a_i}^{b_i}\overset{(i)}{k_{21}}(x_i,t_i)\,\overset{(i)}{f_1}(t_i)\,dt_i
$$

$$
-\frac{1}{\pi}\int_{a_i}^{b_i}\frac{\beta_i}{2}Ci(U_i(t_i-x_i))]\,\overset{(i)}{f_1}(t_i)\,dt_i
$$

$$
+\frac{1}{\pi}\int_{a_i}^{b_i}\overset{(i)}{k_{22}}(x_i,t_i)\,\overset{(i)}{f_2}(t_i)\,dt_i - \frac{1}{\pi}\int_{a_i}^{b_i}\frac{\delta_i}{2}Ci(U(t_i-x_i))]\,\overset{(i)}{f_2}(t_i)\,dt_i
$$

$$
+\sum_{j=i+1}^{n}\{\frac{1}{\pi}\int_{a_j}^{b_j}[\frac{a_{21}^{(j)}(x_i^2\sin^2\theta-(t_j-x_i\cos\theta)^2)}{(x_i^2\sin^2\theta+(t_j-x_i\cos\theta)^2)^2}
$$

$$
+\frac{c_{21}^{(j)}x_i\sin\theta((t_j-x_i\cos\theta)^2+x_i^2\sin^2\theta)}{(x_i^2\sin^2\theta+(t_j-x_i\cos\theta)^2)^2}
$$

$$
+\frac{2b_{21}^{(j)}x_i\sin\theta(t_j-x_i\cos\theta)}{(x_i^2\sin^2\theta+\cos^2\theta(t-x_i)^2)^2}
$$

$$
+\frac{(t_j-x_i\cos\theta)d_{21}^{(j)}(x_i^2\sin^2\theta+(t_j-x_i\cos\theta)^2)}{(x_i^2\sin^2\theta+(t_j-x_i\cos\theta)^2)^2}]\,\overset{(j)}{f_1}(t_j)\,dt_j
$$

$$
+\frac{1}{\pi}\int_{a_j}^{b_j}\overset{(j)}{k_{21}}(x_i,t_j)\,\overset{(j)}{f_1}(t_j)\,dt_j
$$

$$
+\frac{1}{\pi}\int_{a_j}^{b_j}[\frac{a_{22}^{(j)}(x_i^2\sin^2\theta-(t_j-x_i\cos\theta)^2)}{(x_i^2\sin^2\theta+(t_j-x_i\cos\theta)^2)^2}
$$

$$
+\frac{c_{22}^{(j)}x_i\sin\theta((t_j-x_i\cos\theta)^2+x_i^2\sin^2\theta)}{(x_i^2\sin^2\theta+(t_j-x_i\cos\theta)^2)^2}
$$

$$
+\frac{2b_{22}^{(j)}x_i\sin\theta(t_j-x_i\cos\theta)}{(x_i^2\sin^2\theta+\cos^2\theta(t-x_i)^2)^2}
$$

$$
+\frac{d_{22}^{(j)}(t_j-x_i\cos\theta)(x_i^2\sin^2\theta+(t_j-x_i\cos\theta)^2)}{(x_i^2\sin^2\theta+(t_j-x_i\cos\theta)^2)^2}]\,\overset{(j)}{f_2}(t_j)\,dt_j
$$

$$
+\frac{1}{\pi}\int_{a_j}^{b_j}\overset{(j)}{k_{22}}(x_i,t_j)\,\overset{(j)}{f_2}(t_j)\,dt_j\} \tag{74}
$$

where the constants $\overset{(j)}{a_{mn}}\cdots\overset{(j)}{d_{mn}}$ and $\overset{(j)}{k_{mn}}$ are defined in the Appendix, the kernels $\overset{(i)}{k_{mn}}$ are defined as in (58) through (64) and $\theta=\theta_j-\theta_i$.

The above solution is reduced to the case of two collinear cracks embedded in the isotropic plate to demonstrate high accuracy of the results as shown in Table II.

Parametric Studies

In the following parametric studies the length of all cracks is chosen to be $2c = 2$. The infinite plate is subjected to normal stress along y global direction, $\sigma_{yy}=1$ psi. Cracks are located along their local x_i axes, which can be inclined with respect to the global x axis. All the geometrical dimensions are normalized with respect to c. The parametric studies are presented for the normalized mode-I and mode-II SIF, i.e., k_1/k_0 and k_2/k_0, and normalized SERR, i.e., G_1/G_0 and G_2/G_0, where $k_0 = \sigma_{yy}\sqrt{c}$ and $G_0 = \dfrac{8\mu_0 k_0^2}{\pi(\kappa+1)}$.

First study takes into consideration the problem of two collinear horizontal cracks. The same crack configuration was used to produce the results in Table II for homogeneous materials, see insert in Figure 3 or 4. Here we extended the material properties to the show transition from homogeneous material to FGM. The distance between the inner tips is denoted by r. Figures 3 and 4 show mode I normalized SIF and mode-I normalized SERR versus a normalized crack tip distance r/c. The curves are shown using the logarithmic scale for the crack distance variable.

It can be noticed that as the distance between the cracks becomes smaller the SIF and SERR become larger for every power of the exponential variation coefficient γ. Note, that the homogeneous case is represented by making the coefficient $\gamma = 0$. Both driving forces increase as γ increases for materials becoming more nonhomogeneous. The increase is especially significant for the crack tip distance less than 0.01.

The case of collinear inclined cracks at 30 degrees from the horizontal axis is shown in Figures 5 to 8. Mode-I SIF shown in Figure 5 has larger magnitudes for both the homogeneous and FGM materials than the corresponding Mode-II SIF shown in Figure 6. Both modes show increase of SIF for decreasing of the crack tip distance and for increasing of the coefficient γ. Figures 7 and 8 display mode-I and II SERR. Magnitudes of Mode-I SERR are almost three times larger than the corresponding mode-II SERR. Both SERR modes increase for decreasing of the crack tip distance r (same as SIF) and for decreasing of γ (opposite to SIF). It should be pointed out that the material stiffness at each crack tip dominates the results for SERR to the point of reversing the trend in comparison with SIF with respect to γ.

The cases when two cracks are located along two different local radial axes distance d = 1 from the origin is shown in the remaining figures. The location of the first crack is kept constant at 30 degrees while orientation of the second crack is changed from 45 to 90 degrees. Both SIF and SERR are shown for each crack tip versus the orientation angle of the second crack.

Figures 9 and 10 display mode-I and II SIF while Figures 11 and 12 represent both modes of SERR at the left crack tip of the stationary crack 1. It can be noticed that when crack 2 comes closer to crack 1, the tip a_1 is shielded and all driving force components are significantly reduced. Mode-II SIF has its maximum for the orientation angle of the second crack close to 70 degrees. By increasing γ higher magnitudes for mode-I SIF and lower magnitudes for mode-II SIF are produced.

Both modes of the SERR depend not only on square of the SIF but also on the material stiffness at the crack tip. This influence is especially shown in Figure 11 where the homogeneous case produces the smallest SERR for angle 45 degrees similar to the corresponding SIF. However, homogeneous SERR curve becomes largest for higher angles, which is opposite to the corresponding SIF.

The opposite character of SIF and SERR is even better shown in Figures 13 to 16. Here the shielding effect does not exhibit itself. Both modes of SIF for the homogeneous material are smallest (see Figures 13 and 14 for mode-I and II SIF), while both components of SERR are largest (see Figures 15 and 16 for mode-I and II SERR), because of the crack tip material stiffness influence.

The results for the left crack tip of the second crack are shown in Figures 17 to 20. Mode-I SIF depends on the crack orientation and for homogeneous case is smallest. The negative values of the mode-I SIF should be interpreted as the crack closure and SERR for such case is zero.

Mode-II SIF at the tip of the second crack versus its orientation is shown in Figure 18. For the homogeneous case the maximum k_2 is at 45 degrees. In the cases of higher coefficient γ the maximum k_2 is shifted towards 60 degrees crack orientation. The magnitudes of SIF are larger for nonhomogeneous cases than for the homogeneous case when the orientation angles of the second crack are higher than 60 degree.

Mode-I of SERR monotonically decreases to almost zero at 75 degrees and at about 50 degrees does not depend on γ, see Figure 19. For 45-50 degrees the homogeneous case is the highest and for 50-90 is the smallest. Mode-II SERR is the highest for 45-90 degrees angle orientation and has its maximum at 45 degrees, see Figure 20. The shift of the maximum to 55 degrees can only be observed for $\gamma = 1.0$.

Mode-I SIF for the right crack tip b_2 is very small for all the angles examined or it is negative (crack closure) for the angle more than 65 degrees for the homogeneous case and down to 50 degrees for FGM with $\gamma=1$. Mode-II SIF is shown in Figure 22. The maximum is moved to 60 degrees because of the influence of the crack below. The homogeneous material case is the smallest for all the angles of the crack orientation.

Mode-I SERR at the same crack tip is shown in Figure 23. Homogeneous case starts to be the highest at the orientation of 45 degrees and quickly goes to zero at 62 degrees. The FGM with the highest γ also goes to zero but at the smaller angle because the crack tip remains closed. Mode-II SERR is shown in Figure 24. All the curves have their maximum at 55 degrees. The highest SERR is for $\gamma = 1.0$ and the smallest for $\gamma = 0.25$ at the angle of 45 degrees. The homogeneous material produces the smallest SERR for the orientation close to 75 degrees.

All of the above parametric studies demonstrate that the effect of the material properties, crack orientation, location of the additional crack are interdependent and consequently produce behavior different than isotropic homogeneous materials. The model developed in this work can be used to study fracture problems in FGM and can be used to tailor the properties in order to reduce driving force components and effectively increase live of these materials.

Conclusions

Application of the general solution to the mixed boundary value problem was demonstrated to provide an elegant way of obtaining the fundamental solution for a crack embedded in an infinite nonhomogeneous plate. The fundamental solution was used to address multiple crack problem. Parametric studies for the multiple cracks revealed, that both SIF and SERR highly depend on the crack geometrical parameters such as crack orientation, location, relative distance, etc., but they also depend on the power of the exponent describing the rate of change of the material elastic parameters, γ, and local stiffness of the material at each crack tip.

The results demonstrated that the driving forces can be amplified by the collinear crack orientation or they can be reduced by the shielding effect between cracks above or below. The character of the amplification or shielding remains similar for nonhomogeneous materials but in most cases higher than zero coefficient γ increases SIF and reduces SERR.

The well known one-to-one relation between SIF and SERR curves is not always valid for FGM, because SERR also depends on the material elastic constants. Hence, the SIF curves may have different character than SERR curves. The application of the driving forces to crack propagation criterion need to be further studied to determine which driving force (SIF or SERR) best correlates with appropriate experimental results. However, since SERR includes the influence of SIF and material stiffness at the tip it is recommended that total SERR is used for the driving force parameter for FGM.

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Table I Verification of the Solution.

