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Unsteady Aerodynamics – Subsonic Compressible Inviscid Case

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ABSTRACT

This paper presents a new analytical treatment of Unsteady Aerodynamics — the linear theory covering the subsonic compressible (inviscid) case — drawing on some recent work in Operator Theory and Functional Analysis. The specific new results are:

(a) An existence and uniqueness proof for the Laplace transform version of the Possio integral equation as well as a new closed form solution approximation thereof.

(b) A new representation for the time-domain solution of the subsonic compressible aerodynamic equations emphasizing in particular the role of the initial conditions.

KEYWORDS: Unsteady Aerodynamics
Subsonic Compressible Inviscid Aerodynamics
Operator Theory / Semigroups / Laplace Transform
Possio Equation / Closed Form Solution / Initial Value Problem
Boundary Value Problem / Field Equations

02 Aerodynamics: 02–03 Airfoil and Wing Aerodynamics

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1. Introduction

This paper presents a new analytical treatment of Unsteady Aerodynamics drawing on techniques of Operator Theory and Functional Analysis. By Unsteady Aerodynamics we mean here the time-domain solution of the field equations — thereby also emphasizing the role of initial conditions generally ignored in the literature because of the preoccupation with the oscillatory response. In this paper we consider the subsonic (or linearized transonic) compressible case — inviscid of course — in two space dimensions. The extension to three space dimensions is in progress. This work was initiated as part of the Aeroelastic Stability problem — bending-torsion flutter in compressible flow. The incompressible case is treated in [1] for the 2-D strip model of Goland (see [3]).

To clarify terminology, by the "2-D Linear or Subsonic Compressible Case" we mean the flow characterized by the partial differential equations (being the linearized version of the inviscid TSD, Section 2) for the velocity potential, $\phi(x, z, t)$ in the X-Z plane, $-\infty < x < \infty$ and we take $0 \leq z$:

$$a_\infty^2(1-M^2)\,\frac{\partial^2\phi}{\partial x^2} \;+\; a_\infty^2\,\frac{\partial^2\phi}{\partial z^2} \;=\; \frac{\partial^2\phi}{\partial t^2} \;+\; 2Ma_\infty\,\frac{\partial^2\phi}{\partial t\partial x}\;. \tag{1.1}$$

The main boundary conditions are

i) Flow Tangency

$$\frac{\partial\phi}{\partial z}(x, 0+, t) \;=\; w_a(x, t), \quad |x| < b \tag{1.2}$$

where $w_a(x, t)$ is the downwash, $2b$ the strip width, and

ii) The Kutta-Joukowski conditions (see Section 2). (1.3)

The initial condtions are

$$\phi(x, z, 0), \quad \frac{\partial\phi}{\partial t}\,(x, z, 0).$$

The main interest is of course in the acceleration potential

$$\psi(x, z, t) \;=\; \frac{\partial\phi}{\partial t}\,(x, z, t) \;+\; Ma_\infty\,\frac{\partial\phi}{\partial x}\,(x, z, t)$$

(in particular for $z = 0+$) and its Laplace transform

$$\hat{\psi}(x, z, \lambda) \;=\; \int_0^\infty e^{-\lambda t}\,\psi(x, z, t)\;dt, \quad \operatorname{Re}\lambda > 0.$$

By a “particular” solution of (1.1) we mean a solution of (1.1) satisfying the boundary conditions but leaving the initial conditions unspecified — allowed to be arbitrary. The first such solution was given by Possio [2] in the form of an integral equation for the “oscillatory” case — where the downwash is an oscillation at the frequency ω:

$$w_a(x,t) \;=\; \hat{w}_a(x,i\omega)e^{i\omega t}, \quad t \geq 0$$

and the velocity potential then is also oscillatory,

$$\phi(x,z,t) \;=\; \hat{\phi}(x,z,i\omega)e^{i\omega t}.$$

Several authors (see [3]) have presented various versions of proofs, the clearest perhaps being [4]. In this paper we present the Laplace transform version valid for “artibrary motion” — for arbitrary downwash functions, where unlike in the oscillatory case, close attention has to be paid to the initial conditions; we do this by resurrecting the classical time-domain source-doublet integral of Küssner [5]. New with this paper is the existence and uniqueness proof for the Possio integral equation, as well as a new closed form analytical solution approximation. Numerical computations using this solution for the lift and moment [7] show close agreement with the results obtained by series expansions or numerical approximations (e.g., [8]) in the main region of interest of the parameters.

In Section 2 we provide an existence and uniqueness theorem for the time-domain solution of (1.1) for arbitrary initial conditions and satisfying the flow-tangency condition. Whether it satisfies the Kutta-Joukowski conditions depends on the aerodynamic initial conditions. Of course it is customary to dismiss solutions which don't as “nonphysical” as in the nonlinear case [9] but the fact remains that in the mathematical model we have to reckon with their existence. Indeed it is an open question to characterize the initial conditions that lead to solutions which satisfy the Kutta-Joukowski condition.

If we set the time derivatives in (1.1) to be zero, we have:

$$(1-M^2)\,\frac{\partial^2\phi}{\partial x^2} \;+\; \frac{\partial^2\phi}{\partial z^2} \;=\; 0, \tag{1.4}$$

but if we also retain the boundary conditions, the solution becomes a function of time since the downwash is. Again a particular solution can be obtained by classical techniques (see [3]) and we shall denote this by $\phi_M(x,z,t)$. However we should note that the initial conditions are not the same for $\phi_M(x,z,t)$ and $\phi_P(x,z,t)$, the Possio solution. If in (1.4) we set

$$M \;=\; 0; \quad U \;= a_\infty M \;\neq\; 0,$$

and U enters via the Kutta-Joukowski condition, we have the "incompressible" case which leads to the classical "airfoil" equation [3]. We shall denote this solution $\phi_0(x,z,t)$. It is possible to obtain $\phi_M(x,z,t)$ from $\phi_0(x,z,t)$ by a transformation of coordinates — the Prandtl-Glauert transformation — but we will not need to consider this at all.

One of our main results is that any solution $\phi(x,z,t)$ of (1.1) can be expressed as the superposition

$$\phi(x,z,t) \;=\; \phi_M(x,z,t) \;+\; \phi_R(x,z,t) \tag{1.5}$$

where $\phi_R(x,z,t)$ is a solution of (1.1) satisfying "homogeneous boundary" conditions, but with a forcing term depending on $\phi_M(\cdot)$ (see Section 5), and also satisfies the Kutta-Joukowski if $\phi(x,z,t)$ does. In particular therefore $\phi_P(x,z,t)$, the Possio solution, has such a representation. Note that depending on the initial conditions there are solutions of (1.1) satisfying the flow-tangency condition but not the Kutta-Joukowski conditions — whether "nonphysical" or not. Since the aerodynamic initial conditions are never known, this makes "unsteady aerodynamic" calculations (*e.g.*, [10]) difficult to verify by experiment.

In an ascending scale of accuracy we may put

$$\begin{gathered}\phi_0(x,z,t)\\ \phi_M(x,z,t)\\ \phi_P(x,z,t)\end{gathered}$$

(all satisfying the Kutta-Joukowski conditions) in the sense that each may be used to calculate the lift and moment as a function of U (for $\phi_0(\cdot)$), or of M, for $\phi_M(\cdot)$ and $\phi_P(\cdot)$. Note that the initial conditions are different for the latter two, unless we set

$$w_a(0,x) \;=\; \dot{w}_a(0,x) \;=\; 0.$$

Organization

We begin in Section 2 with an abstract (function space) formulation of (1.1) with an appropriate (L_2) definition of boundary values. We consider first the initial value problem for homogeneous boundary conditions leading to the time-domain semigroup solution (for appropriate definition of energy) and the resolvent, the Laplace domain solution. Unfortunately, some knowledge of the Theory of Semigroups is assumed in Section 2. Basically it provides the machinery for going from the Laplace domain to the time domain, and in our case both are important. In Section 3 we develop the Laplace transform version of the Possio equation, starting with Küssner's form of the time domain solution integral. The bulk of the section is devoted to deriving a constructive existence (and uniqueness) of solution leading to a closed form solution which is accurate to the order

$$M^2 \log \tfrac{1}{M}$$

in terms of M or

$$|\gamma^2 \log \gamma|$$

in terms of

$$\gamma \;=\; \frac{\lambda M}{U(1-M^2)}\;.$$

We show that this solution reduces to the known solution for $M = 0$.

In Section 4 we treat (1.4). We obtain a particular time-domain solution, satisfying the Kutta-Joukowski condition, for arbitrary downwash function without invoking the Prandtl-Glauert coordinate transformation. It is a more rigorous mathematical treatment than in [3] with a few new, more general results as well. Finally in Section 5 we assemble the complete solution — combining boundary value and inital value, connecting with the results in Section 2. In particular we show that the Possio solution can be written

$$\phi_P(t,x,z) \;=\; \phi_M(t,x,z) \;+\; \phi_R(t,x,z)$$

where $\phi_R(\cdot)$ satisfies (1.1) with homogeneous boundary condition and zero initial condition but a forcing term depending on $\phi_M(\cdot)$. We also derive an alternate expression for the Laplace transform $\hat{\phi}_P(\lambda,x,z)$ providing an alternate to the Possio equation.

2. The Field Equations

The basic Transonic Small Disturbance or TSD equations for the (perturbation) velocity potential in compressible flow are, following for example the derivation in [9], with $\phi(x, y, z, t)$ denoting the velocity potential:

$$\frac{\partial^2 \phi}{\partial t^2} + 2Ma_\infty \frac{\partial^2 \phi}{\partial t \partial x}$$
$$= a_\infty^2 (1 - M^2) \frac{\partial^2 \phi}{\partial x^2} - \varepsilon \frac{\partial \phi}{\partial x} \frac{\partial^2 \phi}{\partial x^2} + a_\infty^2 \frac{\partial^2 \phi}{\partial y^2} + a_\infty^2 \frac{\partial^2 \phi}{\partial z^2}, \tag{2.0}$$
$$-\infty < x < \infty, \quad -\infty < y < \infty, \quad 0 < z < \infty$$

where, in the usual notation, M is the Mach number and a_∞ the speed of sound, and

$$\varepsilon = (1 + \gamma) M^2 a_\infty^2, \quad \gamma \text{ being the adiabatic constant.}$$

In this paper we shall only consider the linear case where we take $\varepsilon = 0$; moreover we restrict ourselves to the "planar" case where the potential is independent of the variable y, so that the partial derivative with respect to y is set equal to zero. Thus we have for $\phi(x, z, t)$:

$$a_\infty^2 (1 - M^2) \frac{\partial^2 \phi}{\partial x^2} + a_\infty^2 \frac{\partial^2 \phi}{\partial z^2} = \frac{\partial^2 \phi}{\partial t^2} + 2Ma_\infty \frac{\partial^2 \phi}{\partial t \partial x}. \tag{2.1}$$

The boundary conditions that need to be imposed are:

i) *Flow Tangency:*

$$\frac{\partial \phi}{\partial z}(x, 0+, t) = w_a(t, x), \quad |x| < b \tag{2.2}$$

the right side being the "downwash" function which vanishes outside the interval $[-b, b]$, $2b$ being the "strip" width.

ii) *Kutta-Joukowski:*

This condition is stated in terms of the acceleration potential:

$$\psi(x, z, t) = \frac{\partial \phi(x, z, t)}{\partial t} + U \frac{\partial}{\partial x} \phi(x, z, t).$$

We require that

$$\psi(x, 0+, t) = 0, \quad x = b- \text{ and } x \geq b$$

($\equiv$ zero pressure jump at trailing edge).

iii) *Far-field Conditions:*

$$\frac{\partial \phi}{\partial x} = 0, \qquad \frac{\partial \phi}{\partial z} = 0$$

as $|x| \to \infty$ for each nonzero z and $z \to \infty$ for each x. See [3], [9] for more on these conditions.

In addition of course considering (2.1) as an "initial value" problem in the time coordinate t, we will need to specify initial conditions (at $t = 0$) as well.

Abstract Formulation

The first question to be decided is the choice of the function space in which to work. Let R_+^2 denote the half-plane:

$$-\infty < x < \infty, \qquad 0 \le z < \infty.$$

Our basic space will be $L_2\left[R_+^2\right]^3$ with "weights," norm (squared) defined by:

$$||f||^2 = ||f_1||^2 + (1 - M^2)a_\infty^2 ||f_2||^2 + a_\infty^2 ||f_3||^2 \tag{2.3}$$

where

$$f = \begin{vmatrix} f_1 \\ f_2 \\ f_3 \end{vmatrix}, \qquad f_i \in L_2\left[R_+^2\right].$$

We shall denote this Hilbert space by $\mathcal{H}$.

Boundary Values

Since the domain is not bounded we cannot use the usual "trace" definition for boundary values. Here we shall define them in the L^2 sense. We begin with the boundary $z = 0$. We shall say that $f(x)$, $-\infty < x < \infty$, $f(\cdot) \in L_2[R']$ is the boundary value on the boundary at $z = 0$ of a function $f(x, z)$, $|x| < \infty$, $0 < z < \infty$, in $L_2\left[R_+^2\right]$ if

$$\int_{-\infty}^{\infty} |f(x) - f(x, z)|^2 \, dx \tag{2.4}$$

which is defined a.e. in $0 < z < \infty$, goes to zero as $z \to 0+$.

Lemma

Suppose $f(\cdot,\cdot) \in L_2\left[R^2_+\right]$ and so does $\frac{\partial f(\cdot,\cdot)}{\partial z}$ (L_2-partial derivative). Then there exists $h(\cdot) \in L_2[R^1]$ such that

$$\int_{-\infty}^{\infty} |h(x) - f(x,z)|^2 \, dx \;\to\; 0 \quad \text{ as } \; z \to 0+ \; .$$

Proof

We note that

$$\int_0^{\infty} \left| \frac{\partial}{\partial z} \, f(x,z) \right|^2 \, dz \;<\; \infty, \qquad \text{a.e. in } \; -\infty < x < \infty.$$

Let $0 \leq z_n \to 0$. Then

$$\int_{-\infty}^{\infty} |f(x,z_n) - f(x,z_m)|^2 \, dx \;=\; \int_{-\infty}^{\infty} \left| \int_{z_m}^{z_n} \frac{\partial f(x,z)}{\partial z} \, dz \right|^2 \, dx.$$

$$\leq \; |z_n - z_m| \int_{-\infty}^{\infty} \int_{z_m}^{z_n} \left| \frac{\partial f(x,z))}{\partial z} \right|^2 \, dz \, dx \;\leq\; |z_n - z_m| \; \left\| \frac{\partial f}{\partial z} \right\|^2 .$$

Hence $f(\cdot, z_n)$ defines a Cauchy sequence in $L_2[R']$ which converges to a function, denote it h, in $L_2[R^1]$. And of course

$$\int_{-\infty}^{\infty} |h(x) - f(x,z)|^2 \;\to\; 0, \qquad \text{as } z \to 0.$$

The limit $h(\cdot)$ is clearly independent of the sequence chosen. Hence $h(\cdot)$ is the boundary value at $z = 0$, and we may use the notation

$$h(x) \;=\; f(x, 0+)$$

since it would be consistent with the pointwise limit, should there be one.

The situation is different on the boundary $x = +\infty$, $x = -\infty$ in that if, as we shall, we define the boundary value to be $f(+\infty, z)$, if

$$\int_0^{\infty} |f(x,z) - f(+\infty,z)|^2 \, dz \;\to\; 0 \quad \text{ as } \; x \to \infty$$

(and similarly $f(-\infty, z)$ as the boundary value at $x = -\infty$, if

$$\int_0^{\infty} |f(x,z) - f(-\infty,z)|^2 \, dz \;\to\; 0 \quad \text{ as } x \to -\infty \;\;)$$

in that

$$f_x(\cdot, \cdot) \;\in\; L_2\left[R^2_+\right]$$

does not assure this. Although we do have from

$$|f(L,z)|^2 \;=\; 2\,\mathrm{Re}\int_0^L \frac{\partial}{\partial x} f(x,t)\,\bar{f}(x,z)\;dx \;-\; |f(0,z)|^2$$

and $f \in L_2\left[R^2_+\right]$ that

$$\begin{aligned} f(L,z) &\;\to\; 0 \qquad \text{as} \quad L \to \infty \\ f(L,z) &\;\to\; 0 \qquad \text{as} \quad L \to -\infty, \quad \text{a.e. in} \quad z\,. \end{aligned}$$

Similarly we define the boundary value $f(x, \infty)$ by

$$\int_{-\infty}^{\infty} |f(x,\infty) - f(x,z)|^2\;dx \;\to\; 0 \qquad \text{as} \quad z \to \infty \tag{2.5}$$

assuming of course the limit exists.

We note that (2.1) is an initial-value plus boundary-value problem, albeit linear. The technique for solving such problems using the Theory of Semigroups of Operators is outlined in [7]. Thus we can construct the solution as part due to the initial value problem setting the boundary value to be zero and part due to the boundary problem without regard to the initial value. The former leads to a Cauchy problem and that is what we shall deal with first.

The Initial Value Problem: Semigroup Solution

We introduce the operator A with domain and range in $\mathcal{H}$ by:

$$\mathcal{D}(A) \;=\; \left[f \;=\; \begin{vmatrix} f_1 \\ f_2 \\ f_3 \end{vmatrix}, \quad \begin{matrix} \frac{\partial f_1}{\partial x} \text{ and } \frac{\partial f_1}{\partial z} \;\in\; L_2\left[R^2_+\right] \\ \frac{\partial f_2}{\partial x} \text{ and } \frac{\partial f_3}{\partial z} \;\in\; L_2\left[R^2_+\right] \end{matrix} \right]$$

with the "homogeneous" boundary conditions:

i) $\displaystyle\int_{-\infty}^{\infty} |f_3(x,\varepsilon)|^2\;dx \;\to\; 0 \quad \text{as} \;\; \varepsilon \to 0$

and the "regularity" conditions:

ii) $\displaystyle\int_0^{\infty} |f_i(L,z)|^2\;dz \;\to\; 0 \quad \text{as} \;\; |L| \to \infty, \;\; i = 1,2,3, \;\; \text{a.e. in} \;\; x$

iii) $\displaystyle\int_{-\infty}^{\infty} |f_i(x,L)|^2 \, dx \;\to\; 0 \quad \text{as } \; L \to \infty, \;\; \text{a.e. in } \; x$

$$Af \;=\; \left|\begin{array}{ccc} -2U\frac{\partial}{\partial x} & (1-M^2)a_\infty^2\frac{\partial}{\partial x} & a_\infty^2\frac{\partial}{\partial z} \\ \\ \frac{\partial}{\partial x} & 0 & 0 \\ \\ \frac{\partial}{\partial z} & 0 & 0 \end{array}\right| \begin{array}{c} f_1 \\ \\ f_2 \\ \\ f_3 \end{array} \;. \tag{2.6}$$

Note that the Kutta-Joukowski condition is omitted. Thus defined, A has clearly a dense domain. Moreover:

Lemma

A is dissipative (on its domain).

