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OPTIMAL CONTROL OF DYNAMIC SYSTEMS AND

ITS APPLICATION TO SPLINE APPROXIMATION

by

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CONTENTS

ABSTRACT

Generally, classical polynomial splines tend to exhibit unwanted undulations. In this work, we discuss a technique, based on control principles, for eliminating these undulations and increasing the smoothness properties of the spline interpolants. We give a generalization of the classical polynomial splines and show that this generalization is, in fact, a family of splines that covers the broad spectrum of polynomial, trigonometric and exponential splines. A particular element in this family is determined by the appropriate control data. It is shown that this technique is easy to implement.

Several numerical and curve-fitting examples are given to illustrate the advantages of this technique over the classical approach. Finally, we discuss the convergence properties of the interpolant.

LIST OF FIGURES

CHAPTER I

INTRODUCTION

In this work, we study the control problem

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + B\vec{u}(t), \qquad \vec{y}(t) = C\vec{x}(t) \tag{1.1}$$

with the cost function

$$J(u) = \int_\Omega \sum_{k=0}^{n} \beta_k^2 |u^{(k)}|^2 \, dt. \tag{1.2}$$

Here, $\vec{x} \in \mathbb{R}^m$, $\vec{y} \in \mathbb{R}^p$, $\vec{u} \in L^2(\Omega, \mathbb{R}^l)$, $t \in \Omega \subset \mathbb{R}$, and $A \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^m)$, $B \in \mathcal{L}(\mathbb{R}^l, \mathbb{R}^m)$ and $C \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^p)$. The vectors $\vec{x}$ and $\vec{u}$ are the state and control vectors of the system, respectively. A is called the state matrix, B the control matrix and C the observation matrix.

Our goal is to find the control $\vec{u}$ that will drive the system from one point to the other in the state space $\mathbb{R}^m$ and at the same time minimizes the cost function $J(u)$. We will also establish controllability conditions of the system (given the cost function $J(u)$) and then apply the results to spline approximation problems.

It has been shown [13] that the system (1.1) is controllable if and only if the matrix

$$Z = \begin{pmatrix} B & AB & \dots & A^{m-1}B \end{pmatrix} \tag{1.3}$$

has rank m and that

$$M = \{(A, B) : \dot{\vec{x}} = A\vec{x} + B\vec{u} \ \ is \ \ controllable\} \tag{1.4}$$

is a manifold in $\mathbb{R}^{m(m+l)}$ [14]. Anderson and Moore [2], Luenberger [23], and Sage [28] have dealt extensively with the optimal control problem when the cost function is $J(u) = \int_\Omega u^2 \, dt$ (the minimum energy problem). Conditions for controllability of the system are also given [23].

Spline interpolation constitutes a class of piecewise polynomial approximation that is commonly used when approximating many of the functions that arise in actual physical processes. Spline approximations of functions are preferred to most approximation and interpolation methods because of their inherent smoothness properties [10]. In [21], the *minimal property* associated with spline approximation is shown. A great deal of work has been done on polynomial splines, particularly cubic splines [11]. In [6] the convergence properties of a special class of quintic splines are discussed. There is a small amount of literature on exponential splines [7, 15, 24, 26]. McCartin [24] has given an excellent theortical discussion of exponential splines and also studied its convergence rates and extremal properties. Pruess [26, 27] asserts that exponential splines can produce co-convex and co-montone interpolants.

In their paper [30], Zhang, Tomlinson, and Martin show the relationship between control theory and spline approximation by studying the minimum energy problem, namely: minimize

$$J(u) = \int_0^T u^2(s)\, ds$$

subject to

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + \vec{b}u(t), \quad t \in [0, T]$$

or equivalently

$$\vec{x}(t) = e^{At}\vec{x}(0) + \int_0^t e^{A(t-s)}\vec{b}u(s)\, ds.$$

In this case, they obtained the optimal control law by observing that the operator

$$K : L_2[0, T] \longrightarrow \mathbb{R}^m$$

defined by

$$Ku = \int_0^T e^{A(T-s)}\vec{b}u(s)\, ds$$

has an adjoint given by

$$K^*\vec{z} = \vec{b}^T e^{A^T(T-s)}\vec{z}$$

for any $\vec{z} \in \mathbb{R}^m$. Thus, the optimal control can be written as

$$u = K^*(KK^*)^{-1}(\vec{x}(T) - e^{AT}\vec{x}(0)).$$

(The interested reader should see Luenberger [23] or any other standard text on functional analysis for more details.) By imposing certain smoothness requirements on $u(t)$, they were able to obtain the spline functions. However, this approach does not eliminate the undulations associated with classical polynomial splines. In an attempt to overcome this problem, we introduce into the cost function, $J(u)$, derivatives of the control law $u(t)$ and hence, formulate our problem in the sobolev space $H^n(\Omega)$.

In this work, we intend to use optimal control theory to develop methodology for spline approximations. As an illustration, suppose that the write-head of a computer is required to move from a certain position, $\vec{x}(t_0)$, to another position, $\vec{x}(T)$, then some control $u(t)$ is needed to drive the write-head from the initial state, $\vec{x}(0)$, to the final state, $\vec{x}(T)$. Thus, the write-head (the system) must go through a certain set of points, namely,

$$(t_0, \vec{x}(t_0)), (t_1, \vec{x}(t_1)), \ldots, (t_{n-1}, \vec{x}(t_{n-1})), (t_n, \vec{x}(t_n))$$

$$0 = t_0 < t_1 < \ldots < t_{n-1} < t_n = T$$

at given times. A spline curve can be fitted through these data points and then a control that takes the system through this trajectory determined. We hope to find the set of controls

$$u_i(t): \quad \vec{x}(t_{i-1}) \xrightarrow{u_i(t)} \vec{x}(t_i)$$

that achieves this while minimizing the functional $J(u)$. Then, by applying the appropriate smoothness requirements of $u(t)$ at the endpoints of each subinterval, $[t_{i-1}, t_i]$, we will obtain and characterize the class of spline approximations. Numerical examples will be given to demonstrate the advantages of this technique.

The work has been divided into several parts. In Chapter II, we formulate the control problem in the space

$$H^n(\Omega) = \{u \in L_2(\Omega) | D^j u \in L_2(\Omega), \quad j \in \mathbb{Z}, \quad 0 \leq j \leq n\}.$$

We also discuss some of the basic concepts in systems theory and control. Furthermore, a brief discussion of some of the properties of the space $H^n(\Omega)$, including the relevant embedding theorems, is given. In Chapter III, optimal control is discussed. The control problem is transformed into a system of boundary value problems and, by applying standard techniques for solving $BVPs$, the desired optimal control law is obtained. Chapter IV deals with the derivation of spline functions by imposing appropriate smoothness conditions on the optimal control law obtained in Chapter III. The splines are then classified by studying the structure of the basis functions. Finally, in Chapter V, results of computer simulations and a discussion of rates of convergence are given.

CHAPTER II

FORMULATION OF THE PROBLEM

2.1 Motivation

In order to motivate the importance of developing the relationship between control theory and spline functions, we first present an example that is of great practical importance.

Example 2.1 *The dynamics of a computer disk drive.*

Consider the disk-drive of a computer system. It can be modelled as an inertia system with governing equation given by:

$$M(\theta)\ddot{\theta} + C(\theta,\dot{\theta})\dot{\theta} + K(\theta)\theta = f(\theta) \tag{2.1}$$

where $M(\theta)$ represents a generalized inertia term and is positive definite, $C(\theta,\dot{\theta})$ is generalized damping function, $K(\theta)$ is a generalized stiffness function, and $f(\theta)$ is a forcing function. Linearizing the system at 0, we obtain

$$\ddot{x} + H_1\dot{x} + H_2 x = M^{-1}(0)f. \tag{2.2}$$

Transforming equation (2.2) into a system of first-order equations, we obtain

$$\dot{\vec{z}} = A\vec{z} + Bf \tag{2.3}$$

where $\vec{z} = (x_1, x_2)^T$, $x_1 = x$, $x_2 = \dot{x}$

$$A = \begin{pmatrix} 0 & I \\ -H_2 & -H_1 \end{pmatrix}$$

and

$$B = \begin{pmatrix} 0 \\ I \end{pmatrix}.$$

Equation (2.3) expresses the system dynamics in state-space form. Now, suppose that the write-head is required to move from a certain (initial) position, $\vec{z}(t_0)$, to another (final) position, $\vec{z}(t_f)$. Then, the input function, f, must be chosen appropriately to drive the system from the initial state, $\vec{z}(t_0)$, to the final state, $\vec{z}(t_f)$, following a certain trajectory. Thus, the write-head must pass through a certain set of points

$$(t_0, \vec{z}(t_0)), \ldots, (t_n, \vec{z}(t_n)),$$

$$0 = t_0 < t_1 < \ldots < t_{n-1} < t_n = T$$

at given times. A spline curve can be fitted through this set of points and then a control that takes the system through this trajectory determined. We can find several such functions, f, that will drive the system through the specified set of points. However, a more interesting problem is to find the function, f, that not only drives the system from one point to another in state-space but also minimizes a certain fuctional, $J(f)$. This kind of problem will be solved in a general setting and by forcing f to satisfy certain smoothness conditions at specified points, we will obtain and characterize the class of spline functions.

2.2 Basic Systems Theory Concepts in Finite-Dimensions

In general, by time-invariant, finite-dimensional linear system $\sum(A, B, C, D)$ on the state-space, X, we mean that X, U, and Y are finite-dimensional linear vector spaces and A, B, C, and D are bounded linear maps: $A \in \mathcal{L}(X)$, $B \in \mathcal{L}(U, X)$, $C \in \mathcal{L}(X, Y)$ and $D \in \mathcal{L}(U, Y)$. $X = \mathbb{R}^m$, $U = \mathbb{R}^m$, $Y = \mathbb{R}^k$ are called the state, input, and output spaces, respectively. Furthermore, the state $\vec{x}(t) \in X$, the input $u(t) \in U$, and the output $\vec{y}(t) \in Y$, are related by the equations

$$\dot{\vec{x}}(t) = A\vec{x}(t) + B\vec{u}(t), \qquad t \geq 0, \quad \vec{x}(0) = \vec{x}_0 \tag{2.4}$$

$$\vec{y}(t) = C\vec{x}(t) + D\vec{u}(t), \tag{2.5}$$

where $\vec{x}_0 \in X$ is an arbitrary initial condition. If $\vec{u} \in L^2([0, T]; U)$, then $\vec{x} \in C([0, T]; X)$ and $\vec{y} \in L^2([0, T]; Y)$ are given by

$$\vec{x}(t) = e^{At}\vec{x}_0 + \int_0^t e^{A(t-s)}B\vec{u}(s)ds \tag{2.6}$$

and

$$\vec{y}(t) = Ce^{At}\vec{x}_0 + \int_0^t Ce^{A(t-s)}B\vec{u}(s)ds + D\vec{u}(t) \tag{2.7}$$

The frequency-domain representation of equation (2.7) is given by

$$\hat{y}(s) = D\hat{u}(s) + C(sI - A)^{-1}B\hat{u}(s). \tag{2.8}$$

The above representation is obtained by letting $\vec{x}_0 = 0$ in equation (2.7) and then taking Laplace transforms. Equation (2.8) can be written as

$$\hat{y}(s) = G(s)\hat{u}(s) \tag{2.9}$$

where

$$G(s) = D + C(sI - A)^{-1}B \tag{2.10}$$

is called the *transfer function* of the finite-dimensional system $\sum(A, B, C, D)$ defined by equation (2.4). The transfer functions are proper rational matrices with complex coefficients. A theory for control design based on a transfer matrix description has been developed using the algebraic properties of the finite-dimensional transfer functions. In this work, we will base our analysis entirely on state-space theory.

2.3 The Problem Statement

Consider the linear system:

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + \vec{b}u(t), \qquad t \in [0, T], \tag{2.11}$$

with

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ a_1 & a_2 & a_3 & \dots & a_m \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \quad \vec{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_{m-1}(t) \\ x_m(t) \end{pmatrix}. \tag{2.12}$$

and the observation function

$$y(t) = \vec{c}^T\vec{x}(t), \qquad \vec{c}^T = (1, 0, \dots, 0). \tag{2.13}$$

Let the interval $[0, T]$ be partitioned into p subintervals

$$P : 0 = t_0 < t_1 < \dots < t_{p-1} < t_p = T,$$

and set $h_i = t_i - t_{i-1}$. Our objective is to determine the control element $u(t) \in C^{m-2}[0,T]$ that drives the system (2.11) from $\vec{x}(0)$ to $\vec{x}(T)$ such that the observed function $y(t)$ satisfies the interpolation conditions

$$y(t_i) = \alpha_i, \qquad i = 0, 1, \ldots, p-1, p. \tag{2.14}$$

Moreover, we require that $u(t)$ minimize the cost function

$$J(u) = \int_0^T \left(\sum_{k=0}^{n} (u^{(k)})^2 \right) dt. \tag{2.15}$$

(Notice that equation (2.15) is a special case of equation (1.2) where $\beta = 1$.) A control that achieves this objective is called optimal. Here, $\vec{x}, \vec{b} \in \mathbb{R}^m$, $u(t) \in \mathbf{L}^2[0,T]$ and A is an $m \times m$ matrix. We want to find the control law $u(t)$ that drives the system (2.11) from $\vec{x}(0) = \vec{x}^0$ to $\vec{x}(T) = \vec{x}^T$ and minimizes the functional $J(u)$. Before we go any further, the following definition and theorem are in order:

Definition 2.1 *The linear system (2.11) is said to be controllable if for every pair of vectors $(\vec{x}^0, \vec{x}^T) \in \mathbb{R}^m$, there exists a finite time T and a control $u(t)$ such that,*

$$\vec{x}(T) = e^{AT}\vec{x}^0 + \int_0^T e^{A(T-s)}\vec{b}u(s)\,ds.$$

Theorem 2.1 *The given linear system (2.11) is controllable if and only if the controllability matrix*

$$\mathbf{M} = \left(\begin{array}{cccc} \vec{b} & A\vec{b} & \ldots & A^{m-1}\vec{b} \end{array} \right) \tag{2.16}$$

has rank m. We then say that the pair $(A, \vec{b})$ is controllable.

Example 2.2 *Consider the system $\dot{\vec{x}}(t) = A\vec{x}(t) + \vec{b}u(t)$, $\vec{x}(0) = \vec{x}_0$. Let*

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

and $\vec{b} = (0\,0\,1)^T$. Then

$$rank(b,\ Ab,\ A^2b) = rank \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = 3. \tag{2.17}$$

Hence, the pair (A, b) is controllable.

The system (2.11) , with A, $\vec{b}$ as in equation (2.12), is controllable. This is a direct consequence of Theorem (2.1) since, in this case, the matrix $\mathbf{M} = (\ \vec{b} \ \ A\vec{b} \ \ \dots \ \ A^{m-1}\vec{b} \ )$ has full rank. Now, our problem is to minimize the functional $J(u)$ subject to the constraint

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + \vec{b}u(t), \qquad t \in [0,T].$$

We may replace this equation by the equivalent constraint

$$\vec{x}(t) = e^{At}\vec{x}_0 + \int_0^T e^{A(t-s)}\vec{b}u(s)\,ds. \tag{2.18}$$

Then the control problem may be formulated in the space

$$H^n(\Omega) = \{u \in L^2(\Omega) | \ \ D^j u \in L^2(\Omega), \quad \Omega \subset \mathbb{R}, \quad j \in \mathbb{Z}, 0 \le j \le n\}. \tag{2.19}$$

$H^n(\Omega)$ is the Sobolev space of order n on $\Omega \subset \mathbb{R}$ with inner product defined by

$$(u,v)_{H^n(\Omega)} = \int_0^T \left(\sum_{k=0}^n u^{(k)} v^{(k)} \right) dt \tag{2.20}$$

and the corresponding norm

$$\|u\|^2_{H^n(\Omega)} = \int_0^T \sum_{k=0}^n |\, u^{(k)} \,|^2 \, dt = \sum_{k=0}^n \|u^{(k)}\|^2_{L_2(\Omega)}. \tag{2.21}$$

Since our problem is formulated in the space $H^n(\Omega)$, it is appropriate that we state some of the properties of this space that will be most useful in solving our problem.

2.4 Sobolev Spaces and Embedding Theorems

In this section we will give some of the important results about the Sobolev space $H^n(\Omega)$ that will prove useful in this work. Sobolev spaces are very useful when a higher degree of smoothness is desired. On the other hand, a major problem with this space is that many of the operators that occur frequently in applications are not self-adjoint with respect to the Sobolev inner product.

Definition 2.2 *Let $l \ge 0$ be an integer;* $1 \le p < \infty$, $\Omega \subset \mathbb{R}^m$.

$$\begin{aligned} W_p^n(\Omega) &= \{u \in L_p(\Omega) | \ \exists \ \mathcal{D}^\alpha u \in L_p(\Omega), \quad \forall \alpha \ni |\alpha| \le n\} \\ &\subset L_p(\Omega) \end{aligned}$$

with norm

$$\|u\|_{W_p^n(\Omega)} = \left(\int_\Omega \sum_{|\alpha| \le n} |\mathcal{D}^\alpha u(t)|^p \, dt \right)^{\frac{1}{p}} .$$

If $p = 2$ and $\Omega \subset \mathbb{R}$, then

$$W_p^n(\Omega) = W_2^n(\Omega) = H^n(\Omega),$$

$$H^n(\Omega) = \{u \in L_2(\Omega) | D^j u \in L_2(\Omega), \ \ j \in \mathbb{Z}, \ \ 0 \le j \le n\}.$$

with norm

$$\|u\|^2_{H^n(\Omega)} = \sum_{k=0}^{n} \|D^k u\|^2_{L_2(\Omega)}$$

and

$$H_0^n(\Omega) = \{u \in H^n(\Omega) | D^j u = 0 \ \ on \ \ \partial\Omega, 0 < j\}.$$

Definition 2.3 *Suppose that $\mathbb{X}_1$, $\| \bullet \|_1$ and $\mathbb{X}_2$, $\| \bullet \|_2$ are Banach spaces and that $\exists$ a positive constant $c < \infty$ such that $\|x\|_1 \le c\|x\|_2$, $\forall x \in \mathbb{X}_2$ implies $\mathbb{X}_2 \subset \mathbb{X}_1$, then we say that $\mathbb{X}_2$ is embedded in $\mathbb{X}_1$. Furthermore, the embedding is called compact if the unit ball*

$$B_1^{\mathbb{X}_2}(0) = \{x \in \mathbb{X}_2 | \|x\|_2 \le 1\}$$

is compact in the space $\mathbb{X}_1$.

Definition 2.4 *Compactness Criterion in Hilbert Spaces.*
Let H be a Hilbert space and $\{\phi_k\}_{k=1}^\infty$ an orthonormal basis in H. Let $B \subset H$ be a bounded subset of H. Define

$$P_n : H \to H_n = span\{\phi_k\}_{k=1}^\infty$$

by

$$P_n x = \sum_{k=1}^{n} (x, \phi_k)\phi_k$$

$$P_n^\perp : H \to H_n^\perp = \bar{span}\{\phi_k\}_{k=n+1}^\infty$$

$$P_n^\perp x = \sum_{k=n+1}^{\infty} (x, \phi_k)\phi_k$$

Furthermore, define

$$\epsilon_n(B) = \sup_{x \in B} \|P_n^{\perp} x\|.$$

Then $\bar{B}$ is compact iff

$$\lim_{n \to \infty} \epsilon_n(B) = 0.$$

Theorem 2.2 *Let H be a Hilbert space with orthonormal basis $\{\phi_k\}_{k=1}^{\infty}$, $H_1 \subset H$ and $H_2 \subset H$. Suppose that*

$$0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \lambda_n \to \infty$$

$$0 \leq \mu_1 \leq \mu_2 \leq \cdots \mu_n \to \infty$$

and

$$\frac{\mu_n}{\lambda_n} \xrightarrow{n} \infty$$

Furthermore, let $\|u\|_1^2 = \sum_{k=1}^{\infty} \lambda_k |(u, \phi_k)|^2$ and $\|u\|_2^2 = \sum_{k=1}^{\infty} \mu_k |(u, \phi_k)|^2$ be norms in H_1 and H_2, respectively. Then $H_2 \subset H_1$ and the embedding is compact. ■

Lemma 2.1 *Let*

$$\|u\|_1 = \sum_{k=0}^{n} \|u^{(k)}\|_{L^2(\Omega)}. \tag{2.22}$$

Then (2.21) and (2.22) are equivalent.