$c\gamma$	Konda and Erdogan (1994) $k_1(a)/\sqrt{c}$	Present Study $k_1(a)/\sqrt{c}$	Konda and Erdogan (1994) $k_2(a)/\sqrt{c}$	Present Study $k_2(a)/\sqrt{c}$
0.25	1.036	1.036	0.065	0.062
0.50	1.101	1.101	0.129	0.122
1.0	1.258	1.260	0.263	0.243

$c=(b-a)/2$

Table II Two Collinear Cracks In Isotropic Plate

$\frac{r_d}{c}$	From literature				Present Method	
	Horri and Nemat-Nasser (1985)		Erdogan (1962)			
	Inner	Outer	Inner	Outer	Inner	Outer
0.22	---	---	1.45387	1.11741	1.45736	1.11786
0.50	1.2289	1.0811	1.22894	1.08107	1.22894	1.08107
0.857	1.1333	1.0579	1.13326	1.05786	1.13329	1.05787

APPENDIX

$$a_{11}^{(1)} = \mathrm{Re}\left[\frac{4\,E^{2I\theta}\,\alpha\,\mathrm{Sinh}[a]\,\sqrt{\frac{1}{1+\kappa_2}}}{(\beta - I\delta)\,\sqrt{3-\kappa_2}}\right]$$

$$a_{12}^{(1)} = \mathrm{Re}\left[-\frac{4\,I\,E^{2I\theta}\,\alpha\,\mathrm{Sinh}[a]\,\sqrt{\frac{1}{1+\kappa_2}}}{(\beta - I\delta)\,\sqrt{3-\kappa_2}}\right]$$

$$a_{21}^{(1)} = \mathrm{Re}\left[\frac{4\,E^{2I\theta}\,\alpha\,\mathrm{Sinh}[a]\,\sqrt{\frac{1}{1+\kappa_2}}}{(-I\beta - \delta)\,\sqrt{3-\kappa_2}}\right]$$

$$a_{22}^{(1)} = \mathrm{Re}\left[\frac{4\,E^{2I\theta}\,\alpha\,\mathrm{Sinh}[a]\,\sqrt{\frac{1}{1+\kappa_2}}}{(\beta - I\delta)\,\sqrt{3-\kappa_2}}\right]$$

$$b_{11}^{(1)} = \mathrm{Im}\left[\frac{4\,E^{2I\theta}\,\alpha\,\mathrm{Sinh}[a]\,\sqrt{\frac{1}{1+\kappa_2}}}{(\beta - I\delta)\,\sqrt{3-\kappa_2}}\right]$$

$$b_{12}^{(1)} = \mathrm{Im}\left[-\frac{4\,I\,E^{2I\theta}\,\alpha\,\mathrm{Sinh}[a]\,\sqrt{\frac{1}{1+\kappa_2}}}{(\beta - I\delta)\,\sqrt{3-\kappa_2}}\right]$$

$$b_{21}^{(1)} = \mathrm{Im}\left[\frac{4\,E^{2I\theta}\,\alpha\,\mathrm{Sinh}[a]\,\sqrt{\frac{1}{1+\kappa_2}}}{(-I\beta - \delta)\,\sqrt{3-\kappa_2}}\right]$$

$$b_{22}^{(1)} = \mathrm{Im}\left[\frac{4\,E^{2I\theta}\,\alpha\,\mathrm{Sinh}[a]\,\sqrt{\frac{1}{1+\kappa_2}}}{(\beta - I\delta)\,\sqrt{3-\kappa_2}}\right]$$

$$c_{11}^{(1)} = \mathrm{Re}\left[\frac{2\,(-I\,\mathrm{Cos}[\theta] + \mathrm{Sin}[\theta])\,\sqrt{\frac{1}{1+\kappa_2}}\left(E^{-I\theta}\,\mathrm{Sinh}[a] - 2\,\mathrm{Cosh}[a]\,\mathrm{Sin}[\theta]\,\sqrt{-1+\frac{4}{1+\kappa_2}}\right)}{\sqrt{3-\kappa_2}}\right]$$

$$c_{12}^{(1)} = \mathrm{Re}\left[2\,I\left(\frac{(2\,I\,\mathrm{Sin}[\theta]^2 + \mathrm{Sin}[2\theta])\,\mathrm{Sinh}[a]\,\sqrt{\frac{1}{1+\kappa_2}}}{\sqrt{3-\kappa_2}} + \frac{\mathrm{Cosh}[a]}{1+\kappa_2}\right)\right]$$

$$\overset{(i)}{c_{21}} = \mathrm{Re}\left[\frac{2\,I\,E^{2\,I\,\theta}\,\mathrm{Cosh}[a]}{1+\kappa_2}\right]$$

$$\overset{(i)}{c_{22}} = \mathrm{Re}\left[\frac{2\,I\,E^{2\,I\,\theta}\,\mathrm{Sinh}[a]\,\sqrt{\frac{1}{1+\kappa_2}}}{\sqrt{3-\kappa_2}}\right]$$

$$\overset{(i)}{d_{11}} = \mathrm{Im}\left[\frac{2\,(-I\,\mathrm{Cos}[\theta]+\mathrm{Sin}[\theta])\,\sqrt{\frac{1}{1+\kappa_2}}\,\left(E^{-I\,\theta}\,\mathrm{Sinh}[a]-2\,\mathrm{Cosh}[a]\,\mathrm{Sin}[\theta]\,\sqrt{-1+\frac{4}{1+\kappa_2}}\right)}{\sqrt{3-\kappa_2}}\right]$$

$$\overset{(i)}{d_{12}} = \mathrm{Im}\left[2\,I\left(\frac{(2\,I\,\mathrm{Sin}[\theta]^2+\mathrm{Sin}[2\,\theta])\,\mathrm{Sinh}[a]\,\sqrt{\frac{1}{1+\kappa_2}}}{\sqrt{3-\kappa_2}}+\frac{\mathrm{Cosh}[a]}{1+\kappa_2}\right)\right]$$

$$\overset{(i)}{d_{21}} = \mathrm{Im}\left[\frac{2\,I\,E^{2\,I\,\theta}\,\mathrm{Cosh}[a]}{1+\kappa_2}\right]$$

$$\overset{(i)}{d_{22}} = \mathrm{Im}\left[\frac{2\,I\,E^{2\,I\,\theta}\,\mathrm{Sinh}[a]\,\sqrt{\frac{1}{1+\kappa_2}}}{\sqrt{3-\kappa_2}}\right]$$

$$\overset{(i+1)}{a_{11}} = \mathrm{Re}\left[\frac{4\,E^{2\,I\,\theta}\,\alpha\,\mathrm{Sinh}[b]\,\sqrt{\frac{1}{1+\kappa_2}}}{(\beta+I\,\delta)\,\sqrt{3-\kappa_2}}\right]$$

$$\overset{(i+1)}{a_{12}} = \mathrm{Re}\left[\frac{4\,I\,E^{2\,I\,\theta}\,\alpha\,\mathrm{Sinh}[b]\,\sqrt{\frac{1}{1+\kappa_2}}}{(\beta+I\,\delta)\,\sqrt{3-\kappa_2}}\right]$$

$$\overset{(i+1)}{a_{21}} = \mathrm{Re}\left[\frac{4\,E^{2\,I\,\theta}\,\alpha\,\mathrm{Sinh}[b]\,\sqrt{\frac{1}{1+\kappa_2}}}{(I\,\beta-\delta)\,\sqrt{3-\kappa_2}}\right]$$

$$\overset{(i+1)}{a_{22}} = \mathrm{Re}\left[\frac{4\,E^{2\,I\,\theta}\,\alpha\,\mathrm{Sinh}[b]\,\sqrt{\frac{1}{1+\kappa_2}}}{(\beta+I\,\delta)\,\sqrt{3-\kappa_2}}\right]$$

$$\overset{(i+1)}{b_{11}} = \mathrm{Im}\left[\frac{4\,E^{2\,I\,\theta}\,\alpha\,\mathrm{Sinh}[b]\,\sqrt{\frac{1}{1+\kappa_2}}}{(\beta+I\,\delta)\,\sqrt{3-\kappa_2}}\right]$$

$$\overset{(i+1)}{b_{12}} = \mathrm{Im}\left[\frac{4\,I\,E^{2\,I\,\theta}\,\alpha\,\mathrm{Sinh}[b]\,\sqrt{\frac{1}{1+\kappa_2}}}{(\beta+I\,\delta)\,\sqrt{3-\kappa_2}}\right]$$

$$b_{21}^{(i+1)} = \mathrm{Im}\left[\frac{4\,E^{2I\theta}\,\alpha\,\mathrm{Sinh}[b]\sqrt{\frac{1}{1+\kappa_2}}}{(I\beta-\delta)\sqrt{3-\kappa_2}}\right]$$

$$b_{22}^{(i+1)} = \mathrm{Im}\left[\frac{4\,E^{2I\theta}\,\alpha\,\mathrm{Sinh}[b]\sqrt{\frac{1}{1+\kappa_2}}}{(\beta+I\delta)\sqrt{3-\kappa_2}}\right]$$

$$c_{11}^{(i+1)} = \mathrm{Re}\left[\left(\frac{4\,\mathrm{Cosh}[b]\,\mathrm{Sin}[\theta]\,(-I\,\mathrm{Cos}[\theta]+\mathrm{Sin}[\theta])}{1+\kappa_2}+\frac{2\,I\,\mathrm{Sinh}[b]\sqrt{\left(\frac{3-\kappa_2}{1+\kappa_2}\right)}}{-3+\kappa_2}\right)\right]$$

$$c_{12}^{(i+1)} = \mathrm{Re}\left[\left(-\frac{2\,(-1+E^{2I\theta})\,\mathrm{Sinh}[b]\sqrt{\frac{1}{1+\kappa_2}}}{\sqrt{3-\kappa_2}}+\frac{2\,I\,\mathrm{Cosh}[b]}{1+\kappa_2}\right)\right]$$

$$c_{21}^{(i+1)} = \mathrm{Re}\left[\frac{2\,I\,E^{2I\theta}\,\mathrm{Cosh}[b]}{1+\kappa_2}\right]$$

$$c_{22}^{(i+1)} = \mathrm{Re}\left[\frac{2\,I\,E^{2I\theta}\,\mathrm{Sinh}[b]\sqrt{\frac{1}{1+\kappa_2}}}{\sqrt{3-\kappa_2}}\right]$$

$$d_{11}^{(i+1)} = \mathrm{Im}\left[\left(\frac{4\,\mathrm{Cosh}[b]\,\mathrm{Sin}[\theta]\,(-I\,\mathrm{Cos}[\theta]+\mathrm{Sin}[\theta])}{1+\kappa_2}+\frac{2\,I\,\mathrm{Sinh}[b]\sqrt{\left(\frac{3-\kappa_2}{1+\kappa_2}\right)}}{-3+\kappa_2}\right)\right]$$

$$d_{12}^{(i+1)} = \mathrm{Im}\left[\left(-\frac{2\,(-1+E^{2I\theta})\,\mathrm{Sinh}[b]\sqrt{\frac{1}{1+\kappa_2}}}{\sqrt{3-\kappa_2}}+\frac{2\,I\,\mathrm{Cosh}[b]}{1+\kappa_2}\right)\right]$$

$$d_{21}^{(i+1)} = \mathrm{Im}\left[\frac{2\,I\,E^{2I\theta}\,\mathrm{Cosh}[b]}{1+\kappa_2}\right]$$

$$d_{22}^{(i+1)} = \mathrm{Im}\left[\frac{2\,I\,E^{2I\theta}\,\mathrm{Sinh}[b]\sqrt{\frac{1}{1+\kappa_2}}}{\sqrt{3-\kappa_2}}\right]$$

where,

$$\theta = \theta_{i+1} - \theta_i$$

$$a = -\frac{1}{2} x_i \,\mathrm{Sin}[\theta]\; (\beta - i\,\delta) \sqrt{\frac{3-\kappa_2}{\kappa_2+1}}$$

$$b = \frac{1}{2} x_{i+1} \,\mathrm{Sin}[\theta]\; (\beta + i\,\delta) \sqrt{\frac{3-\kappa_2}{\kappa_2+1}}$$