Proof

With [,] denoting inner products in $\mathcal{H}$, as well as $L^2\left[R_+^2\right]$, we have:

$$\begin{aligned} [Af,f] \;&=\; \left[\frac{-2U\partial f_1}{\partial x} \;+\; (1-M^2)a_\infty^2\frac{\partial f_2}{\partial x} \;+\; a_\infty^2\frac{\partial f_3}{\partial z}\,,\;\; f_1\right] \\ &\qquad + (1-M^2)a_\infty^2\left[\frac{\partial f_1}{\partial x}\,,\; f_2\right] \;+\; a_\infty^2\left[\frac{\partial f_1}{\partial z}\,,\; f_3\right] \end{aligned}$$

and

$$\begin{aligned} 2\,\mathrm{Re}\,[Af,f] \;&=\; -2U\left(\left[\frac{\partial f_1}{\partial x}\,,\; f_1\right] \;+\; \left[f_1\,,\; \frac{\partial f_1}{\partial x}\right]\right) \\ &\quad + (1-M^2)a_\infty^2\left(\left[\frac{\partial f_2}{\partial x}\,,\; f_1\right] + \left[f_1\,,\; \frac{\partial f_2}{\partial x}\right] + \left[\frac{\partial f_1}{\partial x}\,,\; f_2\right] + \left[f_2\,,\; \frac{\partial f_1}{\partial x}\right]\right) \\ &\quad + a_\infty^2\left(\left[\frac{\partial f_1}{\partial z}\,,\; f_3\right] \;+\; \left[f_3\,,\; \frac{\partial f_1}{\partial z}\right] \;+\; \left[\frac{\partial f_3}{\partial z}\,,\; f_1\right] \;+\; \left[f_1\,,\; \frac{\partial f_3}{\partial z}\right]\right). \end{aligned}$$

Now

$$\left[\frac{\partial f_1}{\partial x}\,,\ f_1\right] \ +\ \left[f_1\,,\ \frac{\partial f_1}{\partial x}\right]$$
$$= \lim_{L\to\infty}\int_0^\infty dz\ \int_{-L}^{L}\left(\frac{\partial f_1(x,z)}{\partial x}\ \overline{f_1(x,z)}\ +\ f_1(x,z)\ \overline{\frac{\partial f_1(x,z)}{\partial x}}\right)\ dx$$
$$= \lim_{L\to\infty}\int_0^\infty \left(|f_1(L,z)|^2\ -\ |f_1(-L,z)|^2\right)\ dz$$
$$= 0$$

$$\left[\frac{\partial f_2}{\partial x}\,,\ f_1\right] \ +\ \left[f_2\,,\ \frac{\partial f_1}{\partial x}\right] \ +\ \left[\frac{\partial f_1}{\partial x}\,,\ f_2\right] \ +\ \left[f_1\,,\ \frac{\partial f_2}{\partial x}\right]$$
$$= \lim_{L\to\infty}\int_0^\infty \Big[\ f_2(x,z)\,\bar f_1(x,z)\ +\ f_1(x,z)\,\bar f_2(x,z)\Big]_{-L}^{L}\ dz$$
$$= 0$$

by virtue of the boundary conditions. Next

$$\left[\frac{\partial f_1}{\partial z}\,,\ f_3\right] \ +\ \left[f_1\,,\ \frac{\partial f_3}{\partial z}\right] \ =\ \lim_{\substack{L\to\infty\\ \varepsilon\to 0+}}\int_{-\infty}^{\infty} \Big[\ f_1(x,z)\,\bar f_3(x,z)\Big]_{-\varepsilon}^{L}\ dx$$
$$= 0.$$

Similarly

$$\left[\frac{\partial f_3}{\partial z}\,,\ f_1\right] \ +\ \left[f_3\,,\ \frac{\partial f_1}{\partial z}\right] \ =\ 0$$

where we have used the vanishing boundary conditions at $x=\pm\infty$ and $z=\infty$. As for $z=0$ we note that

$$\int_{-\infty}^{\infty}\Big|f_1(x,z)\,\bar f_3(x,z)\Big|\ dz\ \le\ \sqrt{\int_{-\infty}^{\infty}|f_1(x,z)|^2\ dx}\ \sqrt{\int_{-\infty}^{\infty}|f_3(x,z)|^2\ dz}$$

and as $z\to 0+$ since $\frac{\partial f_1}{\partial z}\ \in L^2[R^2_+]$

$$\int_{-\infty}^{\infty}|f_1(x,z)|^2\ dx\ \to\ \int_{-\infty}^{\infty}|f_1(x,0+)|^2\ dx\ <\ \infty$$

and

$$\int_{-\infty}^{\infty} |f_3(x,z)|^2 \, dx \;\to\; 0 \quad \text{as} \;\; z \to 0+ \; .$$

Hence it follows that A is dissipative and has a dense domain, and A can be closed with the closure defined by

$$\bar{A} \;=\; (A^*)^*.$$

But using the usual arguments involving distributional derivatives, it is readily seen that:

$$A^* \;=\; -A \quad \text{on the domain of } A$$

and

$$A^* \;=\; -\bar{A}; \quad \left(\bar{A}\right)^* \;=\; -\bar{A}.$$

Hence $\bar{A}$ is dissipative and hence is the generator of a contraction C_0-semigroup which we shall denote by $S(t)$, $t \geq 0$. We hasten to remark that

$$(S(t))^{-1} \;=\; S(t)^*$$

and hence $S(t)$ becomes a group by defining

$$S(-t) \;=\; S(t)^{-1}$$

and of course

$$\|S(t)\| \;=\; 1 \qquad \text{for each } t, \; -\infty < t < \infty.$$

What is important for us is to note that the point spectrum of $\bar{A}$ is empty. ($\bar{A}$ has no eigenvalues.) The resolvent set of $\bar{A}$ is the set of those complex numbers λ such that the nonhomogeneous equation:

$$\lambda g \;-\; \bar{A} g \;=\; f$$

has a unique solution for every f in H, and we use the notation

$$g \;=\; R\left(\lambda, \bar{A}\right) f.$$

From the dissipativity of $\bar{A}$, it follows that the resolvent set of $\bar{A}$ includes all λ such that $\operatorname{Re}\lambda > 0$, and we actually have the construction

$$R\left(\lambda, \bar{A}\right) f \;=\; \int_0^{\infty} e^{-\lambda t} \, S(t) f \; dt, \quad \operatorname{Re}\lambda > 0.$$

In fact it can be recognized as the Laplace transform version of (2.1). Let

$$f \;=\; \begin{vmatrix} f_1 \\ f_2 \\ f_3 \end{vmatrix}, \qquad g \;=\; \begin{vmatrix} g_1 \\ g_2 \\ g_3 \end{vmatrix}.$$

Then (assuming that f is such that g is actually in $\mathcal{D}(A)$), we have

$$\begin{aligned} g_1 &= \lambda\hat{\phi}(x,z,\lambda) \\ g_2 &= \frac{\partial}{\partial x}\,\hat{\phi}(x,z,\lambda) \\ g_3 &= \frac{\partial}{\partial z}\,\hat{\phi}(x,z,\lambda) \end{aligned}$$

where $\hat{\phi}(x,z,\lambda)$ is the solution of

$$\begin{aligned} \lambda^2\hat{\phi}(\lambda,x,z) \;+\; 2Ma_\infty\lambda\,\frac{\partial\hat{\phi}(\lambda,x,z)}{\partial x} \;-\; a_\infty^2\,(1-M^2)\,\frac{\partial^2\hat{\phi}(\lambda,x,z)}{\partial x^2} & \\ -\,a_\infty^2\,\frac{\partial^2\hat{\phi}(\lambda,x,z)}{\partial z^2} \;&=\; f_1(\lambda,x,z), \quad -\infty<x<\infty,\; 0<z<\infty. \end{aligned}$$

Resolvent

To evaluate the resolvent, it is convenient to work with Fourier transforms. Thus for $f(\cdot,\cdot)$ in $L_2\left[R_+^2\right]$, we define (in the usual sense):

$$\hat{f}(i\omega_1\,,\,i\omega_2) \;=\; \int_{-\infty}^{\infty}\int_0^{\infty} e^{-i\omega_1x-i\omega_2z}f(x,z)\;dx\;dz, \quad -\infty<\omega_1\,,\,\omega_2<\infty. \tag{2.7}$$

Also since for $\sigma>0$

$$\begin{aligned} \int_{-\infty}^{\infty}\left(\int_0^{\infty} e^{-\sigma z}\,|f(x,z)|\;dz\right)^2\;dx \;&\le\; \int_{-\infty}^{\infty}\frac{1}{\sigma}\int_0^{\infty} e^{-\sigma z}\,|f(x,z)|\;dz\;dx \\ &\le\; \frac{1}{\sigma}\int_{-\infty}^{\infty}\int_0^{\infty}|f(x,z)|^2\;dz\;dx \;<\;\infty, \end{aligned}$$

we can also define the Fourier-Laplace transform

$$\hat{f}(\omega_1,\mu) \;=\; \int_{-\infty}^{\infty} e^{-i\omega_1x}\;dx\;\int_0^{\infty} e^{-\mu z}f(x,z)\;dz, \quad \mathrm{Re}\,\mu>0. \tag{2.8}$$

In particular we see that for f in $\mathcal{D}(A)$,

$$f \;=\; \left|\begin{array}{c} f_1 \\ f_2 \\ f_3 \end{array}\right|, \quad \mu>0,$$

$$\lim_{\mu\to\infty} \mu \hat{f}_1(\omega_1\,,\,\mu) \;=\; \int_{-\infty}^{\infty} e^{-i\omega_1 x}\, f_1(x,0+)\; dx \tag{2.9}$$

which we can also express as:

$$= \; \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}_1(i\omega_1\,,\, i\omega_2)\; d\omega_2\,. \tag{2.10}$$

Let

$$g \;=\; Af.$$

Then noting that a.e. in x:

$$\int_0^\infty e^{-i\omega_2 z}\, \frac{\partial f_1(x,z)}{\partial z}\; dz \;=\; -f_1(x,0+) \;+\; i\omega_2 \int_0^\infty e^{-i\omega_2 z} f_2(x,z)\; dz$$

we have:

$$\hat{g} \;=\; H \begin{vmatrix} f_1 \\ f_2 \\ f_3 \end{vmatrix} - \begin{vmatrix} 0 \\ 0 \\ \int_{-\infty}^{\infty} e^{-i\omega_1 x} f_1(x,0+)\; dx \end{vmatrix} \tag{2.11}$$

where H is the multiplier matrix:

$$H \;=\; \begin{vmatrix} -2Ui\omega_1 & a_\infty^2(1-M^2)(i\omega_1) & a_\infty^2 i\omega_2 \\ i\omega_1 & 0 & 0 \\ i\omega_2 & 0 & 0 \end{vmatrix}. \tag{2.12}$$

Note that we can use (2.11) to define $\bar{A}$ as a "multiplier" operator. Moreover

$$\lambda f \;-\; \bar{A} f \;=\; g$$

becomes

$$\hat{g} \;=\; (\lambda I_3 \;-\; H)\hat{f} \;+\; \begin{vmatrix} 0 \\ 0 \\ \int_{-\infty}^{\infty} e^{-i\omega_1 x} f_1(x,0+)\; dx \end{vmatrix}$$

and hence

$$f \;=\; R\left(\lambda, \bar{A}\right) g, \quad \operatorname{Re}\lambda \neq 0,$$

becomes

$$\hat{f} \; + \; (\lambda I_3 \; - \; H)^{-1} \left| \begin{array}{c} 0 \\ \\ 0 \\ \\ \frac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} \hat{f}_1(i\omega_1\,,\, i\omega_2)\; d\omega_2 \end{array} \right| \; = \; (\lambda I_3 \; - \; H)^{-1} \hat{g}. \qquad (2.13)$$

Noting that

$$(\lambda I_3 - H)^{-1} \;=\; \frac{1}{d(\lambda)} \left| \begin{array}{ccc} \lambda^2 & \lambda c_1 i\omega_1 & \lambda c_2 i\omega_2 \\ \\ \lambda i\omega_1 & \lambda^2 + 2\lambda U i\omega_1 + c_2\omega_2^2 & c_2\omega_1\omega_2 \\ \\ -\lambda i\omega_2 & c_2\omega_1\omega_2 & \lambda^2 + 2\lambda U i\omega_1 + c_1\omega_1^2 \end{array} \right|$$

where

$$c_1 \;=\; (1 - M^2)a_\infty^2\,, \quad c_2 \;=\; a_\infty^2$$

$$\begin{aligned} d(\lambda) \;&=\; \lambda\left(\lambda^2 \;+\; 2Ui\omega_1\lambda \;+\; c_1\omega_1^2 \;+\; c_2^2\omega_2^2\right) \\ &\neq\; 0 \qquad\qquad\qquad\qquad \text{for} \;\; \lambda \neq i\omega, \;\; \omega \text{ real}\;, \end{aligned}$$

we have, writing μ in place of $i\omega_2$:

$$\hat{f} \;=\; \left| \begin{array}{c} \hat{f}_1 \\ \hat{f}_2 \\ \hat{f}_3 \end{array} \right| \,,$$

$$\hat{f}_1 \;=\; \frac{\lambda^2\hat{g}_1 \;+\; \lambda c_1(i\omega_1)\hat{g}_2 \;+\; \lambda\mu c_2\left(\hat{g}_3 - \hat{f}_{1b}\right)}{\lambda\left(\lambda^2 + 2Ui\omega_1\lambda + c_1\omega_1^2 - c_2\mu^2\right)}\,, \quad \mathrm{Re}\,\mu \geq 0 \qquad (2.13a)$$

where $\hat{f}_{1b}$ is a function of $i\omega_1$ defined by

$$\hat{f}_{1b}(i\omega_1) \;=\; \lim_{\mu\to\infty} \mu\hat{f}_1\,(i\omega_1\,,\,\mu)$$

$$\hat{f}_2 \;=\; \frac{1}{d(\lambda)}\left(\lambda i\omega_1\hat{g}_1 \;+\; (\lambda^2 + 2\lambda U i\omega_1 - c_2\mu^2)\hat{g}_2 \;-\; c_2\mu(i\omega_1)\left(\hat{g}_3 - \hat{f}_{1b}\right)\right), \quad \mathrm{Re}\,\mu \geq 0, \qquad (2.13b)$$

$$\hat{f}_3 \;=\; \frac{1}{d(\lambda)}\left(\lambda\mu\hat{g}_1 \;+\; c_2(i\omega_1)\mu\hat{g}_2 \;+\; (\lambda^2 + 2\lambda U i\omega_1 + c_1\omega_1^2)\left(\hat{g}_3 - \hat{f}_{1b}\right)\right), \quad \operatorname{Re}\mu \geq 0. \tag{2.13c}$$

We should also note the relationships:

$$f_1(x,z) \;=\; \lambda\int_{-\infty}^{x} f_2(y,z)\;dy \;-\; \int_{-\infty}^{x} g_2(y,z)\;dy$$

and when $g_2(y, 0+)$ is defined:

$$f_1(x,0+) \;=\; \lambda\int_{-\infty}^{x} f_2(y,0+)\;dy \;-\; \int_{-\infty}^{x} g_2(y,0+)\;dy.$$

We know of course from the general theory that $R(\lambda, \bar{A}\,)$ is defined for λ not pure imaginary. But (2.13a)–(2.13c) may well continue to be defined for λ imaginary as well for some g since the range of

$$\left(i\omega I \;-\; \bar{A}\right)$$

is dense in $\mathcal{H}$. Indeed if

$$d(\lambda) \;=\; 0,$$

then

$$\begin{aligned} \lambda \;&=\; \frac{-2Ui\omega_1 \;\pm\; \sqrt{-4\omega_1^2U^2 \;-\; 4(c_1\omega_1^2 + c_2\omega_2^2)}}{2} \\ &=\; i\left[-U\omega_1 \;\pm\; \sqrt{\omega_1^2U^2 + c_1\omega_1^2 + c_2\omega_2^2}\,\right]. \end{aligned}$$

We need only require that the numerators in (2.13a)–(2.13c) are also zero for this value of λ, so that $\hat{f}_i$ defined therein with $i\omega_2$ in place of μ are square-integrable.

Spectrum of $\bar{A}$

The resolvent of $\bar{A}$ is clearly not compact being characterizable as a "multiplier" operator. Hence we cannot expect that $\bar{A}$ has eigenvalues. In fact the point spectrum of $\bar{A}$ is empty, and it has a pure continuous spectrum: $\operatorname{Re}\lambda \leq 0$. Again this follows from the "multiplier" characterization. Indeed

$$Af \;=\; \lambda f$$

yields from (2.13a) that

$$\hat{f}_1(i\omega_1\,,\,i\omega_2) \;=\; \frac{-i\omega_2\, c_2\, \hat{f}_{1b}}{(\lambda^2 + 2\nu i\omega_1\lambda + c_1\omega_1^2 + c_2\omega_2^2)}$$

and integrating with respect to ω_2 yields:

$$\hat{f}_{1b} \;=\; \int_{-\infty}^{\infty} \hat{f}_1(i\omega_1\,,\,i\omega_2)\, d\omega_2 \;=\; 0.$$

Hence $\hat{f}_1$ is zero also, and so is f, in turn.

Velocity Potential

Given the semigroup solution

$$F(t) \;=\; S(t)\, F(0), \quad F(0) \in \mathcal{D}(A)$$

where $F(t)$ denotes

$$\begin{vmatrix} f_1(t,x,z) \\ f_2(t,x,z) \\ f_3(t,x,z) \end{vmatrix}$$

we can relate the solution to (2.1) in the following way. Define the velocity potential $\phi(t,x,z)$ by:

$$\phi(t,x,z) \;=\; \phi(0,x,z) \;+\; \int_0^t f_1(\sigma,x,z,)\, d\sigma \tag{2.14}$$

where $\phi(0,x,z)$ is required to be such that

$$\frac{\partial\phi}{\partial x}\,,\,\frac{\partial\phi}{\partial z} \;\in\; L_2\left[R_+^2\right].$$

Then we can calculate that

$$\frac{\partial^2 \phi}{\partial t^2} \;+\; 2U\,\frac{\partial^2 \phi}{\partial t \partial x} \;=\; (1 - M^2) a_\infty^2\,\frac{\partial f_2}{\partial x} \;+\; a_\infty^2\,\frac{\partial f_3}{\partial z}$$

$$\frac{\partial^2 \phi}{\partial t \partial x} \;=\; \frac{\partial^2 \phi}{\partial x \partial t} \;=\; \frac{\partial f_1}{\partial x} \;=\; \frac{\partial f_2}{\partial t}\,.$$

Hence

$$\frac{\partial}{\partial t}\left[\frac{\partial \phi}{\partial x} \;-\; f_2\right] \;=\; 0. \tag{2.14a}$$

We define

$$\frac{\partial \phi(0, x, z)}{\partial x} \;=\; f_2(0, x, z)$$

so that

$$\frac{\partial \phi}{\partial x} \;=\; f_2\,.$$

Similarly

$$\frac{\partial^2 \phi}{\partial t \partial z} \;=\; \frac{\partial^2 \phi}{\partial z \partial t} \;=\; \frac{\partial f_1}{\partial z} \;=\; \frac{\partial f_3}{\partial t}\,.$$

Hence

$$\frac{\partial}{\partial t}\left[\frac{\partial \phi}{\partial z} \;-\; f_3\right] \;=\; 0.$$

We define

$$\frac{\partial \phi(0, x, z)}{\partial z} \;=\; f_3(0, x, z)$$

and hence

$$\frac{\partial \phi}{\partial z} \;=\; f_3\,.$$

Hence it follows that $\phi(t, x, z)$ satisfies

$$\frac{\partial^2 \phi}{\partial t^2} \;+\; 2U\,\frac{\partial^2 \phi}{\partial t \partial x} \;=\; (1 - M^2) a_\infty^2\,\frac{\partial^2 \phi}{\partial x^2} \;+\; a_\infty^2\,\frac{\partial^2 \phi}{\partial z^2} \tag{2.15}$$

with

$$\phi(0, x, z) \qquad \text{given} \tag{2.16}$$

$$\frac{\partial}{\partial t}\,\phi(t, x, z)\bigg|_{t=0} \;\text{given} \;=\; f_1(0, x, z) \tag{2.17}$$

$$\frac{\partial \phi}{\partial x}\,(t,x,z,)\bigg|_{t=0} \;=\; f_2(0,x,z) \tag{2.18}$$

$$\frac{\partial \phi}{\partial z}\,(t,x,z,)\bigg|_{t=0} \;=\; f_3(0,x,z) \tag{2.19}$$

where

$$F(0,x,z) \;=\; \left|\begin{array}{c} f_1(0,x,z) \\ f_2(0,x,z) \\ f_3(0,x,z) \end{array}\right| \;\in\; \mathcal{D}(A).$$

Also we see that the solution is unique, subject to the initial conditions (2.16)–(2.19) and the boundary conditions imposed in describing the domain of A. Note also that from (2.14a) we have

$$\frac{\partial f_2}{\partial t} \;=\; \frac{\partial}{\partial t}\,\frac{\partial \phi}{\partial x} \;=\; \frac{\partial}{\partial x}\,\frac{\partial \phi}{\partial t} \;=\; \frac{\partial f_1}{\partial x}\;.$$

Hence

$$f_1(t,x,z) = \int_{-\infty}^{x} \frac{\partial f_2(t,y,z)}{\partial t}\, dy \tag{2.20}$$

Kutta-Joukowski Condition

So far we have not addressed the Kutta-Joukowski condition. In view of (2.14) where we may think of $f_1(t,x,z)$ as the time-derivative of the velocity potential, and $f_2(t,x,z)$ as the derivative

$$\frac{\partial \phi(t,x,z)}{\partial x}$$

we may state the Kutta-Joukowski condition as

$$\begin{aligned} f_1(t,x,0+) \;+\; Uf_2(t,x,0+) \;&=\; 0 \qquad \text{at } x = b- \\ &=\; 0 \qquad \text{at } \; x \geq b. \end{aligned} \tag{2.21}$$

We may then include it as a restriction of the domain of A:

$$D(A_0) \subset D(A)$$

$$D(A_0) = [f \mid f \in \mathcal{D}(A) \text{ and } f \text{ satisfies } (2.21)]$$

$$A_0 f = Af.$$

The restriction A_0 thus defined still has a dense domain and is of course dissipative thereon but the closure of A_0 is readily seen to be $\bar{A}$.

We can also give a "dynamic" Kutta-Joukowski condition in the following way. The acceleration potential defined by

$$\psi(t,x,z) = \frac{\partial \phi(t,x,z)}{\partial t} + U \frac{\partial \phi(t,x,z)}{\partial x}$$

can by virtue of (2.20) be expressed in the form

$$\psi(t,x,z) = \int_{-\infty}^{x} \frac{\partial f_2(t,y,z)}{\partial t}\, dy + U f_2(t,x,z). \tag{2.22}$$

The Kutta-Joukowski condition can then be expressed entirely in terms of $f_2(t,x,z)$ as a "dynamic" condition since it involves the time-derivative. Thus $\psi(t,x,z)$ defined by (2.22) is required to satisfy

$$\psi(t,x,0+) = 0 \quad \text{at } x = -b \text{ and for } x \geq b.$$

We shall refer to this as the "dynamic" Kutta-Joukowski condition.

3. Solving the Boundary Value Problem: Possio's Integral Equation, Laplace Transform Version

In this section we obtain one solution $\phi(t, x, z)$ to the boundary-value problem:

$$\frac{\partial^2\phi}{\partial t^2} + 2U\frac{\partial^2\phi}{\partial t\partial x} = a_\infty^2(1-M^2)\frac{\partial^2\phi}{\partial x^2} + a_\infty^2\frac{\partial^2\phi}{\partial z^2} \tag{3.1}$$

satisfying the flow tangency condition, the Kutta-Joukowski condition and the far field conditions, but without regard to any initial conditions.