Theorem 2.3 *The Sobolev space $H^n(\Omega)$ is a Hilbert space.*

Proof: The proof of this Lemma is found in Aubin [3]. ■

Theorem 2.4 *The space $H^n(\Omega)$ is separable and reflexive.[1]*

Theorem 2.5 *Extension Theorem for Sobolev Spaces.[1, 16]*
Let $\Omega \subset \mathbb{R}^m$ and $\partial\Omega$ piecewise smooth. Assume also that $\bar{\Omega} \subset \Omega_1$. For any fixed p, l, $(1 \leq p < \infty, l \geq 0, l \in \mathbb{Z})$, $\exists$ a bounded linear operator

$$E : W_p^l(\Omega) \to W_{p,0}^l(\Omega)$$

such that $\forall\ u \in W_p^l(\Omega)$, $(Eu)(x) = u(x)$, $x \in \Omega$, and $\|Eu\|_{W_p^l(\Omega_1)} \leq C\|u\|_{W_p^l(\Omega)}$. ■

Now, consider the space $L_2(\Omega)$, where $\Omega = [0,T] \subset \mathbb{R}^1$.

$$\phi_k(t) = \frac{1}{\sqrt{T}} e^{\frac{2\pi i k t}{T}}, \quad k \in \mathbb{Z}$$

is an orthonormal basis in $L_2(\Omega)$. So $u \in L_2(\Omega)$ implies that

$$\begin{aligned} u(t) &= \sum_{k\in\mathbb{Z}} c_k \phi_k(t) \\ &= \frac{1}{\sqrt{T}} \sum_{k\in\mathbb{Z}} c_k e^{\frac{2\pi i k t}{T}} \end{aligned} \tag{2.23}$$

where

$$\begin{aligned} c_k &= (u, \phi_k) \\ &= \int_\Omega u(t)\bar{\phi}_k(t)\, dt \\ &= \frac{1}{\sqrt{T}} \int_\Omega u(t) e^{-\frac{2\pi i k t}{T}}\, dt \end{aligned} \tag{2.24}$$

If $u \in H_0^n(\Omega)$, then

(i)

$$\|u\|^2_{H^n(\Omega)} = \sum_{k=0}^{n} \|D^k u\|^2_{L_2(\Omega)}$$

with

$$\|u\|^2_{L_2(\Omega)} = \sum_{k=0}^{n} |c_k|^2,$$

by Parseval's equality.

(ii) $\frac{du}{dt} \in L_2(\Omega)$; therefore, since $u \in H_0^n(\Omega)$,

$$\frac{du}{dt} = \frac{1}{\sqrt{T}} \sum_{k\in\mathbb{Z}} d_k e^{\frac{2\pi i k t}{T}},$$

$$d_k = (\frac{du}{dt}, \phi_k) = \frac{2\pi i k}{T} c_k \tag{2.25}$$

and

$$\|\frac{du}{dt}\|^2 = \sum_{k\in\mathbb{Z}} |\frac{2\pi i k}{T}|^2 |c_k|^2. \tag{2.26}$$

(iii) Finally,

$$\|D^l u\|^2_{L_2(\Omega)} = \sum_{k\in\mathbb{Z}} |\frac{2\pi i k}{T}|^{2l} |c_k|^2. \tag{2.27}$$

This is true since $D^l u \in L_2(\Omega)$. Hence,

$$D^l u = \frac{1}{\sqrt{T}} \sum_{k\in\mathbb{Z}} A_k e^{\frac{2\pi ikt}{T}},$$

where

$$A_k = (D^l u, \phi_k) = (\frac{2\pi ik}{T})^l c_k, \tag{2.28}$$

$\forall u \in H_0^n(\Omega)$.

$$\|u\|^2_{H^n(\Omega)} = \sum_{k\in\mathbb{Z}} \left(\sum_{l=0}^{n} k^{2l}\right) |c_k|^2 = \sum_{l=0}^{n} \|D^l u\|^2_{L_2(\Omega)} \tag{2.29}$$

Thus, there exist constants $\alpha_1, \alpha_2 > 0$ such that

$$\alpha_1(1+k^2)^n \le \sum_{l=0}^{n} k^{2l} \le \alpha_2(1+k^2)^n. \tag{2.30}$$

This establishes that the norm defined by

$$|||u|||^2_{H^n(\Omega)} = \sum_{k\in\mathbb{Z}} (1+k^2)^l |c_k|^2$$

is equivalent to the original norm

$$\|u\|^2_{H^n(\Omega)} = \sum_{k=0}^{n} \|D^k u\|^2_{L_2(\Omega)}.$$

Lemma 2.2 *The embedding $H_0^n(\Omega) \subset H_0^{n-1}(\Omega)$ is compact.*

Proof: By the above discussion, we have

$$\begin{aligned} \|u\|_{H^{n-1}(\Omega)} &= \sum_{l\in\mathbb{Z}} (1+k^2)^{l-1} |c_l|^2 \\ &= \sum_{l\in\mathbb{Z}} \lambda_l |c_l|^2 \end{aligned} \tag{2.31}$$

and

$$\begin{aligned} \|u\|_{H^{n}(\Omega)} &= \sum_{l\in\mathbb{Z}} (1+k^2)^{l} |c_l|^2 \\ &= \sum_{l\in\mathbb{Z}} \mu_l |c_l|^2, \end{aligned} \tag{2.32}$$

where $\lambda_k = (1+k^2)^{n-1}$ and $\mu_k = (1+k^2)^n$. Thus, by Theorem (2.2) the embedding is compact since

$$\frac{\mu_n}{\lambda_n} = \frac{(1+k^2)^n}{(1+k^2)^{n-1}} \to \infty.$$

■

Theorem 2.6 *The embedding $H^n(\Omega) \subset H^{n-1}(\Omega)$ is compact.*

Proof: It is obvious that $H^n(\Omega)$ is embedded in $H^{n-1}(\Omega)$ since

$$\|u\|_{H^{n-1}(\Omega)} \leq \|u\|_{H^n(\Omega)}.$$

However, the compactness of the embedding is not so obvious. To prove this, we apply lemma (2.2). Let $\Omega \subset \Omega_1$ with $\bar{\Omega} \subset \Omega_1$. By the extension theorem,

$$E : H^n(\Omega) \to H_0^n(\Omega_1)$$

is bounded. By lemma (2.2), the embedding $C : H_0^n(\Omega_1) \subset H_0^{n-1}(\Omega_1)$ is compact. Now, let

$$R : H_0^{n-1}(\Omega_1) \to H_0^{n-1}(\Omega_1),$$
$$u \in H_0^{n-1}(\Omega) \mapsto Ru = u|_\Omega$$

Clearly,

$$\|Ru\|_{H^{n-1}(\Omega)} \leq \|u\|_{H_0^{n-1}(\Omega_1)}.$$

Thus, the embedding

$$H^n(\Omega) \xrightarrow{E} H_0^n(\Omega_1) \overset{C}{\subset} H_0^{n-1}(\Omega_1) \xrightarrow{R} H^{n-1}(\Omega)$$

is compact and the proof is complete. ■

Theorem 2.7 *Let $\Omega \subset \mathbb{R}^m$ be a bounded domain with $\partial\Omega \in C^1$. (a) If $2n > m$, then $H^n(\Omega)$ is embedded in $C(\bar{\Omega})$ and the embedding is bounded; that is, there exists a constant C such that $\|u\|_{C(\Omega)} \leq C\|u\|_{H^n(\Omega)}$ for all $u \in H^n(\Omega)$. Furthermore, the embedding is compact. (b) If $n - \frac{m}{2} > r$, then $H^n(\Omega) \subset C^r(\Omega)$ and the embedding is compact.*

Proof: The proof of this Theorem is found in references [1, 16].

2.5 Existence and Uniqueness of a Solution

The set of elements $u \in H^n(\Omega)$ satisfying the constraint (2.18) is a linear variety $\bar{V}$ in $H^n(\Omega)$, that is,

$$\begin{aligned} V &= \{u \in H^n(\Omega) : \vec{x}(t) = \int_0^t e^{A(t-s)}\vec{b}u(s)ds\}, \\ V^\perp &= \{w \in H^n(\Omega) : (u,w) = 0 \quad \forall\, u \in V\}, \\ \bar{V} &= \{u \in H^n(\Omega) : \quad \vec{x}(t) = e^{At}\vec{x}_0 + \int_0^t e^{A(t-s)}\vec{b}u(s)ds\}. \end{aligned} \tag{2.33}$$

Therefore, the control problem is equivalent to finding the element $u \in \bar{V}$ of minimum norm.

Theorem 2.8 *The control problem (2.11), with A, $\vec{b}$ as in equation (2.12), has a unique solution.*

Proof. To establish the existence and uniqueness of a solution, it suffices to show that the linear variety $\bar{V}$, as defined in (2.33), is closed. First, observe that $\bar{V}$ is nonempty since the system is controllable (by Theorem 2.1). Let $\{u_n\}$ be a sequence of elements from $\bar{V}$ converging to an element u. To show that $\bar{V}$ is closed, we need to prove that $u \in \bar{V}$. Let

$$\vec{y}(t) = e^{At}\vec{x}_0 + \int_0^t e^{A(t-s)}\vec{b}u(s)ds. \tag{2.34}$$

We must show that $\vec{x}(t) = \vec{y}(t)$.

$$\vec{y}(t) - \vec{x}_n(t) = \int_0^t e^{A(t-s)}\vec{b}[u(s) - u_n(s)]ds. \tag{2.35}$$

By Cauchy-Schwarz inequality, we have

$$\begin{aligned} |\, \vec{y}(t) - \vec{x}_n(t)\,|^2 &\le (\int_0^t |\, e^{A(t-s)}\vec{b}\,|^2\, ds)(\int_0^t |\, u(s) - u_n(s)\,|^2\, ds) \\ &\le \int_0^t |\, (\sum_{k=0}^{\infty} \frac{A^k(t-s)^k}{k!}\vec{b})\,|^2\, ds \int_0^t |\, u(s) - u_n(s)\,|^2 ds \\ &\le \int_0^t \left(\sum_{k=0}^{\infty} \frac{\|A\|^k\, |\,(t-s)\,|^k}{k!}\|\vec{b}\|\right)^2 ds\|u - u_n\|^2 \\ &\le \int_0^t e^{2\|A\|(t-s)}ds\|u - u_n\|^2 \\ &\le \frac{\|A\|^{-1}}{2}\left(e^{2\|A\|t} - 1\right)\|u - u_n\|^2 \end{aligned} \tag{2.36}$$

Since $A \in L(\mathbb{R}^m)$, there exists a constant $M < \infty$ such that $\|A\| \leq M$. Thus,

$$| \vec{y}(t) - \vec{x}_n(t) |^2 \leq \frac{M^{-1}}{2} \left(e^{2Mt} - 1\right) \|u - u_n\|^2. \tag{2.37}$$

Therefore, integrating from 0 to T, we get

$$\|\vec{y} - \vec{x}_n\| \leq C\|u - u_n\| \tag{2.38}$$

where $C = \left(\frac{M^{-1}}{2}\left(\frac{M^{-1}}{2}\left(e^{2MT} - 1\right) - T\right)\right)^{\frac{1}{2}}$. Thus,

$$\|\vec{y} - \vec{x}\| \leq \|\vec{y} - \vec{x}_n\| + \|\vec{x}_n - \vec{x}\| \leq C\|u - u_n\| + \|\vec{x}_n - \vec{x}\|. \tag{2.39}$$

Since $u \to u_n$ and $\vec{x} \to \vec{x}_n$ as $n \to \infty$, we get $\vec{y} = \vec{x}$. ∎

CHAPTER III
OPTIMAL CONTROL

Optimal control is a branch of modern control that provides analytical designs of a special type. In this case, the system is required to be the best possible system of a particular kind in addition to satisfying stability requirements and all the desirable constraints associated with classical control. Linear optimal control is a special type of optimal control in which the controlled system is assumed linear and the control element is forced to be linear. This leads to an output that is linearly dependent on the input. The linear nature of many engineering plants justifies the study and analysis of linear optimal control systems.

To obtain the optimal control law we first reduce the control problem to a boundary-value problem. The resulting boundary-value problem is then solved using standard techniques. The embedding theorems of section (2.4) assures us that the solution of this boundary value problem is bounded.

3.1 The Optimal Control Law

In this section, we will determine the optimal control law for our system

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + \vec{b}u(t), \quad t \in \Omega,$$

with cost function

$$J(u) = \int_\Omega \sum_{k=0}^{n} |u^{(k)}|^2 \, dt.$$

The optimal control $u(t)$ is the element $u \in \bar{V}$ of minimum norm, where $\bar{V}$ is given by equation (2.33). What follows is a construction of $V^\perp$. For $u, v \in H^n([0,T])$, consider the inner product $(u,v)_{H^n(\Omega)}$. If we integrate this by parts, we obtain

$$\begin{aligned}
(u,v)_{H^n(\Omega)} &= \int_0^T \sum_{k=0}^{n} u^{(k)} v^{(k)} \, ds = \sum_{k=0}^{n} \int_0^T u^{(k)} v^{(k)} \, ds \qquad (3.1)\\
&= \int_0^T uv \, ds + \sum_{k=1}^{n} \left[(-1)^k \int_0^T v^{(2k)} u \, ds + \sum_{j=1}^{k} (-1)^{k-j} v^{(2k-j)} u^{(j-1)} \Big|_0^T \right]\\
&= \int_0^T \sum_{k=0}^{n} (-1)^k v^{(2k)} u + \sum_{k=1}^{n} \left(\sum_{j=1}^{k} (-1)^{k-j} v^{(2k-j)} u^{(j-1)} \right) \Big|_0^T
\end{aligned}$$

$$= \int_0^T \sum_{k=0}^{n} (-1)^k v^{(2k)} u\, ds + [v^{(1)}u + v^{(2)}u^{(1)} - v^{(3)}u + \ldots + v^{(n)}u^{(n-1)} - v^{(n+1)}u^{(n-2)} + \ldots + (-1)^{(n-1)} v^{(2n-1)} u] \,|_0^T$$

Now, we want to minimize

$$J(u) = (u, u)_{H^n(\Omega)}$$

subject to

$$g(u) = e^{-AT}x(T) - x(0) - \int_0^T e^{-As}\vec{b}u(s)\, ds = 0.$$

To this end, we let

$$L(u, \vec{\lambda}) = J(u) + \vec{\lambda}^T g(u)$$

where $\vec{\lambda}$ is the optimizing vector. Therefore, the optimal control law is obtained by solving the following equations[1]:

$$\frac{\partial L(u, \vec{\lambda})}{\partial u} = 0$$

$$\frac{\partial L(u, \vec{\lambda})}{\partial \lambda_j} = 0, \qquad j = 1, 2, \ldots, m$$

Using equation (3.1) in the above equations, we obtain the following boundary-value problem:

$$\sum_{k=0}^{n} (-1)^k u^{(2k)} \;\; = \;\; \vec{\lambda}^T e^{-At}\vec{b}$$

---

[1]Notice that

$$L(u, \vec{\lambda}) = \int_0^T \left(\sum_{k=0}^{n} (-1)^k u^{(2k)} - \vec{\lambda}^T e^{-As}\vec{b} \right) u(s)\, ds + \sum_{k=1}^{n} \left(\sum_{j=1}^{k} (-1)^{k-j} u^{(2k-j)} u^{(j-1)} \right) |_0^T + \vec{\lambda}^T \left(e^{-AT}x(T) - x(0) \right)$$

and, hence,

$$\frac{\partial L(u, \vec{\lambda})}{\partial u} = \int_0^T \left(\sum_{k=0}^{n} (-1)^k u^{(2k)} - \vec{\lambda} e^{-As}\vec{b} \right) ds$$

$$\frac{\partial L(u, \vec{\lambda})}{\partial \lambda_j} = g_j(u), \qquad j = 1, 2, \ldots, m$$

$$\begin{aligned}
\sum_{k=1}^{n}(-1)^{k-1}u^{(2k-1)}\mid_{0,T} &= 0\\
\sum_{k=1}^{n-1}(-1)^{k-1}u^{(2k)}\mid_{0,T} &= 0\\
&\vdots\\
\sum_{k=1}^{2}(-1)^{k-1}u^{(2k+n-3)}\mid_{0,T} &= 0\\
v^{(n)}\mid_{0,T} &= 0.
\end{aligned} \tag{3.2}$$

We will show that the BVP (3.2) has a unique solution. This is not surprising since we are dealing with a system that is controllable. Furthermore, it will be shown that the solution of equation (3.2) is a function of $\vec{\lambda}$ and t namely $u(\vec{\lambda}, t)$. Then, the desired optimal control law will be given by

$$u^*(t) = u(\vec{\lambda}^*, t), \tag{3.3}$$

where $\vec{\lambda}^*$ is the vector $\vec{\lambda}$ that satisfies

$$\int_0^t e^{-As}\vec{b}u(\vec{\lambda}, s)\,ds = e^{-AT}\vec{x}(T) - e^{-At_0}\vec{x}(t_0). \tag{3.4}$$

Now, if we let

$$\begin{aligned}
u &= \psi_1\\
u^{(1)} &= \psi_1' = \psi_2\\
&\vdots\\
u^{(2n-1)} &= \psi_{2n-1}' = \psi_{2n}\\
u^{(2n)} &= \psi_{2n}' = (-1)^{2n}u^{2(n-1)} + \ldots + (-1)^{n+3}u^{(4)} +\\
&\quad (-1)^{n+2}u^{(2)} + (-1)^{n+1}u + (-1)^{n}\vec{\lambda}^T e^{-At}\vec{b},
\end{aligned} \tag{3.5}$$

then the differential equation reduces to the following system of first-order linear differential equations:

$$\Psi'(t) = F\Psi(t) + \vec{\sigma}(t) \tag{3.6}$$

$$B\Psi(t)\mid_0 = B_0\Psi(t) = 0 \tag{3.7}$$

$$B\Psi(t)\mid_T = B_T\Psi(t) = 0, \tag{3.8}$$

where

$$\Psi(t) = (\psi_1(t), \psi_2(t), \ldots, \psi_{2n}(t))^T,$$

$$\vec{\sigma}(t) = (-1)^n \vec{b}^T e^{-A^T t} \vec{\lambda} \vec{e}_{2n}$$

and $\vec{e}_{2n}$ is the $2n - vector$ with 1 at the $2n - th$ position and zero elsewhere.

$$F = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ (-1)^{n+1} & 0 & (-1)^{n+2} & \ldots & (-1)^{2n} & 0 \end{pmatrix}, \tag{3.9}$$

and

$$B = \begin{pmatrix} 0 & 1 & 0 & -1 & 0 & \ldots & \ldots & \ldots & (-1)^{n-2} & 0 & (-1)^{n-1} \\ 0 & 0 & 1 & 0 & -1 & \ldots & \ldots & \ldots & 0 & (-1)^{n-2} & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ldots & \ldots & \ldots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ldots & \ldots & \ldots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \ldots & 1 & \ldots & 0 & 0 & 0 \end{pmatrix}. \tag{3.10}$$

By elementary row operation, the matrix, B, may be expressed in the equivalent form

$$B = \{b_{i,j}\}_{i=1,\ldots,n; j=1,\ldots,2n}$$

where

$$b_{i,j} = \begin{cases} 1 & \text{if } j = i+1,\ i = 1, \ldots, n \\ (-1)^{n-i} & \text{if } j = 2n - (i-1),\ i = 1, \ldots, n-1 \\ 0 & \text{otherwise.} \end{cases} \tag{3.11}$$

Now, let $\Phi(t)$ be a fundamental matrix solution of the homogeneous BVP

$$\Psi'(t) = F\Psi(t) \tag{3.12}$$

$$B_0\Psi(t) = B\Psi(0) = 0 \tag{3.13}$$

$$B_T\Psi(t) = B\Psi(T) = 0 \tag{3.14}$$

with $\Phi(0) = I$. The general solution of equation (3.6), when it exists, then satisfies

$$\Psi(t) = \Phi(t)\vec{c} + \Psi^o(t) \tag{3.15}$$

where

$$\Psi^o(t) = \Phi(t)\int_0^t \Phi^{-1}(s)\sigma(s)\,ds \tag{3.16}$$

is a solution of equation (3.12) with $\Psi^o(0) = 0$ and $\vec{c}$ is an arbitrary element of $\mathbb{R}^{2n}$. Therefore, the boundary conditions (3.13), (3.14) become

$$B_0(\Phi(t)\vec{c} + \Psi^o(t)) = B\vec{c} = 0 \tag{3.17}$$

$$B_T(\Phi(t)\vec{c} + \Psi^o(t)) = (B_T\Phi(t))\vec{c} + B_T\Psi^o(t) = 0. \tag{3.18}$$

This gives the following system of equations

$$\begin{bmatrix} B \\ B_T\Phi(t) \end{bmatrix} \vec{c} = \begin{bmatrix} 0 \\ \Gamma_1 \end{bmatrix} \tag{3.19}$$

where $\Gamma_1 = -B_T\Psi^o(t)$. For a convenient notation we let

$$H = \begin{bmatrix} B \\ B_T\Phi(t) \end{bmatrix}$$

and

$$\Gamma = \begin{bmatrix} 0 \\ \Gamma_1 \end{bmatrix}.$$

Thus,

$$H\vec{c} = \Gamma. \tag{3.20}$$

Let us discuss the following important lemmas before going any further.

Lemma 3.1 *The matrix F has $2n$ linearly independent eigenvectors.*

Proof: From

$$det(\beta I - F) = 0 \tag{3.21}$$

we obtain the characteristic polynomial of F as

$$\beta^{2n} - \beta^{2n-2} + \ldots + (-1)^n = \frac{\beta^{2n+2} + (-1)^n}{\beta^2 + 1} = 0. \tag{3.22}$$

Solving equation (3.22), we see that the eigenvalues of the matrix F are

$$\beta_k = e^{(\frac{1}{2}+\frac{k}{n+1})\pi i}, \qquad k = 1, \ldots, n, n+2, \ldots, 2n+1. \tag{3.23}$$

Thus, F has $2n$ distinct eigenvalues. Now, solving

$$(\beta_k I - F)p_k = 0 \tag{3.24}$$

gives the eigenvectors of F as

$$p_k = \begin{pmatrix} 1 \\ \beta_k \\ \beta_k^2 \\ \vdots \\ \beta_k^{j-1} \\ \vdots \\ \beta_k^{2n-2} \\ \beta_k^{2n-1} \end{pmatrix} p_{k,1} \qquad k = 1, 2, \ldots, n, n+2, \ldots, 2n+1, \;\; j = 1, \ldots, 2n \tag{3.25}$$

where $p_{k,1}$ is the first component of the eigenvector p_k. Since the eigenvalues are all distinct and the eigenvectors are as in equation (3.25), we then have that the matrix F has $2n$ linearly independent eigenvectors.