and,

$$\overset{(i)}{k_{11}} = \frac{\kappa + 1}{4\,\mathrm{Exp}[\beta\, x_i]} \Big(\Big(\mathrm{x112tc} + \mathrm{x112t} - 2 * \Big(\overset{(i)}{a_{11}} + \overset{(i)}{c_{11}}\Big) * \mathrm{Exp}[-\alpha\, x_i\, \mathrm{Sin}[\theta]]\Big)\, \mathrm{Cos}[\alpha\, (t_{i+1} - x_i\, \mathrm{Cos}[\theta])] + i \Big(\mathrm{x112tc} - \mathrm{x112t} + 2\, i\, \Big(\overset{(i)}{b_{11}} + \overset{(i)}{d_{11}}\Big) * \mathrm{Exp}[-\alpha\, x_i\, \mathrm{Sin}[\theta]]\Big)\, \mathrm{Sin}[\alpha\, (t_{i+1} - x_i\, \mathrm{Cos}[\theta])]\Big)$$

$$\overset{(i)}{k_{12}} = \frac{\kappa_2 + 1}{4\,\mathrm{Exp}[\beta\, x_i]} \Big(\Big(\mathrm{x122tc} + \mathrm{x122t} - 2 * \Big(\overset{(i)}{a_{12}} + \overset{(i)}{c_{12}}\Big) * \mathrm{Exp}[-\alpha\, x_i\, \mathrm{Sin}[\theta]]\Big)\, \mathrm{Cos}[\alpha\, (t_{i+1} - x_i\, \mathrm{Cos}[\theta])] + i \Big(\mathrm{x122tc} - \mathrm{x122t} + 2\, i\, \Big(\overset{(i)}{b_{12}} + \overset{(i)}{d_{12}}\Big) * \mathrm{Exp}[-\alpha\, x_i\, \mathrm{Sin}[\theta]]\Big)\, \mathrm{Sin}[\alpha\, (t_{i+1} - x_i\, \mathrm{Cos}[\theta])]\Big)$$

$$\overset{(i)}{k_{21}} = \frac{\kappa_2 + 1}{4\,\mathrm{Exp}[\beta\, x_i]} \Big(\Big(\mathrm{x212tc} + \mathrm{x212t} - 2 * \Big(\overset{(i)}{a_{21}} + \overset{(i)}{c_{21}}\Big) * \mathrm{Exp}[-\alpha\, x_i\, \mathrm{Sin}[\theta]]\Big)\, \mathrm{Cos}[\alpha\, (t_{i+1} - x_i\, \mathrm{Cos}[\theta])] + i \Big(\mathrm{x212tc} - \mathrm{x212t} + 2\, i\, \Big(\overset{(i)}{b_{21}} + \overset{(i)}{d_{21}}\Big) * \mathrm{Exp}[-\alpha\, x_i\, \mathrm{Sin}[\theta]]\Big)\, \mathrm{Sin}[\alpha\, (t_{i+1} - x_i\, \mathrm{Cos}[\theta])]\Big)$$

$$\overset{(i)}{k_{22}} = \frac{\kappa_2 + 1}{4\,\mathrm{Exp}[\beta\, x_i]} \Big(\Big(\mathrm{x222tc} + \mathrm{x222t} - 2 * \Big(\overset{(i)}{a_{22}} + \overset{(i)}{c_{22}}\Big) * \mathrm{Exp}[-\alpha\, x_i\, \mathrm{Sin}[\theta]]\Big)\, \mathrm{Cos}[\alpha\, (t_{i+1} - x_i\, \mathrm{Cos}[\theta])] + i \Big(\mathrm{x222tc} - \mathrm{x222t} + 2\, i\, \Big(\overset{(i)}{b_{22}} + \overset{(i)}{d_{22}}\Big) * \mathrm{Exp}[-\alpha\, x_i\, \mathrm{Sin}[\theta]]\Big)\, \mathrm{Sin}[\alpha\, (t_{i+1} - x_i\, \mathrm{Cos}[\theta])]\Big)$$

$$\overset{(i+1)}{k_{11}} = \frac{\kappa_2 + 1}{4\,\mathrm{Exp}[\beta\, x_{i+1}]} \Big(\Big(\mathrm{x111tc} + \mathrm{x111t} - 2 * \Big(\overset{(i+1)}{a_{11}} + \overset{(i+1)}{c_{11}}\Big) * \mathrm{Exp}[-\alpha\, x_{i+1}\, \mathrm{Sin}[\theta]]\Big)\, \mathrm{Cos}[\alpha\, (t_i - x_{i+1}\, \mathrm{Cos}[\theta])] + i \Big(\mathrm{x111tc} - \mathrm{x111t} + 2\, i\, \Big(\overset{(i+1)}{b_{11}} + \overset{(i+1)}{d_{11}}\Big) * \mathrm{Exp}[-\alpha\, x_{i+1}\, \mathrm{Sin}[\theta]]\Big)\, \mathrm{Sin}[\alpha\, (t_i - x_{i+1}\, \mathrm{Cos}[\theta])]\Big)$$

$$\overset{(i+1)}{k_{12}} = \frac{\kappa_2 + 1}{4\,\mathrm{Exp}[\beta\, x_{i+1}]} \Big(\Big(\mathrm{x121tc} + \mathrm{x121t} - 2 * \Big(\overset{(i+1)}{a_{12}} + \overset{(i+1)}{c_{12}}\Big) * \mathrm{Exp}[-\alpha\, x_{i+1}\, \mathrm{Sin}[\theta]]\Big)\, \mathrm{Cos}[\alpha\, (t_i - x_{i+1}\, \mathrm{Cos}[\theta])] + i \Big(\mathrm{x121tc} - \mathrm{x121t} + 2\, i\, \Big(\overset{(i+1)}{b_{12}} + \overset{(i+1)}{d_{12}}\Big) * \mathrm{Exp}[-\alpha\, x_{i+1}\, \mathrm{Sin}[\theta]]\Big)\, \mathrm{Sin}[\alpha\, (t_i - x_{i+1}\, \mathrm{Cos}[\theta])]\Big)$$

$$\overset{(i+1)}{k_{21}} = \frac{\kappa_2 + 1}{4\,\mathrm{Exp}[\beta\, x_{i+1}]} \Big(\Big(\mathrm{x211tc} + \mathrm{x211t} - 2 * \Big(\overset{(i+1)}{a_{21}} + \overset{(i+1)}{c_{21}}\Big) * \mathrm{Exp}[-\alpha\, x_{i+1}\, \mathrm{Sin}[\theta]]\Big)\, \mathrm{Cos}[\alpha\, (t_i - x_{i+1}\, \mathrm{Cos}[\theta])] + i \Big(\mathrm{x211tc} - \mathrm{x211t} + 2\, i\, \Big(\overset{(i+1)}{b_{21}} + \overset{(i+1)}{d_{21}}\Big) * \mathrm{Exp}[-\alpha\, x_{i+1}\, \mathrm{Sin}[\theta]]\Big)\, \mathrm{Sin}[\alpha\, (t_i - x_{i+1}\, \mathrm{Cos}[\theta])]\Big)$$

$$\overset{(i+1)}{k_{22}} = \frac{\kappa_2 + 1}{4\,\mathrm{Exp}[\beta\, x_{i+1}]} \Big(\Big(\mathrm{x221tc} + \mathrm{x221t} - 2 * \Big(\overset{(i+1)}{a_{22}} + \overset{(i+1)}{c_{22}}\Big) * \mathrm{Exp}[-\alpha\, x_{i+1}\, \mathrm{Sin}[\theta]]\Big)\, \mathrm{Cos}[\alpha\, (t_i - x_{i+1}\, \mathrm{Cos}[\theta])] + i \Big(\mathrm{x221tc} - \mathrm{x221t} + 2\, i\, \Big(\overset{(i+1)}{b_{22}} + \overset{(i+1)}{d_{22}}\Big) * \mathrm{Exp}[-\alpha\, x_{i+1}\, \mathrm{Sin}[\theta]]\Big)\, \mathrm{Sin}[\alpha\, (t_i - x_{i+1}\, \mathrm{Cos}[\theta])]\Big)$$

where,

$$x112t = \mathrm{Sin}[\theta]^2 \left(\frac{n_3^2 N_{22} e^{-n_3 x_i \mathrm{Sin}[\theta]} - n_4^2 N_{21} e^{-n_4 x_i \mathrm{Sin}[\theta]}}{N_{11} N_{22} - N_{12} N_{21}} \right) + \mathrm{Cos}[\theta]^2 \left(-\alpha^2 \frac{N_{22} e^{-n_3 x_i \mathrm{Sin}[\theta]} - N_{21} e^{-n_4 x_i \mathrm{Sin}[\theta]}}{N_{11} N_{22} - N_{12} N_{21}} \right) +$$
$$2\,\mathrm{Sin}[\theta]\,\mathrm{Cos}[\theta] \left(-i\,\alpha \frac{n_3 N_{22} e^{-n_3 x_i \mathrm{Sin}[\theta]} - n_4 N_{21} e^{-n_4 x_i \mathrm{Sin}[\theta]}}{N_{11} N_{22} - N_{12} N_{21}} \right)$$

$$x122t = \mathrm{Sin}[\theta]^2 \left(\frac{n_4^2 N_{11} e^{-n_4 x_i \mathrm{Sin}[\theta]} - n_3^2 N_{12} e^{-n_3 x_i \mathrm{Sin}[\theta]}}{N_{11} N_{22} - N_{12} N_{21}} \right) + \mathrm{Cos}[\theta]^2 \left(-\alpha^2 \frac{N_{11} e^{-n_4 x_i \mathrm{Sin}[\theta]} - N_{12} e^{-n_3 x_i \mathrm{Sin}[\theta]}}{N_{11} N_{22} - N_{12} N_{21}} \right) +$$
$$2\,\mathrm{Sin}[\theta]\,\mathrm{Cos}[\theta] \left(-i\,\alpha \frac{n_4 N_{11} e^{-n_4 x_i \mathrm{Sin}[\theta]} - n_3 N_{12} e^{-n_3 x_i \mathrm{Sin}[\theta]}}{N_{11} N_{22} - N_{12} N_{21}} \right)$$

$$x212t = \mathrm{Sin}[\theta]\,\mathrm{Cos}[\theta] \left(\frac{n_3^2 N_{22} e^{-n_3 x_i \mathrm{Sin}[\theta]} - n_4^2 N_{21} e^{-n_4 x_i \mathrm{Sin}[\theta]}}{N_{11} N_{22} - N_{12} N_{21}} \right) -$$
$$\mathrm{Sin}[\theta]\,\mathrm{Cos}[\theta] \left(-\alpha^2 \frac{N_{22} e^{-n_3 x_i \mathrm{Sin}[\theta]} - N_{21} e^{-n_4 x_i \mathrm{Sin}[\theta]}}{N_{11} N_{22} - N_{12} N_{21}} \right) +$$
$$\left(\mathrm{Cos}[\theta]^2 - \mathrm{Sin}[\theta]^2\right) \left(-i\,\alpha \frac{n_3 N_{22} e^{-n_3 x_i \mathrm{Sin}[\theta]} - n_4 N_{21} e^{-n_4 x_i \mathrm{Sin}[\theta]}}{N_{11} N_{22} - N_{12} N_{21}} \right)$$