We begin with the construction for a particular solution of (3.1) due to Küssner [5, p.7] (see [18], for a direct proof that (3.1) is satisfied), using doublets at the source:

$$\phi(x,y,z,t) = \frac{1}{4\pi}\int_{-b}^{b} d\xi \int_{-\infty}^{\infty} d\eta \int_{-\infty}^{x-\xi} dx'$$

$$\cdot \frac{\partial}{\partial z}\left[\frac{A\left(\xi,\ \eta,\ t + \frac{x'}{U(1-M^2)} - \frac{x-\xi}{U} - \frac{\sqrt{x'^2+(1-M^2)((y-\eta)^2+z^2)}}{a_\infty(1-M^2)}\right)}{\sqrt{x'^2 + (1-M^2)((y-\eta)^2+z^2)}}\right]$$

where $A(\xi, \eta, t)$ is the doublet intensity on the airfoil which is to be determined from the boundary conditions. Note that

$$\phi(x, y, -z, t) = -\phi(x, y, z, t).$$

And further,

$$\phi(x, y, 0+, 0+) = 0.$$

Since $\phi(x, y, z, t)$ does not, in our case, depend on y, we have:

$$\phi(x,z,t) = \frac{1}{4\pi}\int_{-b}^{b} d\xi \int_{-\infty}^{\infty} d\eta \int_{-\infty}^{x-\xi} dx'$$

$$\cdot \frac{\partial}{\partial z}\left[\frac{A\left(\xi,\ \eta,\ t - \frac{x-\xi}{U} + \frac{x'}{U(1-M^2)} - \frac{\sqrt{x'^2+(1-M^2)((\eta^2+z^2)}}{a_\infty(1-M^2)}\right)}{\sqrt{x'^2 + (1-M^2)(\eta^2+z^2)}}\right] \tag{3.2}$$

where $A(\xi, \eta, t)$ does not depend on η either, so that we may denote it $A(\xi, t)$.

Initial Conditions

We note that (3.2) defines the initial ($t = 0$) values of the velocity field. It is easy to verify from

$$A(\xi, t) = 0, \quad t < 0$$

that

$$\phi(x,z,0) \;=\; 0$$

$$\dot{\phi}(x,z,0) \;=\; 0$$

from the fact that the set

$$x' \;<\; x-\xi$$

$$\frac{-(x-\xi)}{U} \;+\; \frac{x'}{U(1-M^2)} \;-\; \frac{\sqrt{x'^2 \;+\; (1-M^2)(\eta^2+z^2)}}{a_\infty(1-M^2)} \;>\; 0$$

is actually empty, for $M<1$.

Since

$$w_a(x,t) \;=\; \frac{\partial}{\partial z}\,\phi(x,z,t)\bigg|_{z=0}$$

we have from (3.2) that

$$\begin{aligned} w_a(x,t) \;&=\; \frac{1}{4\pi}\int_{-b}^{b} d\xi \;\int_{-\infty}^{\infty} d\eta \;\int_{-\infty}^{x-\xi} dx' \\ &\quad\cdot\; \frac{\partial^2}{\partial z^2}\left[\frac{A\left(\xi,\; t \;-\; \frac{x-\xi}{U} \;+\; \frac{x'}{U(1-M^2)} \;-\; \frac{-\sqrt{x'^2+(1-M^2)(\eta^2+z^2)}}{a_\infty((1-M^2))}\right)}{\sqrt{x'^2 \;+\; (1-M^2)(\eta^2+z^2)}}\right]\Bigg|_{z=0} \end{aligned} \tag{3.2a}$$

Unfortunately this is as far as we can go in the time domain. We therefore invoke either Laplace or Fourier transforms. Traditionally the Fourier transform is preferred because one may think of an "oscillating" doublet corresponding to an oscillatory downwash input. The first such derivation is due to Possio [2] leading to the Possio integral equation, the most lucid derivation of which may be found in [4]. As we show later, the Laplace transform can formally be obtained from the "oscillatory" version, and also vice versa. Here however we shall actually derive the Laplace transform version of the Possio equation, emphasizing the role of initial conditions. It will turn out that this is actually more useful than the time-domain solution since generally the primary interest is in the stability of an aeroelastic system [1]. A Laplace transform version is given in [11, p.3] but it involves divergent integrals which must be interpreted carefully. In fact the author derives a formula ([11, p.4, eq.31), for the initial conditions therefrom which is invalid in the present (2-D) case.

We note that

$$A(\xi,\tau) \;=\; 0, \quad \tau < 0.$$

Let

$$a \;=\; \frac{x-\xi}{U} \;-\; \frac{x'}{U(1-M^2)} \;+\; \frac{\sqrt{x'^2+(1-M^2)(\eta^2+z^2)}}{a_\infty(1-M^2)}\;.$$

Then since we must have that

$$t \;-\; a \;\geq\; 0 \quad \text{for all } t>0$$

we need only consider

$$a \;\leq\; 0.$$

Hence defining the Laplace transforms:

$$\hat{A}(\xi,\lambda) \;=\; \int_0^\infty e^{-\lambda t}\,A(\xi,t)\;dt; \quad \hat{\phi}(x,z,\lambda) \;=\; \int_0^\infty e^{-\lambda t}\,\phi(x,z,t)\;dt, \quad \operatorname{Re}\lambda > 0$$

we have, taking Laplace transforms in (3.2):

$$\hat{\phi}(x,z,\lambda) \;=\; \frac{1}{4\pi}\int_{-b}^{b} d\xi \int_{-\infty}^{x-\xi} dx' \cdot \exp\left(\lambda\left(\frac{x'}{U(1-M^2)} - \frac{x-\xi}{U}\right)\right) \cdot \hat{A}(\xi,\lambda)$$

$$\cdot\; \frac{\partial}{\partial z}\int_{-\infty}^{\infty} \frac{\exp\left(-\lambda\,\frac{\sqrt{x'^2+(1-M^2)(\eta^2+z^2)}}{a_\infty(1-M^2)}\right)}{\sqrt{x'^2\;+\;(1-M^2)(\eta^2+z^2)}}\;d\eta.$$

Making the change of variable,

$$\zeta^2 \;-\; 1 \;=\; \frac{(1-M^2)\eta^2}{a^2}\;, \quad a^2 \;=\; x'^2 \;+\; (1-M^2)z^2$$

we have:

$$\zeta^2 \;\geq\; 1, \quad \zeta^2 \;=\; \frac{a^2+(1-M^2)\eta^2}{a^2}\;, \quad \zeta \;=\; \frac{\sqrt{x'^2\;+\;(1-M^2)(\eta^2+z^2)}}{a}$$

$$d\eta \;=\; \frac{a^2}{1-M^2}\,\frac{\zeta}{\eta}\,d\zeta\;, \quad \eta^2 \;=\; \frac{a^2(\zeta^2-1)}{(1-M^2)}$$

$$\int_{-\infty}^{\infty} \frac{\exp\left(-\lambda \frac{\sqrt{x'^2 + (1-M^2)(\eta^2+z^2)}}{a_\infty(1-M^2)}\right)}{\sqrt{x'^2 \ + \ (1-M^2)(\eta^2 + z^2)}} \ d\eta$$

$$= \ 2\int_1^{\infty} \exp\left(\frac{-\lambda a\zeta}{a_\infty(1-M^2)}\right) \cdot \frac{1}{\zeta a} \cdot \frac{\zeta}{\eta} \cdot \frac{a^2}{(1-M^2)} \ d\zeta$$

$$= \ 2\int_1^{\infty} \exp\left(\frac{-\lambda a\zeta}{a_\infty(1-M^2)}\right) \cdot \frac{1}{\zeta a} \cdot \frac{\zeta\sqrt{1-M^2}}{a\sqrt{\zeta^2-1}} \cdot \frac{a^2}{(1-M^2)} \ d\zeta$$

$$= \ \frac{2}{\sqrt{1-M^2}} \ \int_1^{\infty} \exp\left(\frac{-\lambda a\zeta}{a_\infty(1-M^2)}\right) \ \frac{1}{\sqrt{\zeta^2-1}} \ d\zeta$$

$$= \ \frac{2}{\sqrt{1-M^2}} \ K_0\left(\frac{\lambda\sqrt{x'^2 \ + \ (1-M^2)z^2}}{a_\infty(1-M^2)}\right).$$

Hence

$$\hat{\phi}(x,z,\lambda) \ = \ \frac{1}{2\pi}\int_{-b}^{b} d\xi \cdot \hat{A}(\xi,\lambda)$$

$$\cdot \int_{-\infty}^{x-\xi}\left(\frac{1}{\sqrt{1-M^2}}\right) \ \exp\left(\frac{\lambda}{U}\left(\frac{x'}{1-M^2} \ - \ (x-\xi)\right)\right)$$

$$\cdot \ \frac{\partial}{\partial z} \ K_0\left(\frac{\lambda\sqrt{x'^2 \ + \ (1-M^2)z^2}}{a_\infty(1-M^2)}\right) \ dx' \qquad (3.3)$$

which is our basic formula in what follows. Note that:

$$\frac{\partial}{\partial z} \ K_0\left(\frac{\lambda\sqrt{x'^2 + (1-M^2)z^2}}{a_\infty(1-M^2)}\right)$$

$$= \ -K_1\left(\frac{\lambda\sqrt{x'^2 \ + \ (1-M^2)z^2}}{a_\infty(1-M^2)}\right) \cdot \frac{\lambda}{a_\infty(1-M^2)} \cdot \frac{(1-M^2)z}{\sqrt{x'^2 + (1-M^2)z^2}}$$

$$= \ \frac{\lambda}{a_\infty} \ (-1)K_1\left(\frac{\lambda\sqrt{x'^2 + (1-M^2)z^2}}{a_\infty(1-M^2)}\right) \cdot \frac{z}{\sqrt{x'^2 + (1-M^2)z^2}} \ . \qquad (3.3a)$$

We invoke next the boundary condition

$$w_a(x,t) \ = \ \frac{\partial}{\partial z} \ \phi(x,0+,t).$$

Denoting the Laplace transform

$$\int_0^\infty e^{-\lambda t}\, w_a(x,t)\; dt \quad \text{ by } \;\; \hat{w}_a(x,\lambda)$$

and noting that

$$\int_0^\infty e^{-\lambda t}\, \frac{\partial}{\partial z}\, \phi(x,0+,t)\; dt \;=\; \frac{\partial}{\partial z}\, \hat{\phi}(x,0+,\lambda),$$

we have from (3.3) that

$$\hat{w}_a(x,\lambda) \;=\; \frac{1}{2\pi}\,\lim_{z\to 0+}\int_{-\infty}^{\infty} d\xi\;\; \hat{A}(\xi,\lambda)\cdot \frac{1}{\sqrt{1-M^2}}\int_{-\infty}^{x-\xi} \exp\left(\frac{\lambda}{U}\left(\frac{y}{1-M^2} \;-\; (x-\xi)\right)\right)$$
$$\cdot\, \frac{\partial^2}{\partial z^2}\; K_0\left(\frac{\lambda\sqrt{y^2+(1-M^2)z^2}}{a_\infty(1-M^2))}\right)\; dy \qquad (3.4)$$

or, alternately,

$$\tilde{w}_a(x,\lambda) \;=\; \lim_{z\to 0+}\int_{-b}^{b} \frac{\hat{A}(\xi,\lambda)}{2\pi}\;\frac{d\xi}{\sqrt{1-M^2}}$$
$$\cdot\int_{-\infty}^{x-\xi} \exp\left(\frac{-\lambda}{U}\,(x-\xi-y) \;+\; \frac{\lambda}{U}\,y\left(\frac{M^2}{1-M^2}\right)\right)$$
$$\cdot\,\frac{\partial^2}{\partial z^2}\; K_0\left(\frac{\lambda}{a_\infty(1-M^2)}\;\sqrt{y^2+(1-M^2)z^2}\right)\; dy$$
$$=\; \lim_{z\to 0+}\;\frac{1}{2\pi}\int_{-b}^{b}\hat{A}(\xi,\lambda)\;\frac{d\xi}{\sqrt{1-M^2}}\cdot\;\int_{-\infty}^{x-\xi}\exp\left(\frac{-\lambda(x-\xi)}{U}\right)$$
$$\cdot\;\left[\exp\left(\frac{\lambda}{U}\;\frac{y}{(1-M^2)}\right)\;\frac{\partial^2}{\partial z^2}\;K_0\left(\frac{\lambda}{a_\infty(1-M^2)}\;\sqrt{y^2+(1-M^2)z^2}\right)\right]\; dy \qquad (3.5)$$

where we note that

$$\frac{\partial^2}{\partial z^2}\,K_0(\cdot) \;=\; \frac{\lambda^2}{a_\infty^2(1-M^2)}\;K_0\left(\frac{\lambda}{a_\infty(1-M^2)}\;\sqrt{y^2+(1-M^2)z^2}\right)$$
$$-\;(1-M^2)\;\frac{\partial^2}{\partial y^2}\;K_0\left(\frac{\lambda}{a_\infty(1-M^2)}\;\sqrt{y^2+(1-M^2)z^2}\right)$$

where we have used the identity [12, p.79]:

$$\begin{aligned} K_1'(x) &= \frac{-K_1(x)}{x} \; - \; K_0(x) \\ K_0'(x) &= -K_1(x). \end{aligned}$$

Relation to Pressure Distribution

The acceleration potential is defined by

$$\psi(x,z,t) \;=\; \frac{\partial \phi}{\partial t} \;+\; U\,\frac{\partial \phi}{\partial x}\,.$$

Denoting the Laplace transform

$$\int_0^\infty e^{-\lambda t}\,\psi(x,z,t)\;dt \quad \text{by} \;\; \hat{\psi}(x,z,\lambda)$$

we have

$$\hat{\psi}(x,z,\lambda) \;=\; \lambda\hat{\phi}(x,z,\lambda) \;+\; U\,\frac{\partial}{\partial x}\,\hat{\phi}(x,z,\lambda)$$

since

$$\phi(x,z,0+) \;=\; 0.$$

Using (3.3) for $\hat{\phi}(x,z,\lambda)$ we obtain

$$\begin{aligned} \hat{\psi}(x,z,\lambda) \;&=\; \frac{U}{2\pi}\,\frac{1}{\sqrt{1-M^2}}\int_{-b}^{b}\hat{A}(\xi,\lambda)\,\exp\left[\frac{-\lambda(x-\xi)}{U} \;+\; \frac{\lambda}{U}\,\frac{(x-\xi)}{(1-M^2)}\right] \\ &\qquad\qquad \frac{\partial}{\partial z}\,K_0\left(\frac{\lambda M}{U(1-M^2)}\,\sqrt{(x-\xi)^2 \;+\; (1-M^2)z^2}\right) \\ &=\; \frac{U}{2\pi}\,\frac{1}{\sqrt{1-M^2}}\int_{-b}^{b} d\xi \;\; \hat{A}(\xi,\lambda)\;\exp\left(\frac{\lambda(x-\xi)}{U}\,\frac{M^2}{(1-M^2)}\right) \\ &\qquad \cdot\,\frac{-\lambda}{U}\,\frac{Mz}{\sqrt{(x-\xi)^2+(1-M^2)z^2}}\,K_1\left(M\,\frac{\lambda}{U}\,\frac{\sqrt{(x-\xi)^2+(1-M^2)z^2}}{(1-M^2)}\right). \end{aligned}$$

We see that for any $\varepsilon > 0$,

$$\begin{aligned} \lim_{z\to 0}\hat{\psi}_0(x,z,\lambda) \;&=\; \lim_{z\to 0}\frac{U}{2\pi}\,\frac{1}{\sqrt{1-M^2}}\int_{|x-\xi|<\varepsilon} d\xi \;\; \hat{A}(\xi,\lambda)\;\exp\left(\frac{\lambda(x-\xi)}{U}\,\frac{M^2}{(1-M^2)}\right) \\ &\qquad \cdot\,\frac{-\lambda}{U}\,\frac{Mz}{\sqrt{(x-\xi)^2+(1-M^2)z^2}}\,K_1\left(\frac{M\lambda}{U}\,\frac{\sqrt{(x-\xi)^2+(1-M^2)z^2}}{(1-M^2)}\right) \end{aligned}$$

for any $\varepsilon > 0$. Further for ε and z small enough,

$$K_1\left(\frac{M\lambda}{U}\,\frac{\sqrt{(x-\xi)^2+(1-M^2)z^2}}{(1-M^2)}\right)$$

can be replaced by

$$\frac{U(1-M^2)}{\lambda M\sqrt{(x-\xi)^2+(1-M^2)z^2}}\ .$$

Hence for ε and z small enough:

$$\begin{aligned}
\lim_{z\to 0}\hat{\psi}(x,z,\lambda) &= (-1)\,\frac{U}{2\pi}\,\frac{1}{\sqrt{1-M^2}}\int_{-\varepsilon}^{\varepsilon}\frac{(1-M^2)z}{\sigma^2+(1-M^2)z^2}\,d\sigma\ \hat{A}(x,\lambda)\\
&= (-1)\,\frac{U}{2\pi}\int_{-\varepsilon}^{\varepsilon}\frac{d}{d\sigma}\ \tan^{-1}\left[\frac{\sigma}{z\sqrt{1-M^2}}\right]\ d\sigma.
\end{aligned}$$

Hence it follows that

$$\lim_{z\to 0}\hat{\psi}(x,z,\lambda) \ = \ (-1)\,\frac{U}{2}\ \hat{A}(x,\lambda).$$

The pressure distribution defined by

$$\begin{aligned}
\hat{\Delta}P(x,\lambda) &= (-\rho)\lim_{z\to 0}\left[\hat{\psi}(x,z,\lambda)\ -\ \hat{\psi}(x,-z,\lambda)\right]\\
&= (-\rho)\lim_{z\to 0}\left[\hat{\psi}(x,z,\lambda)\ -\ \hat{\psi}(x,-z,\lambda)\right]\\
&= U\hat{A}(x,\lambda)\rho.
\end{aligned}$$

Hence

$$\hat{A}(x,\lambda) \ = \ \frac{\Delta\hat{P}(x,\lambda)}{\rho U}\ .$$

In particular, by the Kutta-Joukowski condition we must require $\hat{A}(x,\lambda)$ is continuous at $x = b-$ and

$$\hat{A}(b-,\lambda) \ = \ 0.$$

Possio's Integral Equation

To derive the Laplace transform version of the Possio integral, we begin with (3.4) where

$$\int_{-\infty}^{x-\xi} \exp\left(\frac{\lambda y}{U(1-M^2)}\right) \frac{\partial^2}{\partial y^2} K_0\left(\frac{\lambda}{a_0(1-M^2)} \sqrt{y^2+(1-M^2)z^2}\right) dy$$

$$= \left[\exp\left(\frac{\lambda y}{U(1-M^2)}\right) \frac{\partial}{\partial y} K_0\left(\frac{\lambda}{a_0(1-M^2)} \sqrt{y^2+(1-M^2)z^2}\right)\right]_{-\infty}^{x-\xi}$$

$$- \frac{\lambda}{U(1-M^2)} \int_{-\infty}^{x-\xi} \exp\left(\frac{\lambda y}{U(1-M^2)}\right)$$

$$\cdot \frac{\partial}{\partial y} K_0\left(\frac{\lambda}{a_0(1-M^2)} \sqrt{y^2+(1-M^2)z^2}\right) dy$$

$$= -\exp\left(\frac{\lambda}{U}\,\frac{(x-\xi)}{(1-M^2)}\right) \cdot K_1\left(\frac{\lambda}{a_\infty(1-M^2)} \sqrt{(x-\xi)^2+(1-M^2)z^2}\right)$$

$$\cdot \frac{\lambda}{a_\infty(1-M^2)} \frac{(x-\xi)}{\sqrt{(x-\xi))^2+(1-M^2)z^2}}$$

$$- \frac{\lambda}{U(1-M^2)} \left[\exp\left(\frac{\lambda y}{U(1-M^2)}\right) K_0\left(\frac{\lambda}{a_0(1-M^2)} \sqrt{(x-\xi)^2+(1-M^2)z^2}\right)\right]_{-\infty}^{x-\xi}$$

$$+ \frac{\lambda^2}{U^2(1-M^2)^2} \int_{-\infty}^{(x-\xi)} \exp\left(\frac{\lambda y}{U(1-M^2)}\right)$$

$$\cdot K_0\left(\frac{\lambda}{a_0(1-M^2)} \sqrt{(x-\xi)^2+(1-M^2)z^2}\right) dy$$

which as $z \to 0$

$$= \left\{\exp\left(\frac{\lambda(x-\xi)}{U(1-M^2)}\right) K_1\left(\frac{\lambda}{a_\infty(1-M^2)} |x-\xi|\right) \cdot \frac{\lambda}{a_\infty(1-M^2)} \frac{-(x-\xi)}{|x-\xi|}\right.$$

$$- \frac{\lambda}{U(1-M^2)} \exp\left(\frac{\lambda(x-\xi)}{U(1-M^2)}\right) K_0\left(\frac{\lambda}{a_\infty(1-M^2)} |x-\xi|\right)$$

$$\left.+ \frac{\lambda^2}{U^2(1-M^2)^2} \int_{-\infty}^{x-\xi} \exp\left(\frac{\lambda y}{U(1-M^2)}\right) K_0\left(\frac{\lambda}{a_\infty(1-M^2)} |y|\right) dy\right\}$$