Lemma 3.2 *(Gantmacher [17]) For a nonsingular operator A, $Ax = 0$ implies $x = 0$.*

Lemma 3.3 *When the system (2.11) is controllable, then the matrix H has full rank for all $T > 0$.*

Proof : Recall that

$$H = \begin{bmatrix} B \\ B\Phi(T) \end{bmatrix}.$$

To prove this lemma, it suffices to show that $null(H) = \vec{0}$, by lemma (3.2). Without loss of generality, we take $T = 1$. Therefore, let $\vec{x} \in null(H)$. Then

$$B\vec{x} = 0 \tag{3.26}$$

and

$$Be^F\vec{x} = 0 \tag{3.27}$$

where F and B are given by equations (3.9) and (3.10), respectively. By using equation (3.11), we see that $B\vec{x} = 0$ implies that

$$\begin{pmatrix} x_2 + (-1)^{n-1} x_{2n} \\ x_3 + (-1)^{n-2} x_{2n-1} \\ \vdots \\ x_j + (-1)^{n-(j-1)} x_{2n-(j-1)} \\ \vdots \\ x_{n+1} \end{pmatrix} = \bar{0}. \tag{3.28}$$

If we denote the columns of e^F by P_j, $j = 1, 2, \ldots, 2n$, then equation (3.27) may be written as

$$(BP_1 \;\; BP_2 \;\; \ldots BP_{2n})\,\vec{x} = 0$$

where

$$BP_j = \begin{pmatrix} p_{2,j} + (-1)^{n-1} p_{2n,j} \\ p_{3,j} + (-1)^{n-2} p_{2n-1,j} \\ \vdots \\ p_{i,j} + (-1)^{n-(i-1)} p_{2n-(i-1),j} \\ \vdots \\ p_{n+1} \end{pmatrix}. \tag{3.29}$$

Thus,

$$\begin{aligned} 0 &= (BP_1 \;\; BP_2 \;\; \ldots BP_{2n})\,\vec{x} \\ &= \sum_{j=1}^{2n} x_j BP_j \\ &= \sum_{j=1}^{2n} x_j \left(p_{i,j} + (-1)^{n-(i-1)} p_{2n-(i-1),j}\right); \quad i = 1, 2, \ldots, n. \end{aligned} \tag{3.30}$$

From equation (3.28)

$$x_{2n-(j-1)} = (-1)^{-n+j+2} x_j \quad j = 1, \ldots, n. \tag{3.31}$$

Thus, equation (3.30) becomes

$$0 = \sum_{j=1}^{2n} x_j \left(p_{i,j} + (-1)^{n-(i-1)} p_{2n-(i-1),j}\right)$$

$$
\begin{aligned}
&= \sum_{j=1}^{n} x_j \left(p_{i,j} + (-1)^{n-(i-1)} p_{2n-(i-1),j}\right) + \sum_{j=n+1}^{2n} x_j \left(p_{i,j} + (-1)^{n-(i-1)} p_{2n-(i-1),j}\right) \\
&= \sum_{j=1}^{n} x_j \left(p_{i,j} + (-1)^{n-(i-1)} p_{2n-(i-1),j}\right) + \\
&\quad \sum_{j=1}^{n} (-1)^{-(n-j+1)} x_{j+1} \left(p_{i,2n-j+1} + (-1)^{n-(i-1)} p_{2n-(i-1),2n-j+1}\right) \\
&= \left(p_{i,1} + (-1)^{n-(i-1)} p_{2n-(i-1),1}\right) x_1 + \\
&\quad \sum_{j=2}^{n} \left(p_{i,j} + (-1)^{n-(i-1)} p_{2n-(i-1),j} + (-1)^{-(n-j+1)} p_{i,2n-j+2} + p_{2n-(i-1),2n-j+2}\right) x_j \\
&= m_{i,1} x_1 + m_{i,j} x_j, \quad i = 1, \ldots, n; j = 2, \ldots, n. \qquad (3.32)
\end{aligned}
$$

This yields the set of equations

$$M\vec{y} = 0$$

where $\vec{y} = (x_1, x_2, \ldots, x_n)^T$ and $M = \{m_{i,j}\}_{i,j=1}^{n}$

$$m_{i,1} = p_{i,1} + (-1)^{n-(i-1)} p_{2n-(i-1),1} \quad i = 1, \ldots, n$$

$$m_{i,j} = p_{i,j} + (-1)^{n-(i-1)} p_{2n-(i-1),j} + (-1)^{-(n-j+1)} p_{i,2n-j+2} + p_{2n-(i-1),2n-j+2}$$

$$i = 1, \ldots, n; j = 2, \ldots, n.$$

Since P_j, $P_j = (p_{1,j}, \ldots, p_{2n,j})^T$, are linearly independent, it follows that the columns of the $n \times n$ matrix M are also linearly independent. Hence,

$$M\vec{y} = 0$$

implies

$$\vec{y} = 0,$$

that is, $x_1 = x_2 = \ldots = x_n = 0$. But from equation (3.32)

$$x_{2n-(j-1)} = (-1)^{-n+j+2} x_j \quad j = 1, \ldots, n.$$

Thus, $x_{n+1} = \ldots = x_{2n} = 0$ and $\vec{x} = \bar{0}$. Therefore, $null(H) = \emptyset$. ∎

Theorem 3.1 *Consider the linear nonhomogeneous system of differential equations*

$$\vec{x}' = A(t)\vec{x} + \vec{b}(t) \qquad (3.33)$$

where $A(t) \in L_1(M)$ *and* $b(t) \in L_1$. *Let* $T : C \to \mathbb{R}^n$ *be continuous and linear. Further, let the solutions of (3.33) satisfy*

$$T\vec{x}(t) = r, \tag{3.34}$$

for any given $r \in \mathbb{R}^n$. *Then the BVP (3.33), (3.34) has a unique solution for every* $r \in \mathbb{R}^n$ *and every* $\vec{b}(t) \in L_1$ *if and only if the corresponding homogeneous linear BVP*

$$\vec{x}' = A(t)\vec{x} \tag{3.35}$$

$$T\vec{x}(t) = 0 \tag{3.36}$$

has only the trivial solution $\vec{x}(t) = 0$.

Proof : The proof of this Theorem can be found in Bernfeld/Lakshmikantham [8].

■

We now state the following:

Theorem 3.2 *The BVP (3.12), (3.13), (3.14) has a unique solution.*

Proof : The proof follows from Lemma (3.3) and Theorem (3.1). ■

3.2 Determination of the Optimizing Vector $\vec{\lambda}^*$

Let us now determine the vector $\vec{\lambda}^*$ that yields the optimal control $u(t)$. First, we observe that when the conditions of theorem (3.1) hold and F given by equation (3.9), the unique solution of (3.12), (3.13), (3.14) is expressible in the form

$$\begin{aligned}\Psi(t) &= \Phi(t-t_0)H^{-1}\Gamma + \Psi^o(t) \\ &= e^{F(t-t_0)}H^{-1}\Gamma + \int_{t_0}^{t} e^{F(t-s)}\vec{\sigma}(s)\,ds. \end{aligned} \tag{3.37}$$

Thus, from equation (3.3), the optimal control is represented by

$$u(t,\vec{\lambda}^*) = \vec{e_1}^T\Psi(t,\vec{\lambda}^*) \tag{3.38}$$

where the $2n$-vector, $\vec{e}_1$, is the coordinate vector given by $\vec{e_1} = (1,0,\ldots,0,0)^T$. The vector $\vec{\lambda}^*$, (hereafter, denoted simply $\vec{\lambda}$), that yields the optimal control is obtained by applying equation (3.4):

$$e^{-At_f}\vec{x}(t_f) - e^{-At_0}\vec{x}(t_0) = \int_{t_0}^{t_f} e^{-As}\vec{b}u(s,\vec{\lambda})\,ds. \tag{3.39}$$

For ease of notation, let

$$\Lambda(t_0,t_f) = e^{-At_f}\vec{x}(t_f) - e^{-At_0}\vec{x}(t_0). \tag{3.40}$$

Thus

$$\begin{aligned}\Lambda(t_0,t_f) &= \int_{t_0}^{t_f} e^{-As}\vec{b}\vec{e}_1^T\Psi(s)\,ds \\ &= \int_{t_0}^{t_f} e^{-As}\vec{b}\vec{e}_1^T\left[e^{F(s-t_0)}H^{-1}\Gamma + \int_{t_0}^{s} e^{F(s-r)}\vec{\sigma}(r)\,dr\right]ds \\ &= \int_{t_0}^{t_f} e^{-As}\vec{b}\vec{e}_1^T e^{F(s-t_0)}H^{-1}\Gamma + \\ &\quad \int_{t_0}^{t_f} e^{-As}\vec{b}\vec{e}_1^T\left(\int_{t_0}^{s} e^{F(s-r)}\vec{\sigma}(r)\,dr\right)ds \end{aligned} \tag{3.41}$$

Now,

$$\begin{aligned}&\int_{t_0}^{t_f} e^{-As}\vec{b}\left(\vec{e}_1^T\int_{t_0}^{s} e^{F(s-r)}\vec{\sigma}(r)dr\right)ds \\ &= \int_{t_0}^{t_f} e^{-As}\vec{b}\left(\vec{e}_1^T\int_{t_0}^{s} e^{F(s-r)}(-1)^n\vec{e}_{2n}\vec{\lambda}^T e^{-Ar}\vec{b}\,dr\right)ds\end{aligned}$$

$$\begin{aligned}
&= \int_0^h e^{-A(s'+t_0)}\vec{b}\left(\vec{e}_1^T e^{Fs'} \int_0^{s'} e^{-Fr'}\vec{e}_{2n}(-1)^n \vec{\lambda}^T e^{-A(t_0+r')}\vec{b}\,dr'\right) ds' \\
&= e^{-At_0}(-1)^n \int_0^h e^{-As'}\vec{b}\left(\vec{e}_1^T e^{Fs'} \int_0^{s'} e^{-Fr'}\vec{e}_{2n}\vec{\lambda}^T e^{-A(t_0+r')}\vec{b}\,dr'\right) ds' \\
&= e^{-At_0}(-1)^n \int_0^h e^{-As'}\vec{b}\left(\vec{e}_1^T e^{Fs'} \int_0^{s'} e^{-Fr'}\vec{e}_{2n}\vec{b}^T e^{-A^T r'}\,dr'\right) ds'\, e^{-A^T t_0}\vec{\lambda} \\
&= e^{-At_0}G(h)e^{-A^T t_0}\vec{\lambda} \qquad (3.42)
\end{aligned}$$

where

$$G(h) = (-1)^n \int_0^h e^{-As'}\vec{b}\left(\vec{e}_1^T e^{Fs'} \int_0^{s'} e^{-Fr'}\vec{e}_{2n}\vec{b}^T e^{-A^T r'}\,dr'\right) ds' \qquad (3.43)$$

$h = t_f - t_0$; $r = t_0 + r'$, and $s = t_0 + s'$. Similarly,

$$\begin{aligned}
\int_{t_0}^{t_f} e^{-As}\vec{b}\left(\vec{e}_1^T e^{F(s-t_0)}H^{-1}\Gamma\right) ds &= \int_0^h e^{-A(t_0+s')}\vec{b}\left(\vec{e}_1^T e^{Fs'}H^{-1}\Gamma\right) ds \\
&= \left(\int_0^h e^{-A(t_0+s')}\vec{b}\left(\vec{e}_1^T e^{Fs'}\right) ds'\right) H^{-1}\Gamma \\
&= e^{-At_0}\left(\int_0^h e^{-As'}\vec{b}\left(\vec{e}_1^T e^{Fs'}\right) ds'\right) H^{-1}\Gamma \\
&= e^{-At_0}K(h)H^{-1}\Gamma \\
&= e^{-At_0}K(h)W_2\Gamma_1 \qquad (3.44)
\end{aligned}$$

where

$$K(h) = \int_0^h e^{-As'}\vec{b}\left(\vec{e}_1^T e^{Fs'}\right) ds' \qquad (3.45)$$

and W_2 is the $2n \times n$ submatrix of H^{-1} given by $[W_1 : W_2] = H^{-1}$. Now, $\Gamma = (\vec{0}, \Gamma_1)^T$ and the n-vector Γ_1 is, (from 3.19), given by

$$\begin{aligned}
\Gamma_1 &= -Be^{Ft_f}\int_{t_0}^{t_f} e^{-Fs}\vec{\sigma}(s)\,ds \\
&= -Be^{Ft_f}\int_0^h e^{-F(t_0+s')}\vec{\sigma}(t_0+s')\,ds' \\
&= -Be^{Fh}\int_0^h e^{-Fs'}(-1)^n\vec{e}_{2n}\vec{\lambda}^T e^{-A(t_0+s')}\vec{b}\,ds \\
&= (-1)^{n+1}Be^{Fh}\left(\int_0^h e^{-Fs'}\vec{e}_{2n}\vec{b}^T e^{-A^T s'}\,ds'\right) e^{-A^T t_0}\vec{\lambda} \\
&= (-1)^{n+1}B\int_0^h e^{F(h-s')}\vec{e}_{2n}\vec{b}^T e^{-A^T s'}\,ds\, e^{-A^T t_0}\vec{\lambda} \\
&= B\Omega(h)e^{-A^T t_0}\vec{\lambda} \qquad (3.46)
\end{aligned}$$

where

$$\Omega(h) = (-1)^{n+1} \int_0^h e^{F(h-s')} \vec{e}_{2n} \vec{b}^T e^{-A^T s'} ds' \tag{3.47}$$

is a $2n \times m$ constant matrix. Substituting (3.46) in (3.44), we obtain

$$\int_{t_0}^{t_f} e^{-As} \vec{b} \left(\vec{e}_1^T e^{Fs} H^{-1} \Gamma \right) ds = e^{-At_0} K(h) \eta(h) e^{-A^T t_0} \vec{\lambda} \tag{3.48}$$

where

$$\eta(h) = W_2 B \Omega(h). \tag{3.49}$$

Thus, equation (3.41) simplifies to

$$e^{-At_0} G(h) e^{-A^T t_0} \vec{\lambda} + e^{-At_0} K(h) \eta(h) e^{-A^T t_0} \vec{\lambda} = e^{-At_f} \vec{x}(t_f) - e^{-At_0} \vec{x}(t_0). \tag{3.50}$$

That is

$$e^{-At_0} [G(h) + K(h)\eta(h)] e^{-A^T t_0} \vec{\lambda} = [e^{-At_f} \vec{x}(t_f) - e^{-At_0} \vec{x}(t_0)]. \tag{3.51}$$

From equation (3.51), we obtain

$$e^{-At_0} C e^{-A^T t_0} \vec{\lambda}(t_0, t_f) = e^{-At_f} \vec{x}(t_f) - e^{-At_0} \vec{x}(t_0) \tag{3.52}$$

where

$$C = (G(h) + K(h)\eta(h)). \tag{3.53}$$

Equation (3.52) is a system of linear equations in λ_i, $i = 1, 2, \ldots, m$. Hence, the optimal control law is obtained by substituting the solution $\vec{\lambda}$ of (3.52) into (3.38).

Theorem 3.3 *The system (3.52) of linear equations has a unique solution. Furthermore, the solution of (3.52) is given by* $\vec{\lambda} = (e^{-At_0} C e^{-A^T t_0})^{-1} \vec{\Lambda}(t_0, t_f)$.

Proof : This follows from the existence of the optimal control law.

The preceding analysis leads to the following:

Theorem 3.4 *When the system (2.11) is controllable, the control that drives the system from* $\vec{x}(t_0)$ *to* $\vec{x}(t_f)$ *and minimizes the cost function* $J(u) = \int_{t_0}^{t_f} \sum_{k=0}^{n} (u^k)^2 \, ds$ *is given by*

$$u(t, \vec{\lambda}) = \vec{e}_1^T \left(e^{F(t-t_0)} H^{-1} \Gamma + \int_{t_0}^{t} e^{F(t-s)} \vec{\sigma}(s) \, ds \right) \tag{3.54}$$

where $t \in [t_0, t_f]$, $\vec{\sigma}(s) = (-1)^n \vec{e_{2n}} \vec{\lambda}^T e^{-As} \vec{b}$, *and* F, H, Γ *are as given in (3.9) and (3.10).*

Theorem 3.5 *The following are equivalent:*

1. *There exists a control* $u(t, \vec{\lambda})$ *such that*

$$x(T) = e^{AT}x(0) + \int_0^T e^{A(T-s)} b u(s, \lambda)\, ds$$

is satisfied for all $x(T)$*,* $x(0)$ *and all* $T > 0$.

2. *The BVP*

$$\begin{cases} \frac{d}{dt}\vec{\psi}(t) = F\vec{\psi} + \vec{e}_{2n}\sigma \\ B\vec{\psi}(0) = 0 \\ B\vec{\psi}(T) = 0 \end{cases}$$

has a unique solution. Here, $\vec{\psi} = (u, u^{(1)}, \ldots, u^{(2n)})^T$, $\sigma = (-1)^n \vec{b}^T e^{-A^T t}\vec{\lambda}$, *and* F *and* B *are as in equations (3.9) and (3.10).*

3. *The matrix*[2]

$$C = G(h) + K(h)\eta(h)$$

is nonsingular.

[2]From equations (3.43), (3.45), and (3.49),

$$G(h)+K(h)\eta(h) = (-1)^n \int_{t_0}^{t_f} e^{-As}\vec{b}\vec{e}_1^T e^{Fs} \left(\int_{t_0}^{s} e^{-F\tau}\vec{e}_{2n}\vec{b}^T e^{-A^T\tau}\, d\tau - W_2 B \int_{t_0}^{t_f} e^{F(h-p)} \vec{e}_{2n}\vec{b}^T e^{-A^T p}\, dp \right) ds$$

CHAPTER IV

APPLICATION OF CONTROL THEORY TO SPLINE APPROXIMATION

In this section, we describe a procedure for constructing spline functions from control principles.

4.1 Splines and Control Theory

Theorem (3.2) implies that the optimal control law for the system (2.11) is unique and this control element is given by equation (3.54). Since a control law for the system exists, there exists a set of points $\vec{x}^1, \ldots, \vec{x}^{p-1}$ with $x_1^i = \alpha_i$, $i = 0, 1, \ldots, p$ such that the solution of the system (2.11) satisfies $\vec{x}(t_i) = \vec{x}^i$, $i = 0, 1, \ldots, p-1, p$. By theorem (3.4), the control element that drives the system (2.11) from $\vec{x}^{i-1}$ to $\vec{x}^i$, $i = 1, \ldots, p$, and satisfies equation (2.13) is given by

$$u(t)|_{[t_{i-1}, t_i]} = u_i(t), \tag{4.1}$$

where $u_i(t)$ is the restriction of $u(t)$ on $[t_{i-1}, t_i]$. Now, we need to determine the unknowns $\vec{x}^i$, $i = 0, 1, \ldots, p$. However, since $x_1^i = \alpha_i$, $i = 0, 1, \ldots, p$, we only have $(m-1)(p+1)$ unknowns to determine. This is realized from the $(m-1)(p-1)$ continuity conditions on the control $u(t)$, namely,

$$u_i^{(r)}(t_i) = u_{i+1}^{(r)}(t_i), \qquad r = 0, \ldots, m-2, \qquad i = 1, \ldots, p-1 \tag{4.2}$$

and $2(m-1)$ conditions at $t = 0$ and at $t = T$. Now, from equation (3.54),

$$u(t) = \vec{e}_1^T \left(e^{F(t-t_0)} H^{-1} \Gamma + e^{Ft} \int_{t_0}^{t} e^{-Fs} \vec{\sigma}(s)\, ds \right) \tag{4.3}$$

$$u^{(r)}(t) = \vec{e}_1^T \left(e^{F(t-t_0)} F^r H^{-1} \Gamma + e^{Ft} F^r \int_{t_0}^{t} e^{-Fs} \vec{\sigma}(s)\, ds + \sum_{j=0}^{r-1} F^j \vec{\sigma}^{(r-1-j)}(t) \right) \tag{4.4}$$

$$r = 0, \ldots, m-2.$$

Thus,

$$u_i^{(r)}(t) = \vec{e}_1^T \left(e^{F(t-t_{i-1})} F^r H_i^{-1} \Gamma_i + e^{Ft} F^r \int_{t_{i-1}}^{t} e^{-Fs} \vec{\sigma}_i(s)\, ds + \sum_{j=0}^{r-1} F^j \vec{\sigma}_i^{(r-1-j)}(t) \right) \tag{4.5}$$

$$r = 0, \ldots, m-2, \qquad t \in [t_{i-1}, t_i].$$

Similarly,

$$u_{i+1}^{(r)}(t) = \vec{e}_1^T \left(e^{F(t-t_i)} F^r H_{i+1}^{-1} \Gamma_{i+1} + e^{Ft} F^r \int_{t_i}^{t} e^{-Fs} \vec{\sigma}_{i+1}(s)\, ds + \sum_{j=o}^{r-1} F^j \vec{\sigma}_{i+1}^{(r-1-j)}(t) \right) \tag{4.6}$$

$$r = 0, \ldots, m-2, \qquad t \in [t_i, t_{i+1}]$$

where, from equation (3.6)

$$\vec{\sigma}_i(t) = (-1)^n \vec{e}_{2n} \vec{\lambda}^T e^{-At} \vec{b} \tag{4.7}$$

and from equations (3.17) and (3.18)

$$H_i = \begin{bmatrix} B\Phi(t_{i-1}) \\ B\Phi(t_i) \end{bmatrix} \tag{4.8}$$

and $\Phi(t) = e^{F(t-t_{i-1})}$. Thus,

$$u_i^{(r)}(t_i) = \vec{e}_1^T \left(e^{Fh_i} F^r H_i^{-1} \Gamma(t_i) + F^r \int_{t_{i-1}}^{t_i} e^{F(t_i - s)} \vec{\sigma}_i(s)\, ds + \sum_{j=0}^{r-1} F^r \vec{\sigma}_i^{(r-1-j)}(t_i) \right) \tag{4.9}$$

$$u_{i+1}^{(r)}(t_i) = \vec{e}_1^T \left(F^r H_{i+1}^{-1} \Gamma(t_{i+1}) + \sum_{j=0}^{r-1} F^r \vec{\sigma}_{i+1}^{(r-1-j)}(t_i) \right) \tag{4.10}$$