$$x222t = \mathrm{Sin}[\theta]\,\mathrm{Cos}[\theta] \left(\frac{n_4^2 N_{11} e^{-n_4 x_i \mathrm{Sin}[\theta]} - n_3^2 N_{12} e^{-n_3 x_i \mathrm{Sin}[\theta]}}{N_{11} N_{22} - N_{12} N_{21}} \right) -$$
$$\mathrm{Sin}[\theta]\,\mathrm{Cos}[\theta] \left(-\alpha^2 \frac{N_{11} e^{-n_4 x_i \mathrm{Sin}[\theta]} - N_{12} e^{-n_3 x_i \mathrm{Sin}[\theta]}}{N_{11} N_{22} - N_{12} N_{21}} \right) +$$
$$\left(\mathrm{Cos}[\theta]^2 - \mathrm{Sin}[\theta]^2\right) \left(-i\,\alpha \frac{n_4 N_{11} e^{-n_4 x_i \mathrm{Sin}[\theta]} - n_3 N_{12} e^{-n_3 x_i \mathrm{Sin}[\theta]}}{N_{11} N_{22} - N_{12} N_{21}} \right)$$

$$x111t = \mathrm{Sin}[\theta]^2 \left(\frac{n_1^2 h_{22} e^{n_1 x_{i+} \mathrm{Sin}[\theta]} - n_2^2 h_{21} e^{n_2 x_{i+} \mathrm{Sin}[\theta]}}{h_{11} h_{22} - h_{12} h_{21}} \right) + \mathrm{Cos}[\theta]^2 \left(-\alpha^2 \frac{h_{22} e^{n_1 x_{i+1} \mathrm{Sin}[\theta]} - h_{21} e^{n_2 x_{i+1} \mathrm{Sin}[\theta]}}{h_{11} h_{22} - h_{12} h_{21}} \right) -$$
$$2\,\mathrm{Sin}[\theta]\,\mathrm{Cos}[\theta] \left(-i\,\alpha \frac{n_1 h_{22} e^{n_1 x_{i+1} \mathrm{Sin}[\theta]} - n_2 h_{21} e^{n_2 x_{i+1} \mathrm{Sin}[\theta]}}{h_{11} h_{22} - h_{12} h_{21}} \right)$$

$$x121t = \mathrm{Sin}[\theta]^2 \left(\frac{n_2^2 h_{11} e^{n_2 x_{i+1} \mathrm{Sin}[\theta]} - n_1^2 h_{12} e^{n_1 x_{i+1} \mathrm{Sin}[\theta]}}{h_{11} h_{22} - h_{12} h_{21}} \right) + \mathrm{Cos}[\theta]^2 \left(-\alpha^2 \frac{h_{11} e^{n_2 x_{i+1} \mathrm{Sin}[\theta]} - h_{12} e^{n_1 x_{i+1} \mathrm{Sin}[\theta]}}{h_{11} h_{22} - h_{12} h_{21}} \right) -$$
$$2\,\mathrm{Sin}[\theta]\,\mathrm{Cos}[\theta] \left(-i\,\alpha \frac{n_2 h_{11} e^{n_2 x_{i+1} \mathrm{Sin}[\theta]} - n_1 h_{12} e^{n_1 x_{i+1} \mathrm{Sin}[\theta]}}{h_{11} h_{22} - h_{12} h_{21}} \right)$$

$$x211t = -\mathrm{Sin}[\theta]\,\mathrm{Cos}[\theta] \left(\frac{n_1^2 h_{22} e^{n_1 x_{i+1} \mathrm{Sin}[\theta]} - n_2^2 h_{21} e^{n_2 x_{i+1} \mathrm{Sin}[\theta]}}{h_{11} h_{22} - h_{12} h_{21}} \right) +$$
$$\mathrm{Sin}[\theta]\,\mathrm{Cos}[\theta] \left(-\alpha^2 \frac{h_{22} e^{n_1 x_{i+1} \mathrm{Sin}[\theta]} - h_{21} e^{n_2 x_{i+1} \mathrm{Sin}[\theta]}}{h_{11} h_{22} - h_{12} h_{21}} \right) +$$
$$\left(\mathrm{Cos}[\theta]^2 - \mathrm{Sin}[\theta]^2\right) \left(-i\,\alpha \frac{n_1 h_{22} e^{n_1 x_{i+1} \mathrm{Sin}[\theta]} - n_2 h_{21} e^{n_2 x_{i+1} \mathrm{Sin}[\theta]}}{h_{11} h_{22} - h_{12} h_{21}} \right)$$

$$x221t = -\mathrm{Sin}[\theta]\,\mathrm{Cos}[\theta]\left(\frac{n_2^2\,h_{11}\,e^{n_2 x_{i+1}\,\mathrm{Sin}[\theta]} - n_1^2\,h_{12}\,e^{n_1 x_{i+1}\,\mathrm{Sin}[\theta]}}{h_{11}\,h_{22} - h_{12}\,h_{21}}\right) + \mathrm{Sin}[\theta]\,\mathrm{Cos}[\theta]\left(-\alpha^2\,\frac{h_{11}\,e^{n_2 x_{i+1}\,\mathrm{Sin}[\theta]} - h_{12}\,e^{n_1 x_{i+1}\,\mathrm{Sin}[\theta]}}{h_{11}\,h_{22} - h_{12}\,h_{21}}\right) + \left(\mathrm{Cos}[\theta]^2 - \mathrm{Sin}[\theta]^2\right)\left(-i\,\alpha\,\frac{n_2\,h_{11}\,e^{n_2 x_{i+1}\,\mathrm{Sin}[\theta]} - n_1\,h_{12}\,e^{n_1 x_{i+1}\,\mathrm{Sin}[\theta]}}{h_{11}\,h_{22} - h_{12}\,h_{21}}\right)$$

x112tc = Conjugate[x112t], x122tc = Conjugate[x122t], x212tc = Conjugate[x212t], x222tc = Conjugate[x222t], x111tc = Conjugate[x111t], x121tc = Conjugate[x121t], x211tc = Conjugate[x211t] and x221tc = Conjugate[x221t].

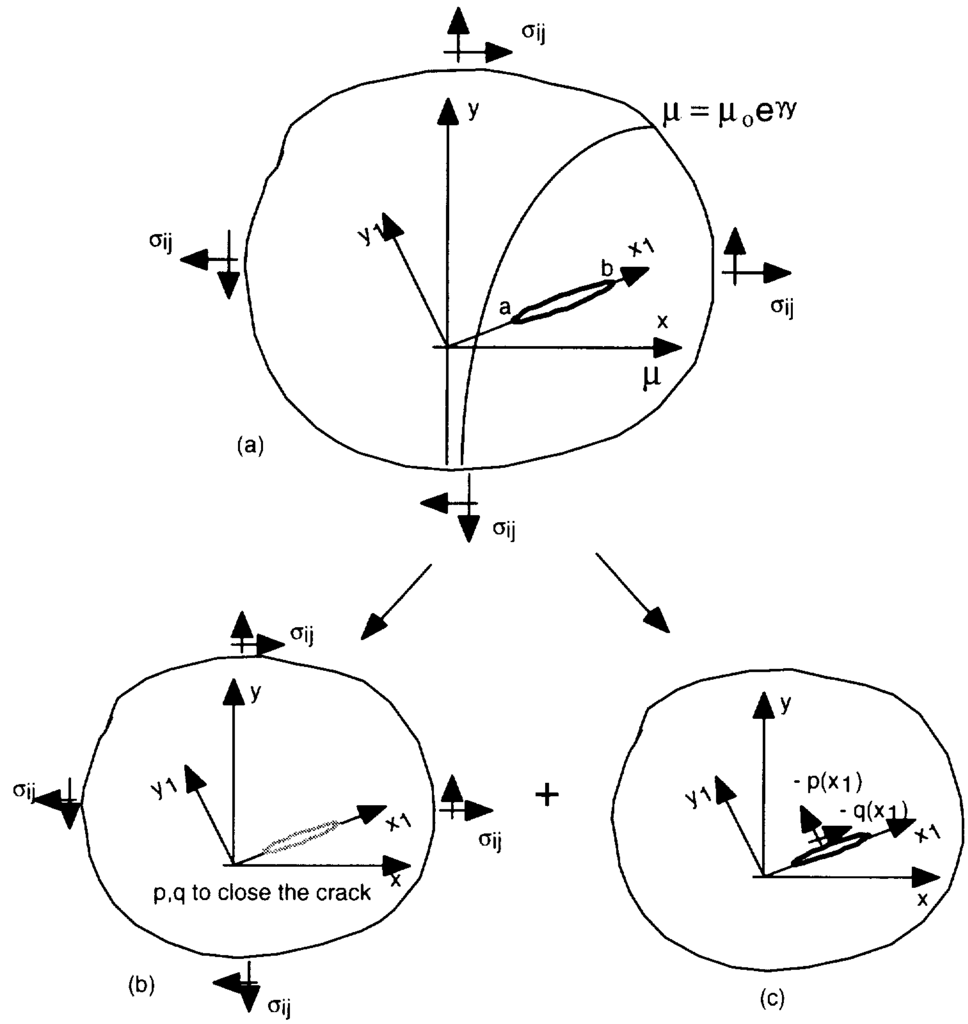

Figure 1. Methodology of Solution for the Fundamental Problem.

(a) Mixed Boundary Value Problem for the FGM.

(b) Infinite FGM Plate without Crack.

(c) Perturbation Problem of a Crack loaded by Surface Tractions.

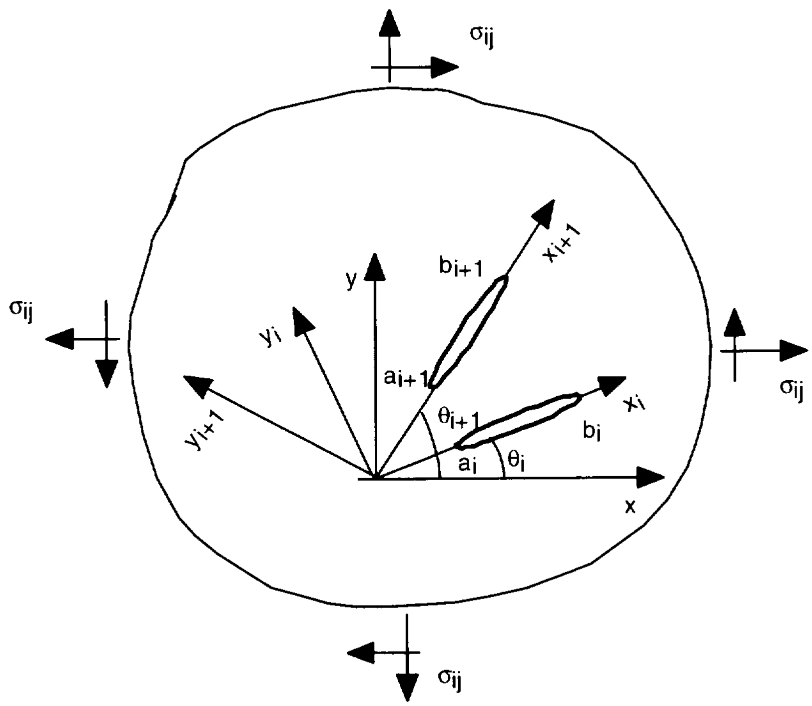

Figure 2. Multiple Cracks Embedded in the Infinite FGM Plate.