Hence (3.4) can be expressed:

$$\frac{1}{2\pi}\int_{-b}^{b} \frac{\hat{A}(\xi,\lambda)}{\sqrt{1-M^2}} \exp\left(\frac{-\lambda(x-\xi)}{U}\right)$$

$$\cdot \left[(-1)(1-M^2)\left\{\exp\frac{\lambda(x-\xi)}{U(1-M^2)}\left(K_1\left(\frac{\lambda}{a_\infty(1-M^2)}\ |x-\xi|\right)\cdot\frac{\lambda}{a_\infty(1-M^2)}\ \frac{-(x-z)}{|x-\xi|}\right.\right.\right.$$

$$\left.- \ \frac{\lambda}{U(1-M^2)}\ K_0\left(\frac{\lambda}{a_\infty(1-M^2)}\ |x-\xi|\right)\right)$$

$$\left.+ \ \frac{\lambda^2}{U^2(1-M^2)^2}\int_{-\infty}^{(x-\xi)}\left(\exp\ \frac{\lambda y}{U(1-M^2)}\right)\ K_0\left(\frac{\lambda|y|}{a_\infty(1-M^2)}\right)\ dy\right\}$$

$$\left.+ \ \frac{\lambda^2}{a_\infty^2(1-M^2)}\int_{-\infty}^{(x-\xi)}\left(\exp\ \frac{\lambda y}{U(1-M^2)}\ K_0\left(\frac{\lambda|y|}{a_\infty(1-M^2)}\right)\right)\ dy\right]$$

$$= \ \frac{1}{2\pi}\int_{-b}^{b}\frac{\hat{A}(\xi,\lambda)}{\sqrt{1-M^2}}\ \left[\left(\exp\ \frac{\lambda(x-\xi)M^2}{U(1-M^2)}\right)\right.$$

$$\cdot\ \left(K_1\left(\frac{\lambda}{a_\infty(1-M^2)}\ |x-\xi|\right)\ \frac{\lambda}{a_\infty}\ \frac{+(x-\xi)}{|x-\xi|}\ +\ \frac{\lambda}{U}\ K_0\left(\frac{\lambda}{a_\infty(1-M^2)}\ |x-\xi|\right)\right)$$

$$\left.-\ \frac{\lambda^2}{U^2}\int_{-\infty}^{(x-\xi)}\left(\exp\left(\frac{-\lambda(x-\xi)}{U}\ +\ \frac{\lambda y}{U(1-M^2)}\right)\right)K_0\left(\frac{\lambda|y|}{a_\infty(1-M^2)}\right)\ dy\right]. \tag{3.6}$$

$$= \ \frac{1}{2\pi}\int_{-b}^{b}\frac{\hat{A}(\xi,\lambda)}{\sqrt{1-M^2}}\ d\xi$$

$$\cdot\ \left[\left(\exp\ \frac{\lambda(x-\xi)M^2}{U(1-M^2)}\right)\ K_1\left(\frac{\lambda M}{U(1-M^2)}\ |x-\xi|\right)\ \frac{\lambda}{U}\ \frac{M|x-\xi|}{(x-\xi)}\right.$$

$$-\ \frac{\lambda^2}{U^2}\int_{-\infty}^{(x-\xi)}\exp\left(\frac{\lambda y}{U(1-M^2)}\ -\ \frac{\lambda(x-\xi)}{U}\right)$$

$$\left.\cdot\ \left(K_0\left(\frac{\lambda M|y|}{U(1-M^2)}\right)\ -\ \frac{K_0}{(1-M^2)}\left(\frac{\lambda M|x-\xi|}{U(1-M^2)}\right)\right)\ dy\right] \tag{3.7}$$

We can simplify this a bit further by noting that [13, p.708]:

$$\int_{-\infty}^{0} \left(\exp \frac{\lambda y}{U(1-M^2)}\right) K_0\left(\frac{\lambda |y|}{a_\infty(1-M^2)}\right) dy \;=\; \frac{U}{\lambda} \sqrt{1-M^2} \log\left(\frac{1+\sqrt{1-M^2}}{M}\right).$$

Hence finally we have:

$$\hat{w}_a(x,\lambda) \;=\; \frac{1}{2\pi} \int_{-b}^{b} G(\lambda,\, x-\xi)\, \hat{A}(\xi,\lambda)\, d\xi, \quad |x| \le b, \tag{3.8}$$

where the kernel $G(\lambda,\cdot)$ is defined by:

$$\begin{aligned} G(\lambda,y) \;=\; \frac{1}{\sqrt{1-M^2}} \frac{\lambda}{U} \Bigg[& \frac{M|y|}{y} K_1\left(\frac{\lambda}{U} \frac{M}{(1-M^2)} |y|\right) \exp\left(\frac{\lambda}{U} \frac{M^2}{(1-M^2)} y\right) \\ & + K_0\left(\frac{\lambda}{U} \frac{M}{(1-M^2)} |y|\right) \exp\left(\frac{\lambda}{U} \frac{M^2}{(1-M^2)} y\right) \\ & - \frac{\lambda}{U} \int_0^y K_0\left(U \frac{\lambda|\sigma| M}{(1-M^2)}\right) \exp\left(\frac{-\lambda y}{U} + \frac{\lambda\sigma}{U(1-M^2)}\right) d\sigma \\ & - \sqrt{1-M^2} \log\left(\frac{1+\sqrt{1-M^2}}{M}\right) \exp\left(\frac{-\lambda y}{U}\right) \Bigg]. \end{aligned} \tag{3.9}$$

The integral in (3.8) has again to be interpreted in the Cauchy sense.

Remark

For $M = 0$, (3.8) reduces to

$$\hat{w}_a(x,\lambda) \;=\; \frac{1}{2\pi} \int_{-b}^{b} \hat{A}(\xi,\lambda)\, G(\lambda, x-\xi)\, d\xi$$

where

$$\begin{aligned} G(\lambda,y) &= \frac{1}{y} - \frac{\lambda^2}{U^2} \int_0^\infty e^{-\lambda\sigma/U} \log \frac{|\sigma - y|}{|y|}\, d\sigma \\ &= \frac{1}{y} - \frac{\lambda}{U} e^{-\lambda y/U} \left[\mathrm{Chi}\, \frac{\lambda y}{U} + \mathrm{Shi}\, \frac{\lambda y}{U}\right]. \end{aligned}$$

For the definition of $\mathrm{Chi}\,(\cdot)$, $\mathrm{Shi}\,(\cdot)$ see [13]. This is a known result for the "oscillatory case": $\lambda = i\omega$; see [3,4,5]. Note that unlike the "oscillatory case" we do not use divergent integrals and "retain only finite parts" as in [4].

Remark

We do have that

$$\lambda \hat{w}_a(x,\lambda) \; + \; U \frac{\partial}{\partial x} \, \hat{w}_a(x,\lambda)$$

$$= \frac{\partial \hat{\psi}_0(x,0,\lambda)}{\partial z}$$

$$= \lim_{z\to 0} \frac{U}{2\pi} \, \frac{1}{\sqrt{1-M^2}} \int_{-b}^{b} \hat{A}(\xi,\lambda) \, \exp\left(\frac{\lambda(x-\xi)}{U} \, \frac{M^2}{(1-M^2)} \right)$$

$$\cdot \, \frac{\partial^2}{dz^2} \, K_0 \left(\frac{\lambda M}{U(1-M^2)} \sqrt{(x-\xi)^2 + (1-M^2)z^2} \right) \, d\xi$$

which we could use in place of (3.8) since the left side is given, but we *cannot* take the limit inside (in the integrand) because it leads to a kernel with a singularity of

$$\frac{1}{(x-\xi)^2}$$

and the corresponding Cauchy integral is *not* defined! This is even more evident for $M = 0$. Solving for $\hat{w}_a(x,\lambda)$ essentially "smooths" this to make the singularity of order one: that is,

$$\sim \; \frac{1}{x-\xi} \; .$$

Possio's Equation: Oscillatory Case

We can obtain the "oscillatory" case from (3.8) as follows. Let

$$w_a(x,t) \; = \; \bar{w}_a(x, i\omega_0) e^{i\omega_0 t}, \quad t \geq 0, \; \omega_0 \text{ real} \neq 0.$$

Then

$$\hat{w}_a(x,\lambda) \; = \; \frac{\bar{w}_a(x,i\omega_0)}{\lambda - i\omega_0} \; .$$

Hence (3.8) yields

$$\frac{\bar{w}_a(x,i\omega_0)}{\lambda - i\omega_0} \; = \; \frac{1}{2\pi} \int_{-b}^{b} \hat{A}(\xi,\lambda) \, G(\lambda, x-\xi) \, d\xi.$$

Hence the right side has a simple pole at $\lambda = i\omega_0$. We note that $G(\lambda,\cdot)$ has no poles on the imaginary axis, if we omit zero. Hence

$$\hat{A}(\xi,\lambda) \; = \; \frac{\bar{A}(\xi,i\omega_0)}{\lambda - i\omega_0} \; .$$

Substituting and letting $\lambda \to i\omega_0$, we obtain

$$\bar{w}_a(x, i\omega_0) \;=\; \frac{1}{2\pi}\int_{-b}^{b} \bar{A}(\xi, i\omega_0)\, G(i\omega_0, x-\xi)\; d\xi. \tag{3.10}$$

This is the Possio equation. See [3,4] where Hankel functions are used in place of the Bessel K functions. Conversely given (3.10), we may formally replace $i\omega$ by λ to obtain (3.8), provided the kernel is analytic in the right half-plane. In other words, we may replace $i\omega$ by λ in the oscillatory Possio equation, and we could get (3.8) in this manner from the oscillatory versions in [3,4] using the Bessel K functions.

Existence and Uniqueness of Solution

Uniqueness

We begin with Uniqueness. Suppose then that for some nonzero λ, $\operatorname{Re}\lambda \geq 0$, (3.8) has two solutions in $L^{\frac{4}{3}-\varepsilon}$, for some ε, with the property that each vanishes at $1-$. Then the difference, denoted $\hat{A}(\cdot\,, \lambda)$ will satisfy

$$0 \;=\; \frac{1}{2\pi}\int_{-b}^{b} G(\lambda,\, x-\xi)\, \hat{A}(\xi, \lambda)\; d\xi, \quad |x| < b. \tag{3.11}$$

Going back then to (3.3), $\hat{\phi}(x, z, \lambda)$ will satisfy the Laplace transform of (2.1),

$$\lambda^2\hat{\phi} \;+\; 2U\lambda\,\frac{\partial\hat{\phi}}{\partial x} \;=\; a_\infty^2(1-M^2)\,\frac{\partial^2\hat{\phi}}{\partial x^2} \;+\; a_\infty^2\,\frac{\partial^2\hat{\phi}}{\partial z^2}\,. \tag{3.12}$$

Moreoever the $\hat{\phi}(\cdot)$ satisfies the boundary condition

$$\frac{\partial\hat{\phi}}{\partial z}\,(x, 0+, \lambda) \;=\; 0, \quad |x| < b.$$

Hence as we have seen

$$F \;=\; \begin{vmatrix} \lambda\hat{\phi} \\ \frac{\partial\hat{\phi}}{\partial x} \\ \frac{\partial\hat{\phi}}{\partial z} \end{vmatrix}$$

satisfies

$$\lambda F \;=\; AF.$$

but the point spectrum of $\bar{A}$ is empty. Hence $F = 0$, and $\hat{\phi}(\cdot) = 0$, proving uniqueness of solution to (3.8). In other words the uniqueness is a consequence of the fact that $\bar{A}$ has no eigenvalues.

Existence

The question of existence (or solution to (3.8) for each λ, $\mathrm{Re}\,\lambda \geq 0$) is not an idle one — the existence usually comes with some means of constructing the solution. This is indeed the case here, as we shall show, drawing on the work of Tricomi [14] and Sohngen [15] on the airfoil equation.

We begin with some Lemmas.

Lemma 1

Let $f \in L^p[-b, b]$ (L^p for short in what follows). Then the finite Hilbert transform H, defined by:

$$\begin{aligned} Hf &= g \\ g(x) &= \sqrt{1-M^2}\,\frac{1}{2\pi}\int_{-b}^{b}\frac{f(\xi)}{(x-\xi)}\,d\xi, \qquad \text{a.e. } |x| < b \end{aligned} \tag{3.13}$$

where the integral is defined in the Cauchy sense, defines a bounded linear operator on L^p into L^p for any $p > 1$.

Proof

Titchmarsh [16, p.132].

Lemma 2

Let $g(\cdot) \in L^\infty[-b, b]$ (L^∞ hereinafter). Then

$$\begin{aligned} Rg &= f \\ f(x) &= \left(\frac{2}{\pi\sqrt{1-M^2}}\right)\sqrt{\frac{b-x}{b+x}}\int_{-b}^{b}\sqrt{\frac{b+\xi}{b-\xi}}\,\frac{g(\xi)}{\xi - x}\,d\xi, \qquad \text{a.e. } |x| < b \end{aligned} \tag{3.14}$$

defines a linear bounded transformation on L^∞ into $L^{\frac{4}{3}-}$.

Proof

Sohngen [14] , Tricomi [15, p.175 *et seq.*].

Lemma 3

Suppose $g \in L^\infty$. Then the integral equation:

$$Hf \;=\; g$$

is satisfied by any $f(\cdot)$ of the form:

$$f(x) \;=\; (Rg)(x) \;+\; \frac{c}{\sqrt{b^2 - x^2}} \tag{3.15}$$

where c is an arbitrary constant. Furthermore if Rg is bounded at $x = b-$, then Rg is the only solution with that property.

Remark

(3.15) enables us to construct solutions of (3.12) which do *not* satisfy the Kutta-Joukowski condition.

Proof

Sohngen [14] , Tricomi [15].

Corollary

As an application of Lemma 3, consider the example (setting $b = 1$):

$$g(x) \;=\; \frac{1}{2\pi}\int_{-1}^{1} \frac{d\xi}{(x-\xi)} \;=\; \frac{1}{2\pi}\left(\log(1+x) \;-\; \log(1-x)\right), \quad |x| < 1. \tag{3.16}$$

It follows that

$$(Rg)(x) \;=\; 1, \quad -1 \leq x \leq 1.$$

Let

$$g_+(x) \;=\; \log(1+x)$$
$$g_-(x) \;=\; -\log(1-x).$$

Then straightforward analysis shows that

$$(Rg_+)(x) \;\rightarrow\; 0 \quad \text{as} \;\; x \rightarrow 1.$$

Hence it follows that

$$(Rg_-)(x) \;\rightarrow\; 1 \quad \text{as} \;\; x \rightarrow 1.$$

Lemma 4

Suppose $g(\cdot)$ satisfies a Lipschitz condition for some α, $+\frac{1}{2} < \alpha \leq 1$:

$$|g(t) - g(s)| \;\leq\; M|t-s|^{\alpha}, \quad -b \leq s,\ t \leq b. \tag{3.17}$$

Let

$$f \;=\; Rg.$$

Then

$$|f(x)| \;\to\; 0 \quad \text{as} \;\; x \to b- \;.$$

Proof

We begin by noting that if

$$g_1(x) \;=\; 1, \quad |x| < b,$$

then

$$(Rg_1)(x) \;=\; \frac{1}{\sqrt{1{-}M^2}}\;\frac{2}{\pi}\sqrt{\frac{b-x}{b+x}}\;\cdot\;(\pi)$$

and $\to 0$ as $x \to b-$. Now

$$\left|\int_{-b}^{b}\sqrt{\frac{b+\xi}{b-\xi}}\;\frac{(g(\xi)-g(x))}{\xi - x}\;d\xi\right| \;\leq\; M\int_{-b}^{b}\sqrt{b+\xi}\;\frac{|\xi - x|^{\alpha-1}}{\sqrt{b-\xi}}\;d\xi$$

and

$$\int_{-b}^{b}|\xi - x|^{\alpha-1}\cdot(b-\xi)^{-1/2}\,d\xi \;<\; \infty$$

for $|x| < b$, by a simple application of the Hölder inequality, for

$$\alpha \,-\, 1 \;=\; \tfrac{-1}{2} \;+\; \varepsilon, \quad \varepsilon > 0.$$

It follows that

$$(Rg - Rg_1)(x) \;\to\; 0- \quad \text{as} \;\; x \to b-$$

and hence so does

$$(Rg)(x) \;\to\; 0 \quad \text{as} \;\; x \to b-\,,$$

as required.

Corollary

$$(Rg)(x) \ \rightarrow \ 0 \quad \text{ as } \ x \rightarrow b-$$

if g is absolutely continuous with derivative in L^∞. Moreover under this condition,

$$f \ = \ Rg$$

is the only solution of the integral equation:

$$Hf \ = \ g$$

such that $f(\cdot)$ is bounded at $x = b-$.

It is convenient to rewrite (3.8) for a more detailed analysis of $G(\lambda, y)$. Since it is only a matter of scaling we will take $b = 1$, from now on. Let

$$\tilde{w}(x,\lambda) \ = \ \hat{w}(x,\lambda) \exp\left(\frac{-\lambda M^2 x}{U(1-M^2)}\right)$$

$$\tilde{A}(x,\lambda) \ = \ \hat{A}(x,\lambda) \exp\left(\frac{-\lambda M^2 x}{U(1-M^2)}\right)$$

$$K_{11}(z) \ = \ zK_1(z).$$

Then (3.8) becomes:

$$\begin{aligned} \tilde{w}(x,\lambda) \ = \ & \int_{-1}^{1} d\xi \ \tilde{A}(\xi,\lambda) \\ & \cdot \left\{ \frac{\sqrt{1-M^2}}{2\pi} K_{11}\left(\frac{\lambda}{U}\frac{M}{(1-M^2)} \ |(x-\xi)|\right) \cdot \frac{1}{(x-\xi)} \right. \\ & + \frac{1}{2\pi\sqrt{1-M^2}} \frac{\lambda}{U} K_0\left(\frac{\lambda}{U}\frac{M}{(1-M^2)} \ |(x-\xi)|\right) \\ & \left. - \frac{1}{2\pi\sqrt{1-M^2}} \left(\frac{\lambda^2}{U^2}\right) \int_0^\infty \exp\left(\frac{-\lambda}{U}\frac{t}{1-M^2}\right) K_0\left(\frac{\lambda}{U}\frac{M}{1-M^2}|t-x+\xi|\right) dt \right\}. \end{aligned} \tag{3.18}$$

The crucial point of departure of our analysis is to rewrite the kernel in parenthesis in the integral above by

$$\frac{\sqrt{1-M^2}}{2\pi}\,\frac{1}{x-\xi}\ +\ G_0(\lambda;\,x-\xi)$$

where, defining

$$\begin{aligned} R_{11}(z) &= K_{11}(z)\ -\ 1 \\ &= zK_1(z)\ -\ 1 \end{aligned} \tag{3.19}$$

we have that

$$\begin{aligned} G_0(\lambda;\,y) \;=\; & \frac{\sqrt{1-M^2}}{2\pi}\,R_{11}\left(\frac{\lambda M}{U(1-M^2)}\,|y|\right)\frac{1}{y}\ +\ \frac{1}{2\pi\sqrt{1-M^2}}\,\frac{\lambda}{U}\,K_0\left(\frac{\lambda M}{U(1-M^2)}\,|y|\right) \\ & -\ \frac{1}{2\pi\sqrt{1-M^2}}\,\frac{\lambda^2}{U^2}\int_0^\infty \exp\left(\frac{-\lambda t}{U(1-M^2)}\right)K_0\left(\frac{\lambda M}{U(1-M^2)}|t-y|\right)\,dt. \end{aligned} \tag{3.20}$$

Define the operator $G_0(\lambda)$ by

$$\begin{aligned} G_0(\lambda)f &= g, \quad \operatorname{Re}\lambda \geq 0 \\ g(x) &= \int_{-1}^{1} G_0(\lambda,\,x-\xi)\,f(\xi)\,d\xi, \quad |x|<1. \end{aligned} \tag{3.21}$$

Then $G_0(\lambda)$ is readily verified to be a linear bounded operator on $L^{\frac{4}{3}-}$ into L^∞, and (3.10) can be rewritten in operator form as:

$$\tilde{w}(\cdot\,,\lambda)\ =\ H\left(\tilde{A}(\cdot\,,\lambda)\right)\ +\ G_0(\lambda)\,\tilde{A}(\cdot\,,\lambda). \tag{3.22}$$

What we have done is simply to isolate the "singular part" of the kernel as the first term 3).

From (3.13) we can obtain our first breakdown of the solution by operating on both sides by R and obtaining

$$R\tilde{w}(\cdot\,,\lambda)\ =\ \tilde{A}(\cdot\,,\lambda)\ +\ RG_0(\lambda)\,\tilde{A}(\cdot\,,\lambda). \tag{3.23}$$

Existence Theorem

We can now prove existence of a solution of (3.23), which is actually an easy consequence of (3.23) where the main feature to be noted is that $RG_0(\lambda)$ is compact on $L^{\frac{4}{3}-}$ into $L^{\frac{4}{3}-}$. By the uniqueness of solution argument we see that

$$(I + RG_0(\lambda))A = 0 \quad \text{for some } A \text{ in } L^{\frac{4}{3}-},$$

would imply that

$$HA + G_0(\lambda)A = 0$$

or that (3.11) holds. But in that case we have seen that A must be zero. But $RG_0(\lambda)$ being compact by the "Fredholm alternative" we have that $(I + RG_0(\lambda))$ has a bounded inverse. Hence from (3.23) we have that

$$\tilde{A}(\cdot\,,\lambda) = (I + RG_0(\lambda))^{-1}R\tilde{w}(\cdot\,,\lambda). \tag{3.24}$$

Or, we have both existence and uniqueness of solution.

A Constructive Solution

We can obtain a constructive solution for $|\lambda|$ small. Thus from (3.13) it can be seen that for each $M < 1$,

$$\|G_0(\lambda)\| \to 0 \quad \text{as } |\lambda| \to 0. \tag{3.25}$$

Hence for fixed M,

$$\|RG_0(\lambda)\| < 1 \quad \text{for } |\lambda| < |\lambda_M|. \tag{3.26}$$

Hence we have our first result on existence:

Lemma 5

For each M, $0 \leq M < 1$, we can find $|\lambda_M|$, $0 < |\lambda_M| < \infty$, such that (3.15) has a unique solution given by

$$\begin{aligned} \tilde{A}(\cdot\,,\lambda) &= (I + RG_0(\lambda))^{-1}R\tilde{w}(\cdot\,,\lambda) \\ &= R\tilde{w}(\cdot\,,\lambda) + \sum_{1}^{\infty}(-1)^k(RG_0(\lambda))^k R\tilde{w}(\cdot\,,\lambda) \end{aligned} \tag{3.27}$$

for all $|\lambda| < |\lambda_M|$.