Substituting equations (4.9) and (4.10) in (4.2), we have

$$\begin{aligned} \vec{e}_1^T \left(e^{Fh_i} F^r H_i^{-1} \Gamma(t_i) + F^r \int_{t_{i-1}}^{t_i} e^{F(t_i - s)} \vec{\sigma}_i(s)\, ds + \sum_{j=0}^{r-1} F^r \vec{\sigma}_i^{(r-1-j)}(t_i) \right) &= \\ \vec{e}_1^T \left(F^r H_{i+1}^{-1} \Gamma(t_{i+1}) + \sum_{j=0}^{r-1} F^r \vec{\sigma}_{i+1}^{(r-1-j)}(t_i) \right) & \end{aligned} \tag{4.11}$$

$$r = 0, \ldots, m-2, \qquad i = 1, \ldots, p-1$$

The first term on the right side of equation (4.9) is:

$$\begin{aligned} \vec{e}_1^T e^{Fh_i} F^r H_i^{-1} \Gamma(t_i) &= \vec{e}_1^T e^{Fh_i} F^r H_i^{-1} \Gamma(t_i) \\ &= \vec{e}_1^T e^{Fh_i} F^r H_i^{-1} \begin{pmatrix} \vec{0} \\ \Gamma_1(t_i) \end{pmatrix} \end{aligned} \tag{4.12}$$

If we set

$$H_i^{-1} = (H_i' \vdots H_i'') \tag{4.13}$$

where H_i', H_i'' are $2n \times n$ submatrices of H_i^{-1}, we obtain

$$\begin{aligned}
\vec{e}_1^T e^{Fh_i} F^r H_i^{-1} \Gamma(t_i) &= \vec{e}_1^T e^{Fh_i} F^r (H_i' \vdots H_i'') \begin{pmatrix} \vec{0} \\ B\Omega(t_{i-1}, t_i)\vec{\lambda}(t_i) \end{pmatrix} \\
&= \vec{e}_1^T e^{Fh_i} F^r H_i'' B\Omega(h_i) e^{-A^T t_{i-1}} \vec{\lambda}(t_{i-1}, t_i) \qquad (4.14)
\end{aligned}$$

The second term in equation (4.9) is:

$$\begin{aligned}
\vec{e}_1^T F^r e^{Ft_i} \int_{t_{i-1}}^{t_i} e^{-Fs} \vec{\sigma}_i(s)\, ds &= \vec{e}_1^T F^r e^{Ft_i} \int_{t_{i-1}}^{t_i} e^{-Fs} \vec{\sigma}_i(s)\, ds \\
&= \vec{e}_1^T F^r e^{Ft_i} \int_{t_{i-1}}^{t_i} e^{-Fs} \vec{e}_{2n} (-1)^n \vec{b}^T e^{-A^T s} \vec{\lambda}(t_{i-1}, t_i)\, ds \\
&= (-1)^n \vec{e}_1^T F^r e^{Ft_i} \left(\int_{t_{i-1}}^{t_i} e^{-Fs} \vec{e}_{2n} \vec{b}^T e^{-A^T s}\, ds \right) \vec{\lambda}(t_{i-1}, t_i) \\
&= (-1)^n \vec{e}_1^T F^r e^{Fh} \left(\int_0^h e^{-Fs'} \vec{e}_{2n} \vec{b}^T e^{-A^T s'}\, ds' \right) e^{-A^T t_{i-1}} \vec{\lambda}(t_{i-1}, t_i) \\
&= -\vec{e}_1^T F^r \Omega(h_i) e^{-A^T t_{i-1}} \vec{\lambda}(t_{i-1}, t_i) \qquad (4.15)
\end{aligned}$$

Finally, the third term in equation (4.9) is:

$$\begin{aligned}
\vec{e}_1^T \sum_{j=0}^{r-1} F^j \vec{\sigma}_i^{(r-1-j)}(t_i) &= \vec{e}_1^T \sum_{j=0}^{r-1} F^j \vec{\sigma}_i^{(r-1-j)}(t_i) \\
&= (-1)^{n+r-1} \vec{e}_1^T \sum_{j=0}^{r-1} (-1)^j F^j \vec{e}_{2n} \vec{b}^T (A^{r-1-j})^T \lambda(t_i) \qquad (4.16)
\end{aligned}$$

On substituting equations (4.12), (4.15), and (4.16) into equation (4.11) we obtain

$$\begin{aligned}
&\vec{e}_1^T [e^{Fh_i} F^r H_i'' B_i \Omega(h_i) e^{-A^T t_{i-1}} - \\
&\quad F^r \Omega(h_i) e^{-A^T t_{i-1}} + (-1)^{n+r-1} \sum_{j=0}^{r-1} (-1)^j F^j \vec{e}_{2n} \vec{b}^T (A^{r-1-j})^T] \vec{\lambda}^i = \\
&\quad \vec{e}_1^T [F^r H_{i+1}'' B_{i+1} \Omega(h_{i+1}) e^{-A^T t_i} + \\
&\quad (-1)^{n+r-1} \sum_{j=0}^{r-1} (-1)^j F^j \vec{e}_{2n} \vec{b}^T (A^{r-1-j})^T] \vec{\lambda}^{i+1} \qquad (4.17)
\end{aligned}$$

From equation (3.52),

$$\vec{\lambda}^i = e^{A^T t_{i-1}} C^{-1}(h_i)[e^{-Ah_i}\vec{x}^i - \vec{x}^{i-1}]$$

Thus, equation (4.17) simplifies to

$$\begin{aligned}
&-\vec{e}_1^T M_i(h_i) e^{A^T t_{i-1}} C^{-1}(h_i)\vec{x}^{i-1} + \\
&\quad \vec{e}_1^T[M_i(h_i)e^{A^T t_{i-1}} C^{-1}(h_i)e^{-Ah_i} + M_{i+1}(h_{i+1})e^{A^T t_i} C^{-1}(h_{i+1})]\vec{x}^i - \\
&\quad \vec{e}_1^T M_{i+1}(h_{i+1}) e^{A^T t_i} C^{-1}(h_{i+1}) e^{-Ah_{i+1}} \vec{x}^{i+1} = 0
\end{aligned} \tag{4.18}$$

where

$$h_i = t_i - t_{i-1}$$

$$M_i = e^{Fh_i} F^r H_i'' B_i \Omega(h_i) e^{-A^T t_{i-1}} - F^r \Omega(h_i) e^{-A^T t_{i-1}} + (-1)^{n+r-1} \sum_{j=0}^{r-1} (-1)^j F^j \vec{e}_{2n} \vec{b}^T (A^{r-1-j})^T \tag{4.19}$$

and

$$M_{i+1} = F^r H_{i+1}'' B_{i+1} \Omega(h_{i+1}) e^{-A^T t_i} + (-1)^{n+r-1} \sum_{j=0}^{r-1} (-1)^j F^j \vec{e}_{2n} \vec{b}^T (A^{r-1-j})^T \tag{4.20}$$

$$i = 1, \ldots, p-1, \qquad r = 0,\ 1,\ \ldots m-2.$$

Therefore, we can obtain the unknowns $\vec{x}^1, \ldots, \vec{x}^{p-1}$ by solving the linear system (4.18). This enables us to define the control $u(t)$ piecewise on the interval $[0, T]$. Further, the solution of the system (2.11) is given by

$$\vec{x}(t) = e^{At}\vec{x}^0 + \int_0^t e^{A(t-s)} b u(s)\, ds \tag{4.21}$$

Now, from the structure of the state matrix A, we observe that $x_i'(t) = x_{i+1}(t), i = 1, \ldots, m-1$. Thus, the continuity of $x_i'(t)$ implies the continuity of $x_{i+1}(t)$ for $i = 1, \ldots, m-1$. Also, the continuity of $u^{(r)}(t)$ implies the continuity of $x_1^{(m+r)}(t), r = 0, \ldots, m-2$. This leads to the conclusion that the observation function $y(t) = x_1(t)$ is an element of $C^{2m-2}[0, T]$ and satisfies the boundary conditions

$$y^{(r)}(0) = x_{r+1}^0, \qquad y^{(r)}(T) = x_{r+1}^T, \qquad r = 0, \ldots, m-1 \tag{4.22}$$

and

$$y(t_i) = x_1(t_i), \qquad i = 1, \ldots, p-1. \tag{4.23}$$

Thus, $y(t)$ is a spline function and the above discussion proves the following:

Theorem 4.1 *Let $\vec{x}(t)$ be the solution of the system (2.11), (2.12), and (2.15) with $x_1(t_i) = \alpha_i$, $i = 0, \ldots, p$. Then there exists a unique function $y(t) \in C^{2m-2}[0,T]$ that satisfies equations (4.22) and (4.23).*

4.2 Classification of Splines

The type of spline is determined by a set of basis functions. By control principles, we can construct these basis functions. Now, suppose the interval $[0,T]$ is subdivided into p subintervals. In this analysis, it suffices to consider just one subinterval to determine the kind of interpolation functions of the state $\vec{x}(t)$. Thus, on the subinterval $[t_0, t_1]$,

$$\vec{x}(t) = e^{A(t-t_0)}\vec{x}(t_0) + \int_{t_0}^{t} e^{A(t-s)}\vec{b}u(s,\vec{\lambda})ds \tag{4.24}$$

The control law is given by Theorem (3.4):

$$u(t,\vec{\lambda}) = \vec{e}_1^T \left(e^{F(t-t_0)}H^{-1}\Gamma + \int_{t_0}^{t} e^{F(t-s)}\vec{\sigma}(s,\vec{\lambda})\, ds \right) \tag{4.25}$$

From equations (3.44) and (3.46)

$$G(t) = (-1)^n \int_0^{t-t_0} e^{-As'}\vec{b} \left(\int_0^{s'} \vec{e}_1^T e^{F(s'-r')}\vec{e}_{2n}\vec{b}^T e^{-A^T r'}\, dr' \right) ds' \tag{4.26}$$

$$K(t) = \int_0^{t-t_0} e^{-As'}\vec{b}\left(\vec{e}_1^T e^{Fs'}\right) ds' \tag{4.27}$$

and

$$C(t) = K(t)\eta(h) + G(t) \tag{4.28}$$

If we substitute equation (4.25) into (4.24), we obtain

$$\begin{aligned}
\vec{x}(t) &= e^{A(t-t_0)}\vec{x}(t_0) + \int_{t_0}^{t} e^{A(t-s)}\vec{b}\left(\vec{e}_1^T\left(e^{F(t-t_0)}H^{-1}\Gamma + \int_{t_0}^{s} e^{F(t-r)}\vec{\sigma}(r)\, dr\right)\right) ds \\
&= e^{A(t-t_0)}\left[\vec{x}^0 + K(t)W_2\eta(h)e^{-A^T t_0}\vec{\lambda} + G(t)e^{-A^T t_0}\vec{\lambda}\right] \\
&= e^{A(t-t_0)}\left[\vec{x}^0 + (K(t)\eta(h) + G(t))e^{-A^T t_0}e^{A^T t_0}C^{-1}e^{At_0}(e^{-At_f}\vec{x}^1 - e^{-At_0}\vec{x}^0)\right] \\
&= e^{A(t-t_0)}\left[\vec{x}^0 + C(t)C^{-1}(h)e^{At_0}(e^{-At_f}\vec{x}^1 - e^{-At_0}\vec{x}^0)\right] \\
&= e^{A(t-t_0)}\left[\vec{x}^0 + C(t)C^{-1}(h)(e^{-Ah}\vec{x}^1 - \vec{x}^0)\right]
\end{aligned} \tag{4.29}$$

Now, since the matrix F has linearly independent eigenvectors $(p_1, p_2, \ldots, p_{2n})$, then F is similar to a diagonal matrix $D = diag(\beta_1, \ldots, \beta_{2n})$ [19]; that is, there exists a nonsingular matrix P such that

$$F = PDP^{-1} \tag{4.30}$$

Let

$$P = \begin{pmatrix} p_{1,1} & p_{1,2} & \cdots & p_{1,2n} \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ p_{2n,1} & p_{2n,2} & \cdots & p_{2n,2n} \end{pmatrix}, \quad P^{-1} = \begin{pmatrix} p'_{1,1} & \cdots & \cdots & p'_{1,2n} \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ p'_{2n,1} & \cdots & \cdots & p'_{2n,2n} \end{pmatrix}.$$

Here, the columns of P are the eigenvectors p_k, $k = 1, 2, \ldots, 2n$, of the matrix F. Let E be the Jordan canonical form of the matrix A. Then $A = QEQ^{-1}$, where Q is the matrix such that $AQ = QE$. Thus, further analysis is simplified by replacing the matrix F with PDP^{-1} and the state matrix A with its Jordan matrix E. Hence, we may write

$$\vec{x}(t) = Qe^{E(t-t_0)}Q^{-1}\left[\vec{x}^0 + C(t)C^{-1}(h)(Qe^{-Eh}Q^{-1}\vec{x}^1 - \vec{x}^0)\right]. \tag{4.31}$$

Let the first row of the matrix $Qe^{E(t-t_0)}Q^{-1}(I - C(t)C^{-1}(h))$ be $(\phi_1(t), \ldots, \phi_m(t))$, and the first row of the matrix $Qe^{E(t-t_0)}Q^{-1}C(t)C^{-1}(h)Qe^{-Eh}Q^{-1}$ be $(\psi_1(t), \ldots, \psi_m(t))$. Now, the system (2.11) is controllable since $rank(\, b\, Ab\, \ldots\, A^{m-1}b\,) = m$. The output of the system is

$$\begin{aligned} y(t) &= \vec{e}_1^{\,T}\vec{x}(t) \\ &= \vec{e}_1^{\,T}(e^{A(t-t_0)}\vec{x}^0 + \int_{t_0}^{t} e^{A(t-s)}\vec{b}u(s)\,ds) \\ &= Qe^{E(t-t_0)}Q^{-1}\left[\vec{x}^0 + C(t)C^{-1}(h)(Qe^{-Eh}Q^{-1}\vec{x}^1 - \vec{x}^0)\right] \\ &= (\phi_1(t), \ldots, \phi_m(t))\vec{x}^0 + (\psi_1(t), \ldots, \psi_m(t))\vec{x}^1 \\ &= \sum_{k=1}^{m} x_k^0\phi_k(t) + \sum_{l=1}^{m} x_l^1\psi_l(t). \end{aligned} \tag{4.32}$$

By choosing $\vec{x}^0 = \vec{e}_k$ and $\vec{x}^1 = \vec{0}$, we obtain $\phi_k^{(r)} = y^{(r)}(t), \quad r = 0, \ldots, m-1$. Thus, for $r = 0, \ldots, m-1$,

$$\phi_k^{(r)}(0) = y^{(r)}(0) = x_1^{(r)}(0) = x_{r+1}(0) = x_{r+1}^0 = \delta_{k,r+1}$$

$$\phi_k^{(r)}(h) = y^{(r)}(h) = x_1^{(r)}(h) = x_{r+1}(h) = x_{r+1}^1 = 0,$$
$$k = 1, \ldots, m.$$

In a similar manner, if we choose $\vec{x}^0 = \vec{0}$ and $\vec{x}^1 = \vec{e}_j$, then we obtain

$$\psi_j^{(r)}(t) = y^{(r)}(t), \qquad r = 1, \ldots, m.$$

Hence,

$$\psi_j^{(r)}(0) = y^{(r)} = 0$$
$$\psi_j^{(r)}(h) = y^{(r)}(h) = \delta_{j,r+1}$$
$$j = 1, \ldots, m.$$

The above discussion proves the following theorem.

Theorem 4.2 *Let A be given by equation (2.12). Furthermore, let the first row of the matrix $Qe^{E(t-t_0)}Q^{-1}(I - C(t)C^{-1}(h))$ be $(\phi_1(t), \ldots, \phi_m(t))$, and the first row of the matrix $Qe^{E(t-t_0)}Q^{-1}C(t)C^{-1}(h)Qe^{-Eh}Q^{-1}$ be $(\psi_1(t), \ldots, \psi_m(t))$. Then, for $r = 0, \ldots, m$,*
$\phi_k^{(r)}(0) = \delta_{k,r+1}, \quad \phi_k^{(r)}(h) = 0, \quad k = 1, \ldots, m$
$\psi_l^{(r)}(t) = 0, \quad \psi_l^{(r)}(h) = \delta_{l,r+1}, \quad l = 1, \ldots, m.$ ■

Therefore, ϕ_k, ψ_k, $k = 1, \ldots, m$ are basis functions for $y(t)$. These basis functions depend on the entries of the matrices e^{Et}, $K(t)$, and $G(t)$. Hence, we can determine the type of spline function by carefully examining the entries of these matrices.

Proposition 4.1 *Let the state matrix A be nilpotent of order m and the cost function $J(u) = \int_0^T \sum_{k=0}^n (u^{(k)}(s))^2\,ds$. Then $y(t)$ is a polynomial spline if and only if $n = 0$.*

Proof. To prove this proposition, we observe that the spline function $y(t)$ is expressed in terms of the basis functions ϕ_k and ψ_k, $k = 1, \ldots, m$, equation (4.31). However, the basis functions themselves are dependent on the entries of the matrices e^{Et} and e^{Dt}, where E and D are the Jordan forms of A and F, respectively. Now, if A is nilpotent, as in the proposition, then A is already a Jordan matrix and hence the entries of e^{Et} are all powers of t. It then follows that the spline function $y(t)$, which is a linear combination of the entries of e^{Et} and e^{Dt} is a polynomial if and only if the entries of e^{Dt} are all powers of t. This is the case only if $n = 0$. ∎

From theorem (4.2), we see that the basis functions ϕ_i, and ψ_i are determined by the matrices e^{Et}, and $e^{Et}Q^{-1}C(t)$. But these matrices are themselves characterized by the spectrum of the state matrix A, and the matrix F. Therefore, we will classify the spline functions obtained by control principles by examining the entries of A, and F. In this classification, we consider the case where the state matrix has dimension 2 and the cost function $J(u) = \int_0^T ((u(s))^2 + (u'(s))^2)\, ds$. For higher dimensions and derivatives of higher orders, the procedure is similar but more involved. So let

$$A = \begin{pmatrix} 0 & 1 \\ \alpha_1 & 2\alpha_2 \end{pmatrix}, \qquad \alpha_1, \alpha_2 \in \mathbb{R}^1$$

$$\vec{b} = (0\,1)^T.$$

Then the eigenvalues of A are

$$\xi_1 = \alpha_2 + \sqrt{\alpha_2^2 + \alpha_1}$$

$$\xi_2 = \alpha_2 - \sqrt{\alpha_2^2 + \alpha_1}.$$

Thus the Jordan form of A and the transformation matrix Q are as follows:

$$E = \begin{pmatrix} \xi_1 & 0 \\ 0 & \xi_2 \end{pmatrix},$$

$$Q = \begin{pmatrix} 1 & 1 \\ \xi_1 & \xi_2 \end{pmatrix}, \qquad Q^{-1} = \frac{1}{\xi_2 - \xi_1} \begin{pmatrix} \xi_2 & -1 \\ -\xi_1 & 1 \end{pmatrix}.$$

Since $J(u) = \int_0^T ((u(s))^2 + (u'(s))^2)\, ds$, then, from equation (3.22),

$$F = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

with eigenvalues and eigenvectors

$$\beta_1 = 1, \qquad \beta_2 = -1, \qquad p_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad p_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Thus

$$P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \qquad P^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

$$D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad e^{Dt} = \begin{pmatrix} e^{t} & 0 \\ 0 & e^{-t} \end{pmatrix}.$$

Also, from equation (3.10),

$$B = (0\,1)$$

and on the subinterval $[t_{i-1}, t_i]$,

$$\phi(t) = e^{F(t-t_{i-1})} = \begin{pmatrix} \cosh(t-t_{i-1}) & \sinh(t-t_{i-1}) \\ \sinh(t-t_{i-1}) & \cosh(t-t_{i-1}) \end{pmatrix}.$$