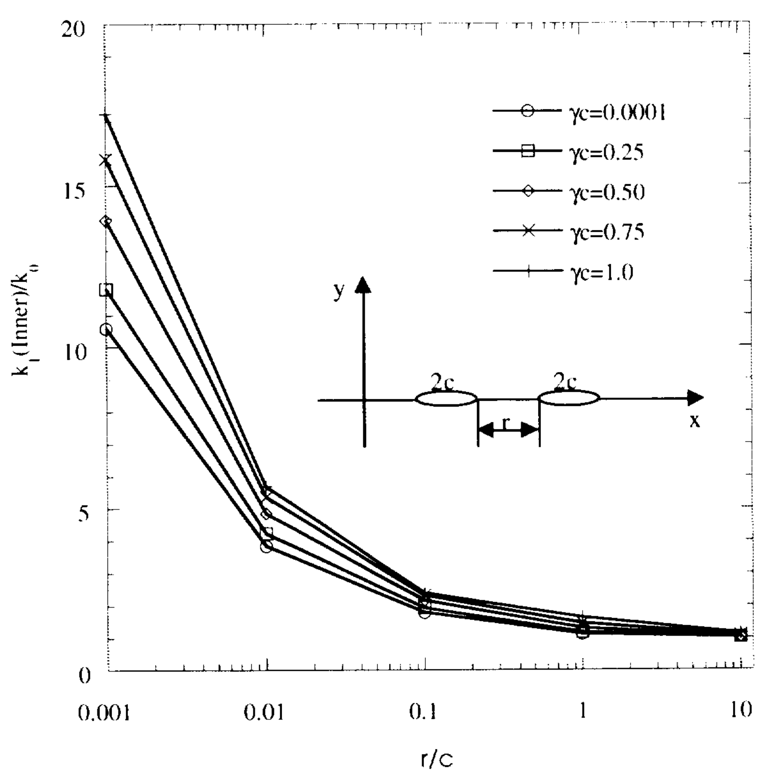

Figure 3. Mode I normalized SIF versus normalized inner crack tips distance for two collinear horizontal cracks. ($\sigma_{yy} = 1.0, \sigma_{xx} = \sigma_{xy} = 0.0$).

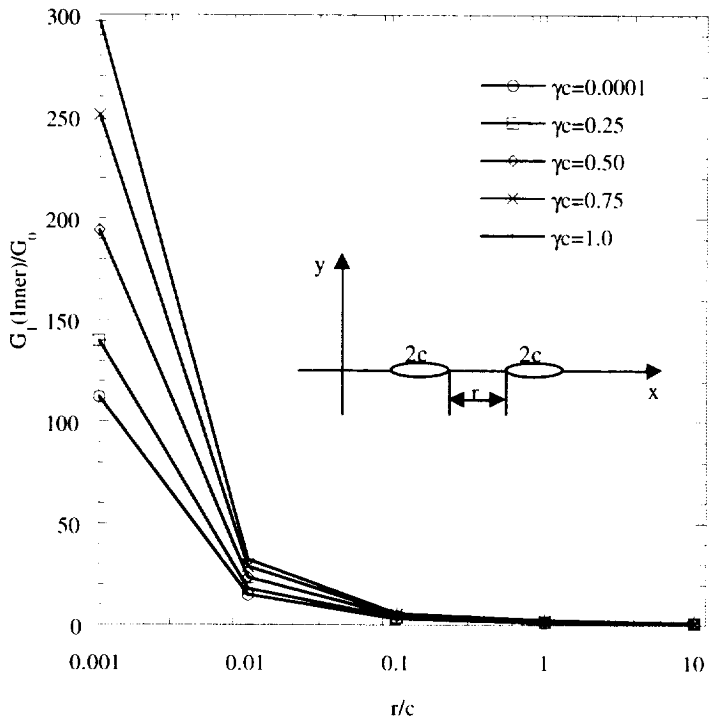

Figure 4. Mode I normalized SERR versus normalized inner crack tips distance for two collinear horizontal cracks. (σ_{yy} = 1.0, σ_{xx} = σ_{xy} =0.0).

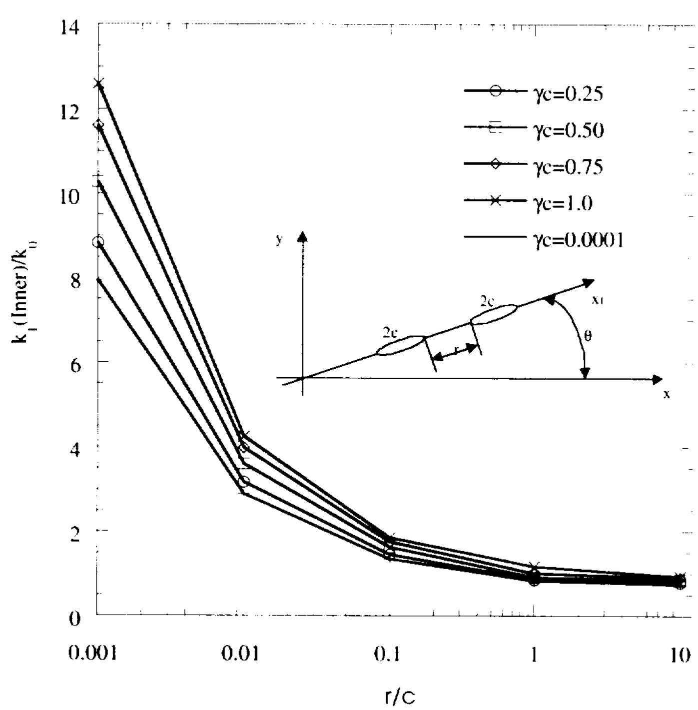

Figure 5. Mode I normalized SIF versus normalized inner crack tips distance for two collinear cracks along the θ=30 deg. (σ_{yy} = 1.0, σ_{xx} = σ_{xy} =0.0).

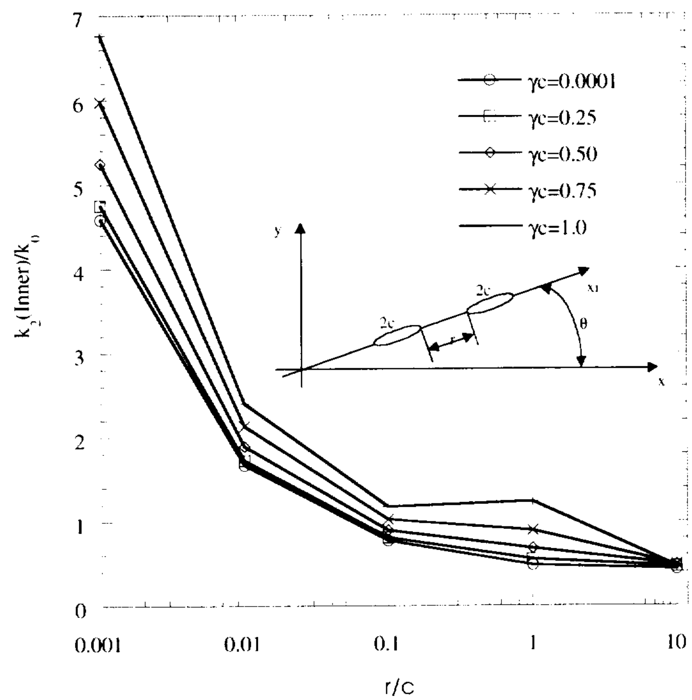

Figure 6. Mode II normalized SIF versus normalized inner crack tips distance for two collinear cracks along the θ=30 deg. (σ_{yy} = 1.0, σ_{xx} = σ_{xy} =0.0).

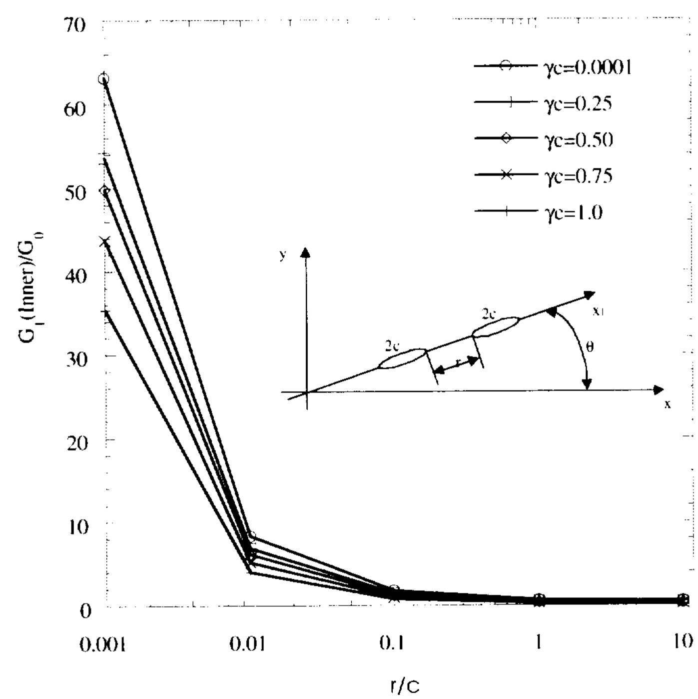

Figure 7. Mode I normalized SERR versus normalized inner crack tips distance for two collinear cracks along the θ=30 deg. (σ_{yy} = 1.0, σ_{xx} = σ_{xy} =0.0).

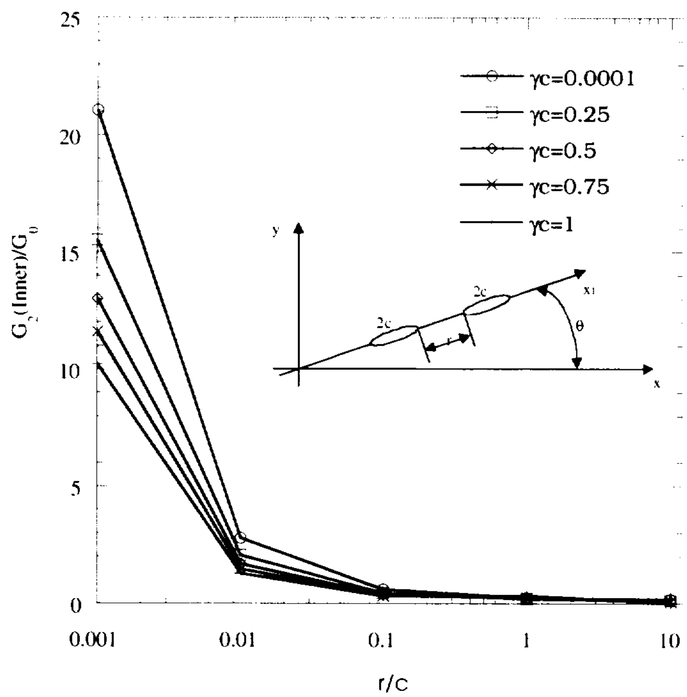

Figure 8. Mode II normalized SERR versus normalized inner crack tips distance for two collinear cracks along the θ=30 deg. (σ_{yy} = 1.0, σ_{xx} = σ_{xy} =0.0).

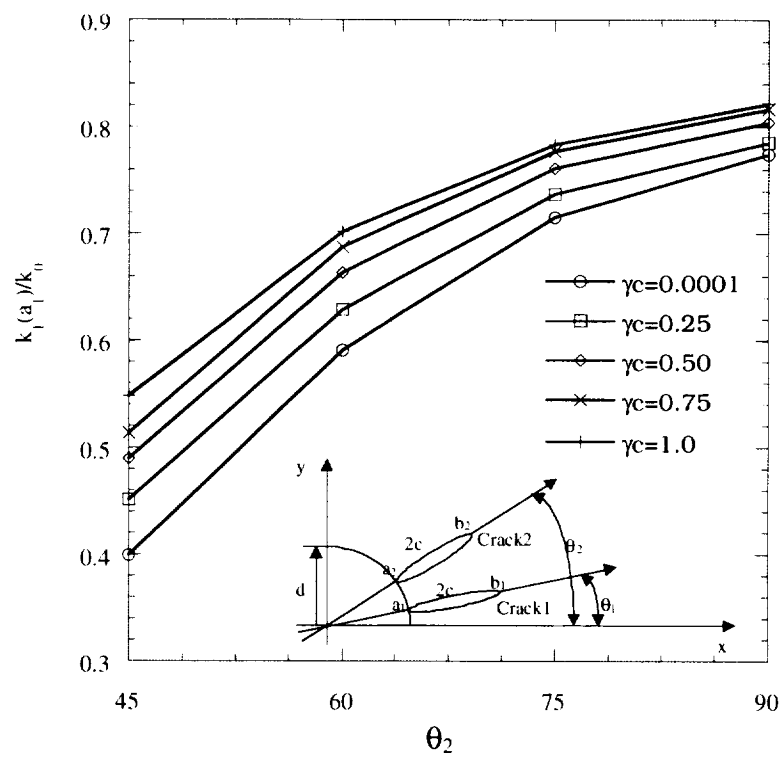

Figure 9. Mode I normalized SIF at the tip a_1 versus crack (2) orientation angle θ_2 for two inclined cracks. (σ_{yy} = 1.0, σ_{xx} = σ_{xy} =0.0, d=1.0, θ_1=30 deg.).