Remark 1

In particular we have from (3.27) that for $\lambda = 0$:

$$\tilde{A}(\cdot, 0) \;=\; R\tilde{w}(\cdot, 0). \tag{3.28}$$

Remark 2

In [3] there is mention of an expansion given by Dietze [6] without proof. It is not clear to this author whether it is the same as (3.27) or not. Note that the expansion in (3.27) is *not* in powers of M.

Remark 3

Since the major interest in the use of the Possio solution is in the range of values of λ such that

$$k \;<\; 1$$

where

$$k \;=\; \frac{|\lambda|\, b}{U}$$

the expansion (3.27) is not totally void of value and we expect that taking the expansion to the first term

$$\hat{A}_1(\cdot\,, \lambda) \;=\; R\hat{w}(\cdot\,, \lambda) \;-\; RG_0(\lambda)\, R\hat{w}(\cdot\,, \lambda)$$

may be useful with further simplification of the kernel $G_0(\lambda)$ as well.

Kutta-Joukowsky Condition

We shall assume from now on that $\hat{w}(\cdot\,, \lambda)$, and hence $\tilde{w}(\cdot\,, \lambda)$, has a bounded derivative in $[-1, 1]$. It would then follow from Lemma 4 that the function

$$(R\hat{w}(\cdot\,, \lambda))x \;\rightarrow\; 0 \quad \text{as} \;\; x \rightarrow 1- \,.$$

We shall now prove:

Theorem 3.1

Let $f \in L^{\frac{4}{3}-}$. Then the function

$$[RG_0(\lambda) f\,](x) \;\rightarrow\; 0 \quad \text{as} \;\; x \rightarrow 1- \,.$$

Proof

Let us follow the breakdown of $G_0(\lambda)$ into the three terms as in (3.20). Let us also use the abbreviated notation:

$$\gamma \;=\; \frac{\lambda}{U}\,\frac{M}{(1{-}M^2)}\;; \qquad \lambda' \;=\; \frac{\lambda}{U(1{-}M^2)}\;.$$

Thus let

$$G_0(\lambda) \;=\; G_{0,1}(\lambda) \;+\; G_{0,2}(\lambda) \;+\; G_{0,3}(\lambda)$$

with corresponding kernels

$$G_{0,1}(\lambda, y) \;=\; \frac{\sqrt{1{-}M^2}}{2\pi}\; R_{11}(\gamma|y|)\;\frac{1}{y} \tag{3.29}$$

$$G_{0,2}(\lambda, y) \;=\; \frac{1}{2\pi\sqrt{1{-}M^2}}\;\frac{\lambda}{U}\;K_0(\gamma|y|) \tag{3.30}$$

$$G_{0,3}(\lambda, y) \;=\; \frac{1}{2\pi\sqrt{1{-}M^2}}\;\frac{\lambda^2}{U^2}\int_0^\infty e^{-\lambda' t}\,K_0(\gamma\,|t-y|)\;dt. \tag{3.31}$$

Define now the operator $\mathcal{R}_{11}(t)$ for each $t \geq 0$ on $L^{\frac{4}{3}-}$ by

$$\mathcal{R}_{11}(t)f \;=\; g(t,\,\cdot), \quad t \geq 0$$

$$g(t,x) \;=\; \frac{\sqrt{1{-}M^2}}{2\pi}\int_{-1}^{1}\frac{R_{11}(\gamma\,|x-\xi-t|)}{x-\xi-t}\;f(\xi)\;d\xi, \quad |x| \leq 1. \tag{3.32}$$

Then $\mathcal{R}_{11}(t)$ is linear bounded on $L^{\frac{4}{3}-}$ into L^∞. This is because the only singularity in the integrand is where

$$x \,-\, \xi \,-\, t \;=\; 0$$

but we can use the known expansion for $K_1(z)$ at $z = 0$ and obtain:

$$\frac{zK_1(z)\,-\,1}{z} \;\approx\; z\left[\tfrac{1}{4}(2\text{ Euler Gamma } - 1) \;+\; \tfrac{1}{2}\log\frac{z}{2}\right], \quad \operatorname{Re} z \geq 0. \tag{3.33}$$

In particular we recognize that

$$G_{0,1}(\lambda) \;=\; \mathcal{R}_{11}(0)$$

and that the corresponding kernel $G_{0,1}(\lambda, y)$ is such that

$$G_{0,1}(\lambda,\,-y) \;=\; -G_{0,1}(\lambda, y), \quad y > 0$$

$$\frac{d}{dy}\left[\frac{R_{11}(\gamma y)}{y}\right] \;=\; \frac{d}{dy}\left[\gamma K_1(\gamma y) \;-\; \frac{1}{y}\right], \qquad\qquad y > 0$$

$$= \;\left[\frac{1}{y^2} \;+\; \gamma^2\left[\frac{-K_1(\gamma y)}{\gamma y} \;-\; K_0(\gamma y)\right]\right]$$

$$= \;\frac{1}{y^2}\Big[[1 \;-\; \gamma y K_1(\gamma y)] \;-\; \gamma^2 K_0(\gamma y)\Big]$$

which at $y = 0$ using the expansion (3.33)

$$= \;\tfrac{1}{4}(2 \text{ Euler Gamma} \;-\; 1) \;+\; \tfrac{1}{2}\log\frac{\gamma y}{2} \;-\; \gamma^2 \log \gamma y$$

which is in $L^p[-1,1]$ for $p \geq 1$ and hence it follows that $\mathcal{R}_{11}(0)f$ has an L^1 derivative in $[-1,1]$ and hence by Lemma 4 we have that the function

$$[R\,\mathcal{R}_{11}(0)f\,](x) \;\to\; 0 \quad \text{ as } \; x \to 1-\,.$$

We shall abbreviate $\mathcal{R}_{11}(0)$ to $\mathcal{R}_{11}$ in what follows. We note also that for $\lambda > 0$, using the expansion at $y = 0$,

$$\frac{R_{11}(\gamma y)}{y} \;=\; \frac{\gamma R_{11}(\gamma y)}{\gamma y} \;\dot{\approx}\; \gamma^2 y\left[\tfrac{1}{4}(2 \text{ Euler Gamma} \;-\; 1) \;+\; \tfrac{1}{2}\log\frac{\gamma y}{2}\right]$$

we have the estimate, as a function of γ that

$$\|\mathcal{R}_{11}\| \;=\; O[\gamma^2\,|\log\gamma|\,], \quad \text{ as } \; \gamma \to 0. \tag{3.34}$$

To handle the remaining two terms (3.30) and (3.31) we define the operator $\mathcal{K}(t)$ on $L^{\frac{4}{3}-}$ for each $t \geq 0$ by:

$$\mathcal{K}(t)\,f \;=\; g(t,\cdot)$$

$$g(t,x) \;=\; \frac{1}{2\pi\sqrt{1-M^2}}\int_{-1}^{1} K_0(\gamma|t-x+\xi|)\,f(\xi)\;d\xi, \quad |x| \leq 1, \tag{3.35}$$

Then $\mathcal{K}(t)$ is a linear bounded operator on $L^{\frac{4}{3}-}$ into L^∞, and we have

$$G_0(\lambda)f \;=\; \mathcal{R}_{11}f \;+\; \frac{\lambda}{U}\,\mathcal{K}(0)f \;-\; \frac{\lambda^2}{U^2}\int_0^\infty e^{-\lambda' t}\,\mathcal{K}(t)f\;dt.$$

Let us consider the second term. Let $g = \mathcal{K}(0)f$. Then integrating by parts, we have:

$$g(x) \;=\; \frac{1}{2\pi}\,\frac{1}{\sqrt{1-M^2}}\,K_0(\gamma(1-x))\,F(1) \;-\; \frac{1}{2\pi}\,\frac{1}{\sqrt{1-M^2}}\int_{-1}^{1}\frac{F(\xi)}{x-\xi}\,d\xi$$

$$+ \;\frac{1}{(1-M^2)}\,(\mathcal{R}_{11}\,F)\,(x), \quad |x| \leq 1, \tag{3.36}$$

where

$$F(x) \;=\; \int_{-1}^{x} f(\xi)\; d\xi.$$

We define a new function $K(\gamma,\cdot)$, for each γ, by

$$K(\gamma,x) \;=\; \frac{1}{\pi^2}\sqrt{\frac{1-x}{1+x}}\;\int_{-1}^{1}\;\sqrt{\frac{1+\xi}{1-\xi}}\;\frac{K_0(\gamma(1-\xi))}{\xi-x}\,d\xi, \quad |x|<1. \tag{3.37}$$

Then it is immediate that

$$R\left[\frac{1}{2\pi}\;\frac{1}{\sqrt{1-M^2}}\;K_0(\gamma(1-\;\cdot\;))\right] \;=\; \frac{1}{(1-M^2)}\;K(\gamma,\cdot)$$

Lemma 5

$$K(\gamma,x) \;\to\; 1 \quad \text{as} \;\; x\to 1- \tag{3.38}$$

Proof

$$K_0(\gamma(1-\xi)) \;+\; \log(\gamma(1-\xi))$$

is Liptschitz in $|\xi|<1$. Hence

$$\sqrt{\frac{1-x}{1+x}}\int_{-1}^{1}\;\sqrt{\frac{1-\xi}{1+\xi}}\;\frac{(K_0(\gamma(1-\xi))\;+\;\log\gamma(1-\xi))}{\xi-x}\,d\xi \;\to\; 0 \quad \text{as} \;\; x\to 1-$$

and so does

$$\sqrt{\frac{1-x}{1+x}}\int_{-1}^{1}\;\sqrt{\frac{1-\xi}{1+\xi}}\;\frac{\log\gamma}{\xi-x}\,d\xi.$$

By the Corollary to Lemma 3 we see that

$$K(\gamma,x) \;\to\; 1 \quad \text{as} \;\; x\to 1-\,.$$

Hence

$$R\left[\mathcal{K}(0)\,f\right] \;=\; \frac{F(1)\,K(\gamma,\cdot)}{(1-M^2)} \;-\; \frac{F(\cdot)}{(1-M^2)} \;+\; \frac{1}{(1-M^2)}\;R\left[\mathcal{R}_{11}F\right]. \tag{3.39}$$

where every term is bounded at $1-$, as required.

Finally we consider the third term in (3.31). The corresponding operator is given by

$$G_{0,3}(\lambda)f \;=\; \frac{-\lambda^2}{U^2}\int_0^\infty e^{-\lambda' t}\,\mathcal{K}(t)f\;dt.$$

Let

$$g(t,\cdot) = \mathcal{K}(t)f.$$

Then integration by parts yields:

$$\begin{aligned} g(t,x) \;&=\; \frac{1}{2\pi}\,\frac{1}{\sqrt{1-M^2}}\,K_0(\gamma(1+t-x))\,F(1) \;-\; \frac{1}{2\pi}\,\frac{1}{\sqrt{1-M^2}}\int_{-1}^{1}\frac{F(\xi)}{x-t-\xi}\,d\xi \\ &\quad -\;\frac{1}{\pi}\,\frac{1}{\sqrt{1-M^2}}\,\left(\mathcal{R}_{11}(t)F\right)(x), \quad |x|\le 1, \end{aligned} \tag{3.40}$$

where

$$F(x) \;=\; \int_{-1}^{x} f(\xi)\,d\xi.$$

We carry out the integration with respect to t, term by term. Let $g_1(t,x)$ denote the first term in (3.40). Then

$$\frac{-\lambda^2}{U^2}\int_0^\infty e^{-\lambda' t}\,g_1(t,x)\,dt \;=\; \frac{-\lambda^2}{U^2}\,\frac{1}{2\pi}\,\frac{F(1)}{\sqrt{1-M^2}}\,\int_0^\infty e^{-\lambda' t}\,K_0(\gamma(1+t-x))\,dt$$

which, by integration by parts,

$$\begin{aligned} &=\; \frac{\lambda}{U}\sqrt{1-M^2}\,\frac{F(1)}{2\pi}\int_0^\infty e^{-\lambda' t}\;\gamma K_1(\gamma(1+t-x))\,dt \\ &\quad -\;\frac{\lambda}{U}\,\frac{F(1)}{2\pi}\sqrt{1-M^2}\;K_0(\gamma(1-x)) \\ &=\; \frac{\lambda}{U}\sqrt{1-M^2}\,\frac{F(1)}{2\pi}\left\{\int_0^\infty e^{-\lambda' t}\,\frac{dt}{1+t-x} \;-\; K_0(\gamma(1-x))\right. \\ &\quad \left. +\;\int_0^\infty e^{-\lambda' t}\,\frac{R_{11}(\gamma(1+t-x))}{1+t-x}\,dt\right\}. \end{aligned} \tag{3.41}$$

The third term in (3.41)

$$\begin{aligned} &\frac{\lambda}{U}\sqrt{1-M^2}\,\frac{F(1)}{2\pi}\int_0^\infty e^{-\lambda' t}\,\frac{R_{11}(\gamma(1+t-x))}{1+t-x}\,dt, \qquad |x|<1 \\ &=\; \frac{\lambda}{U}\sqrt{1-M^2}\,\frac{F(1)}{2\pi}\int_{1-x}^\infty e^{-\lambda'(t-(1-x))}\,\frac{R_{11}(\gamma t)}{t}\,dt \\ &=\; \frac{\lambda}{U}\sqrt{1-M^2}\,\frac{F(1)}{2\pi}\,e^{\lambda'(1-x)}\int_{\gamma(1-x)}^\infty e^{-t/M}\,r_{11}(t)\,dt \end{aligned}$$

where

$$r_{11}(t) \;=\; \frac{|t|K_1(|t|)\,-\,1}{t}\,.$$

Hence defining

$$q_3(x) \;=\; \frac{\sqrt{1-M^2}}{2\pi}\; e^{\lambda'(1-x)} \int_{\gamma(1-x)}^{\infty} e^{-t/M}\, r_{11}(t)\; dt, \quad |x| \le 1$$

the third term can finally be expressed as:

$$\frac{\lambda}{U}\; F(1) q_3(x), \quad |x| \le 1.$$

Now

$$\begin{aligned}
\int_0^\infty e^{-t/M}\, r_{11}(t)\; dt \;&=\; \int_0^\infty e^{-t/M}\; \frac{d}{dt}\; [-K_0(t) \,-\, \log t]\; dt \\
&=\; \frac{1}{M} \int_0^\infty e^{-t/M}\, [-K_0(t) \,-\, \log t]\; dt \\
&=\; \int_0^\infty e^{-t} [-K_0(Mt) \,-\, \log Mt]\; dt \\
&=\; \frac{(-1)}{\sqrt{1-M^2}}\; \log\left[\frac{1}{M} \;+\; \frac{\sqrt{1-M^2}}{M}\right] \,-\, \log M \;+\; \log 2 \\
&=\; \frac{1}{\sqrt{1-M^2}} \left[\left(1 - \sqrt{1-M^2}\right) \log \frac{M}{2} \;-\; \log \frac{(1 + \sqrt{1-M^2}\,)}{2}\right]
\end{aligned} \tag{3.42}$$

and is ≤ 0 for $0 \le M \le 1$. Hence

$$\begin{aligned}
\left|\int_0^\infty e^{-t/M}\, r_{11}(t)\; dt\right| \;&=\; \int_0^\infty e^{-t/M}\, (-r_{11}(t))\; dt \\
&=\; \frac{1}{\sqrt{1-M^2}} \left[\log \frac{(1 + \sqrt{1-M^2}\,)}{2} \;-\; \left(1 - \sqrt{1-M^2}\right) \log \frac{M}{2}\right] \\
&=\; O\left[\frac{M^2}{2}\, \log\, \frac{1}{2M}\right] \qquad \text{as} \;\; M \to 0 \\
&=\; O[1] \qquad\qquad\qquad\quad \text{as} \;\; M \to 1.
\end{aligned}$$

Hence for $\gamma > 0$ (equivalently $\lambda > 0$)

$$\begin{aligned}\left|\frac{\lambda}{U}\, F(1)\, q_3(x)\right| &\leq \frac{\lambda}{U}\, |F(1)|\, e^{2\lambda'} \int_0^\infty e^{-t/M}\, (-r_{11}(t))\, dt \\ &\sim \frac{\lambda}{U}\, e^{2\lambda/U}\, |F(1)|\, \frac{M^2}{2}\, \log \frac{1}{2M} \qquad \text{as} \quad M \to 0 \qquad (3.43) \\ &\sim \frac{\lambda}{U}\, \exp\left(\frac{\lambda(1-x)}{U(1-M^2)}\right)\, |F(1)| \qquad \text{as} \quad M \to 1. \qquad (3.44)\end{aligned}$$

Note also that the function $q_3(\cdot)$ does not depend on $f(\cdot)$. It is also immediate that

$$\left(Rq_3\right)(x) \; = \; 0, \quad x = 1- .$$

Next we need to consider the first term in (3.41). Letting

$$q_1(x) \; = \; \frac{\lambda}{U}\, \sqrt{1-M^2}\, \frac{F(1)}{2\pi} \int_0^\infty e^{-\lambda' t}\, \frac{dt}{1+t-x}\, , \quad |x| \leq 1,$$

we need to calculate

$$Rq_1 \, .$$

For this purpose we note that

$$\int_{-1}^{1} \sqrt{\frac{1+\xi}{1-\xi}}\, \frac{1}{\xi - t - 1}\, \frac{d\xi}{\xi - x} \; = \; \frac{\pi}{x-t-1} \sqrt{\frac{2+t}{t}}\, , \quad t > 0.$$

Hence we see that

$$(Rq_1)(x) \; = \; \frac{-\lambda}{U}\, \frac{F(1)}{\pi} \sqrt{\frac{1-x}{1+x}}\; h(\lambda', x)$$

where

$$h(\lambda, x) \; = \; \int_0^\infty e^{-\lambda t}\, \frac{1}{x-t-1} \sqrt{\frac{2+t}{t}}\, dt. \qquad (3.45)$$

Hence

$$R\left[\frac{-\lambda^2}{U^2} \int_0^\infty e^{-\lambda' t}\, g(t, \cdot)\, dt\right]$$

is given by the function:

$$\frac{-\lambda}{U}\; F(1) \left[K(\gamma, x) \;+\; \frac{1}{\pi} \sqrt{\frac{1-x}{1+x}}\; h(\lambda', x) \right] \;+\; \frac{\lambda}{U}\; F(1)(R\, q_3(\cdot))(x). \tag{3.46}$$

Lemma 6

Let

$$\tilde{h}(\lambda, x) \;=\; \frac{1}{\pi} \sqrt{\frac{1-x}{1+x}}\;\; h(\lambda', x), \quad |x| < 1. \tag{3.47}$$

Then

$$\tilde{h}(\lambda, x) \;\to\; -1 \quad \text{ as } \; x \to 1- .$$

Proof

$$\begin{aligned} \frac{1}{\pi} \sqrt{\frac{1-x}{1+x}}\;\; h(\lambda, x) \;\; &= \;\; \frac{-1}{\pi} \sqrt{\frac{1+x}{1-x}} \int_0^\infty e^{-\lambda t} \frac{1}{t+1-x} \sqrt{\frac{2+t}{t}}\;\; dt \\ &= \;\; \frac{-2}{\pi} \frac{1}{\sqrt{1+x}} \int_0^\infty e^{-\lambda t^2 (1-x)} \sqrt{2 + t^2(1-x)}\;\; \frac{dt}{1+t^2} \\ &\to \;\; -1 \qquad \text{as } \; x \to 1- . \end{aligned}$$

Combining with Lemma 5 we obtain:

Corollary

The function defined by (3.46) goes to zero as $x \to 1-$.