Thus,

$$B\phi(t) = \frac{1}{2}\left(\sinh(t-t_{i-1})\ \cosh(t-t_{i-1})\right)$$

and from equation (3.20), we have

$$H_i = \begin{pmatrix} B\phi(t_{i-1}) \\ B\phi(t_i) \end{pmatrix} \tag{4.33}$$

$$= \begin{pmatrix} 0 & 1 \\ \sinh h_i & \cosh h_i \end{pmatrix} \tag{4.34}$$

$$H_i^{-1} = \frac{-1}{\sinh h_i}\begin{pmatrix} \cosh h_i & -1 \\ -\sinh h_i & 0 \end{pmatrix}$$

where $h_i = t_i - t_{i-1}$. From equation (3.43),

$$\begin{aligned} G(t-t_{i-1}) &= -\int_0^{t-t_{i-1}} e^{-As'}\vec{b}\left(\int_0^{s'} \vec{e}_1^{\,T} e^{F(s'-r')}\vec{e}_2\vec{b}^T e^{-A^T r'}\,dr'\right) ds' \\ &= Q\begin{pmatrix} \hat{g}_{11}(t) & \hat{g}_{12}(t) \\ \hat{g}_{21}(t) & \hat{g}_{22}(t) \end{pmatrix} Q^T \end{aligned}$$

where

$$\hat{g}_{11}(t) = \frac{-1}{2(\xi_2-\xi_1)}\left\{\frac{-2(t-t_{i-1})}{1-\xi_1^2} + \frac{e^{(1-\xi_1)(t-t_{i-1})}-1}{(1-\xi_1)^2} - \frac{e^{-(1+\xi_1)(t-t_{i-1})}-1}{(1+\xi_1)^2}\right\}$$

$$\hat{g}_{12}(t) = \frac{-1}{2(\xi_2-\xi_1)}\left\{\frac{2(e^{(\xi_2-\xi_1)(t-t_{i-1})}-1)}{(1-\xi_2^2)(\xi_2-\xi_1)} - \frac{e^{(1-\xi_1)(t-t_{i-1})}-1}{(1-\xi_1)(1-\xi_2)} + \frac{e^{-(1+\xi_1)(t-t_{i-1})}-1}{(1+\xi_1)(1+\xi_2)}\right\}$$

$$\hat{g}_{21}(t) = \frac{-1}{2(\xi_2-\xi_1)}\left\{\frac{-2(e^{-(\xi_2-\xi_1)(t-t_{i-1})}-1)}{(1-\xi_1^2)(\xi_2-\xi_1)} - \frac{e^{(1-\xi_2)(t-t_{i-1})}-1}{(1-\xi_1)(1-\xi_2)} + \frac{e^{-(1+\xi_2)(t-t_{i-1})}-1}{(1+\xi_1)(1+\xi_2)}\right\}$$

$$\hat{g}_{22}(t) = \frac{-1}{2(\xi_2-\xi_1)}\left\{\frac{-2(t-t_{i-1})}{1-\xi_2^2} + \frac{e^{(1-\xi_2)(t-t_{i-1})}-1}{(1-\xi_2)^2} - \frac{e^{-(1+\xi_2)(t-t_{i-1})}-1}{(1+\xi_2)^2}\right\}.$$

From equation (3.45),

$$\begin{aligned} K(t-t_{i-1}) &= \int_0^{t-t_{i-1}} e^{-As'}\vec{b}\left(\vec{e}_1^{\,T} e^{Fs'}\right) ds' \\ &= Q\begin{pmatrix} \hat{k}_{11}(t) & \hat{k}_{12}(t) \\ \hat{k}_{21}(t) & \hat{k}_{22}(t) \end{pmatrix} \end{aligned}$$

where

$$\hat{k}_{11}(t) = \frac{-1}{2(\xi_2-\xi_1)}\left\{\frac{e^{(1-\xi_1)(t-t_{i-1})}-1}{1-\xi_1} - \frac{e^{-(1+\xi_1)(t-t_{i-1})}-1}{1+\xi_1}\right\}$$

$$\hat{k}_{12}(t) = \frac{-1}{2(\xi_2-\xi_1)}\left\{\frac{e^{(1-\xi_1)(t-t_{i-1})}-1}{1-\xi_1} + \frac{e^{-(1+\xi_1)(t-t_{i-1})}-1}{1+\xi_1}\right\}$$

$$\hat{k}_{21}(t) = \frac{1}{2(\xi_2-\xi_1)}\left\{\frac{e^{(1-\xi_1)(t-t_{i-1})}-1}{1-\xi_1} - \frac{e^{-(1+\xi_1)(t-t_{i-1})}-1}{1+\xi_1}\right\}$$

$$\hat{k}_{22}(t) = \frac{1}{2(\xi_2-\xi_1)}\left\{\frac{e^{(1-\xi_1)(t-t_{i-1})}-1}{1-\xi_1} + \frac{e^{-(1+\xi_1)(t-t_{i-1})}-1}{1+\xi_1}\right\}.$$

Equation (3.47) gives

$$\begin{aligned} \Omega(h_i) &= \int_0^{h_i} e^{F(h_i-s')}\vec{e}_2\vec{b}^{\,T} e^{-A^T s'} ds' \\ &= \begin{pmatrix} \hat{\omega}_{11} & \hat{\omega}_{12} \\ \hat{\omega}_{21} & \hat{\omega}_{22} \end{pmatrix} Q^T \end{aligned}$$

where

$$\hat{\omega}_{11} = -\frac{\xi_1\sinh h_i + \cosh h_i - e^{-\xi_1 h_i}}{(\xi_2-\xi_1)(1-\xi_1^2)}$$

$$\hat{\omega}_{12} = \frac{\xi_2\sinh h_i + \cosh h_i - e^{-\xi_2 h_i}}{(\xi_2-\xi_1)(1-\xi_2^2)}$$

$$\hat{\omega}_{21} = -\frac{\sinh h_i - \xi_1\cosh h_i + \xi_1 e^{-\xi_1} h_i}{(\xi_2-\xi_1)(1-\xi_1^2)}$$

$$\hat{\omega}_{22} = \frac{\sinh h_i - \xi_2 \cosh h_i + \xi_2 e^{-\xi_2 h_i}}{(\xi_2 - \xi_1)(1 - \xi_2^2)}$$

and equation (3.49) gives

$$\begin{aligned} \eta(h) &= H_i'' B \Omega(h_i) \\ &= \begin{pmatrix} \hat{\eta}_{11} & \hat{\eta}_{12} \\ \hat{\eta}_{21} & \hat{\eta}_{22} \end{pmatrix} Q^T \end{aligned} \tag{4.35}$$

where

$$\hat{\eta}_{11} = -\frac{\sinh h_i - \xi_1 \cosh h_i + \xi_1 e^{-\xi_1 h_i}}{(\xi_2 - \xi_1)(1 - \xi_1^2) \sinh h_i}$$

$$\hat{\eta}_{12} = \frac{\sinh h_i - \xi_2 \cosh h_i + \xi_2 e^{-\xi_2 h_i}}{(\xi_2 - \xi_1)(1 - \xi_2^2) \sinh h_i}$$

$$\hat{\eta}_{21} = 0$$

$$\hat{\eta}_{22} = 0.$$

Therefore,

$$K(t)\eta(h_i) = Q \begin{pmatrix} \hat{k}_{11}(t)\hat{\eta}_{11}(h_i) & \hat{k}_{11}(t)\hat{\eta}_{12}(h_i) \\ \hat{k}_{21}(t)\hat{\eta}_{11}(h_i) & \hat{k}_{21}(t)\hat{\eta}_{12}(h_i) \end{pmatrix} Q^T$$

and

$$\begin{aligned} C(t) &= G(t) + K(t)\eta(h_i) \\ &= Q \begin{pmatrix} \hat{g}_{11}(t) + \hat{k}_{11}(t)\hat{\eta}_{11}(h_i) & \hat{g}_{12}(t) + \hat{k}_{11}(t)\hat{\eta}_{12}(h_i) \\ \hat{g}_{21}(t) + \hat{k}_{21}(t)\hat{\eta}_{11}(h_i) & \hat{g}_{22}(t) + \hat{k}_{21}(t)\hat{\eta}_{12}(h_i) \end{pmatrix} Q^T \\ &= \begin{pmatrix} \hat{c}_{11}^i(t) & \hat{c}_{12}^i(t) \\ \hat{c}_{21}^i(t) & \hat{c}_{22}^i(t) \end{pmatrix} \end{aligned} \tag{4.36}$$

where

$$\hat{c}_{11}^i(t) = \hat{g}_{11}(t) + \hat{k}_{11}(t)\hat{\eta}_{11}(h_i)$$

$$\hat{c}_{11}^i(t) = \hat{g}_{12}(t) + \hat{k}_{11}(t)\hat{\eta}_{12}(h_i)$$

$$\hat{c}_{11}^i(t) = \hat{g}_{21}(t) + \hat{k}_{21}(t)\hat{\eta}_{11}(h_i)$$

$$\hat{c}_{11}^i(t) = \hat{g}_{22}(t) + \hat{k}_{21}(t)\hat{\eta}_{12}(h_i)$$

$$
\begin{aligned}
C^{-1}(h_i) &= Q^{-T}\begin{pmatrix} \hat{g}_{22}+\hat{k}_{21}\hat{\eta}_{12} & -(\hat{g}_{12}+\hat{k}_{11}\hat{\eta}_{12} \\ -(\hat{g}_{21}+\hat{k}_{21}\hat{\eta}_{11} & \hat{g}_{11}+\hat{k}_{11}\hat{\eta}_{11} \end{pmatrix} Q^{-1}|C|^{-1} \\
&= \begin{pmatrix} \bar{c}^i_{11}(h_i) & \bar{c}^i_{12}(h_i) \\ \bar{c}^i_{21}(h_i) & \bar{c}^i_{22}(h_i) \end{pmatrix} \qquad (4.37)
\end{aligned}
$$

where

$$
\begin{aligned}
|C| = &(\hat{g}_{11}(h_i)+\hat{k}_{11}(h_i)\hat{\eta}_{11}(h_i))(\hat{g}_{22}(h_i)+\hat{k}_{21}(h_i)\hat{\eta}_{12}(h_i)) - \\
&(\hat{g}_{12}(h_i)+\hat{k}_{11}(h_i)\hat{\eta}_{12}(h_i))(\hat{g}_{21}(h_i)+\hat{k}_{21}(h_i)\hat{\eta}_{11}(h_i)).
\end{aligned}
$$

From equation (4.32),

$$
e^{A(t-t_0)}[I - C(t)C^{-1}(h_i)] = \begin{pmatrix} \phi_1(t) & \phi_2(t) \\ * & * \end{pmatrix} \qquad (4.38)
$$

where

$$
\begin{aligned}
\phi_1(t) = &\frac{1}{\xi_2-\xi_1}\left\{\xi_2 e^{\xi_1(t-t_{i-1})} - \xi_1 e^{\xi_2(t-t_{i-1})}\right\} - \\
&\frac{1}{\xi_2-\xi_1}[\bar{c}_{11}(h_i)\hat{c}_{11}(t)+\bar{c}_{21}(h_i)\hat{c}_{12}(t)]\xi_2 e^{\xi_1(t-t_{i-1})}|C|^{-1} + \\
&\frac{1}{\xi_2-\xi_1}[\bar{c}_{11}(h_i)\hat{c}_{21}(t)+\bar{c}_{21}(h_i)\hat{c}_{22}(t)]\xi_2 e^{\xi_2(t-t_{i-1})}|C|^{-1} - \\
&\frac{1}{\xi_2-\xi_1}[\bar{c}_{12}(h_i)\hat{c}_{11}(t)+\bar{c}_{22}(h_i)\hat{c}_{12}(t)]\xi_1 e^{\xi_1(t-t_{i-1})}|C|^{-1} - \\
&\frac{1}{\xi_2-\xi_1}[\bar{c}_{12}(h_i)\hat{c}_{21}(t)+\bar{c}_{22}(h_i)\hat{c}_{22}(t)]\xi_1 e^{\xi_2(t-t_{i-1})}|C|^{-1} \qquad (4.39)
\end{aligned}
$$

and

$$
\begin{aligned}
\phi_2(t) = &\frac{1}{\xi_2-\xi_1}\left(-e^{\xi_1(t-t_{i-1})} + e^{\xi_2(t-t_{i-1})}\right) + \\
&\frac{1}{\xi_2-\xi_1}[\bar{c}_{11}(h_i)\hat{c}_{11}(t)+\bar{c}_{21}(h_i)\hat{c}_{12}(t)]e^{\xi_1(t-t_{i-1})}|C|^{-1} + \\
&\frac{1}{\xi_2-\xi_1}[\bar{c}_{11}(h_i)\hat{c}_{21}(t)+\bar{c}_{21}(h_i)\hat{c}_{22}(t)]e^{\xi_2(t-t_{i-1})}|C|^{-1} - \\
&\frac{1}{\xi_2-\xi_1}[\bar{c}_{12}(h_i)\hat{c}_{11}(t)+\bar{c}_{22}(h_i)\hat{c}_{12}(t)]e^{\xi_1(t-t_{i-1})}|C|^{-1} - \\
&\frac{1}{\xi_2-\xi_1}[\bar{c}_{12}(h_i)\hat{c}_{21}(t)+\bar{c}_{22}(h_i)\hat{c}_{22}(t)]e^{\xi_2(t-t_{i-1})}|C|^{-1}. \qquad (4.40)
\end{aligned}
$$

Also

$$e^{A(t-t_0)}C(t)C^{-1}(h_i)e^{-Ah_i} = \begin{pmatrix} \psi_1(t) & \psi_2(t) \\ * & * \end{pmatrix} \tag{4.41}$$

where

$$\begin{aligned}\psi_1(t) = {} & \frac{1}{\xi_2-\xi_1}[\bar{c}_{11}(h_i)\hat{c}_{11}(t)+\bar{c}_{21}(h_i)\hat{c}_{12}(t)]\xi_2 e^{\xi_1(t-t_{i-1}-h_i)}|C|^{-1}+ \\ & \frac{1}{\xi_2-\xi_1}[\bar{c}_{11}(h_i)\hat{c}_{21}(t)+\bar{c}_{21}(h_i)\hat{c}_{22}(t)]\xi_2 e^{(\xi_2(t-t_{i-1})-\xi_1 h_i)}|C|^{-1}- \\ & \frac{1}{\xi_2-\xi_1}[\bar{c}_{12}(h_i)\hat{c}_{11}(t)+\bar{c}_{22}(h_i)\hat{c}_{12}(t)]\xi_1 e^{(\xi_1(t-t_{i-1})-\xi_2 h_i)}|C|^{-1}- \\ & \frac{1}{\xi_2-\xi_1}[\bar{c}_{12}(h_i)\hat{c}_{21}(t)+\bar{c}_{22}(h_i)\hat{c}_{22}(t)]\xi_1 e^{\xi_2(t-t_{i-1}-h_i)}|C|^{-1}.\end{aligned} \tag{4.42}$$

and

$$\begin{aligned}\psi_2(t) = {} & -\frac{1}{\xi_2-\xi_1}[\bar{c}_{11}(h_i)\hat{c}_{11}(t)+\bar{c}_{21}(h_i)\hat{c}_{12}(t)]e^{\xi_1(t-t_{i-1}-h_i)}|C|^{-1}- \\ & \frac{1}{\xi_2-\xi_1}[\bar{c}_{11}(h_i)\hat{c}_{21}(t)+\bar{c}_{21}(h_i)\hat{c}_{22}(t)]e^{(\xi_2(t-t_{i-1})-\xi_1 h_i)}|C|^{-1}+ \\ & \frac{1}{\xi_2-\xi_1}[\bar{c}_{12}(h_i)\hat{c}_{11}(t)+\bar{c}_{22}(h_i)\hat{c}_{12}(t)]e^{(\xi_1(t-t_{i-1})-\xi_2 h_i)}|C|^{-1}+ \\ & \frac{1}{\xi_2-\xi_1}[\bar{c}_{12}(h_i)\hat{c}_{21}(t)+\bar{c}_{22}(h_i)\hat{c}_{22}(t)]e^{\xi_2(t-t_{i-1}-h_i)}|C|^{-1}.\end{aligned} \tag{4.43}$$

Now, let us consider the various cases that arise from the various forms of the eigenvalues of the state matrix A.

Case 1: $\alpha_2^2+\alpha_1>0$. In this case the matrix A has two distinct eigenvalues.

Case 1(i): If $\xi_1 \neq \xi_2$, then the spline obtained is an exponential spline with basis functions $\phi_1(t)$, $\phi_2(t)$, $\psi_1(t)$ and $\psi_2(t)$ given by equations (4.39), (4.40), (4.42), and (4.43). Here, $\phi_1(t)$, $\phi_2(t)$, $\psi_1(t)$ and $\psi_2(t)$ are linear combinations of e^t, e^{-t}, $e^{\xi_1 t}$, $te^{\xi_1 t}$, $e^{\xi_2 t}$, and $te^{\xi_2 t}$.

Case 1(ii): If $\alpha_2 = 0$, and $\alpha_1 > 0$, then $\xi_1 = -\xi_2$ and we again obtain an exponential spline. The basis functions are linear combinations of e^t, e^{-t}, $e^{\xi_1 t}$, $e^{-\xi_1 t}$, $te^{\xi_1 t}$, and $te^{-\xi_2 t}$.

Case 1(iii) If $\alpha_1 = 0$, and $\alpha_2 \neq 0$, then $\xi_1 = 0$, and $\xi_2 = -2\alpha_2$ if $\alpha_2 > 0$; $\xi_1 = -2\alpha_2$, and $\xi_2 = 0$ if $\alpha_2 < 0$. The resulting spline function is a linear combination of the functions 1, t, e^t, e^{-t}, $e^{\xi_2 t}$, and $te^{-\xi_2 t}$ if $\alpha_2 > 0$; or 1, t, e^t, e^{-t}, $e^{\xi_1 t}$, and $te^{-\xi_1 t}$ if $\alpha_2 < 0$.

Case 2: $\alpha_2^2 + \alpha_1 \leq 0$. This case leads to two complex eigenvalues:

$$\xi_1 = \alpha_2 + i\omega, \quad \xi_2 = \alpha_2 - i\omega$$

where $\omega = \sqrt{-(\alpha_2^2 + \alpha_1)}$.
Case 2(i): $\alpha_2 \neq 0$, $\alpha_1 < 0$. The resulting spline has basis fucntions which are linear combinations of e^t, e^{-t}, $e^{\alpha_2 t}\text{cos}t$, $e^{\alpha_2 t}\text{sin}t$, $te^{\alpha_2 t}\text{cos}t$, and $te^{\alpha_2 t}\text{sin}t$.
Case 2(ii): $\alpha_2 = 0$, $\alpha_1 < 0$. This gives

$$\xi_1 = i\sqrt{-\alpha_1} = i\omega$$

$$\xi_2 = -i\sqrt{-\alpha_1} = -i\omega$$

and a spline function whose basis functions are linear combinations of 1, t, e^t, e^{-t}, cos$t$ and sin$t$.
Case 3: $\alpha_2^2 + \alpha_1 = 0$. This implies $\xi_1 = \xi_2 = \alpha_2$.
Case 3(i): $\alpha_2 \neq 0$. Then

$$E = \begin{pmatrix} \alpha_2 & 1 \\ 0 & \alpha_2 \end{pmatrix}$$

$$e^{Et} = e^{\alpha_2 t}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

$$Q = \begin{pmatrix} 1 & 0 \\ \alpha_2 & 1 \end{pmatrix}$$

$$Q^{-1} = \begin{pmatrix} 1 & 0 \\ -\alpha_2 & 1 \end{pmatrix}$$

$$\begin{aligned} G(t - t_{i-1}) &= -\int_0^{t-t_{i-1}} e^{-As'}\vec{b}\left(\vec{e}_1^T e^{Fs'} \int_0^{s'} e^{-Fr'}\vec{e}_2\vec{b}^T e^{-A^T r'}\, dr'\right) ds' \\ &= -Q\begin{pmatrix} \hat{g}_{11}(t) & \hat{g}_{12}(t) \\ \hat{g}_{21}(t) & \hat{g}_{22}(t) \end{pmatrix} \end{aligned} \tag{4.44}$$

where

$$\hat{g}_{11}(t) = A_1(t)e^{-2\alpha_2(t-t_{i-1})} + A_2 e^{-\alpha_2(t-t_{i-1})} + A_3$$

$$\hat{g}_{12}(t) = B_1(t)e^{-2\alpha_2(t-t_{i-1})} + B_2(t)e^{-\alpha_2(t-t_{i-1})} + B_3$$

$$\hat{g}_{21}(t) = C_1(t)e^{-2\alpha_2(t-t_{i-1})} + C_2(t)e^{-\alpha_2(t-t_{i-1})} + C_3$$

$$\hat{g}_{22}(t) = D_1(t)e^{-2\alpha_2(t-t_{i-1})} + D_2(t)e^{-\alpha_2(t-t_{i-1})} + D_3$$

where

$$A_1(t) = \frac{(t-t_{i-1})^2}{\alpha_2(1-\alpha_2^2)} + \frac{t-t_{i-1}}{2\alpha_2^2(1-\alpha_2^2)} + \frac{1}{4\alpha_2^2(1-\alpha_2^2)} - \frac{1}{\alpha_2(1-\alpha_2^2)^2}$$

$$A_2(t) = \frac{2}{(1-\alpha_2^2)^2}$$

$$A_3(t) = -\frac{1}{4\alpha_2^3(1-\alpha_2^2)} + \frac{1}{(1-\alpha_2^2)^2}$$

$$B_1(t) = -\frac{t-t_{i-1}}{2\alpha_2(1-\alpha_2^2)} - \frac{1}{4\alpha_2^2(1-\alpha_2^2)}$$

$$B_2(t) = \frac{t-t_{i-1}}{\alpha_2(1-\alpha_2^2)} + \frac{1}{\alpha_2^2(1-\alpha_2^2)}$$

$$B_3(t) = \frac{1}{4\alpha_2^2(1-\alpha_2^2)} + \frac{1}{\alpha_2^2(1-\alpha_2^2)}$$