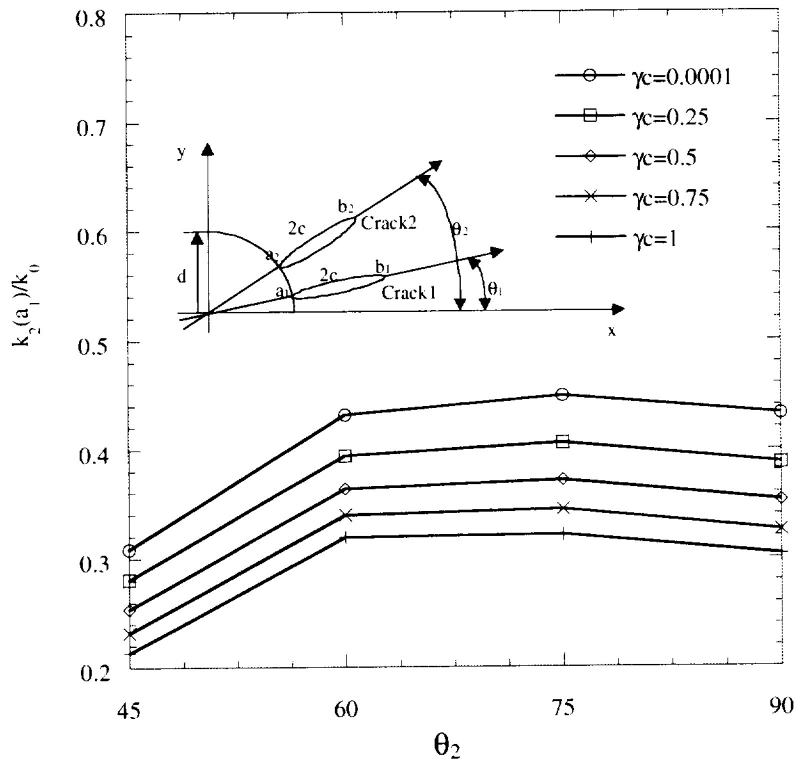

Figure 10. Mode II normalized SIF at the tip a_1 versus crack (2) orientation angle θ_2 for two inclined cracks. (σ_{yy} = 1.0, $\sigma_{xx} = \sigma_{xy}$ =0.0, d=1.0, θ_1=30 deg.).

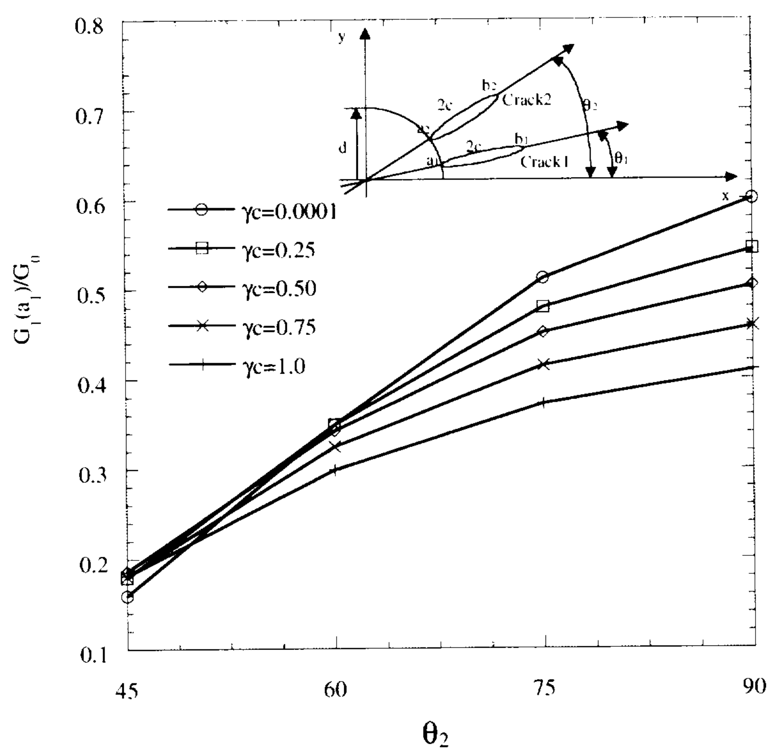

Figure 11. Mode I normalized SERR at the tip a_1 versus crack (2) orientation angle θ_2 for two inclined cracks. (σ_{yy} = 1.0, $\sigma_{xx} = \sigma_{xy}$ =0.0, d=1.0, θ_1=30 deg.).

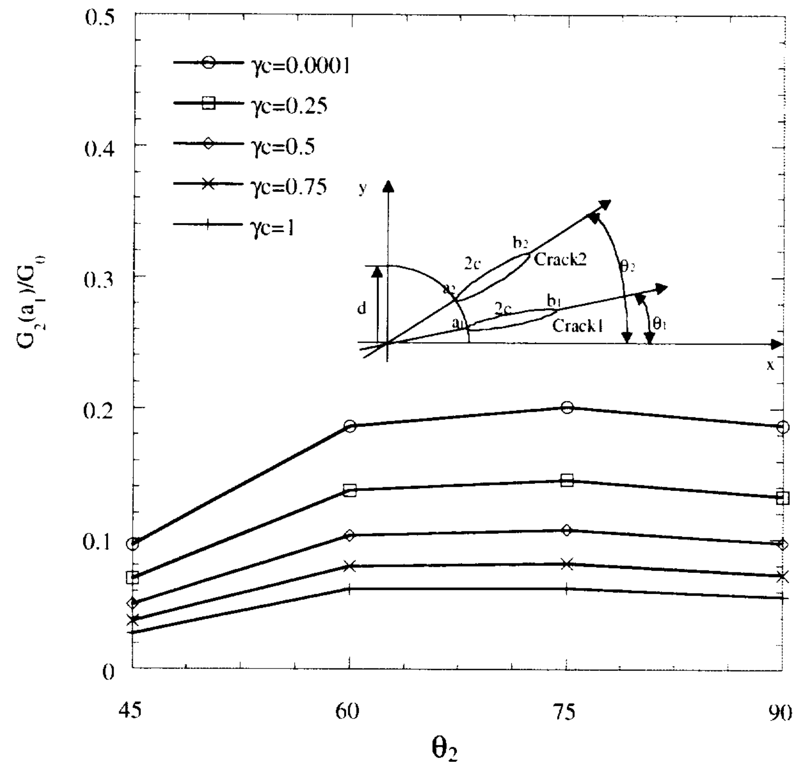

Figure 12. Mode II normalized SERR at the tip a_1 versus crack (2) orientation angle θ_2 for two inclined cracks. (σ_{yy} = 1.0, σ_{xx} = σ_{xy} =0.0, d=1.0, θ_1=30 deg.).

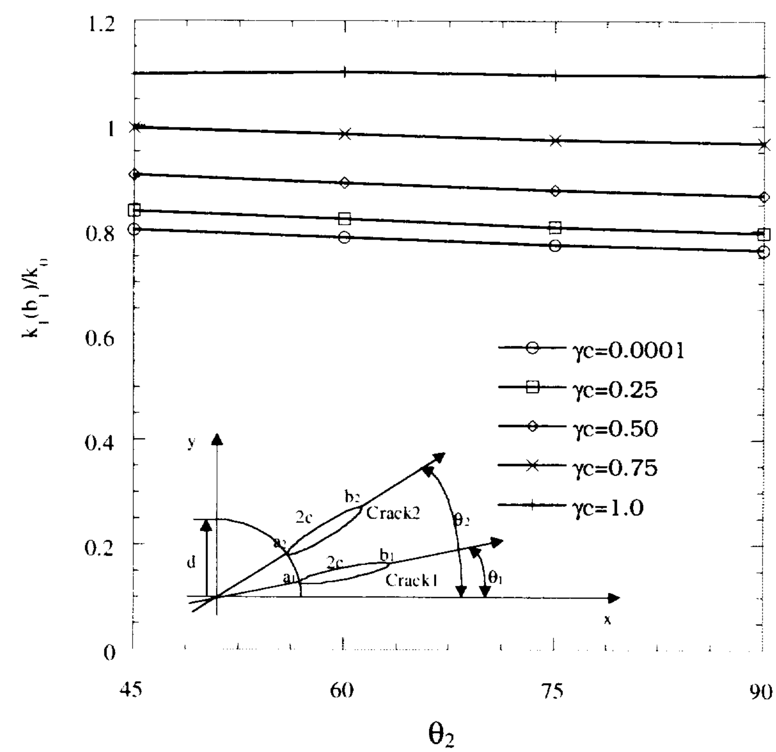

Figure 13. Mode I normalized SIF at the tip b_1 versus crack (2) orientation angle θ_2 for two inclined cracks. (σ_{yy} = 1.0, σ_{xx} = σ_{xy} =0.0, d=1.0, θ_1=30 deg.).

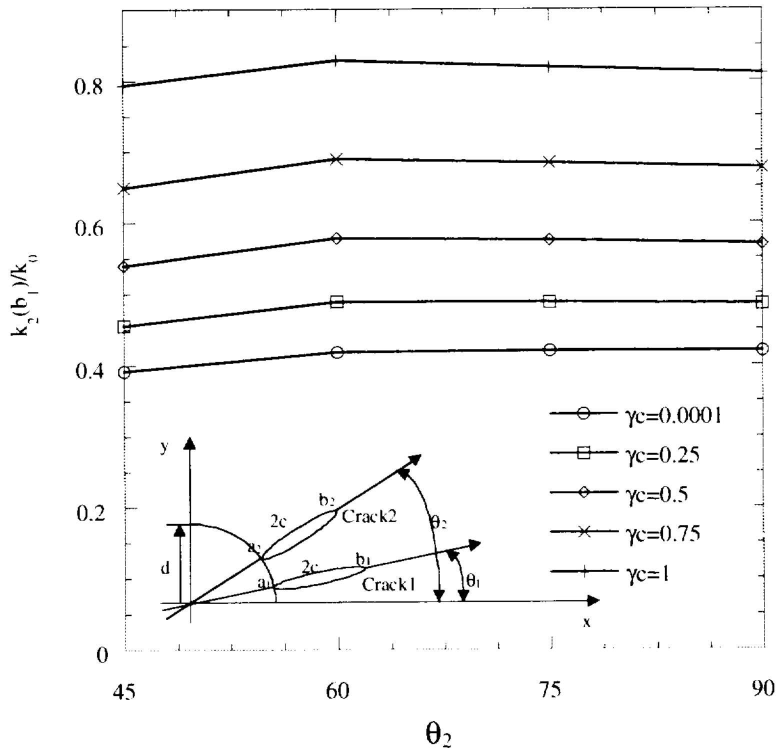

Figure 14. Mode II normalized SIF at the tip b_1 versus crack (2) orientation angle θ_2 for two inclined cracks. (σ_{yy} = 1.0, $\sigma_{xx} = \sigma_{xy}$ =0.0, d=1.0, θ_1=30 deg.).