Next let us tackle the second term in (3.40). Here we note first that we can write:

$$\int_{-1}^{1} \frac{F(\xi)}{x - t - \xi}\; d\xi \;=\; \int_{-1}^{1-t} \frac{F(\xi)}{x - t - \xi}\; d\xi \;+\; \int_{1-t}^{1} \frac{F(\xi)}{x - t - \xi}\; d\xi. \tag{3.48}$$

The first term on the right can be expressed:

$$\int_{t-1}^{1} \frac{F(\xi - t)}{(x - \xi)}\; d\xi$$

which in turn can be expressed as

$$q(t,x) \;=\; \int_{-1}^{1} \frac{F(\xi - t)}{(x-\xi)}\; d\xi,$$

defining

$$F(\xi) \;=\; 0, \quad |\xi| > 1.$$

Hence

$$\left(R\left[\frac{1}{2\pi}\, \frac{1}{\sqrt{1-M^2}}\; q(t,\cdot)\right]\right)(x) \;=\; \frac{F(x-t)}{(1-M^2)}\;.$$

Let $r(t,x)$ denote the second term in (3.48). Then we can write

$$r(t,x) \;=\; \int_0^t \frac{F(1-t+\sigma)}{x-1-\sigma}\; d\sigma$$

and

$$\begin{aligned}
&\frac{1}{2\pi}\, \frac{1}{\sqrt{1-M^2}}\; R[r(t,\cdot)](x) \\
&\quad = \;\frac{1}{2\pi}\, \frac{1}{\sqrt{1-M^2}}\, \frac{2}{\pi}\sqrt{\frac{1-x}{1+x}}\;\cdot\; \int_0^t d\sigma \int_{-1}^{1} \sqrt{\frac{1+\xi}{1-\xi}}\; \frac{1}{\xi - x}\; \frac{1}{\xi-1-\sigma}\; F(1-t+\sigma)\; d\xi \\
&\quad = \;\frac{1}{1-M^2}\, \frac{1}{\pi}\sqrt{\frac{1-x}{1+x}}\; \int_0^t \frac{1}{x-1-\sigma}\sqrt{\frac{2+\sigma}{\sigma}}\; F(1-t+\sigma)\; d\sigma.
\end{aligned}$$

Hence

$$\frac{-\lambda^2}{U^2}\; \frac{1}{\sqrt{1-M^2}} \int_0^\infty e^{-\lambda' t}\; R\left[\frac{-1}{2\pi} \int_{-1}^{1} \frac{F(\xi)}{-t-\xi+\cdot}\; d\xi\right]\; dt$$

is given by the function

$$\begin{aligned}
&\frac{\lambda^2}{U^2}\, \frac{1}{1-M^2} \int_0^\infty e^{-\lambda' t}\, F(x-t)\; dt \\
&\qquad + \frac{\lambda^2}{U^2}\, \frac{1}{1-M^2} \left(\int_0^\infty e^{-\lambda' t}\, F(1-t)\; dt\right) h(\lambda', x)\, \frac{1}{\pi}\sqrt{\frac{1-x}{1+x}}\;, \quad |x| < 1 \qquad (3.49)
\end{aligned}$$

which upon integration by parts in both integrals

$$\begin{aligned}
&= \;\frac{-\lambda}{U}\left[-F(x) \;+\; \int_{-1}^{x} e^{\lambda'(x-\sigma)}\, f(\sigma)\; d\sigma\right] \\
&\qquad\qquad - \frac{\lambda}{U}\, \tilde{h}(\lambda', x)\left[-F(1) \;+\; \int_{-1}^{1} e^{\lambda'(1-\sigma)}\, f(\sigma)\; d\sigma\right]
\end{aligned}$$

and clearly goes to zero as $x \to 1-$.

Finally let us consider the third term in (3.40) and calculate

$$R\left[\frac{-\lambda^2}{U^2(1-M^2)}\int_0^\infty e^{-\lambda' t}\,\mathcal{R}_{11}(t)\,F\;dt\right].$$

Let $q_4(\cdot)$ denote the function inside square brackets:

$$\begin{aligned} q_4(x) &= \int_{-1}^{1}\frac{1}{2\pi}\,\frac{F(\xi)}{\sqrt{1-M^2}}\,d\xi\;\frac{\lambda^2}{U^2}\int_0^\infty e^{-\lambda' t}\,\frac{R_{11}(\gamma|t-(x-\xi)|)}{t-(x-\xi)}\,dt, \quad |x|<1 \\ &= \int_{-1}^{1}\frac{1}{2\pi}\,\frac{\lambda^2}{U^2}\,\frac{F(\xi)}{\sqrt{1-M^2}}\;L(x-\xi)\,e^{-\lambda'(x-\xi)}\;d\xi \end{aligned}$$

where,

$$\begin{aligned} L(x) &= \int_{-x}^{\infty} e^{-\lambda' t}\left(\frac{\gamma|t|\,K_1(\gamma|t|)\;-\;1}{t}\right)\,dt \\ &= \int_0^\infty e^{-\lambda' t}\left(\frac{\gamma|t|K_1(\gamma|t|)\;-\;1}{t}\right)\,dt\;+\;\int_{-x}^{0} e^{-\lambda' t}\left(\frac{\gamma|t|\,K_1(\gamma|t|)\;-\;1}{t}\right)\,dt \end{aligned}$$

where the first integral, denoted $c(M)$, is given by (3.42). We can decompose $q_4(\cdot)$ correspondingly

$$q_4(x)\;=\;q_{4,1}(x)\;+\;q_{4,2}(x) \tag{3.50}$$

where

$$q_{4,1}(x)\;=\;\frac{1}{2\pi}\,\frac{c(M)}{\sqrt{1-M^2}}\,e^{-\lambda' t}\left(\frac{\lambda^2}{U^2}\int_{-1}^{1}F(\xi)\,e^{\lambda\xi}\;d\xi\right) \tag{3.51}$$

$$\begin{aligned} q_{4,2}(x) &= \frac{1}{2\pi}\,\frac{\lambda^2}{U^2}\,e^{-\lambda' t}\,\frac{1}{\sqrt{1-M^2}} \\ &\quad\cdot\int_{-1}^{1}e^{\lambda\xi}\,F(\xi)\;d\xi\int_{-(x-\xi)}^{0}e^{-\lambda' t}\left(\frac{\gamma|t|\,K_1(\gamma|t|)\;-\;1}{t}\right)\,dt \qquad (3.52) \\ &\approx \frac{\gamma^2}{2}\log\gamma \qquad\qquad \text{as}\;\;\gamma\to 0\,. \qquad (3.53) \end{aligned}$$

It is immediate that

$$\begin{aligned} (Rq_{4,1})(x) &\;\to\; 0 \qquad \text{as}\;\; x\to 1- \\ (Rq_{4,2})(x) &\;\to\; 0 \qquad \text{as}\;\; x\to 1-\,. \end{aligned}$$

Hence, finally:

$$R\left(\int_0^\infty \frac{-\lambda^2}{U^2}\,e^{-\lambda' t}\,\mathcal{K}(t)f\;dt\right)$$

is given by the function

$$\frac{-\lambda}{U}\, F(1)\left[K(\gamma,x) \;+\; \tilde h(\lambda',x)\right] \;-\; \frac{\lambda}{U}\left[-F(x) \;+\; \int_{-1}^{x} e^{\lambda'(x-\sigma)}\, f(\sigma)\; d\sigma\right]$$

$$- \frac{\lambda}{U}\left[-F(1) \;+\; \int_{-1}^{1} e^{-\lambda'(x-\sigma)}\, f(\sigma)\; d\sigma\right]\tilde h(\lambda',x) \;+\; (Rq_4(\cdot))(x)$$

$$= \frac{-\lambda}{U}\, F(1)\, K(\gamma,x) \;-\; \frac{\lambda}{U}\left[-F(x) \;+\; \int_{-1}^{x} e^{-\lambda'(x-\sigma)}\, f(\sigma)\; d\sigma\right]$$

$$- \frac{\lambda}{U}\left[\int_{-1}^{1} e^{-\lambda(1-\sigma)}\, f(\sigma)\; d\sigma\right]\tilde h(\lambda',x) \;+\; (Rq_4(\cdot))(x).$$

Hence

$$RG_0(\lambda)f \;=\; R\mathcal{R}_{11}\, f \;+\; R\left[\frac{\lambda}{U}\,\mathcal{K}(0)f \;-\; \frac{\lambda^2}{U^2}\int_0^\infty e{-}\lambda' t\,\mathcal{K}(t)f\; dt\right] \tag{3.54}$$

where

$$R\left[\frac{\lambda}{U}\,\mathcal{K}(0)\, f \;-\; \frac{\lambda^2}{U^2}\;\int_0^\infty e^{-\lambda' t}\,\mathcal{K}(t)\, f\; dt\right]$$

is given by the function

$$\frac{\lambda}{U(1{-}M^2)}\;[K(\gamma,x)\, F(1) \;-\; F(x)] \;-\; \frac{\lambda}{U}\; F(1)\, K(\gamma,x)$$

$$- \frac{\lambda}{U}\left[-F(x) \;+\; \int_{-1}^{x} e^{-\lambda'(x-\sigma)}\, f(\sigma)\; d\sigma\right]$$

$$- \frac{\lambda}{U}\left(\int_{-1}^{1} e^{-\lambda(1-\sigma)}\, f(\sigma)\; d\sigma\right)\tilde h(\lambda',x) \;+\; (Rq_{11}(\cdot))(x) \;+\; F(1)(Rr_{11}(\cdot))(x)$$

$$= \frac{-\lambda}{U}\int_{-1}^{x} e^{-\lambda'(x-\sigma)}\, f(\sigma)\; d\sigma$$

$$- \frac{\lambda}{U}\;\tilde h(\lambda',x)\int_{-1}^{1} e^{-\lambda'(1-\sigma)}\, f(\sigma)\; d\sigma$$

$$+ \frac{\lambda}{U}\;\frac{M^2}{(1{-}M^2)}\;[F(1)\, K(\gamma,x) \;-\; F(x)]$$

$$+ (Rq_4(\cdot))(x) \;+\; \frac{\lambda}{U}\; F(1)(Rq_3(\cdot))(x), \quad |x| \le 1. \tag{3.55}$$

Using Lemma 4 (or otherwise) we see that the function defined by (3.55) goes actually to zero as $x \to 1-$. This concludes the proof of the Theorem 3.1. $\square$

Closed Form Solution Approximation

We now show how we can derive a closed form solution with error of the order

$$|\gamma^2 \log \gamma| \quad \text{as} \;\; \gamma \to 0$$

or

$$|M^2 \log M| \quad \text{as} \;\; M \to 0.$$

For this purpose we note that by (3.55) we can break $RG_0(\lambda)$ as

$$RG_0(\lambda)f \;=\; T(\lambda)f \;+\; \Big(R\mathcal{R}_{11}(0)f \;+\; R\,q_{4,2}\Big) \tag{3.56}$$

where the part in parentheses, as we have seen,

$$O[\,|\gamma^2 \log \gamma|\,] \quad \text{as} \;\; \gamma \to 0$$

and $T(\lambda)$ is defined by

$$g \;=\; T(\lambda)f$$

$$\begin{aligned} g(x) \;&=\; \frac{-\lambda}{U}\int_{-1}^{x} e^{-\lambda'(x-\sigma)}\, f(\sigma)\, d\sigma \;-\; \frac{\lambda}{U}\,\tilde{h}(\lambda', x)\int_{-1}^{1} e^{-\lambda'(1-\sigma)}\, f(\sigma)\, d\sigma \\ &\quad + \frac{\lambda}{U}\,\frac{M^2}{1-M^2}\,[F(1)K(\gamma, x) - F(x)] \;+\; \frac{\lambda}{U}\,F(1)(Rq_3)(x) \\ &\quad + \Big(Rq_{4,1}\Big)(x) \end{aligned} \tag{3.57}$$

where the last term can be simplified further, noting that

$$\int_{-1}^{1} e^{\lambda'\xi}\, F(\xi)\, d\xi \;=\; \frac{1}{\lambda'}\left(e^{\lambda'}\, F(1) \;-\; \int_{-1}^{1} e^{\lambda'\xi}\, f(\xi)\, d\xi\right)$$

so that

$$\Big(Rq_{4,1}\Big)(x) \;=\; c(M)\,\frac{\lambda}{U}\left[F(1)\,q_5(x) \;-\; q_5(x)\int_{-1}^{1} e^{-\lambda(1-\xi)}\, f(\xi)\, d\xi\right] \tag{3.58}$$

where

$$q_5(x) \;=\; \frac{1}{\pi^2}\sqrt{\frac{1-x}{1+x}}\;\left(\int_{-1}^{1}\sqrt{\frac{1+\xi}{1-\xi}}\;\frac{e^{\lambda'(1-\xi)}}{\xi - x}\, d\xi\right) \tag{3.59}$$

both of which are special functions, and do *not* depend on $f(\cdot)$.

Collecting common terms, we can rewrite $T(\lambda)f$ as

$$\begin{aligned} g(x) &= \frac{-\lambda}{U}\int_{-1}^{x} e^{-\lambda'(x-\sigma)}\, f(\sigma)\, d\sigma \;-\; \gamma M\int_{-1}^{x} f(\sigma)\, d\sigma \\ &\quad - \frac{\lambda}{U}\, h(x)\int_{-1}^{1} e^{-\lambda'(1-\sigma)}\, f(\sigma)\, d\sigma \;+\; \gamma M K(x) F(1), \quad |x| \le 1 \end{aligned} \tag{3.60}$$

where

$$\begin{aligned} h(x) &= \tilde{h}(\lambda', x) \;+\; c(M)\, q_5(x) \\ K(x) &= K(\gamma, x) \;+\; \frac{1-M^2}{M^2}\, \Big((Rq_3)(x) \;+\; c(M) q_5(x)\Big). \end{aligned}$$

$H(\cdot)$ and $K(\cdot)$ are in $L^{\frac{4}{3}-}$.

The main point in isolating $T(\lambda)$ is to exploit the fact that we can obtain a closed form solution for

$$(I \;+\; T(\lambda))^{-1}.$$

Since the additional terms in (3.56) are small compared to $T(\lambda)$ for small γ, (or small M for fixed λ), we obtain in this manner a closed form solution for the Possio equation which is particularly suited to the usual range of γ values of interest: $|\gamma| < 1$.

The reason why we can get a closed form solution for g for given f for the equation

$$f \;=\; (I + T(\lambda))g$$

is that $(I + T(\lambda))$ has the form:

$$(I + T(\lambda))f \;=\; g \;+\; Lg \;+\; f_1\mathcal{L}_1(g) \;+\; f_2\mathcal{L}_2(g) \tag{3.61}$$

where

$$Lf \;=\; \frac{-\lambda}{U}\int_{-1}^{x} e^{-\lambda'(x-\sigma)}\, f(\sigma)\, d\sigma \;-\; \gamma M\int_{-1}^{x} f(\sigma)\, d\sigma \tag{3.62}$$

is a Volterra operator, $\mathcal{L}_1(\cdot)$, $\mathcal{L}_2(\cdot)$ are given linear functionals and

$$f_1 \;=\; h, \quad f_2 \;=\; K$$

are given elements in $L^{\frac{4}{3}-}$, and we can evaluate $(I + L)^{-1}$ in closed form. Hence

$$g \;=\; (I + L)^{-1}f \;+\; (I + L)^{-1}f_1\mathcal{L}_1(g) \;+\; (I + L)^{-1}f_2\mathcal{L}_2(g)$$

and we only need to solve an algebraic linear equation in two unknowns for $\mathcal{L}_1(g)$ and $\mathcal{L}_2(g)$, which we can do provided of course the determinant is nonzero.

It is a little easier to proceed directly as we shall show.

Theorem 3.2

The equation

$$f \;=\; g \;+\; T(\lambda)g$$

has the unique solution given by

$$g(x) \;=\; r(x) \;+\; \lambda' \int_{-1}^{x} \cosh\gamma(x-\sigma) r(\sigma)\, d\sigma \;+\; \gamma \int_{-1}^{x} \sinh\gamma(x-\sigma) r(\sigma)\, d\sigma \tag{3.63}$$

where $r(x)$ is given by

$$r(x) \;=\; f(x) \;+\; \gamma S(r) K(x) \;+\; \left(\frac{\lambda}{U}\, h(x) \;-\; \gamma M K(x)\right) C(r) \tag{3.64}$$

where the coefficients (functionals)

$$\left.\begin{aligned} S(r) \;&=\; \int_{-1}^{1} \sinh\gamma(1-\sigma) r(\sigma)\, d\sigma \\ C(r) \;&=\; \int_{-1}^{1} \cosh\gamma(1-\sigma) r(\sigma)\, d\sigma \end{aligned}\right\} \tag{3.65}$$

are given by

$$\begin{vmatrix} C(r) \\ S(r) \end{vmatrix} \;=\; D(\lambda, M)^{-1} \begin{vmatrix} C(f) \\ S(f) \end{vmatrix} \tag{3.66}$$

where the 2×2 matrix

$$D(\lambda, M) \;=\; \begin{vmatrix} 1 \;-\; \frac{\lambda}{U}\, C(h) + \gamma M C(K) & \gamma C(K) \\ & \\ -S(K) & 1 + \gamma S(K) \end{vmatrix}, \tag{3.67}$$

is nonsingular for $\operatorname{Re}\lambda \;\geq\; 0$ excepting possibly a sequence of isolated values $\{\lambda_k\}$ bounded away from zero and $|\lambda_k| \to \infty$ as $k \to \infty$.

Proof

We begin with

$$f(x) \;=\; g(x) \;-\; \frac{\lambda}{U} \int_{-1}^{x} e^{-\lambda'(x-\sigma)}\, g(\sigma)\, d\sigma \;-\; \frac{\lambda}{U}\, h(x) \int_{-1}^{1} e^{-\lambda'(1-\sigma)}\, g(\sigma)\, d\sigma$$

$$+\; M^2 \lambda' \left[K(x) \int_{-1}^{1} g(\sigma)\, d\sigma \;-\; \int_{-1}^{x} g(\sigma)\, d\sigma \right]. \tag{3.68}$$

Let

$$\begin{aligned}\tilde{f}(x) &= e^{\lambda' x} f(x)\\ \tilde{g}(x) &= e^{\lambda' x} g(x)\\ Q(x) &= \int_{-1}^{x} \tilde{g}(\sigma)\, d\sigma.\end{aligned}$$

Multiplying both sides of (3.68), by $e^{\lambda' x}$ we have

$$\begin{aligned}\tilde{f}(x) &= \tilde{g}(x) - \frac{\lambda}{U}\int_{-1}^{x}\tilde{g}(\sigma)\, d\sigma - \frac{\lambda}{U}\, e^{\lambda'(x-1)}\, h(x)\int_{-1}^{x}\tilde{g}(\sigma)\, d\sigma\\ &\quad + M^2\lambda'\left[K(x)\int_{-1}^{1} e^{\lambda'(x-\sigma)}\,\tilde{g}(\sigma)\, d\sigma - \int_{-1}^{x} e^{\lambda'(x-\sigma)}\,\tilde{g}(\sigma)\, d\sigma\right].\end{aligned}$$

Now

$$\int_{-1}^{x} e^{\lambda'(x-\sigma)}\,\tilde{g}(\sigma)\, d\sigma = Q(x) + \lambda'\int_{-1}^{x} e^{\lambda'(x-\sigma)}\,\tilde{g}(x)\, d\sigma$$

$$\int_{-1}^{1} e^{\lambda'(x-\sigma)}\,\tilde{g}(\sigma)\, d\sigma = Q(1)\, e^{\lambda'(x-1)} + \lambda'\int_{-1}^{1} e^{\lambda(x-\sigma)}\, Q(\sigma)\, d\sigma.$$

Hence we can write

$$\begin{aligned}\tilde{f}(x) &= Q'(x) - \frac{\lambda}{U}\, Q(x) - Q(1)\,\frac{\lambda}{U}\, e^{\lambda'(x-1)}\,\tilde{h}(\lambda', x)\\ &\quad + M^2\lambda'\, e^{\lambda'(x-1)}\, Q(1)K(x) + \lambda'^2 M^2\int_{-1}^{1} e^{\lambda'(x-\sigma)}\, Q(\sigma)\, d\sigma\; K(x)\\ &\quad - \lambda' M^2 Q(x) - \lambda'^2 M^2\int_{-1}^{x} e^{\lambda'(x-\sigma)}\,\tilde{g}(\sigma)\, d\sigma\\ &= Q'(x) - \lambda' Q(x) + Q(1)e^{\lambda'(x-1)}\left[\lambda' M^2 K(x) - \frac{\lambda}{U}\, h(x)\right]\\ &\quad + \lambda'^2 M^2\left[K(x)\int_{-1}^{1} e^{\lambda'(x-\sigma)}\, Q(\sigma)\, d\sigma - \int_{-1}^{x} e^{\lambda'(x-\sigma)}\, Q(\sigma)\, d\sigma\right]\end{aligned}$$

$$\begin{aligned}Q'(x) &= \lambda' Q(x) + e^{\lambda' x} f(x)\\ &\quad + Q(1)\, e^{\lambda'(x-1)}\left[\frac{\lambda}{U}\, h(x) - \lambda' M^2 K(x)\right]\\ &\quad + M^2\lambda'^2\left[\int_{-1}^{x} e^{\lambda'(x-\sigma)}\, Q(\sigma)\, d\sigma - K(x)\int_{-1}^{1} e^{\lambda'(x-\sigma)}\, Q(\sigma)\, d\sigma\right]\end{aligned}$$

which yields

$$Q(x) \;=\; \int_{-1}^{x} e^{\lambda' x} f(\sigma)\, d\sigma$$

$$+\, M^2 \lambda'^2 \left[\int_{-1}^{x} e^{\lambda' x}\, d\sigma \int_{-1}^{\sigma} e^{-\lambda' \tau}\, Q(\tau)\, d\tau \;-\; K(\sigma)\, e^{\lambda' x} \int_{-1}^{1} e^{-\lambda' \tau}\, Q(\tau)\, d\tau \right]$$

$$+\, Q(1) \int_{-1}^{x} e^{\lambda' x}\, e^{-\lambda'} \left[\frac{\lambda}{U}\, h(\sigma) \;-\; \lambda' M^2 K(\sigma) \right]\, d\sigma.$$

Hence letting

$$\tilde{Q}(x) \;=\; e^{-\lambda' x}\, Q(x)$$

we have:

$$\tilde{Q}(x) \;=\; \int_{-1}^{x} f(\sigma)\, d\sigma$$

$$+\, M^2 \lambda'^2 \left[\int_{-1}^{x} (x-\tau)\, \tilde{Q}(\tau)\, d\tau \;-\; \int_{-1}^{x} K(\sigma)\, d\sigma \int_{-1}^{1} \tilde{Q}(\tau)\, d\tau \right]$$

$$+\, \tilde{Q}(1) \left[\int_{-1}^{x} \left(\frac{\lambda}{U}\, h(\sigma) \;-\; \lambda' M^2 K(\sigma) \right)\, d\sigma \right]$$

where we have used:

$$\int_{-1}^{x} d\sigma \int_{-1}^{\sigma} e^{-\lambda' \tau}\, Q(\tau)\, d\tau \;=\; \int_{-1}^{x} (x-\tau)\, \tilde{Q}(\tau)\, d\tau.$$