$$C_1(t) = -\frac{t-t_{i-1}}{2\alpha_2(1-\alpha_2^2)} - \frac{1}{4\alpha_2^2(1-\alpha_2^2)} + \frac{1}{(1-\alpha_2^2)^2}$$

$$C_2(t) = -\frac{2}{(1-\alpha_2^2)^2}$$

$$C_3(t) = \frac{1}{4\alpha_2^2(1-\alpha_2^2)} + \frac{1}{(1-\alpha_2^2)^2}$$

$$D_1(t) = \frac{1}{2\alpha_2(1-\alpha_2^2)}$$

$$D_2(t) = -\frac{1}{\alpha_2(1-\alpha_2^2)}$$

$$D_3(t) = \frac{1}{2\alpha_2(1-\alpha_2^2)}$$

$$
\begin{aligned}
K(t-t_{i-1}) &= \int_0^{t-t_{i-1}} e^{-As'}\vec{b}\left(\vec{e}_1^T e^{Fs'}\right) ds' \\
&= Q \begin{pmatrix} \hat{k}_{11}(t) & \hat{k}_{12}(t) \\ \hat{k}_{21}(t) & \hat{k}_{22}(t) \end{pmatrix}
\end{aligned}
\tag{4.45}
$$

where

$$\hat{k}_{11}(t) = E_1(t)e^{(1-\alpha_2)(t-t_{i-1})} + E_2(t)e^{-(1+\alpha_2)(t-t_{i-1})} + E_3(t)$$

$$\hat{k}_{12}(t) = F_1(t)e^{(1-\alpha_2)(t-t_{i-1})} + F_2(t)e^{-(1+\alpha_2)(t-t_{i-1})} + F_3(t)$$

$$\hat{k}_{21}(t) = G_1(t)e^{(1-\alpha_2)(t-t_{i-1})} + G_2(t)e^{-(1+\alpha_2)(t-t_{i-1})} + G_3(t)$$

$$\hat{k}_{22}(t) = H_1(t)e^{(1-\alpha_2)(t-t_{i-1})} + H_2(t)e^{-(1+\alpha_2)(t-t_{i-1})} + H_3(t)$$

and

$$E_1(t) = -\frac{t-t_{i-1}}{2(1-\alpha_2)} + \frac{1}{2(1-\alpha_2)^2}$$

$$E_2(t) = -\frac{t-t_{i-1}}{2(1+\alpha_2)} - \frac{t-t_{i-1}}{2(1+\alpha_2)^2}$$

$$E_3 = -\frac{2\alpha_2}{(1-\alpha_2^2)^2}$$

$$F_1(t) = -\frac{t-t_{i-1}}{2(1-\alpha_2)} + \frac{1}{2(1-\alpha_2)^2}$$

$$F_2(t) = \frac{t-t_{i-1}}{2(1+\alpha_2)} + \frac{1}{2(1+\alpha_2)^2}$$

$$F_3(t) = -\frac{1+\alpha_2^2}{(1-\alpha_2^2)^2}$$

$$G_1(t) = \frac{1}{2(1-\alpha_2)}$$

$$G_2(t) = \frac{1}{2(1+\alpha_2)}$$

$$G_3(t) = \frac{1}{(1-\alpha_2^2)}$$

$$H_1(t) = \frac{1}{2(1-\alpha_2)}$$

$$H_2(t) = -\frac{1}{2(1+\alpha_2)}$$

$$H_3(t) = -\frac{1}{(1-\alpha_2^2)}$$

$$\begin{aligned}
\Omega(h_i) &= \int_0^{h_i} e^{F(h_i-s')}\vec{e}_2\vec{b}^T e^{-A^T s'}\,ds' \\
&= Q\begin{pmatrix} \hat{\omega}_{11} & \hat{\omega}_{12} \\ \hat{\omega}_{21} & \hat{\omega}_{22} \end{pmatrix}
\end{aligned} \tag{4.46}$$

where

$$\omega_{11} = \frac{1-h_i(1-\alpha_2)}{2(1-\alpha_2)^2}e^{(1-\alpha_2)h_i} - \frac{1-h_i(1+\alpha_2)}{2(1+\alpha_2)^2}e^{-(1+\alpha_2)h_i} - \frac{2\alpha_2}{(1-\alpha_2^2)^2}$$

$$\omega_{12} = -\frac{1}{2}\left(\frac{e^{(1-\alpha_2)h_i}-1}{1-\alpha_2} + \frac{e^{-(1+\alpha_2)h_i}-1}{1+\alpha_2}\right)$$

$$\omega_{21} = \frac{h_i-1+h_i\alpha_2}{2(1-\alpha_2)^2}e^{(1-\alpha_2)h_i} - \frac{1+h_i(1+\alpha_2)}{2(1+\alpha_2)^2}e^{-(1+\alpha_2)h_i} + \frac{2(1+\alpha_2^2)}{(1-\alpha_2^2)^2}$$

$$\omega_{22} = \frac{1}{2}\left(\frac{e^{(1-\alpha_2)h_i}-1}{1-\alpha_2} - \frac{e^{-(1+\alpha_2)h_i}-1}{1+\alpha_2}\right)$$

$$\begin{aligned}
H_i &= \begin{pmatrix} B \\ Be^{Fh_i} \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 \\ \sinh h_i & \cosh h_i \end{pmatrix}
\end{aligned}$$

$$H_i^{-1} = \frac{-1}{\sinh h_i}\begin{pmatrix} \cosh h_i & -1 \\ -\sinh h_i & 0 \end{pmatrix}. \tag{4.47}$$

Thus,

$$H_i'' = \frac{1}{\sinh h_i}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{4.48}$$

and

$$\begin{aligned}
\eta(h_i) &= H_i'' Be^{Fh_i}\Omega(h_i) \\
&= \begin{pmatrix} \eta_{11} & \eta_{12} \\ \eta_{21} & \eta 22 \end{pmatrix}
\end{aligned}$$

where

$$\eta_{11} = \left(-\frac{h_i}{2(1-\alpha_2)} - \frac{1}{2(1-\alpha_2)^2} + \frac{h_i \cosh h_i}{2(1-\alpha_2)\sinh h_i} - \frac{h_i \cosh h_i}{2(1-\alpha_2)^2 \sinh h_i}\right) e^{(1-\alpha_2)h_i} +$$
$$\left(-\frac{h_i}{2(1+\alpha_2)} - \frac{1}{2(1+\alpha_2)^2} - \frac{h_i \cosh h_i}{2(1+\alpha_2)\sinh h_i} - \frac{h_i \cosh h_i}{2(1+\alpha_2)^2 \sinh h_i}\right) e^{-(1+\alpha_2)h_i} -$$
$$\left(\frac{1}{2(1-\alpha_2)^2} + \frac{1}{1+\alpha_2)^2} + \frac{h_i \cosh h_i}{2(1-\alpha_2)^2 \sinh h_i} + \frac{h_i \cosh h_i}{2(1+\alpha_2)^2 \sinh h_i}\right) \tag{4.49}$$

$$\eta_{12} = \left(-\frac{1}{2(1-\alpha_2)} + \frac{\cosh h_i}{2(1-\alpha_2)\sinh h_i}\right) e^{(1-\alpha_2)h_i} +$$
$$\left(-\frac{\cdot 1}{2(1+\alpha_2)} - \frac{\cosh h_i}{2(1+\alpha_2)\sinh h_i}\right) e^{-(1+\alpha_2)h_i} -$$
$$\frac{1}{1-\alpha_2^2}\left(1 - \alpha_2 \frac{\cosh h_i}{\sinh h_i}\right) \tag{4.50}$$

$$\eta_{21} = 0$$

$$\eta_{22} = 0$$

$$K^i(t - t_{i-1})\eta(h_i) = Q \begin{pmatrix} \hat{k}_{11}(t)\eta_{11}(h_i) & \hat{k}_{12}(t)\eta_{12}(h_i) \\ \hat{k}_{21}(t)\eta_{11}(h_i) & \hat{k}_{21}(t)\eta_{12}(h_i) \end{pmatrix} Q^T.$$

This leads to

$$\begin{aligned} C^i(t) &= K^i(t)\eta(h_i) + G^i(t) \\ &= Q \begin{pmatrix} \hat{k}_{11}(t)\eta_{11}(h_i) - \hat{g}_{11}(t) & \hat{k}_{11}(t)\eta_{12}(h_i) - \hat{g}_{12}(t) \\ \hat{k}_{21}(t)\eta_{11}(h_i) - \hat{g}_{21}(t) & \hat{k}_{21}(t)\eta_{12}(h_i) - \hat{g}_{22}(t) \end{pmatrix} Q^T \\ &= \begin{pmatrix} c^i_{11}(t) & c^i_{12}(t) \\ c^i_{21}(t) & c^i_{22}(t) \end{pmatrix} \end{aligned} \tag{4.51}$$

where

$$c^i_{11}(t) = \hat{k}_{11}(t)\eta_{11}(h_i) - \hat{g}_{11}(t)$$

$$c^i_{12}(t) = \alpha_2 c_{11}(t) + \hat{k}_{11}(t)\eta_{12}(h_i) - \hat{g}_{12}(t)$$

$$c^i_{21}(t) = \alpha_2 c_{11}(t) + \hat{k}_{21}(t)\eta_{11}(h_i) - \hat{g}_{21}(t)$$

$$c_{22}^i(t) = \alpha_2 c_{21}(t) + \alpha_2(\hat{k}_{11}(t)\eta_{12}(h_i) - \hat{g}_{12}(t)) + \hat{k}_{21}(t)\eta_{12}(h_i) - \hat{g}_{22}(t)$$

$$\begin{aligned} C_i^{-1}(h_i) &= Q^{-T}\begin{pmatrix} \hat{k}_{21}\eta_{12} - \hat{g}_{22} & -\left(\hat{k}_{11}\eta_{12} - \hat{g}_{12}\right) \\ -\left(\hat{k}_{21}\eta_{11} - \hat{g}_{21}\right) & \hat{k}_{11}\eta_{11} - \hat{g}_{11} \end{pmatrix} Q^{-1}|C|^{-1} \\ &= \begin{pmatrix} \bar{c}_{11}^i(h_i) & \bar{c}_{12}^i(h_i) \\ \bar{c}_{21}^i(h_i) & \bar{c}_{22}^i(h_i) \end{pmatrix} \end{aligned} \tag{4.52}$$

where

$$|C| = \left(\hat{k}_{11}(h_i)\eta_{11}(h_i) - \hat{g}_{11}(h_i)\right)\left(\hat{k}_{21}(h_i)\eta_{12}(h_i) - \hat{g}_{22}(h_i)\right) - \\ \left(\hat{k}_{11}(h_i)\eta_{12}(h_i) - \hat{g}_{12}(h_i)\right)\left(\hat{k}_{21}(h_i)\eta_{11}(h_i) - \hat{g}_{21}(h_i)\right).$$

Now

$$\begin{aligned} C(t)C^{-1}(h_i) &= \begin{pmatrix} c_{11}^i(t) & c_{12}^i(t) \\ c_{21}^i(t) & c_{22}^i(t) \end{pmatrix}\begin{pmatrix} \bar{c}_{11}^i(h_i) & \bar{c}_{12}^i(h_i) \\ \bar{c}_{21}^i(h_i) & \bar{c}_{22}^i(h_i) \end{pmatrix} \\ &= \begin{pmatrix} \bar{c}_{11}^i(h)c_{11}(t) + \bar{c}_{21}^i(h_i)c_{12}(t) & \bar{c}_{12}^i(h_i)c_{11}(t) + \bar{c}_{22}^i(h_i)c_{12}(t) \\ \bar{c}_{11}^i(h_i)c_{21}(t) + \bar{c}_{21}^i(h_i)c_{22}(t) & \bar{c}_{12}^i(h_i)c_{21}(t) + \bar{c}_{22}^i(h_i)c_{22}(t) \end{pmatrix} \end{aligned} \tag{4.53}$$

$$e^{A(t-t_{i-1})}\left[I - C(t)C^{-1}(h_i)\right] = \begin{pmatrix} \phi_1(t) & \phi_2(t) \\ * & * \end{pmatrix} \tag{4.54}$$

where

$$\phi_1(t) = e^{\alpha_2(t-t_{i-1})}\left\{(1-\alpha_2(t-t_{i-1}))\left(1 - \bar{c}_{11}^i(h_i)c_{11}(t) - \bar{c}_{21}^i(h_i)c_{12}(t)\right)\right\} - \\ e^{\alpha_2(t-t_{i-1})}\left\{(t-t_{i-1})\left(\bar{c}_{11}^i(h_i)c_{21}(t) + \bar{c}_{21}^i(h_i)c_{22}(t)\right)\right\} \tag{4.55}$$

$$\phi_2(t) = e^{\alpha_2(t-t_{i-1})}\left\{-(1-\alpha_2(t-t_{i-1}))\left(\bar{c}_{12}^i(h_i)c_{11}(t) + \bar{c}_{22}^i(h_i)c_{12}(t)\right)\right\} + \\ e^{\alpha_2(t-t_{i-1})}\left\{(t-t_{i-1})\left(1 - \bar{c}_{12}^i(h_i)c_{21}(t) - \bar{c}_{22}^i(h_i)c_{22}(t)\right)\right\} \tag{4.56}$$

Similarly,

$$e^{A(t-t_{i-1})}C(t)C^{-1}(h_i)e^{-Ah_i} = \begin{pmatrix} \psi_1(t) & \psi_2(t) \\ * & * \end{pmatrix} \tag{4.57}$$

where

$$\psi_1(t) = e^{\alpha_2(t-t_{i-1}-h_i)}\left\{(1-\alpha_2(t-t_{i-1}))(1+\alpha_2 h_i)\left(\bar{c}^i_{11}(h_i)c_{11}(t)+\bar{c}^i_{21}(h_i)c_{12}(t)\right)\right\}+$$
$$e^{\alpha_2(t-t_{i-1}-h_i)}\left\{(t-t_{i-1})(1+\alpha_2 h_i)\left(\bar{c}^i_{11}(h_i)c_{21}(t)+\bar{c}^i_{21}(h_i)c_{22}(t)\right)\right\}+$$
$$e^{\alpha_2(t-t_{i-1}-h_i)}\left\{\alpha_2^2 h_i(1-\alpha_2(t-t_{i-1}))\left(\bar{c}^i_{12}(h_i)c_{11}(t)+\bar{c}^i_{22}(h_i)c_{12}(t)\right)\right\}+$$
$$e^{\alpha_2(t-t_{i-1}-h_i)}\left\{\alpha_2^2 h_i\left(\bar{c}^i_{12}(h_i)c_{21}(t)+\bar{c}^i_{22}(h_i)c_{22}(t)\right)\right\} \tag{4.58}$$

and

$$\psi_2(t) = e^{\alpha_2(t-t_{i-1}-h_i)}\left\{-h_i(1-\alpha_2(t-t_{i-1}))\left(\bar{c}^i_{11}(h_i)c_{11}(t)+\bar{c}^i_{21}(h_i)c_{12}(t)\right)\right\}-$$
$$e^{\alpha_2(t-t_{i-1}-h_i)}\left\{h_i(t-t_{i-1})\left(\bar{c}^i_{11}(h_i)c_{21}(t)+\bar{c}^i_{21}(h_i)c_{22}(t)\right)\right\}+$$
$$e^{\alpha_2(t-t_{i-1}-h_i)}\left\{(1-\alpha_2 h_i)(1-\alpha_2(t-t_{i-1}))\left(\bar{c}^i_{12}(h_i)c_{11}(t)+\bar{c}^i_{22}(h_i)c_{12}(t)\right)\right\}+$$
$$e^{\alpha_2(t-t_{i-1}-h_i)}\left\{(1-\alpha_2 h_i)\left(\bar{c}^i_{12}(h_i)c_{21}(t)+\bar{c}^i_{22}(h_i)c_{22}(t)\right)\right\}. \tag{4.59}$$

Thus, the resulting spline has basis functions which are linear combinations of 1, t, t^2, $e^{-\alpha_2 t}$, $te^{-\alpha_2 t}$, $e^{-2\alpha_2 t}$, $te^{-2\alpha_2 t}$, $t^2e^{-2\alpha_2 t}$, $e^{(1-\alpha_2)t}$, $te^{(1-\alpha_2)t}$, $e^{-(1+\alpha_2)t}$, $te^{-(1+\alpha_2)t}$.
Case 3(ii): $\alpha_2 = 0$. In this case, the state matrix is in Jordan form and hence, $G(t)$ and $K(t)$ may be computed directly as follows:

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

On the subinterval $[t_{i-1}, t_i]$,

$$H_i = \begin{pmatrix} B \\ Be^{Fh_i} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{e^{h_i}-e^{-h_i}}{2} & \frac{e^{h_i}+e^{-h_i}}{2} \end{pmatrix} \tag{4.60}$$

$$H_i^{-1} = \frac{1}{\sinh h_i}\begin{pmatrix} -\cosh h_i & 1 \\ \sinh h_i & 0 \end{pmatrix} \tag{4.61}$$

$$H_i'' = \frac{1}{\sinh h_i}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{4.62}$$

$$\Omega(h_i) = \int_0^{h_i} e^{F(h_i - s')} \vec{e}_2 \vec{b}^T e^{-A^T s'} \, ds'$$
$$= \begin{pmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{pmatrix} \tag{4.63}$$

where

$$\omega_{11}^i = h_i - \sinh h_i$$
$$\omega_{12}^i = \cosh h_i - 1$$
$$\omega_{21}^i = -\omega_{12}^i$$
$$\omega_{22}^i = \sinh h_i.$$

Also

$$\eta(h_i) = H'' B \Omega(h_i)$$
$$= \begin{pmatrix} \eta_{11} & \eta_{12} \\ \eta_{21} & \eta_{22} \end{pmatrix} \tag{4.64}$$

where $\eta_{11} = \frac{1 - \cosh h_i}{\sinh h_i}$, $\eta_{12} = 1$, $\eta_{21} = 0$, and $\eta_{22} = 0$. From equation (3.45),

$$K(\bar{t}) = \int_0^{\bar{t}} e^{-As'} \vec{b} \left(\vec{e}_1^T e^{Fs'} \right) ds' \tag{4.65}$$
$$= \begin{pmatrix} k_{11}(\bar{t}) & k_{12}(\bar{t}) \\ k_{21}(\bar{t}) & k_{22}(\bar{t}) \end{pmatrix} \tag{4.66}$$

where

$$k_{11}(\bar{t}) = -\bar{t} \sinh \bar{t} + \cosh \bar{t} - 1$$
$$k_{12}(\bar{t}) = \bar{t} \cosh \bar{t} - \sinh \bar{t}$$
$$k_{21}(\bar{t}) = \sinh \bar{t}$$
$$k_{22}(\bar{t}) = \cosh \bar{t} - 1$$

$$K^i(h_i) = \begin{pmatrix} -h_i \sinh(h_i) + & -h_i \cosh(h_i) + \\ \cosh(h_i) - 1 & -\sinh(h_i) \\ & \\ \sinh(h_i) & \cosh(h_i) - 1 \end{pmatrix}$$

$$K^i(t)\eta(h_i) = \begin{pmatrix} \hat{k}_{11}(\bar{t}) & \hat{k}_{12}(\bar{t}) \\ \hat{k}_{21}(\bar{t}) & \hat{k}_{22}(\bar{t}) \end{pmatrix} \tag{4.67}$$

where

$$\hat{k}_{11}(\bar{t}) = k_{11}(\bar{t})\eta_{11}$$
$$\hat{k}_{12}(\bar{t}) = k_{11}(\bar{t})\eta_{12}$$
$$\hat{k}_{21}(\bar{t}) = k_{21}(\bar{t})\eta_{11}$$
$$\hat{k}_{22}(\bar{t}) = k_{21}(\bar{t})\eta_{12}$$
$$K^i(h_i)\eta(h_i) = \begin{pmatrix} \hat{k}_{11}(h) & \hat{k}_{12}(h) \\ \hat{k}_{21}(h) & \hat{k}_{22}(h) \end{pmatrix}$$

From equation (3.43),

$$G^i(t) = (-1)\int_0^{t-t_i} e^{-As}\vec{b}\left(\vec{e}_1^T \int_0^s e^{F(s-r)}\vec{e}_2\vec{b}^T e^{-A^T r}\, dr\right) ds \tag{4.68}$$

$$= \begin{pmatrix} g_{11}(\bar{t}) & g_{12}(\bar{t}) \\ g_{21}(\bar{t}) & g_{22}(\bar{t}) \end{pmatrix} \tag{4.69}$$

where

$$g_{11}(\bar{t}) = \frac{1}{3}\bar{t}^3 + \sinh\bar{t} - \bar{t}\cosh\bar{t}$$
$$g_{12}(\bar{t}) = -\frac{\bar{t}^2}{2} - \cosh\bar{t} + \bar{t}\sinh\bar{t} + 1$$
$$g_{21}(\bar{t}) = \cosh\bar{t} - 1 - \frac{\bar{t}^2}{2}$$
$$g_{22}(\bar{t}) = \bar{t} - \sinh\bar{t}$$

$$G^i(h_i) = \begin{pmatrix} \frac{1}{3}h_i^3 + \sinh(h_i) - h_i\cosh(h_i) & -\frac{h_i^2}{2} - \cosh(h_i) + h_i\sinh(h_i) + 1 \\ \cosh(h_i) - 1 - \frac{h_i^2}{2} & h_i - \sinh(h_i) \end{pmatrix}.$$