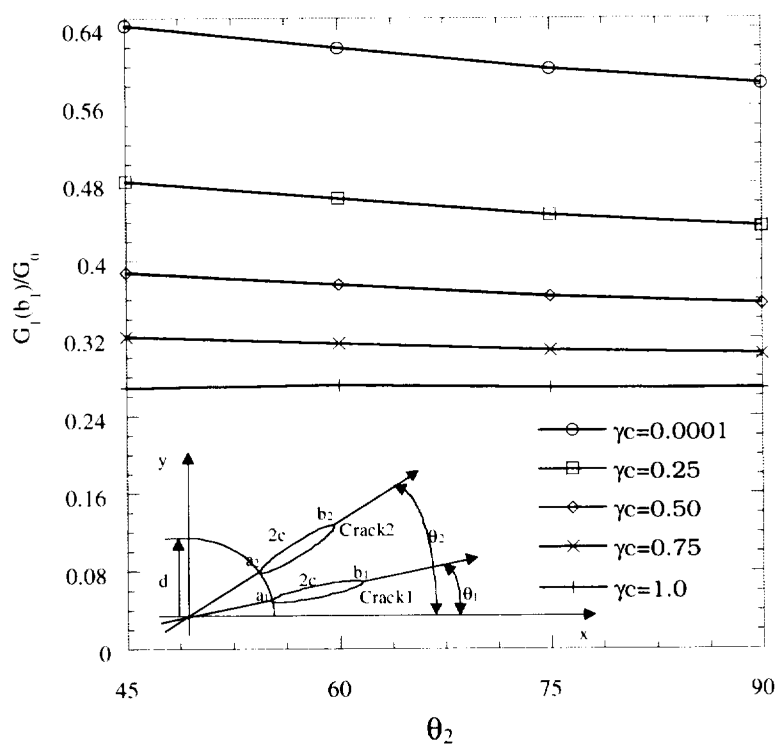

Figure 15. Mode I normalized SERR at the tip b_1 versus crack (2) orientation angle θ_2 for two inclined cracks. (σ_{yy} = 1.0, $\sigma_{xx} = \sigma_{xy}$ =0.0, d=1.0, θ_1=30 deg.).

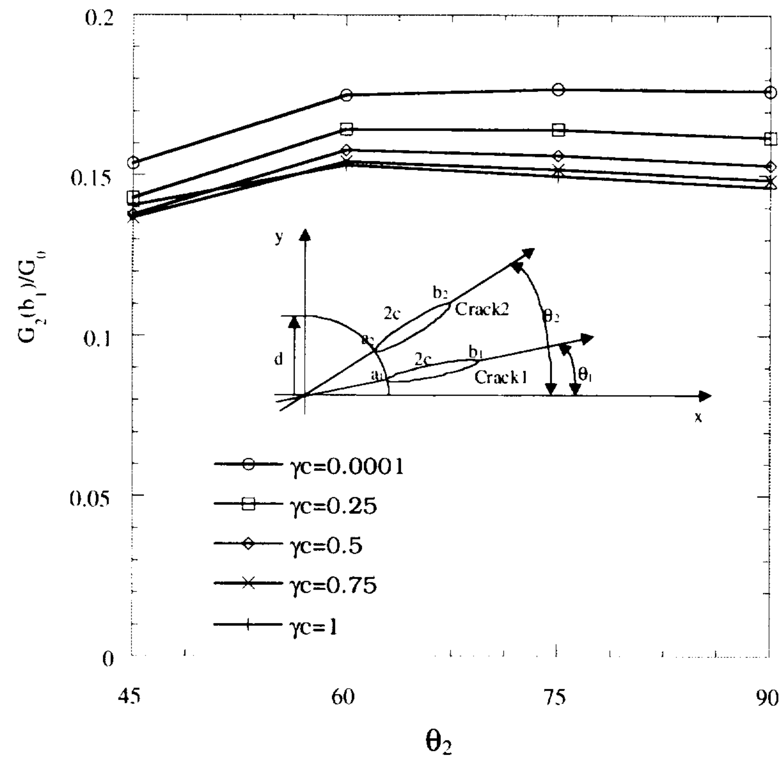

Figure 16. Mode II normalized SERR at the tip b_1 versus crack (2) orientation angle θ_2 for two inclined cracks. (σ_{yy} = 1.0, σ_{xx} = σ_{xy} =0.0, d=1.0, θ_1=30 deg.).

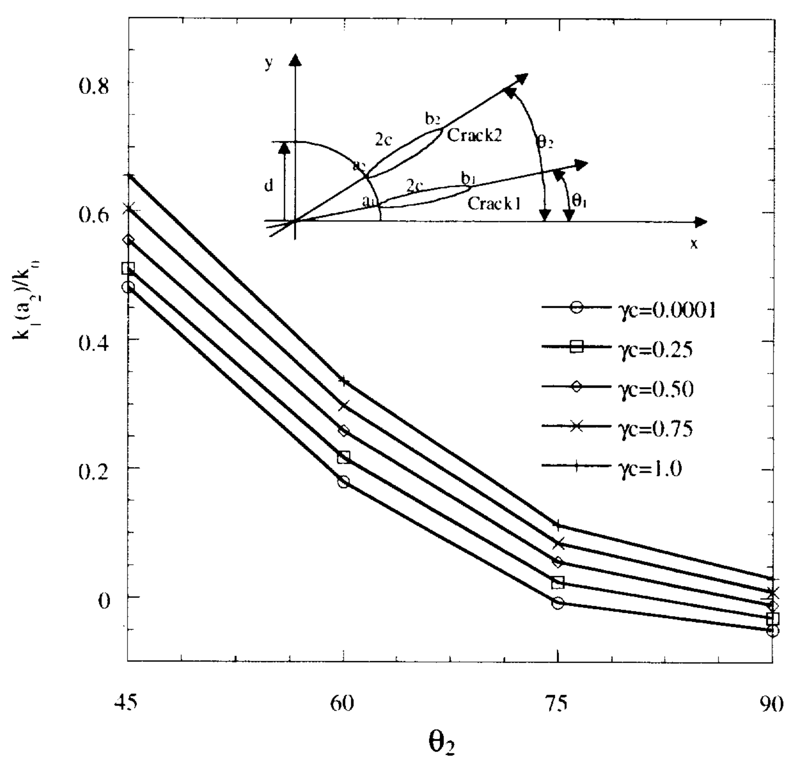

Figure 17. Mode I normalized SIF at the tip a_2 versus crack (2) orientation angle θ_2 for two inclined cracks. (σ_{yy} = 1.0, σ_{xx} = σ_{xy} =0.0, d=1.0, θ_1=30 deg.).

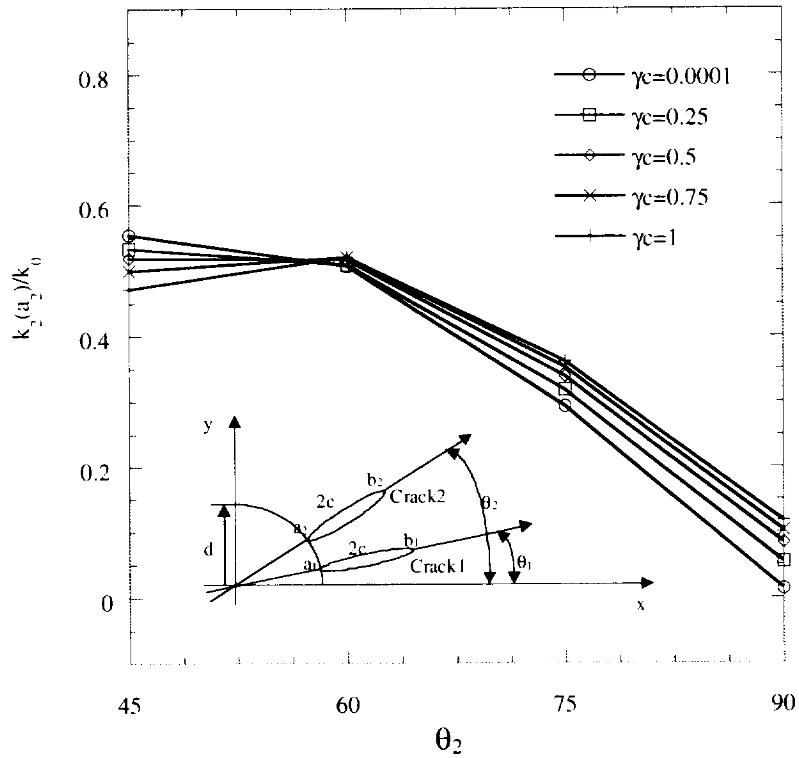

Figure 18. Mode II normalized SIF at the tip a_2 versus crack (2) orientation angle θ_2 for two inclined cracks. (σ_{yy} = 1.0, $\sigma_{xx} = \sigma_{xy}$ =0.0, d=1.0, θ_1=30 deg.).

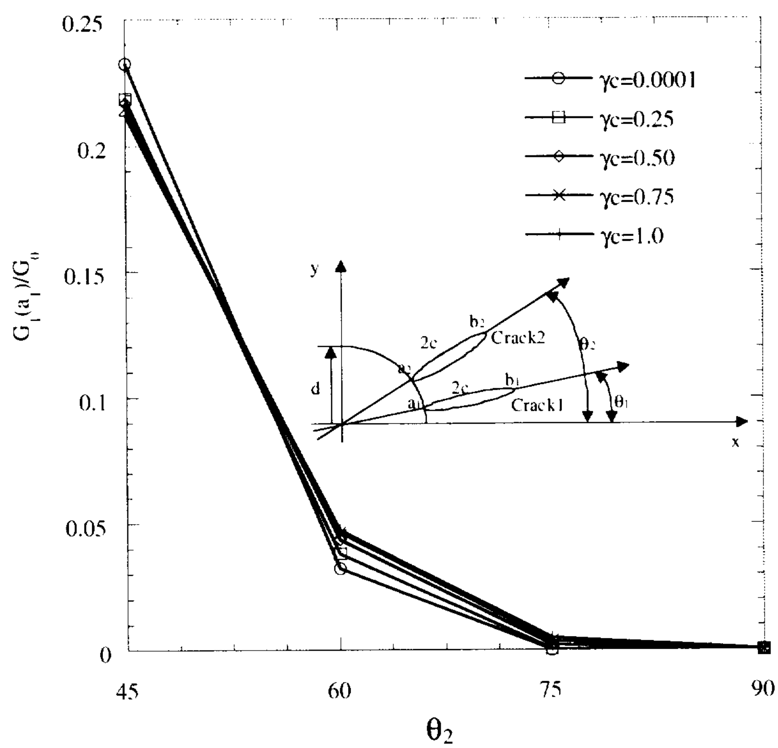

Figure 19. Mode I normalized SERR at the tip a_2 versus crack (2) orientation angle θ_2 for two inclined cracks. (σ_{yy} = 1.0, $\sigma_{xx} = \sigma_{xy}$ =0.0, d=1.0, θ_1=30 deg.).