Hence

$$\tilde{Q}(x) \;-\; \lambda'^2 M^2 \int_{-1}^{x} (x-\tau) \tilde{Q}(\tau)\, d\tau$$

$$=\; \int_{-1}^{x} f(\sigma)\, d\sigma \;-\; \lambda'^2 M^2 \left(\int_{-1}^{x} K(\sigma)\, d\sigma \right) \left(\int_{-1}^{1} \tilde{Q}(\tau)\, d\tau \right)$$

$$+\, \tilde{Q}(1) \int_{-1}^{x} \left(\frac{\lambda}{U}\, h(\sigma) \;-\; \lambda' M^2 K(\sigma) \right)\, d\sigma.$$

This is a Volterra integral equation which has a unique solution. In fact let

$$F(x) \;=\; \int_{-1}^{x} \tilde{Q}(\sigma)\, d\sigma.$$

Then

$$F(-1) \;=\; 0 \;=\; F'(-1)$$

and

$$F''(x) \;-\; \lambda'^2 M^2 F(x) \;=\; r(x)$$

$$r(x) \;=\; f(x) \;-\; \lambda'^2 M^2 K(x) F(1) \;+\; F'(1)\left(\lambda \tilde{h}(\lambda', x) \;-\; \lambda' M^2 K(x)\right)$$

yields

$$\begin{aligned} F(x) &= \int_{-1}^{x} \frac{\sinh\gamma(x-\sigma)}{\gamma}\, r(\sigma)\, d\sigma, \quad \gamma = M\lambda' \\ F'(x) &= \int_{-1}^{x} \cosh\gamma(x-\sigma) r(\sigma)\, d\sigma \end{aligned}$$

or,

$$\begin{aligned} \tilde{Q}(x) &= \int_{-1}^{x} \cosh\gamma(x-\sigma) r(\sigma)\, d\sigma \\ Q(x) &= e^{\lambda' x} \int_{-1}^{x} \cosh\gamma(x-\sigma) r(\sigma)\, d\sigma. \end{aligned}$$

Hence it follows that

$$\begin{aligned} g(x) \;=\; r(x) \;+\; \lambda' \int_{-1}^{x} &\cosh\gamma(x-\sigma) r(\sigma)\, d\sigma \\ &+ \gamma \int_{-1}^{x} \sinh\gamma(x-\sigma) r(\sigma)\, d\sigma \end{aligned} \tag{3.69}$$

where

$$\begin{aligned} r(x) \;=\; f(x) \;-\; \gamma K(x) &\int_{-1}^{1} \sinh\gamma(1-\sigma) r(\sigma)\, d\sigma \\ + \left(\frac{\lambda}{U}\, h(x) \;-\; \gamma\, M K(x)\right) &\int_{-1}^{1} \cosh\gamma(1-\sigma) r(\sigma)\, d\sigma \end{aligned} \tag{3.70}$$

where we need to determine the coefficients $C(r)$, $S(r)$, where $C(\cdot)$, $S(\cdot)$ are functionals defined on $L^{\frac{4}{3}-}$ by:

$$\begin{aligned} C(f) &= \int_{-1}^{1} \cosh\gamma(1-\sigma) f(\sigma)\, d\sigma \\ S(f) &= \int_{-1}^{1} \sinh\gamma(1-\sigma) f(\sigma)\, d\sigma. \end{aligned}$$

But from (3.70) we obtain the algebraic linear equations

$$D(\lambda, M) \begin{vmatrix} C(r) \\ S(r) \end{vmatrix} = \begin{vmatrix} C(f) \\ S(f) \end{vmatrix}$$

where $D(\lambda, M)$ is as defined by (3.67), and has a unique solution provided

$$
\begin{aligned}
d(\lambda, M) &= \det D(\lambda, M) \\
&= \left(1 - \frac{\lambda}{U}\, C(h) + \gamma M C(K)\right)(1 + \gamma S(K)) + \gamma C(K) S(K) \\
&= \left(1 - \frac{\lambda}{U}\, C(h)\right) + \gamma S(K)\left(1 - \frac{\lambda}{U}\, C(h)\right) \qquad (3.71) \\
&\qquad + S(K) C(K)(\gamma + \gamma^2 M)
\end{aligned}
$$

is nonzero.

We note the $d(\lambda, M)$ is continuous in $\operatorname{Re}\lambda \geq 0$, and analytic (no poles or other singularities in the finite part of the plane) in $\operatorname{Re}\lambda > 0$. Moreover

$$
d(M, \lambda) \to \infty \quad \text{as } \operatorname{Re}\lambda \to \infty
$$

and hence it has a finite number of zeros in any finite part of $\operatorname{Re}\lambda \geq 0$, the sequence $\{\lambda_k\}$ of zeros being such that $|\lambda_k| \to \infty$ as $k \to \infty$. It is a continuous function of γ in (3.71) and for $\gamma = 0$ (or $M = 0$):

$$
\begin{aligned}
&= 1 - \frac{\lambda}{U}\, C\left[\tilde{h}(\lambda', \cdot)\right] \\
&= 1 - \frac{\lambda}{U} \int_{-1}^{1} \tilde{h}\left(\frac{\lambda}{U}\,,\ \sigma\right) d\sigma \\
&= \frac{1}{\hat{c}_1(\lambda)}
\end{aligned}
$$

$$
\hat{c}_1(\lambda) = \int_0^\infty e^{-\lambda t}\, c_1(t)\, dt = \frac{U}{\lambda}\, \frac{e^{-\lambda/U}}{K_0\left(\frac{\lambda}{U}\right) + K_1\left(\frac{\lambda}{U}\right)}
$$

and

$$
\tilde{c}_1(\lambda) \to \infty \quad \text{as } \operatorname{Re}\lambda \to \infty.
$$

Hence it follows that the $\{\lambda_k\}$ are bounded away from zero. This concludes the proof. □

Remark 1

We conjecture that $d(\lambda, M)$ has no zeros in any finite part of $\operatorname{Re}\lambda \geq 0$.

Remark 2

Let us summarize our main result. We have derived an approximate closed-form solution to (3.8).

$$\begin{aligned}\hat{A}(x,\lambda) &= \exp\left(\frac{\lambda M^2 x}{U(1-M^2)}\right)\tilde{A}(x,\lambda) \\ \tilde{A}(x,\lambda) &= g(x) \qquad\qquad \text{in} \ \ (3.69)\end{aligned}$$

where we set

$$f(x) = \hat{w}(x,\lambda)\exp\left(\frac{-\lambda M^2 x}{U(1-M^2)}\right).$$

Remark 3

For $M = 0$ (incompressible case) we have that

$$T(\lambda) = RG_0(\lambda)$$

and further (3.42), (3.43) reduce to

$$\begin{aligned}g(x) &= r(x) + \frac{\lambda}{U}\int_{-1}^{x} r(\sigma)\,d\sigma \\ r(x) &= f(x) + \frac{\lambda}{U}\tilde{h}\left(\frac{\lambda}{U}\,,\,x\right)\int_{-1}^{x} r(\sigma)\,d\sigma\end{aligned}$$

and

$$\int_{-1}^{1} r(\sigma)\,d\sigma = \hat{c}_1(\lambda)\int_{-1}^{1} f(\sigma)\,d\sigma.$$

Hence

$$g(x) = r(x) + \frac{\lambda}{U}\int_{-1}^{x} r(\sigma)\,d\sigma \tag{3.72}$$

where

$$r(x) = f(x) + \frac{\lambda}{U}\tilde{h}\left(\frac{\lambda}{U}\,,\,x\right)\hat{c}_1(\lambda)\int_{-1}^{1} f(\sigma)\,d\sigma.$$

This checks with the known result for the incompressible case, as we shall see in the next section.

Remark 4

To simplify calculation without losing too much accuracy we may take

$$\begin{aligned}K(x) &= K(\gamma,x) \\ h(x) &= \tilde{h}(\lambda',x).\end{aligned}$$

Velocity Potential

The Laplace transform of the velocity potential, $\hat{\phi}(x, z, \lambda)$, satisfies

$$\lambda\hat{\phi}(x, z, \lambda) \;-\; \phi(x, z, 0) \;+\; U\,\frac{\partial}{\partial x}\,\hat{\phi}(x, z, \lambda) \;=\; \hat{\psi}(x, z, \lambda). \tag{3.73}$$

Solving this differential equation for $\hat{\phi}(x, z, \lambda)$, we obtain

$$\hat{\phi}(x, z, \lambda) \;=\; \exp\left(\frac{-\lambda(x-a)}{U}\right)\,\hat{\phi}(a, z, \lambda) \;+\; \frac{1}{U}\int_a^x \exp\left(\frac{-\lambda(x-s)}{U}\right)\,\hat{\psi}_0(s, z, \lambda)\,ds$$

where a is arbitrary. From (3.3) we see that

$$\lim_{a\to-\infty}\,\hat{\phi}(a, z, \lambda) \;=\; 0.$$

Hence

$$\hat{\phi}(x, z, \lambda) \;=\; \frac{1}{U}\int_{-\infty}^x \exp\left(\frac{-\lambda(x-s)}{U}\right)\,\hat{\psi}_0(s, z, \lambda)\,ds.$$

Hence in particular for $z = 0+$,

$$\hat{\phi}(x, 0, \lambda) \;=\; \frac{-1}{2}\int_{-b}^x \exp\left(\frac{-\lambda(x-s)}{U}\right)\,\hat{A}(s, \lambda)\,ds. \tag{3.74}$$

Also, differentiating with respect to x:

$$(-1)\,\frac{\partial\hat{\phi}(x, 0, \lambda)}{\partial x} \;=\; \frac{1}{2}\left(\frac{-\lambda}{U}\right)\int_{-b}^x \exp\left(\frac{-\lambda(x-s)}{U}\right)\,\hat{A}(s, \lambda)\,ds \;+\; \tfrac{1}{2}\,\hat{A}(x, \lambda). \tag{3.75}$$

4. Subsonic Incompressible Case

We now specialize the Linearized Compressible Flow equations (2.1) to the case where all the time derivatives therein are set equal to zero, yielding (by cancelling out a_∞, it being nonzero)

$$(1-M^2)\,\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial z^2} = 0 \tag{4.1}$$

but $\phi(\cdot)$ is still a function of time, since the boundary conditions are retained: *viz.,* the flow tangency condition, which then makes the boundary value a function of time through the downwash condition, and the Kutta-Joukowski condition, which also depends on the time coordinate, since the acceleration potential is still defined by

$$\psi(t,x,z) = \frac{\partial\phi(t,x,z)}{\partial t} + U\,\frac{\partial\phi(t,x,z)}{\partial x}\ .$$

However (4.1) is *not* an initial value problem. We are satisfied with any solution. Note that the velocity U has no meaning to (4.1), and enters only in the Kutta-Joukowski condition.

This problem with $M = 0$ is the celebrated "incompressible inviscid flow problem." (See [3,8].) We need to consider it for $M \neq 0$, and this is traditionally done through the so-called Prandtl-Glauert coordinate transformation on the truly "incompressible" solution for $M = 0$. (4.1) is also referred to as the "steady state" equation since time is not present. However it has nothing to do with "steady state" in any sense of asymptotic behavior in time.

The main feature for us is that we can indeed obtain a "time domain" solution unlike the Linearized Compressible Flow equations where we can only obtain the Laplace transform of the solution. Even though no claim is made that the derivation is new, we shall need to be precise mathematically (as compared, say, to the "standard" treatment in [3]). We shall also present a few results which do not appear in [3] or elsewhere.

We shall present a direct approach without recourse to the Prandtl-Glauert transformation. Thus, we shall simply begin, as usual, with the general form of the solution to the "potential equation" (4.1):

$$\begin{aligned}\phi(t,x,z) &= \frac{-1}{2\pi}\,\frac{1}{\sqrt{1-M^2}}\int_{-b}^{b}\gamma_a(t,\xi)\,\tan^{-1}\left(\frac{z\sqrt{1-M^2}}{(x-\xi)}\right)d\xi \\ &\quad - \frac{1}{2\pi}\sqrt{1-M^2}\int_{b}^{b+Ut}\gamma_w(t,\xi)\,\tan^{-1}\left(\frac{z\sqrt{1-M^2}}{(x-\xi)}\right)d\xi \end{aligned} \tag{4.2}$$

where the functions $\gamma_a(t,\cdot)$ and $\gamma_w(t,\cdot)$ are to be determined from the boundary conditions.

From (4.2) we readily deduce that

$$
\begin{aligned}
\frac{\partial\phi(t,x,0+)}{\partial x} &= 0, \qquad\quad x < -b \\
&= 0, \qquad\quad x > b+Ut \\
&= \tfrac{1}{2}\,\gamma_a(t,x), \quad |x| < b \\
&= \tfrac{1}{2}\,\gamma_w(t,x), \quad b < x < b+Ut.
\end{aligned}
$$

Define (the "circulation"):

$$
\Gamma(t) \;=\; \int_{-b}^{b} \gamma_a(t,x)\; dx.
$$

Then, superdots indicating time derivative,

$$
\tfrac{1}{2}\,\dot{\Gamma}(t) \;=\; \frac{\partial}{\partial t}\int_{-b}^{b} \frac{\partial\phi(t,x,0+)}{\partial x}\; dx \;=\; \dot{\phi}(t,b-,0). \tag{4.3}
$$

As usual, we now determined $\gamma_w(t,\cdot)$ from Kutta-Joukowski conditions. Let

$$
F(t,\xi) \;=\; 2\int_{b}^{b+\zeta} \frac{\partial\phi(t,x,0+)}{\partial x}\; dx, \quad -\infty < \zeta < \infty. \tag{4.4}
$$

Then we recognize that

$$
F(t,\zeta) \;=\; \int_{b}^{b+\zeta} \gamma_w(t,x)\; dx \quad \text{ for } \; \zeta \geq 0. \tag{4.5}
$$

The Kutta-Joukowski condition

$$
0 \;=\; \frac{\partial\phi(t,x,0)}{\partial t} \;+\; U\,\frac{\partial\phi(t,x,0)}{\partial x}\;, \quad x \geq b
$$

yields

$$
0 \;=\; \frac{\partial}{\partial t}\left[\int_{-b}^{b+\zeta} \frac{\partial\phi(t,x,0)}{\partial x}\; dx\right] \;+\; U\tfrac{1}{2}\,\gamma_w(t,b{+}\zeta), \quad \zeta \geq 0
$$

or

$$
0 \;=\; \tfrac{1}{2}\,\dot{\Gamma}(t) \;+\; \frac{1}{2}\,\frac{\partial}{\partial t}\,F(t,\zeta) \;+\; \tfrac{U}{2}\,\gamma_w(t,b{+}\zeta), \quad \zeta \geq 0, \tag{4.6}
$$

or,

$$
0 \;=\; \tfrac{1}{2}\,\dot{\Gamma}(t) \;+\; \frac{1}{2}\,\frac{\partial}{\partial t}\,F(t,\zeta) \;+\; \frac{U}{2}\,\frac{\partial}{\partial\zeta}\,F(t,\zeta), \quad \zeta \geq 0,
$$

or, we have the partial differential equation

$$
\frac{\partial F(t,\zeta)}{\partial t} \;=\; -U\,\frac{\partial F(t,\zeta)}{\partial\zeta} \;-\; \dot{\Gamma}(t), \quad t > 0,\; \zeta > 0. \tag{4.7}
$$

We note that

$$F(t,\zeta) \;=\; F(0,\,\zeta - Ut) \;+\; \int_0^t \dot{\Gamma}(\sigma)\, d\sigma \tag{4.8}$$

is a solution of his equation for $\zeta \geq 0$, and by uniqueness of solution of the Cauchy problem associated with this equation, the only solutions such that

$$F(0,\zeta) \;=\; \int_b^{b+\zeta} \gamma_w(0,x)\, dx, \quad \zeta \geq 0.$$

In particular therefore

$$\frac{\partial F(t,\zeta)}{\partial \zeta} \;=\; \frac{\partial F(0,\,\zeta - Ut)}{\partial \zeta}$$

and hence for $\zeta \geq 0$:

$$\frac{\partial F(0,\,\zeta - Ut)}{\partial \zeta} \;=\; \gamma_w(t,\,b+\zeta).$$

Let us use the notation

$$g(\zeta) \;=\; \frac{\partial F(0,\zeta)}{\partial \zeta}\,, \quad -\infty < \zeta < \infty.$$

Then

$$\gamma_w(t,\,b+\zeta) \;=\; g(\zeta - Ut)$$

and going back now to the Kutta-Joukowski condition (4.6), for $\zeta = 0$, therein:

$$\dot{\Gamma}(t) \;=\; -Ug(-Ut), \quad t \geq 0. \tag{4.9}$$

But

$$\gamma_w(t,\,b+\zeta) \;=\; g(\zeta - Ut), \quad \zeta \geq 0$$

or,

$$\gamma_w(t,x) \;=\; g(x-b-Ut), \quad x \geq b.$$

Hence for

$$x \,-\, b \,-\, Ut \;<\; 0 \quad \text{or} \quad x \;<\; b \,+\, Ut,$$

we have from (4.9) that

$$\begin{aligned} g(x-b-Ut) \;&=\; \frac{-1}{U}\,\dot{\Gamma}\left(\frac{Ut+b+x}{U}\right) \\ &=\; \frac{-1}{U}\,\dot{\Gamma}\left(t \,-\, \frac{x-b}{U}\right), \qquad t \geq \frac{x-b}{U}\,, \end{aligned}$$

or,

$$\gamma_w(t,x) \;=\; \frac{-1}{U}\,\dot{\Gamma}\left(t \,-\, \frac{x-b}{U}\right), \quad b < x < b+Ut \tag{4.10}$$

which is then one of the primary results of this theory.

We note that the Kutta Condition at $b-$ requires that

$$\dot{\Gamma}(t) \ + \ U\gamma_a(t, b-) \ = \ 0 \tag{4.10a}$$

since by (4.3):

$$\frac{\partial\phi(t,b-,0+)}{\partial t} \ = \ \frac{\partial}{\partial t}\int_{-b}^{b-} \frac{\partial\phi(t,x,0+)}{\partial x}\, dx \ = \ \tfrac{1}{2}\,\dot{\Gamma}(t)$$

and (4.10a) is consistent with (4.10), upon taking $x = b+$. In particular we have

$$\gamma_w(t, b+) \ = \ \gamma_a(t, b-).$$

Next we exploit the flow tangency condition. We have

$$\begin{aligned} w(t,x) \ &= \ \frac{\partial\phi(t,x,0+)}{\partial z} \\ &= \ \sqrt{1-M^2}\,\left(\frac{-1}{2\pi}\int_{-b}^{b}\frac{\gamma_a(t,\xi)}{x-\xi}\,d\xi \ - \ \frac{1}{2\pi}\int_{b}^{b+Ut}\frac{\gamma_w(t,\xi)}{x-\xi}\,d\xi\right) \end{aligned}$$

where the second term, substituting for $\gamma_w(t,\xi)$ from (4.10):

$$\frac{-1}{2\pi}\int_{b}^{b+Ut}\frac{-1}{U}\,\frac{\dot{\Gamma}\left(t \ - \ \frac{\xi-b}{U}\right)}{x-\xi}\,d\xi \ = \ \frac{1}{2\pi}\int_0^t \frac{\dot{\Gamma}(t-\sigma)}{x-b-U\sigma}\,d\sigma\ .$$

Hence

$$\frac{w(t,x)}{\sqrt{1-M^2}} \ = \ \frac{-1}{2\pi}\int_{-b}^{b}\frac{\gamma_a(t,\xi)}{x-\xi}\,d\xi \ + \ \frac{1}{2\pi}\int_0^t\frac{\dot{\Gamma}(t-\sigma)}{x-b-U\sigma}\,d\sigma\ , \tag{4.11}$$

which is then the "airfoil" equation, a singular integral equation which we need to solve to determine $\gamma_a(t,\xi)$. We have covered the necessary relevant theory in Section 3. Rewriting (4.11) as

$$\frac{1}{2\pi}\,\sqrt{1-M^2}\int_{-b}^{b}\frac{\gamma_a(t,\xi)}{x-\xi}\,d\xi \ = \ \frac{1}{2\pi}\int_0^t\sqrt{1-M^2}\ \frac{\dot{\Gamma}(t-\sigma)}{x-b-U\sigma}\,d\sigma \ - \ w(t,x). \tag{4.12}$$

We have, under the condition of Lipschitzianness, and introducing the operator R, defined by (3.14), for the only solution bounded at $\xi = b-$:

$$\gamma_a(t,\cdot) \ = \ -Rw(t,\cdot) \ + \ q(t,\cdot) \tag{4.13}$$

where we define

$$q(t,x) = \frac{1}{\pi^2}\sqrt{\frac{b-x}{b+x}}\int_{-b}^{b}\sqrt{\frac{b+\xi}{b-\xi}}\;\frac{1}{\xi-x}\int_0^t \frac{\dot{\Gamma}(t-\sigma)}{\xi-b-U\sigma}\,d\sigma\,d\xi$$

$$= \frac{1}{\pi}\sqrt{\frac{b-x}{b+x}}\int_0^t \frac{\dot{\Gamma}(t-\sigma)}{x-b-U\sigma}\sqrt{\frac{2+\tilde{U}}{\tilde{U}\sigma}}\,d\sigma, \qquad \tilde{U} = \frac{U}{b}\,. \tag{4.14}$$

Remark

In the aeroelastic problem (as in [3]), the downwash function

$$w(t,x) = -\dot{h}(t) - (bx-a)\,\dot{\alpha}(t) - U\alpha(t)$$

where $h(\cdot)$ is the "plunge" and $\alpha(\cdot)$ the "pitch" angle.