Thus,

$$\begin{aligned}
C^i(\bar{t}) &= K^i(\bar{t})\eta(h_i) + G^i(\bar{t}) \\
&= \begin{pmatrix} \hat{k}_{11}(\bar{t}) + g_{11}(\bar{t}) & \hat{k}_{12}(\bar{t}) + g_{12}(\bar{t}) \\ \hat{k}_{21}(\bar{t}) + g_{21}(\bar{t}) & \hat{k}_{22}(\bar{t}) + g_{22}(\bar{t}) \end{pmatrix} \\
&= \begin{pmatrix} c_{11}^i(\bar{t}) & c_{12}^i(\bar{t}) \\ c_{21}^i(\bar{t}) & c_{22}^i(\bar{t}) \end{pmatrix} \qquad (4.70)
\end{aligned}$$

where

$$\begin{aligned}
c_{11}^i(\bar{t}) &= \frac{(\bar{t}\sinh\bar{t} - \cosh\bar{t} + 1)(\cosh h_i - 1)}{\sinh h_i} + \frac{\bar{t}^3}{3} - \bar{t}\cosh\bar{t} + \sinh\bar{t} \\
c_{12}^i(\bar{t}) &= -\frac{\bar{t}^2}{2} \\
c_{21}^i(\bar{t}) &= -1 - \frac{\bar{t}^2}{2} - \frac{\sinh\bar{t}}{\sinh h}(\cosh h_i - 1) + \cosh\bar{t} \qquad (4.71) \\
c_{22}^i(t) &= \bar{t}
\end{aligned}$$

$\bar{t} = t - t_{i-1}$, $i = 1, \ldots, n-1$. Hence, when $t = t_i$, we obtain $\bar{t} = h_i$, and

$$\begin{aligned}
C_i(h_i) &= K^i(h_i)\eta(h_i) + G^i(h_i) \\
&= \begin{pmatrix} \hat{k}_{11}(h_i) + g_{11}(h_i) & \hat{k}_{12}(h_i) + g_{12}(h_i) \\ \hat{k}_{21}(h_i) + g_{21}(h_i) & \hat{k}_{22}(h_i) + g_{22}(h_i) \end{pmatrix} \\
&= \begin{pmatrix} c_{11}^i(h_i) & c_{12}^i(h_i) \\ c_{21}^i(h_i) & c_{22}^i(h_i) \end{pmatrix}. \qquad (4.72)
\end{aligned}$$

Furthermore,

$$\begin{aligned}
C_i^{-1}(h_i) &= d(C_i(h_i))^{-1} \begin{pmatrix} c_{22}^i(h_i) & -c_{12}^i(h_i) \\ -c_{21}^i(h_i) & c_{11}^i(h_i) \end{pmatrix} \\
&= \begin{pmatrix} \bar{c}_{11}(h_i) & \bar{c}_{12}(h_i) \\ \bar{c}_{21}(h_i) & \bar{c}_{22}(h_i) \end{pmatrix}
\end{aligned}$$

where $d(C_i(h_i))$ denotes the determinant of $C_i(h_i))$ and is given by

$$d(C_i(h_i)) = c_{11}(h_i)c_{22}(h_i) - c_{12}(h_i)c_{21}(h_i). \qquad (4.73)$$

Therefore, from equation (4.31), we have the following:

$$e^{A\bar{t}} - e^{A\bar{t}}C_i(\bar{t})C_i^{-1}(h_i) = \begin{pmatrix} 1 & \bar{t} \\ 0 & 1 \end{pmatrix} -$$

$$\begin{pmatrix} 1 & \bar{t} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c_{11}(\bar{t})\bar{c}_{11} + c_{12}(\bar{t})\bar{c}_{21} & c_{11}(\bar{t})\bar{c}_{12} + c_{12}(\bar{t})\bar{c}_{22} \\ c_{21}(\bar{t})\bar{c}_{11} + c_{22}(\bar{t})\bar{c}_{21} & c_{21}(\bar{t})\bar{c}_{12} + c_{22}(\bar{t})\bar{c}_{22} \end{pmatrix}$$

$$= \begin{pmatrix} \phi_1(\bar{t}) & \phi_2(\bar{t}) \\ \star & \star \end{pmatrix} \tag{4.74}$$

where

$$\phi_1(\bar{t}) = 1 - [c_{11}(\bar{t})\bar{c}_{11} + c_{12}(\bar{t})\bar{c}_{21} + \bar{t}(c_{21}(\bar{t})\bar{c}_{11} + c_{22}(\bar{t})\bar{c}_{21})] \tag{4.75}$$

and

$$\phi_2(\bar{t}) = \bar{t} - [c_{11}(\bar{t})\bar{c}_{12} + c_{12}(\bar{t})\bar{c}_{22} + \bar{t}(c_{21}(\bar{t})\bar{c}_{12} + c_{22}(\bar{t})\bar{c}_{22})]. \tag{4.76}$$

Substituting for $c_{ij}(t)$, these equations simplify to:

$$\phi_1(\bar{t}) = 1 + \bar{c}_{11}\bar{t} - \bar{c}_{21}\frac{\bar{t}^2}{2} + \bar{c}_{11}\frac{\bar{t}^3}{6} - c_{11}[\sinh\bar{t} + \frac{(1-\cosh\bar{t})(\cosh h_i - 1)}{\sinh h_i}] \tag{4.77}$$

and

$$\phi_2(\bar{t}) = (1 + \bar{c}_{12})\bar{t} - \bar{c}_{22}\frac{\bar{t}^2}{2} + \bar{c}_{12}\frac{\bar{t}^3}{6} - c_{12}[\sinh\bar{t} + \frac{(1-\cosh\bar{t})(\cosh h_i - 1)}{\sinh h_i}]. \tag{4.78}$$

Similarly,

$$e^{A\bar{t}}C_i(\bar{t})C_i^{-1}(h_i)e^{-Ah_i} = \begin{pmatrix} 1 & \bar{t} \\ 0 & 1 \end{pmatrix} \times$$

$$\begin{pmatrix} c_{11}(\bar{t})\bar{c}_{11} + c_{12}(\bar{t})\bar{c}_{21} & c_{11}(\bar{t})\bar{c}_{12} + c_{12}(\bar{t})\bar{c}_{22} \\ c_{21}(\bar{t})\bar{c}_{11} + c_{22}(\bar{t})\bar{c}_{21} & c_{21}(\bar{t})\bar{c}_{12} + c_{22}(\bar{t})\bar{c}_{22} \end{pmatrix} \begin{pmatrix} 1 & -h_i \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \psi_1(t) & \psi_2(t) \\ \star & \star \end{pmatrix} \tag{4.79}$$

where

$$\begin{aligned} \psi_1(\bar{t}) &= [c_{11}(\bar{t})\bar{c}_{11} + c_{12}(\bar{t})\bar{c}_{21} + \bar{t}(c_{21}(\bar{t})\bar{c}_{11} + c_{22}(\bar{t})\bar{c}_{21})] \\ &= 1 - \phi_1(\bar{t}) \end{aligned} \tag{4.80}$$

and

$$\begin{aligned}\psi_2(\bar{t}) &= -h_i[c_{11}(\bar{t})\bar{c}_{11} + c_{12}(\bar{t})\bar{c}_{21} + \bar{t}(c_{21}(\bar{t})\bar{c}_{11} + c_{22}(\bar{t})\bar{c}_{21})]+ \\ &\quad [c_{11}(\bar{t})\bar{c}_{12} + c_{12}(\bar{t})\bar{c}_{22} + \bar{t}(c_{21}(\bar{t})\bar{c}_{12} + c_{22}(\bar{t})\bar{c}_{22})] \\ &= -h_i(1 - \phi_1(\bar{t})) + \bar{t} - \phi_2(t).\end{aligned} \tag{4.81}$$

Thus, the spline obtained has basis functions which are linear combinations of $1, t, t^2, t^3, e^t$, and e^{-t}.

CHAPTER V

CONVERGENCE AND NUMERICAL RESULTS

Here, we will examine convergence rates for the spline approximant for case 3(ii) and then give the results of computer simulations.

5.1 Results for a Nilpotent Matrix

Since the state matrix under consideration is 2×2 (that is $m = 2$), then it follows that $r = 0$, and, from equation (4.18), the requirement $u_i^{(r)}(t_i) = u_{i+1}^{(r)}(t_i)$ yields the following:

$$\begin{aligned} M_i(h_i) &= e^{Fh} H_i'' B \Omega(h_i) e^{-A^T t_{i-1}} - \Omega(h_i) e^{-A^T t_{i-1}} \\ &= (e^{Fh} H_i'' B - I) \Omega(h) e^{-A^T t_{i-1}} \\ &= \begin{pmatrix} -1 & \frac{\cosh h_i}{\sinh h_i} \\ 0 & 0 \end{pmatrix} \Omega(h_i) e^{-A^T t_{i-1}} \end{aligned} \tag{5.1}$$

$$\vec{e}_1^T M_i(h_i) e^{A^T t_{i-1}} C^{-1} = \begin{pmatrix} d_i & e_i \end{pmatrix} \tag{5.2}$$

$$\vec{e}_1^T M_i(h_i) e^{A^T t_{i-1}} C^{-1} e^{-Ah_i} = \begin{pmatrix} d_i & -h_i d_i + e_i \end{pmatrix} \tag{5.3}$$

where

$$d_i = [-\omega_{11}^i + \frac{\cosh h_i}{\sinh h_i} \omega_{21}^i] \bar{c}_{11}^i + [-\omega_{12}^i + \frac{\cosh h_i}{\sinh h_i} \omega_{22}^i] \bar{c}_{21}^i \tag{5.4}$$

$$e_i = [-\omega_{11}^i + \frac{\cosh h_i}{\sinh h_i} \omega_{21}^i] \bar{c}_{12}^i + [-\omega_{12}^i + \frac{\cosh h_i}{\sinh h_i} \omega_{22}^i] \bar{c}_{22}^i \tag{5.5}$$

$$\begin{aligned} M_{i+1}(h_{i+1}) &= H_{i+1}'' B \Omega(h_{i+1}) e^{-A^T t_i} \\ &= \begin{pmatrix} 0 & \frac{1}{\sinh h_{i+1}} \\ 0 & 0 \end{pmatrix} \Omega(h) e^{-A^T t_i} \end{aligned} \tag{5.6}$$

$$\vec{e}^T M_{i+1}(h_{i+1}) e^{A^T t_i} C^{-1} = \begin{pmatrix} d_{i+1} & e_{i+1} \end{pmatrix} \tag{5.7}$$

$$\vec{e}^T M_{i+1}(h_{i+1}) e^{A^T t_i} C^{-1} e^{-Ah_{i+1}} = \begin{pmatrix} d_{i+1} & -hd_{i+1} + e_{i+1} \end{pmatrix} \tag{5.8}$$

where

$$d_{i+1} = \frac{\omega_{21}^{i+1} \bar{c}_{11}^{i+1} + \omega_{22}^{i+1} \bar{c}_{21}^{i+1}}{\sinh h_{i+1}} \tag{5.9}$$

$$e_{i+1} = \frac{\omega_{21}^{i+1} \bar{c}_{12}^{i+1} + \omega_{22}^{i+1} \bar{c}_{22}^{i+1}}{\sinh h_{i+1}} \tag{5.10}$$

$$\vec{e}_1^T M_i(h_i) e^{-A^T h_i} C^{-1} e^{-Ah_i} + \vec{e}^T M_{i+1}(h_{i+1}) e^{-A^T h_{i+1}} C^{-1} = \begin{pmatrix} d_1 + e_1 & -hd_1 + d_2 + e_2 \end{pmatrix} \tag{5.11}$$

Substituting in (4.18), with $\vec{x}^i = \begin{pmatrix} \alpha_i & \beta_i \end{pmatrix}^T$, we obtain

$$-\begin{pmatrix} d_i & e_i \end{pmatrix} \begin{pmatrix} \alpha_{i-1} \\ \beta_{i-1} \end{pmatrix} + \begin{pmatrix} d_i + d_{i+1} & -h_i d_i + e_i + e_{i+1} \end{pmatrix} \begin{pmatrix} \alpha_i \\ \beta_i \end{pmatrix} -$$

$$\begin{pmatrix} d_{i+1} & -h_{i+1} d_{i+1} + e_{i+1} \end{pmatrix} \begin{pmatrix} \alpha_{i+1} \\ \beta_{i+1} \end{pmatrix} = 0 \tag{5.12}$$

$$i = 1, \dots p-1.$$

This gives

$$-e_i \beta_{i-1} + (-h_i d_i + e_i + e_{i+1}) \beta_i - (-h_{i+1} d_{i+1} + e_{i+1}) \beta_{i+1} = d_i \alpha_{i-1} - (d_i + d_{i+1}) \alpha_i + d_{i+1} \alpha_{i+1} \tag{5.13}$$

$$\begin{pmatrix} y & z & 0 & 0 & \dots & 0 \\ x & y & z & 0 & \dots & 0 \\ 0 & x & y & z & \dots & 0 \\ \vdots & & & & & \vdots \\ \vdots & . & & & & \vdots \\ 0 & 0 & \dots & x & y & z \\ 0 & 0 & \dots & 0 & x & y \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \vdots \\ \vdots \\ \beta_{p-2} \\ \beta_{p-1} \end{pmatrix} = \begin{pmatrix} D_1 \\ d_2 \alpha_1 - (d_2 + d_3) \alpha_2 + d_3 \alpha_3 \\ d_3 \alpha_2 - (d_3 + d_4) \alpha_3 + d_4 \alpha_4 \\ \vdots \\ \vdots \\ D_{p-2} \\ D_{p-1} \end{pmatrix} \tag{5.14}$$

where $x = -e_i$, $y = -hd_i + e_i + e_{i+1}$, and $z = -(-hd_{i+1} + e_{i+1})$,

$$D_1 = d_1\alpha_0 - (d_1 + d_2)\alpha_1 + d_2\alpha_2 + e_i\beta_0,$$

$$D_{p-2} = d_{p-2}\alpha_{p-3} - (d_{p-2} + d_{p-1})\alpha_{p-2} + d_{p-1}\alpha_{p-1},$$

and

$$D_{p-1} = d_{p-1}\alpha_{p-2} - (d_{p-1} + d_p)\alpha_{p-1} + d_p\alpha_p + (-hd_{i+1} + e_{i+1})\beta_p.$$

Lemma 5.1 *The coefficient matrix in equation (5.14) is strictly diagonally dominant and hence nonsingular.*

On solving the above system of equations for β, we then apply equation (4.32) to get, on each subinterval $[t_{i-1}, t_i]$, the spline function

$$y(t) = \alpha_{i-1}\phi_1(t) + \beta_{i-1}\phi_2(t) + \alpha_i\psi_1(t) + \beta_i\psi_2(t) \tag{5.15}$$

Here, ϕ_1, ϕ_2, ψ_1, and ψ_2 are given by Theorem (4.2).

5.2 Convergence of the Approximation

In this analysis, we will denote by $s(t)$ the spline approximant, obtained by control principles (given by equation (5.15)), of the function $f(t)$. Error estimates for the classical cubic spline have been discussed extensively; see, for example, references [11, 19, 20]. In this section, we will obtain the convergence rates for our approximation for case 3(ii); similar procedure will yield the error estimates for the other cases. Furthermore, we restrict our analysis to the case of uniform mesh with mesh width h. From equation (5.13), the spline approximation $s(t)$ satisfies the equation

$$x\beta_{i-1} + y\beta_i + z\beta_{i+1} = d_i\alpha_{i-1} - (d_i + d_{i+1})\alpha_i + d_{i+1}\alpha_{i+1} \tag{5.16}$$

where $x = -e_i$, $y = -hd_i + e_i + e_{i+1}$, and $z = -(-hd_{i+1} + e_{i+1})$. Now, $s(t)$ interpolates $f(t)$ at the mesh points $P : t_0 \le t_1 \le \ldots \le t_f$; in other words, $s(t_i) = \alpha_i = f(t_i)$. Clearly, $\beta_i = s'(t_i)$. Hence, equation (5.16) can be written as

$$xs'_{i-1} + ys'_i + zs'_{i+1} = d_1f_{i-1} - (d_1 + e_1)f_i + e_1f_{i+1} \tag{5.17}$$

where $s_i' = s'(t_i)$ and $f_i = f(t_i)$. For h sufficiently small, we have the following:

$$\begin{aligned}
\omega_{11} &= h - \sinh h \approx -\frac{h^3}{6} \\
\omega_{12} &= \cosh h - 1 \approx \frac{h^2}{2} + \frac{h^4}{24} \\
\omega_{21} &= -\cosh h + 1 \approx -\frac{h^2}{2} - \frac{h^4}{24} \\
\omega_{22} &= \sinh h
\end{aligned} \tag{5.18}$$

where, for each truncation, we have omitted terms that are of higher order in h than the ones retained. Also, from equation (4.71):

$$\begin{aligned}
c_{11}(h) &= \frac{h^3}{3} + \sinh h - h - (\cosh h - 1)^2 / \sinh h \\
c_{12}(h) &= -\frac{h^2}{2} \\
c_{21}(h) &= -\frac{h^2}{2} \\
c_{22}(h) &= h.
\end{aligned} \tag{5.19}$$

Thus,

$$\begin{aligned}
detC &= c_{11}c_{22} - c_{12}c_{21} \\
&= h^4/12 + h \sinh h - h^2 - h(\cosh h - 1)^2 / \sinh h \\
&\approx h^6/120
\end{aligned} \tag{5.20}$$

$$\begin{aligned}
\bar{c}_{11} &= c_{22}/det(C) \approx 120/h^5 \\
\bar{c}_{12} &= -c_{12}/det(C) \approx 60/h^4 \\
\bar{c}_{21} &= -c_{21}/det(C) \approx 60/h^4 \\
\bar{c}_{22} &= c_{11}/det(C) \approx 30/h^3
\end{aligned} \tag{5.21}$$

$$\begin{aligned}
d_i &= [-\omega_{11} + \frac{\cosh h}{\sinh h}\omega_{21}]\bar{c}_{11} + [-\omega_{12} + \frac{\cosh h}{\sinh h}\omega_{22}]\bar{c}_{21} \\
&\approx -5/h^2 \\
e_i &= [-\omega_{11} + \frac{\cosh h}{\sinh h}\omega_{21}]\bar{c}_{12} + [-\omega_{12} + \frac{\cosh h}{\sinh h}\omega_{22}]\bar{c}_{22}
\end{aligned}$$

$$
\begin{aligned}
&\approx -3/2h \\
d_{i+1} &= \frac{\omega_{21}\bar{c}_{11} + \omega_{22}\bar{c}_{21}}{\sinh h} \\
&\approx 5/h^2 \\
e_{i+1} &= \frac{\omega_{21}\bar{c}_{12} + \omega_{22}\bar{c}_{22}}{\sinh h} \\
&\approx 7/2h.
\end{aligned} \tag{5.22}
$$

Thus, we obtain the following:

$$
\begin{aligned}
-hd_i + e_i + e_{i+1} &= 14/2h \\
-hd_{i+1} + e_{i+1} &= -3/2h.
\end{aligned} \tag{5.23}
$$

Theorem 5.1 *Let $f(t) \in C^3$ and let $\delta(t) = s(t) - f(t)$ be the error that results when $f(t)$ is interpolated by the spline s, defined above, on the partition $P : 0 = t_0 < t_1 < \ldots < t_p = 1$. Then there exists a constant K such that*

$$
|s - f| \leq K\|f^{(3)}\|h^3 \tag{5.24}
$$

To prove Theorem (5.1), we first discuss the following lemma.

Lemma 5.2 *Let P be any partition of the interval $[a, b]$. If $f(t) \in C^3[a, b]$, then*

$$
|s'(t_i) - f'(t_i)| \leq \frac{h^2}{24}\|f^{(3)}(\xi_i)\| \tag{5.25}
$$

for each node t_i, $i = 0, 1, \ldots, p$ and $t_{i-1} \leq \xi_i \leq t_i$.