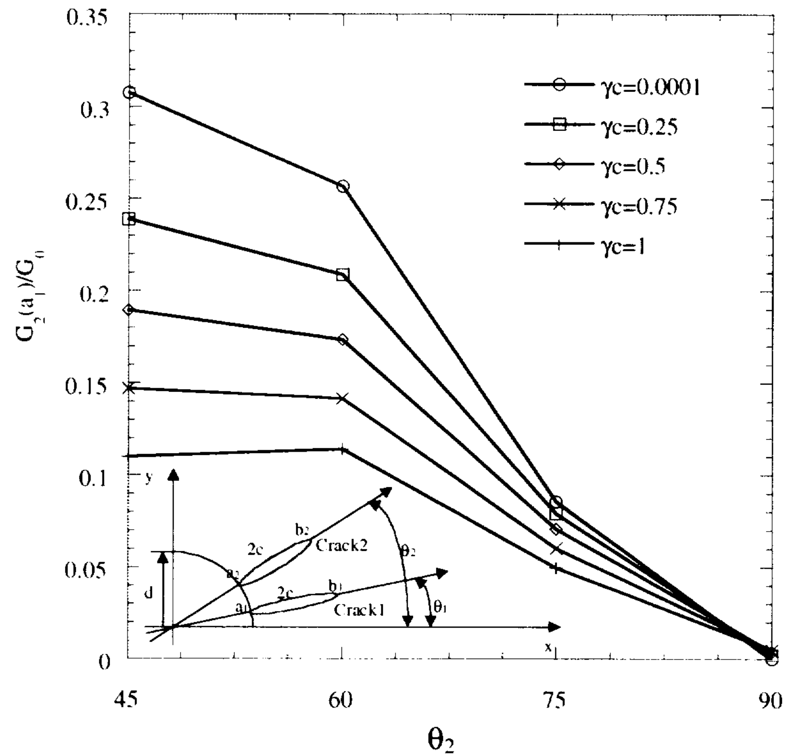

Figure 20. Mode II normalized SERR at the tip a_2 versus crack (2) orientation angle θ_2 for two inclined cracks. (σ_{yy} = 1.0, $\sigma_{xx} = \sigma_{xy}$ =0.0, d=1.0, θ_1=30 deg.).

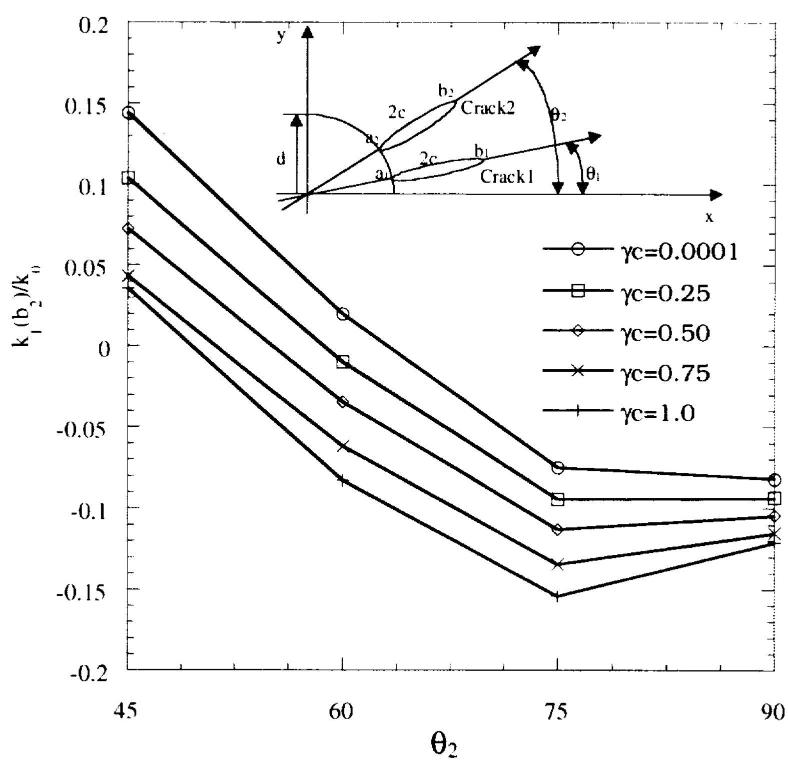

Figure 21. Mode I normalized SIF at the tip b_2 versus crack (2) orientation angle θ_2 for two inclined cracks. (σ_{yy} = 1.0, $\sigma_{xx} = \sigma_{xy}$ =0.0, d=1.0, θ_1=30 deg.).

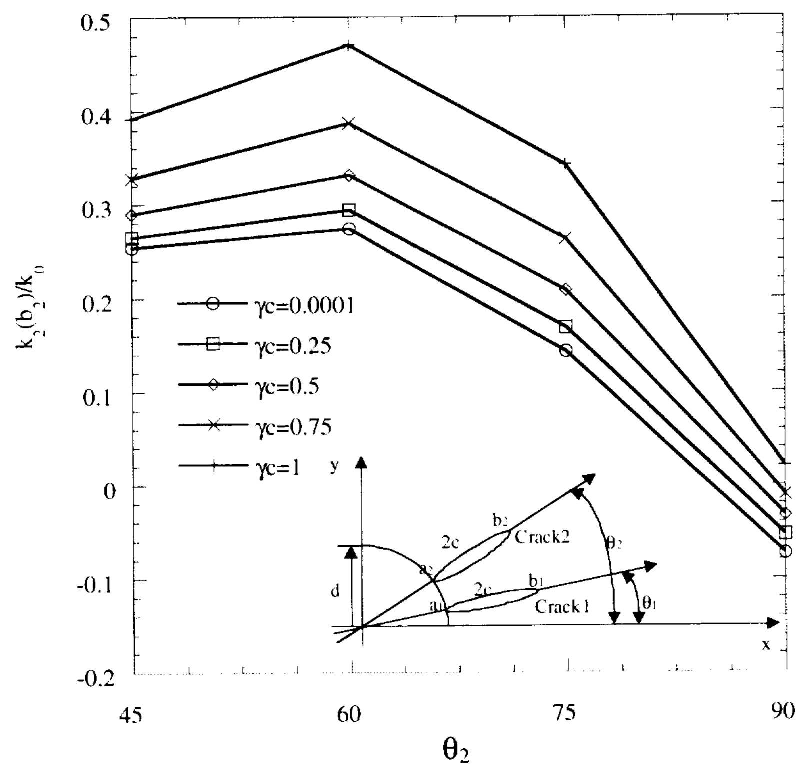

Figure 22. Mode II normalized SIF at the tip b_2 versus crack (2) orientation angle θ_2 for two inclined cracks. (σ_{yy} = 1.0, $\sigma_{xx} = \sigma_{xy}$ =0.0, d=1.0, θ_1=30 deg.).

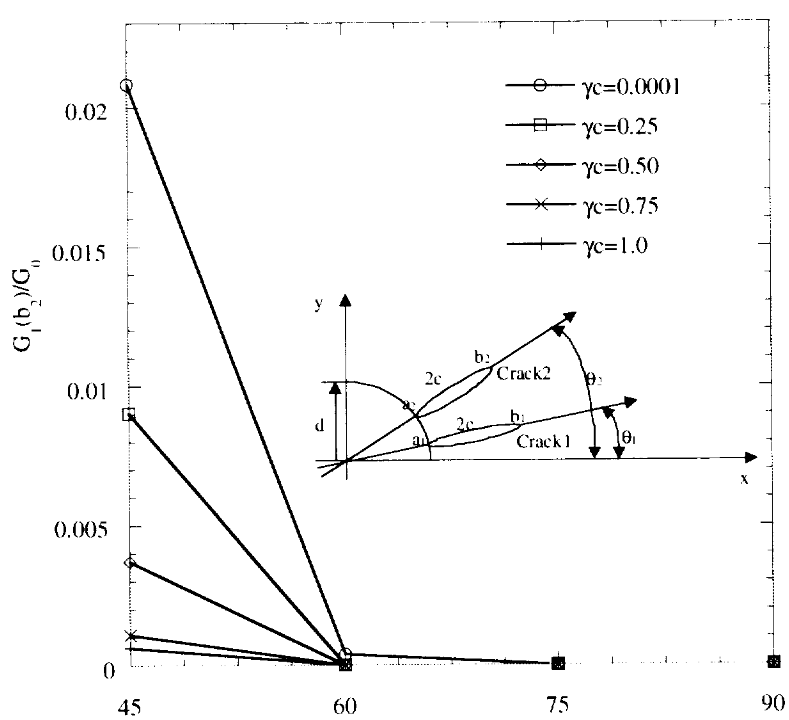

Figure 23. Mode I normalized SERR at the tip b_2 versus crack (2) orientation angle θ_2 for two inclined cracks. (σ_{yy} = 1.0, $\sigma_{xx} = \sigma_{xy}$ =0.0, d=1.0, θ_1=30 deg.).

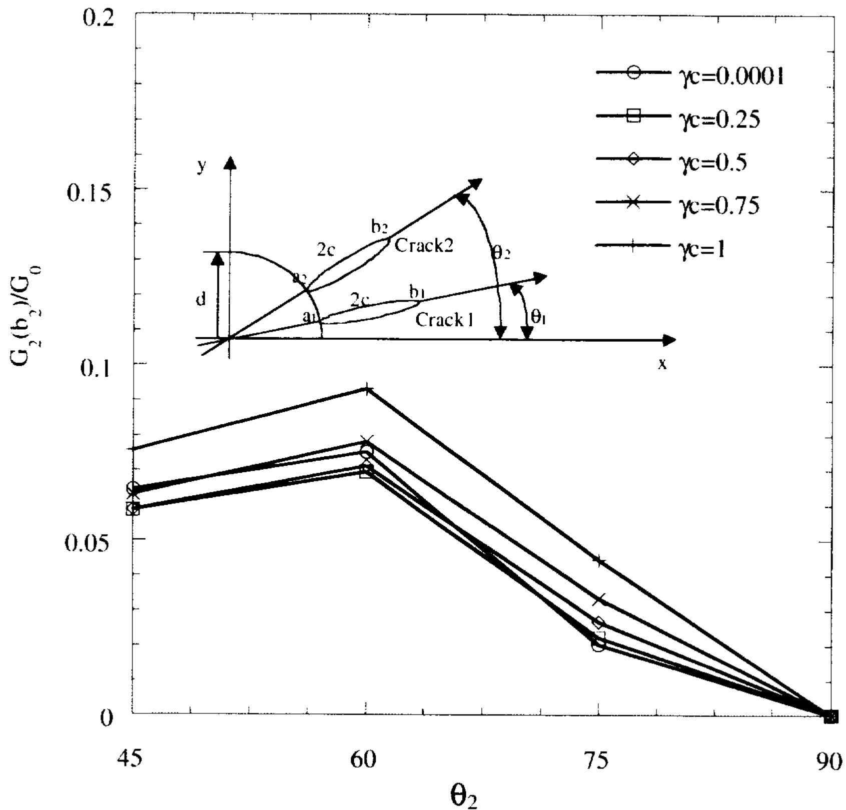

Figure 24. Mode II normalized SERR at the tip b_2 versus crack (2) orientation angle θ_2 for two inclined cracks. (σ_{yy} = 1.0, $\sigma_{xx} = \sigma_{xy}$ =0.0, d=1.0, θ_1=30 deg.).

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A general methodology was constructed to develop the fundamental solution for a crack embedded in an infinite non-homogeneous material in which the shear modulus varies exponentially with the y coordinate. The fundamental solution was used to generate a solution to fully interactive multiple crack problems for stress intensity factors and strain energy release rates. Parametric studies were conducted for two crack configurations. The model displayed sensitivity to crack distance, relative angular orientation, and to the coefficient of nonhomogeneity.

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