Proof:

Using equation (5.23), equation (5.17) may be written as

$$
\frac{3}{2h}s'_{i-1} + \frac{14}{2h}s'_i + \frac{3}{2h}s'_{i+1} = -\frac{5}{h^2}\alpha_{i-1} + \frac{5}{h^2}\alpha_{i+1} \tag{5.26}
$$

This simplifies to

$$
3s'_{i-1} + 14s'_i + 3s'_{i+1} = -\frac{10}{h}\left(f_{i-1} - f_{i+1}\right) \tag{5.27}
$$

where we have used the interpolation data $\alpha_i = f_i$. Suppose that each term of the form $f_{i\pm1}^{(r)}$ is expandable as:

$$
f_{i\pm1}^{(r)} = f_i^{(r)} \pm h f_i^{(r+1)} + \frac{h^2}{2} f_i^{(r+2)} \pm \frac{h^3}{6} f_i^{(r+3)} + \frac{h^4}{24} f_i^{(r+4)} \pm \ldots
$$

Then, it can easily be shown, using Taylor's formula, that

$$3f'_{i-1} + 14f'_i + 3f'_{i+1} = 20f'_i + \frac{3h^2}{2}\left(f'''(\xi_-) + f'''(\xi_+)\right) \tag{5.28}$$

$t_{i-1} \le \xi \le t_{i+1}$. If $f^{(3)}$ is continuous on $[x_{i-1}, x_{i+1}]$, then by the Intemediate Value Theorem we may write

$$3f'_{i-1} + 14f'_i + 3f'_{i+1} = 20f'_i + 3h^2 f'''(\xi) \tag{5.29}$$

Let $s'_i - f'_i = E_i$. Then, subtracting equation (5.28) from equation (5.27) and replacing f'_i with $\frac{1}{2h}[f_{i+1} - f_{i-1}] - \frac{h^2}{6} f^3(\xi)$, we obtain

$$3E_{i-1} + 14E_i + 3E_{i+1} = \frac{h^2}{3} f^{(3)}(\xi) \tag{5.30}$$

That is,

$$GE = H \tag{5.31}$$

where

$$G = \begin{pmatrix} 14 & 3 & 0 & 0 & \dots & 0 \\ 3 & 14 & 3 & 0 & \dots & 0 \\ 0 & 3 & 14 & 3 & \dots & 0 \\ \vdots & & & & & \vdots \\ \vdots & & & & & \vdots \\ 0 & 0 & \dots & 3 & 14 & 3 \\ 0 & 0 & \dots & 0 & 3 & 14 \end{pmatrix}$$

and $H_i = \frac{h^2}{3} f^{(3)}(\xi)$. If we multiply both sides of equation (5.31) by the matrix

$$D = diag(1/14, \dots, 1/14),$$

we obtain

$$DGE = DH \tag{5.32}$$

and

$$DG = I + B$$

$$= \begin{pmatrix} 1 & 3/14 & 0 & 0 & \dots & 0 \\ 3/14 & 1 & 3/14 & 0 & \dots & 0 \\ 0 & 3/14 & 1 & 3/14 & \dots & 0 \\ \vdots & & & & & \vdots \\ \vdots & & & & & \vdots \\ 0 & 0 & \dots & 3/14 & 1 & 3/14 \\ 0 & 0 & \dots & 0 & 3/14 & 1 \end{pmatrix} \tag{5.33}$$

where $\|B\|_\infty = 6/14 \leq 1$. Thus, $(DG)^{-1} = (I+B)^{-1}$ exists, (see reference [31], p61). Hence,

$$\begin{aligned} \|E\|_\infty &= \|(DG)^{-1}DH\|_\infty \\ &\leq \|(DG)^{-1}\|_\infty \|D\|_\infty \|H\|_\infty \\ &\leq \frac{1}{1-\frac{6}{14}} \frac{1}{14} \frac{h^2}{3} \|f^{(3)}\| \\ &\leq \frac{h^2}{24} \|f^{(3)}\| \end{aligned} \tag{5.34}$$

($4/27h^2\|f^{(3)}\|$ for the classical case, see reference [19]). ∎

Now, to prove Theorem 5.1, we observe that $\delta(t) = s(t) - f(t) \in C^2$ and since $s(t_i) = f(t_i)$, $i = 0, 1, \dots, p$, we obtain, by the Mean Value Theorem,

$$\delta(t) = \int_{t_{i-1}}^{t} \delta'(r)\, dr \tag{5.35}$$

This gives

$$\begin{aligned} \|\delta(t)\| &\leq \int_{t_{i-1}}^{t} |\delta'(r)|\, dr \\ &\leq \|\delta'\| \max |(t - t_{i-1})| \end{aligned} \tag{5.36}$$

From equation (5.34),

$$\|\delta'\| \leq \frac{h^2}{24} \|f^{(3)}\|.$$

Hence,

$$\|\delta(t)\| \leq \frac{h^3}{24} \|f^{(3)}\| \tag{5.37}$$

This completes the proof of the theorem.

5.3 Numerical Examples

ALGORITHM FOR CONSTRUCTING THE CUBIC-EXPONENTIAL[1] SPLINES

To construct the spline interpolant $s(t)$ for the function $f(t)$, defined at the nodes $t_0 < t_1 < \ldots < t_n$, satisfying $s'(t_0) = f'(t_0)$ and $s'(t_n) = f'(t_n)$:

INPUT A, $\vec{b}$, F, B, n; t_0, t_1, $\ldots$, t_n; $\alpha_1 = f(t_1)$, $\ldots$, $\alpha_{n-1} = f(t_{n-1})$; $\beta_0 = f'(t_0)$; $\beta_n = f'(t_n)$.

OUTPUT β_k, $k = 1, 2, \ldots, n-1$.

Recall, from equation (5.15),

$$s(t) = \alpha_{i-1}\phi_1(t) + \beta_{i-1}\phi_2(t) + \alpha_i\psi_1(t) + \beta_i\psi_2(t) \quad for \quad t_{i-1} \le t \le t_i$$

Step I For $i = 0, 1, \ldots, n-1$ set $h_i = t_i - t_{i-1}$.

Step II Compute $\Omega(h_i)$, $C(t)$, ϕ_i, and ψ_i, $i = 1, 2$, from equations (4.63), (4.70), (4.77), (4.78), (4.80) and (4.81), respectively.

Step III Set

$$d_i = [-\omega^i_{11} + \frac{\cosh h_i}{\sinh h_i}\omega^i_{21}]\bar{c}^i_{11} + [-\omega^i_{12} + \frac{\cosh h_i}{\sinh h_i}\omega^i_{22}]\bar{c}^i_{21}$$
$$e_i = [-\omega^i_{11} + \frac{\cosh h_i}{\sinh h_i}\omega^i_{21}]\bar{c}^i_{12} + [-\omega^i_{12} + \frac{\cosh h_i}{\sinh h_i}\omega^i_{22}]\bar{c}^i_{22}$$
$$d_{i+1} = \frac{\omega^{i+1}_{21}\bar{c}^{i+1}_{11} + \omega^{i+1}_{22}\bar{c}^{i+1}_{21}}{\sinh h_{i+1}}$$
$$e_{i+1} = \frac{\omega^{i+1}_{21}\bar{c}^{i+1}_{12} + \omega^{i+1}_{22}\bar{c}^{i+1}_{22}}{\sinh h_{i+1}}$$

where ω_{ij} and c_{ij} are the entries of the matrices Ω and C, respectively.

Step IV Set $l_1 = d_1\alpha_0 - (d_1 + d_2)\alpha_1 + d_2\alpha_2 + e_i\beta_0$, and
$l_{n-1} = d_{n-1}\alpha_{n-2} - (d_{n-1} + d_n)\alpha_{n-1} + d_n\alpha_n + e_i\beta_0$,

Step V Set $l_j = d_j\alpha_{j-1} - (d_j + d_{j+1})d_j + d_{j+1}\alpha_{j+1}$
$j = 2, \ldots, n-1$.
Set $L = (l_1, l_2, \ldots, l_{n-2}, l_{n-1})'$.

[1] So called to reflect the fact that it contains cubic polynomials as well as exponential terms

Step VI Form the tridiagonal matrix, M, with:
lower diagonal elements, -e_j,
diagonal elements, $(-h_j d_j + e_j + e_{j+1}$, and
upper diagonal elements, $-(-h_j d_{j+1} + e_{j+1})$.

Step VII Solve the system $M\beta = L$.

Step VIII OUTPUT β_j, $j = 1, 2, \ldots, n-1$.

Example 5.1 $f(t) = \sin(\pi t)$

Consider the test function

$$f(t) = \sin(\pi t), \quad t \in [0, 1]$$

We set the boundary conditions:

$$\beta_0 = -\pi, \quad \beta_p = \pi$$

and then construct the spline function for $h = 0.2, 0.1, 0.05$. Graphs of the spline function $s(t)$ are shown in Figures 5.1 - 5.3.

Example 5.2 $f(t) = e^{-30t^3}$

Consider the test function

$$f(t) = e^{-30t^3}, \quad t \in [0, 1]$$

We set the boundary conditions:

$$\beta_0 = 0, \quad \beta_p = -30e^{-10}$$

and then construct the spline function for $h = 0.2, 0.1, 0.05$. Graphs of the spline function $s(t)$ and its derivative are shown in Figures 5.4 - 5.6.

Example 5.3 *For our third example, we consider the function*

$$f(t) = \begin{cases} 0 & \text{if } -1 < t < 0 \\ 1/2 & \text{if } t = 0 \\ 1 & \text{if } 0 < t < 1. \end{cases}$$

We set the boundary conditions:

$$\beta_0 = 0, \quad \beta_p = 0$$

and then construct the spline function for $n = 80$. Graph of the spline function $s(t)$ is shown in Figure 5.7. In Figure 5.8 is the graph of this same function using the classical quintic spline. Comparing these two figures, we see that there is an improvement over the classical quintic approximant. However, we must realise that the scheme used to obtain Figure 5.7 contains only cubic terms. Further, only the first derivative of the control law is used in this case. Increasing the order of the derivatives of the control law used in the cost function, $J(u)$, will result in greater smoothness of the interpolant.

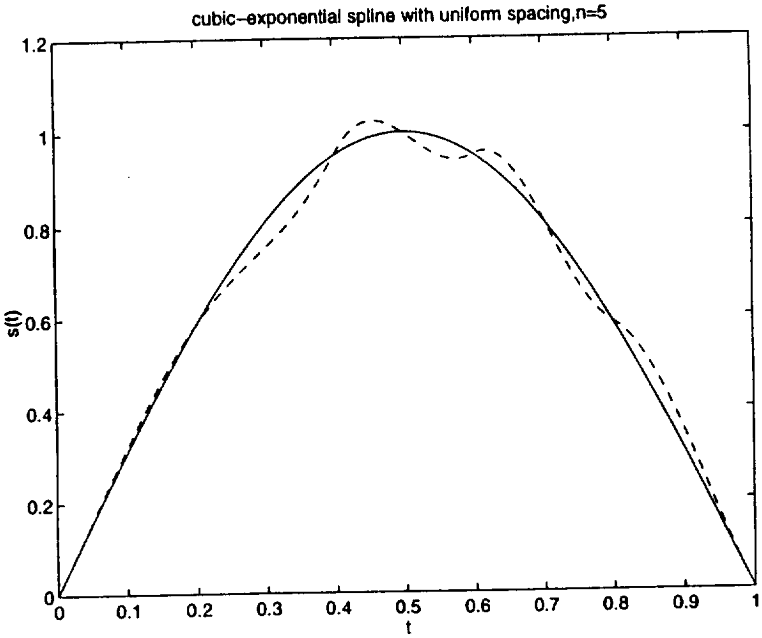

Figure 5.1: Function: $f(t) = \sin(\pi t), n = 5$

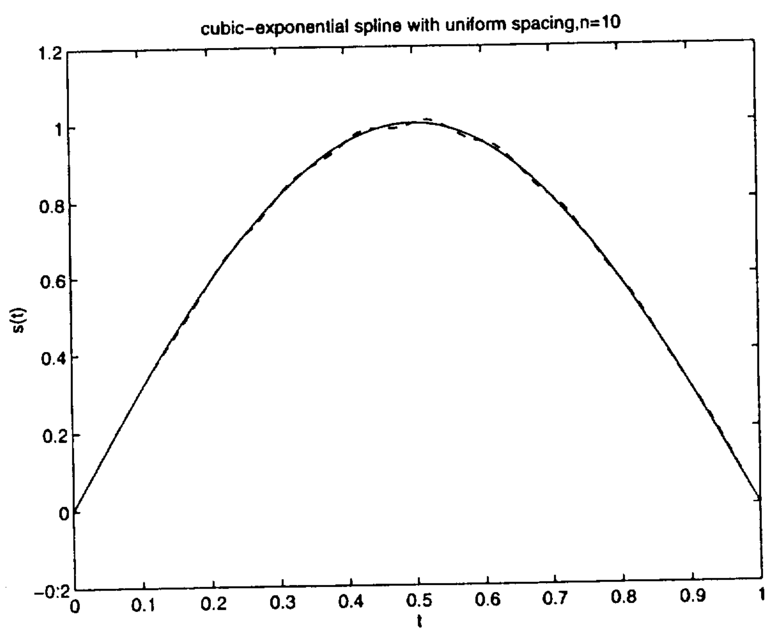

Figure 5.2: Function: $f(t) = \sin(\pi t), n = 10$

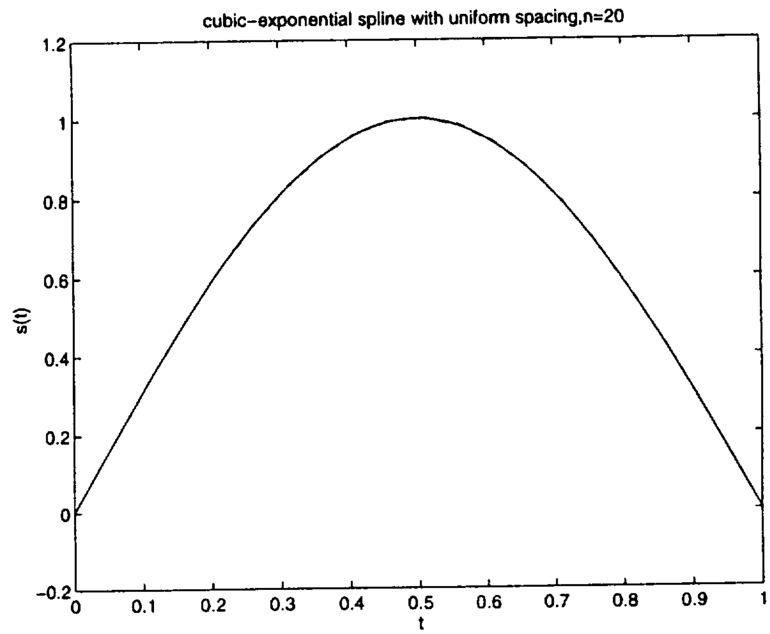

Figure 5.3: Function: $f(t) = \sin(\pi t), n = 20$

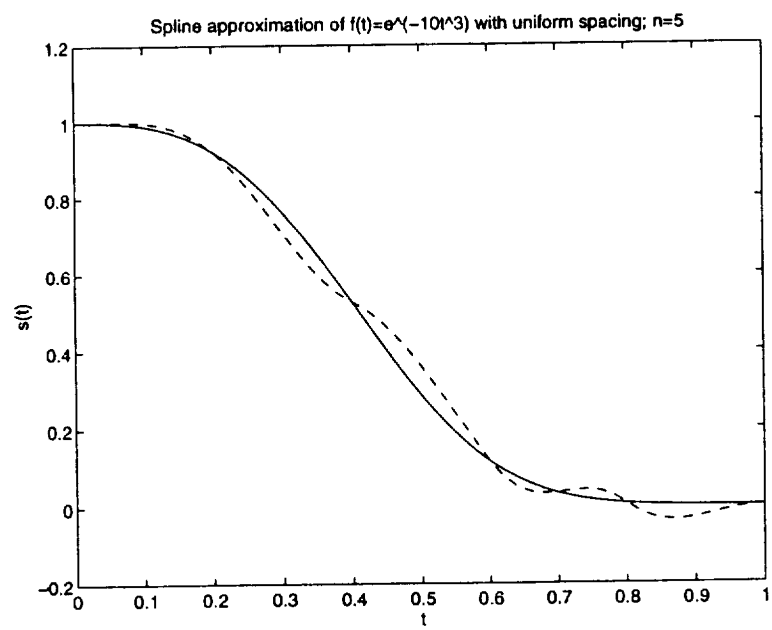

Figure 5.4: Function: $f(t) = e^{-10t^3}, n = 5$

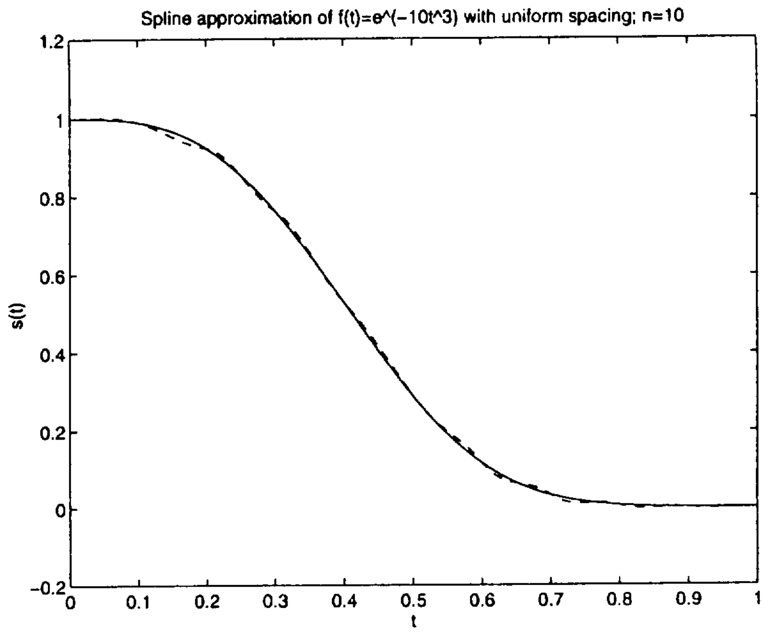

Figure 5.5: Function: $f(t) = e^{-10t^3}, n = 10$

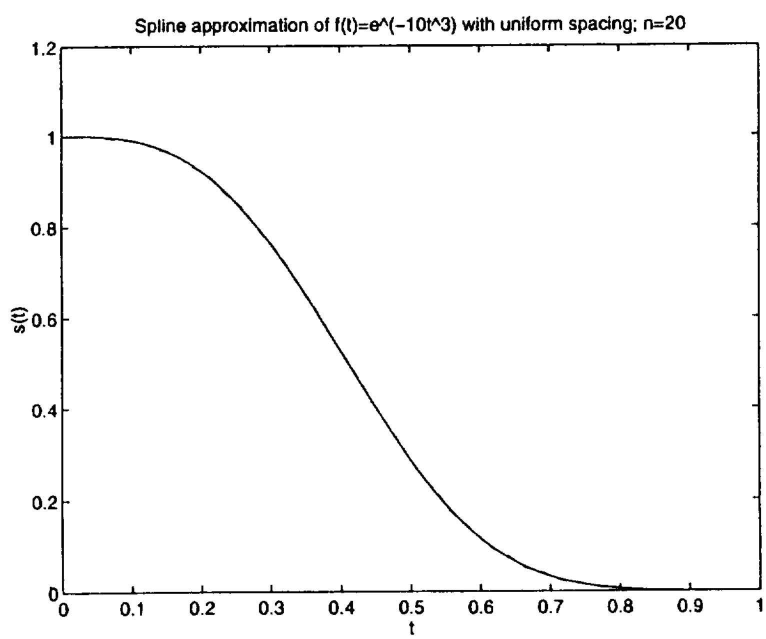

Figure 5.6: Function: $f(t) = e^{-10t^3}, n = 20$

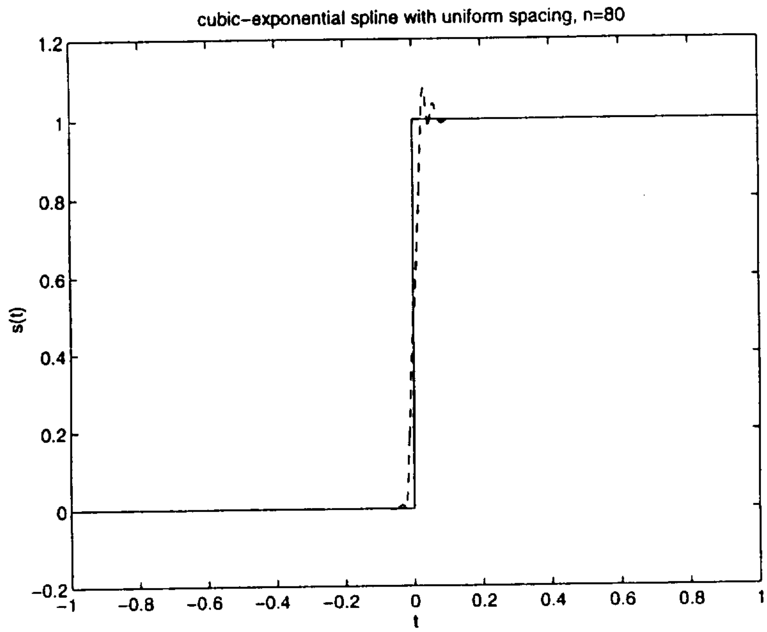

Figure 5.7: Interpolation by control theory approach

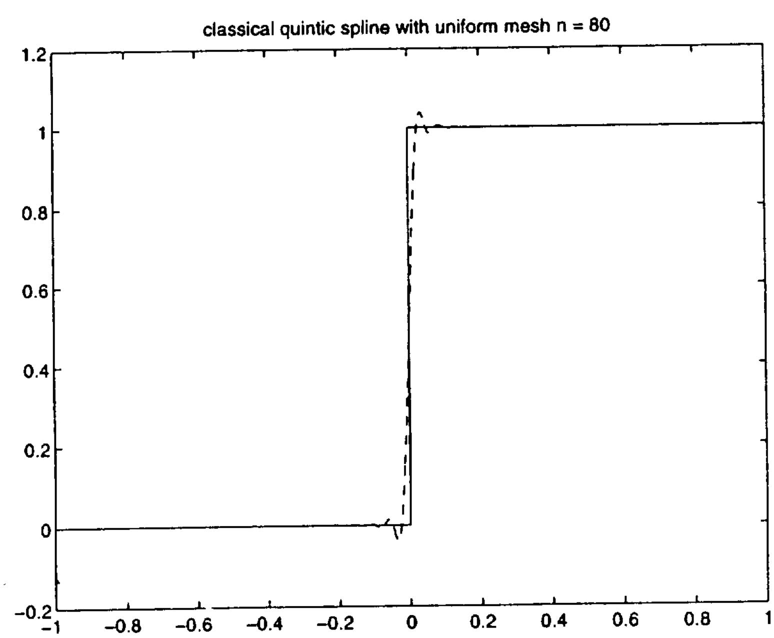

Figure 5.8: Classical quintic interpolation

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