From (4.13) setting $t=0$ we obtain

$$\gamma_a(0,\cdot) = -Rw(0,\cdot) \tag{4.15}$$

showing that the initial aerodynamic flow conditions are determined by the initial conditions of the downwash. Our next step is to determine the circulation function from (4.13). It is convenient to let

$$B(t,\cdot) = -Rw(t,\cdot)$$

and

$$B(t) = \int_{-b}^{b} B(t,x)\,dx,$$

so that

$$\Gamma(0) = B(0)$$

and, integrating (4.13), we obtain

$$\Gamma(t) = B(t) + \int_{-b}^{b} q(t,x)\,dx$$

where

$$\int_{-b}^{b} q(t,x)\,dx = \int_0^t \left(1 - \sqrt{\frac{2+\tilde{U}\sigma}{\tilde{U}\sigma}}\right)\dot{\Gamma}(t-\sigma)\,d\sigma$$

or,

$$0 = B(t) - B(0) - \int_0^t \sqrt{\frac{2+\tilde{U}\sigma}{\tilde{U}\sigma}}\;\dot{\Gamma}(t-\sigma)\,d\sigma$$

which is a linear integral equation for $\Gamma(t)$, where the important observation is that

$$\lambda \int_0^\infty e^{-\lambda t} \sqrt{\frac{2+\tilde{U}t}{\tilde{U}t}}\; dt \;=\; \frac{1}{\hat{c}_1(\lambda)}$$

where

$$\hat{c}_1(\lambda) \;=\; \frac{\tilde{U}}{\lambda}\, \frac{e^{-\lambda/\tilde{U}}}{K_0\left(\frac{\lambda}{\tilde{U}}\right) \,+\, K_1\left(\frac{\lambda}{\tilde{U}}\right)} \;=\; \int_0^\infty e^{-\lambda t} c_1(t)\, dt, \quad \operatorname{Re}\lambda > 0 \tag{4.16}$$

and hence

$$\lambda\hat{\Gamma}(\lambda) \,-\, \Gamma(0) \;=\; \left(\hat{B}(\lambda) \,-\, \frac{B(0)}{\lambda}\right) \frac{1}{h(\lambda)} \;=\; \hat{c}_1(\lambda)\left(\lambda\hat{B}(\lambda) \,-\, B(0)\right) \tag{4.17}$$

yielding a "convolution" interpretation in the time domain. This is another central result in the airfoil theory, the key being (4.16) due to Sears (see [3]). However this is not of particular importance in the Laplace transform theory for (4.13). Thus Laplace transforming (4.13) we have, using (4.17) as an intermediate step:

$$\hat{\gamma}_a(\lambda,\cdot) \;=\; -R\hat{w}(\lambda,\cdot) \;+\; \left(\lambda\hat{\Gamma}(\lambda) - \Gamma(0)\right)\, \hat{h}(\lambda,\cdot)$$

or, using (4.17),

$$\hat{\gamma}_a(\lambda,\cdot) \;=\; -R\hat{w}(\lambda,\cdot) \;+\; \left(\lambda\hat{B}(\lambda) - B(0)\right)\, \hat{c}_1(\lambda)\, \hat{h}(\lambda,\cdot) \tag{4.18}$$

where

$$\hat{h}(\lambda, x) \;=\; \frac{1}{\pi}\sqrt{\frac{b-x}{b+x}}\; \int_0^\infty e^{-\lambda\sigma}\, \frac{1}{x-b-U\sigma}\, \sqrt{\frac{2+\tilde{U}\sigma}{\tilde{U}\sigma}}\; d\sigma$$

$$\hat{\gamma}_a(\lambda, x) \;=\; \int_0^\infty e^{-\lambda t}\, \gamma_a(t,x)\, dx$$

$$\hat{B}(\lambda) \;=\; \int_0^\infty e^{-\lambda t}\, B(t)\, dt$$

To finish the calculations we need next to evaluate $\gamma_w(t,\cdot)$, using (4.10). We have

$$\gamma_w(t,\, b+\zeta) \;=\; \frac{-1}{U}\; \dot{\Gamma}\left(t \,-\, \frac{\zeta}{U}\right), \quad 0 < \zeta < Ut.$$

But the time domain version of (4.17) yields

$$\dot{\Gamma}(t) \;=\; \int_0^t c_1(t-\sigma)\; \dot{B}(\sigma)\; d\sigma$$

and hence

$$\begin{aligned}\gamma_w(t,\, b+\zeta) &= \frac{-1}{U}\int_0^{t-(\zeta/U)} c_1(t-\sigma)\ \dot{B}(\sigma)\ d\sigma \\ &= 0, \qquad \zeta > Ut,\end{aligned}$$

also

$$\begin{aligned}\hat{\gamma}_w(\lambda,\, b+\zeta) &= \int_0^\infty e^{-\lambda t}\gamma_w(t,\, b+\zeta)\ dt, \quad 0 < \zeta < Ut \\ &= \frac{-1}{U}\ e^{-\lambda\zeta/U}\left(\lambda\hat{\Gamma}(\lambda)\ -\ \Gamma(0)\right) \\ &= \frac{-1}{U}\ e^{-\lambda\zeta/U}\,\hat{c}_1(\lambda)\left(\lambda\hat{B}(\lambda)\ -\ B(0)\right).\end{aligned}$$

This yields for the acceleration potential

$$\hat{\psi}(\lambda, x, 0+) \;=\; \hat{\gamma}_a(\lambda, x) \;+\; \int_{-b}^{x}(\lambda\hat{\gamma}_a(\lambda, y)\ -\ \gamma_a(0, y))\ dy \tag{4.19}$$

using

$$\hat{\gamma}_a(\lambda, x) \;=\; (-R\hat{w}(\lambda,\cdot))\,(x) \;+\; \left(\lambda\hat{B}(\lambda)\ -\ B(0)\right)\,\hat{c}_1(\lambda)\,\hat{h}(\lambda, x), \tag{4.20}$$

and (4.15)

$$\gamma_a(0, y) \;=\; -(Rw(0,\cdot))(y).$$

We note that (3.49) is the same as (4.19) if we set the initial conditions

$$\gamma_a(0, y) \;=\; 0; \quad B(0) \;=\; 0$$

and take

$$\begin{aligned}r(x) &= \gamma_a(\lambda, x) \\ f(\cdot) &= -R\,\hat{w}(\lambda,\cdot)\end{aligned}$$

and in fact this analysis provides an interpretation for the function $r(\cdot)$ in (3.72).

Also, finally, we have for the velocity potential:

$$\begin{aligned}\phi(t, x, z) \;=\;& \frac{-1}{2\pi}\int_{-b}^{b} B(t,\xi)\ \tan^{-1}\frac{z\sqrt{1-M^2}}{(x-\xi)}\ d\xi \\ &- \frac{1}{2\pi}\int_{-b}^{b}\frac{1}{\pi}\sqrt{\frac{b-\xi}{b+\xi}}\ \int_0^t \frac{\dot{\Gamma}(t-\sigma)}{\xi-b-U\sigma}\sqrt{\frac{2+\tilde{U}\sigma}{\tilde{U}\sigma}}\ d\sigma\cdot\ \tan^{-1}\frac{z\sqrt{1-M^2}}{(x-\xi)}\ d\xi \\ &+ \frac{1}{2\pi}\int_0^t \dot{\Gamma}(t-\sigma)\ \tan^{-1}\frac{z\sqrt{1-M^2}}{(x-\xi)}\ d\sigma,\end{aligned} \tag{4.21}$$

where

$$\dot{\Gamma}(t) \;=\; \int_0^t c_1(t-\sigma)\; \dot{B}(\sigma)\; d\sigma$$

$$B(t,\cdot) \;=\; -Rw(t,\cdot)$$

$$B(t) \;=\; \int_{-b}^{b} B(t,x)\; dx$$

$$\hat{c}_1(\lambda) \;=\; \int_0^\infty e^{-\lambda t}\, c_1(t)\; dt \;=\; \frac{\tilde{U}}{\lambda}\, \frac{e^{-\lambda/\tilde{U}}}{K_0\left(\frac{\lambda}{\tilde{U}}\right) \;+\; K_1\left(\frac{\lambda}{\tilde{U}}\right)}\;.$$

Note that the initial conditions are:

$$\phi(0,x,z) \;=\; \frac{-1}{2\pi}\int_{-b}^{b} B(0,\xi)\, \tan^{-1}\frac{z\sqrt{1-M^2}}{(x-\xi)}\;\; d\xi.$$

$$\dot{\phi}(0,x,z) \;=\; \frac{-1}{2\pi}\int_{-b}^{b} \dot{B}(0,\xi)\, \tan^{-1}\frac{z\sqrt{1-M^2}}{(x-\xi)}\;\; d\xi.$$

Hence

$$\phi(0,x,0) \;=\; 0 \;=\; \dot{\phi}(0,x,0).$$

On the other hand

$$\frac{\partial\phi}{\partial x}\,(0+,x,0+) \;=\; \gamma_a(0+,x) \;=\; B(0,x)$$

and is not necessarily zero. Hence

$$\begin{aligned}
\psi(0,x,0) \;&=\; \tilde{U}\,\frac{\partial\phi}{\partial x} \;+\; \frac{\partial\phi}{\partial t}\\
&=\; \tilde{U}\,B(0,x) \;=\; 0, \qquad \text{if}\;\; w_a(0,\cdot)=0.
\end{aligned}$$

To connect with the Possio solution (cf. (3.49)), we note that

$$\begin{aligned}
\psi_p(0,x,0) \;&=\; \psi_0(0,x,0) \;+\; \phi_p(0,x,0)\\
&=\; \frac{-U}{2}\,A(x,0) \;=\; 0, \qquad \text{if}\;\; w_a(0,\cdot)=0=\;\dot{w}(0,\cdot).
\end{aligned}$$

Hence $\psi(0,x,0)$ and $\psi_p(0,x,0)$ are equal if the downwash initial conditions are zero, but not otherwise.

5. Solving the Boundary-Value Initial-Value Problem

Solving the Boundary-Value Problem

We first consider the problem of finding a particular solution to the Boundary Value Problem for (2.1) without regard to the initial value. Thus let A_L denote the extension of A where we drop (only) the condition (i) that the boundary value be zero. Thus

$$A_L f = g \qquad f = \begin{vmatrix} f_1 \\ f_2 \\ f_3 \end{vmatrix} \qquad g = \begin{vmatrix} g_1 \\ g_2 \\ g_3 \end{vmatrix}$$

where

$$g = \begin{vmatrix} -2U\frac{\partial}{\partial x} & a_\infty^2(1-M^2)\frac{\partial}{\partial x} & a_\infty^2\frac{\partial}{\partial z} \\ \frac{\partial}{\partial x} & 0 & 0 \\ \frac{\partial}{\partial z} & 0 & 0 \end{vmatrix} f\,. \tag{5.1}$$

The problem then is: Given $w(t,\cdot)$ where for each $t \geq 0$, $w(t,\cdot) \in L_2[R^1]$, find $f(t)$ in $\mathcal{H}$ such that

i) $A_L f(t) = 0$

and

ii)

$$\int_{-\infty}^{\infty} |f_3(t,x,z) - w_a(t,x)|^2 \, dx \to 0, \qquad \text{as} \quad z \to 0+\,. \tag{5.2}$$

It is easy to verify that such a solution is given by

$$f(t) = \begin{vmatrix} f_1(\cdot) \\ f_2(\cdot) \\ f_3(\cdot) \end{vmatrix},$$

$$f_1(t,x,z) = 0$$

$$f_2(t,x,z) = \frac{\partial}{\partial x}\phi(t,x,z,)$$

$$f_3(t,x,z) = \frac{\partial}{\partial z}\phi(t,x,z,)$$

where $\phi(t, x, z)$ is the velocity potential determined explicitly in Section 4, equation (4.21), satisfying (4.1) and the flow tangency condition and the far field conditions. We shall denote this velocity potential by $\phi_M(t, x, z)$ where M is the subscript to indicate that M is not taken to be zero. We can also calculate the velocity potential via the Possio equation as in Section 3, setting $M = 0$ in the acceleration potential (but retaining M in the definition of R). As we have seen, the two determinations differ in the initial conditions. In what follows we may pick either one and still denote it $\phi_M(\cdot)$.

We note while $f(t)$ is not in the domain of A_0, it does satisfy the dynamic Kutta-Joukowski condition by our construction. In other words

$$\psi(t, x, z) \;=\; \int_{-\infty}^{x} \frac{\partial}{\partial t} f_2(t, y, z)\, dy \;+\; U f_2(t, x, z)$$

will satisfy

$$\begin{aligned} \psi(t, b-, 0+) &= 0 \\ \psi(t, x, 0+) &= 0, \quad x \geq b. \end{aligned}$$

Moreover we can write

$$f(t) \;=\; Dw(t, \cdot)$$

where D defines a linear bounded transformation on $L_2[R^1]$ into $\mathcal{H}$ for each $t \geq 0$.

Solving the Combined Problem

We can construct the solution for the problem (2.1) — the combined initial-value/boundary-value problem, following the general technique outlined in [1,2,3].

We claim that the solution is given by defining:

$$x(t) \;=\; x_0(t) \;+\; Dw(t, \cdot), \quad t \geq 0 \tag{5.3}$$

where

$$x_0(t) \;=\; S(t)(x(0) \;-\; Dw(0, \cdot)) \;-\; \int_0^t S(t - \sigma) D\dot{w}(\sigma, \cdot)\, d\sigma \tag{5.4}$$

(the superdot as usual denoting time derivative) which is the generalized (or weak) solution of

$$\dot{x}_0(t) \;=\; Ax_0(t) \;-\; D\dot{w}(t, \cdot) \tag{5.5}$$

(with the tacit asumption on strong differentiability of $Dw(t, \cdot)$) with the initial condition

$$x_0(0) \;=\; x(0) \;-\; Dw(0, \cdot).$$

To verify that $x(t)$ given by (5.3) is the solution sought, we have only to note that

$$\begin{aligned}
\dot{x}(t) &= \dot{x}_0(t) + D\dot{w}(t,\cdot) \\
&= Ax_0(t) \\
&= A_L x_0(t) \\
&= A_L[x(t) - Dw(t,\cdot)] \\
&= A_L x(t).
\end{aligned}$$

Defining the velocity potential as before (Section 2) by

$$\phi(t,x,z) = \phi(0,x,z) + \int_0^t f_1(\sigma,x,z)\, d\sigma$$

in the notation

$$x(t) = \begin{vmatrix} f_1(t,\cdot) \\ f_2(t,\cdot) \\ f_3(t,\cdot) \end{vmatrix}$$

and defining

$$\begin{aligned}
\frac{\partial}{\partial x}\phi(0,x,z) &= f_2(0,x,z) \\
\frac{\partial}{\partial z}\phi(0,x,z) &= f_3(0,x,z)
\end{aligned}$$

we see that $\phi(t,x,z)$ satisfies (2.1) and the boundary conditions. And

$$\begin{vmatrix} \phi(0,x,z) \\ \dot{\phi}(0,x,z) \end{vmatrix}$$

can be specified arbitrarily within our far field conditions and boundary smoothness conditions and of course the differentiability conditions. The primary interest is however in the acceleration potential at $z = 0+$ definable through $f_2(t,x,z)$.

Let $\phi_P(t,x,z)$ denote the Possio solution for the velocity potential. This is a particular solution obtained by the "doublet-at-source" method. But this solution can be expressed in the form (5.3) for an appropriate initial condition. Let

$$x_P(t) = \begin{vmatrix} \frac{\partial\phi_P}{\partial t} \\ \frac{\partial\phi_P}{\partial x} \\ \frac{\partial\phi_P}{\partial z} \end{vmatrix}.$$

Then we have:

$$x_P(t) \;=\; Dw(t,\cdot) \;+\; S(t)(x_P(0) - Dw(0,\cdot)) \;-\; \int_0^t S(t-\sigma)\, D\dot{w}(\sigma,\cdot)\, d\sigma. \qquad (5.6)$$

Let us define the Laplace transform:

$$\hat{x}_P(\lambda) \;=\; \int_0^\infty e^{-\lambda t}\, x_P(t)\; dt, \quad \operatorname{Re}\lambda > 0.$$

Then we have

$$\begin{aligned} \hat{x}_P(\lambda) \;&=\; R(\lambda,A)\,(x_P(0) \,-\, Dw(0)) \;-\; R(\lambda,A)\,(\lambda D\hat{w}(\lambda) \,-\, Dw(0)) \;+\; D\hat{w}(\lambda) \\ &=\; D\hat{w}(\lambda) \;+\; R(\lambda,A)\,(x_P(0) \,-\, \lambda D\hat{w}(\lambda))\,. \end{aligned} \qquad (5.7)$$

But since (see Section 3)

$$\phi_P(x,z,0) \;=\; \dot{\phi}_P(x,z,0) \;=\; 0$$

we have that

$$x_P(0) \;=\; 0$$

and hence

$$\hat{x}_P(\lambda) \;=\; (I \,-\, \lambda R(\lambda,A))\, D\hat{w}(\lambda). \qquad (5.8)$$

We have in (5.8) an alternate technique for finding $\hat{x}_P(\lambda)$, alternate to the Possio equation.

The acceleration potential defined by (2.22) can be expressed as a linear bounded transformation $\mathcal{L}$:

$$\psi(t) \;=\; \mathcal{L}x(t)$$

and hence

$$\hat{\psi}_P(\lambda) \;=\; \hat{\psi}_M(\lambda) \;+\; \mathcal{L}\,R(\lambda,A)(x_P(0) \,-\, \lambda D\hat{w}(\lambda)) \qquad (5.9)$$

which shows in particular that $\hat{\psi}_P(\lambda)$ has an essential singularity along the entire negative real axis just as $D\hat{w}(\lambda)$ has but is otherwise analytic.

Finally we note that

$$\mathcal{L}\,(R(\lambda,A)(x_P(0) \,-\, \lambda D\hat{w}(\lambda)))$$

satisfies the Kutta-Joukowski conditions since both $\hat{\psi}_P(\lambda)$ and $\hat{\psi}_M(\lambda)$ do. But there are clearly many solutions of the field equations which do *not*. Indeed the question of characterizing those initial conditions which lead to solutions which satisfy the Kutta-Jouskowski conditions is open.

Finally we note that

$$x(t) \ - \ x_P(t) \ = \ S(t)(x(0) \ - \ x_P(0))$$

or,

$$x(t) \ = \ x_P(t) \ + \ S(t)(x(0) \ - \ x_P(0)).$$

Thus any two solutions differ by a term which depends only on the initial conditions assumed for the field. Since these conditions in a physical sense rarely are specifiable, the unsteady aerodynamics can never be determined completely. Note that

$$\|S(t)x\|^2 \ = \ \|x\|^2$$

so that there is no energy decay, within this theory! Of course solutions which do not satisfy the Kutta-Joukowski condition may be dismissed as "not physical."

General Representation for Velocity Potential

The representation (5.1) can be recast as a representation for velocity potential. Thus:

Theorem 5.1

Any solution $\phi(t,x,z)$ of (2.1) with the flow-tangency and initial conditions can be represented as

$$\phi(t,x,z) \ = \ \phi_M(t,x,z) \ + \ \phi_R(t,x,z) \tag{5.10}$$

where $\phi_R(t,x,z)$ is the unique solution of (2.1) with a forcing term: *viz.*:

$$\frac{\partial^2 \phi_R}{\partial t^2} \ + \ 2a_\infty M \, \frac{\partial^2 \phi_R}{\partial t \partial x} \ - \ a_\infty^2 (1-M^2) \, \frac{\partial^2 \phi_R}{\partial x^2} \ - \ a_\infty^2 \, \frac{\partial^2 \phi_R}{\partial y^2} \ = \ \frac{\partial^2 \phi_M}{\partial t^2} \ - \ 2a_\infty M \, \frac{\partial^2 \phi_M}{\partial t \partial x} \tag{5.11}$$

with homogeneous boundary conditions and initial conditions consistent with

$$\phi_R(t,x,z) \ = \ \phi(t,x,z) \ - \ \phi_M(t,x,z), \quad t \geq 0.$$

Moreover, $\phi_R(t,x,z)$ satisfies the Kutta-Joukowski conditions if $\phi(t,x,z)$ does.

Proof

We calculate directly that

$$\frac{\partial^2 \phi_R}{\partial t^2} \ + \ 2a_\infty M \, \frac{\partial^2 \phi_R}{\partial t \partial x} \ = \ \frac{\partial^2 \phi}{\partial t^2} \ + \ 2a_\infty M \, \frac{\partial^2 \phi}{\partial t \partial x} \ - \ \frac{\partial^2 \phi_M}{\partial t^2} \ - \ 2a_\infty M \, \frac{\partial^2 \phi_M}{\partial t \partial x} \ .$$

Now

$$\frac{\partial^2\phi}{\partial t^2} + 2a_\infty M\,\frac{\partial^2\phi}{\partial t\partial x} = a_\infty^2(1-M^2)\,\frac{\partial^2\phi}{\partial x^2} + a_\infty^2\,\frac{\partial^2\phi}{\partial z^2}$$

$$= a_\infty^2(1-M^2)\,\frac{\partial^2\phi_R}{\partial x^2} + a_\infty^2\,\frac{\partial^2\phi_R}{\partial z^2}$$

since

$$(1-M^2)\,\frac{\partial^2\phi_M}{\partial x^2} + a_\infty^2\,\frac{\partial^2\phi_M}{\partial z^2} = 0$$

and (5.11) follows. Since both $\phi(\cdot)$ and $\phi_M(\cdot)$ satisfy the flow tangency conditions, we see from (5.10) that $\phi_R(\cdot)$ satisfies homogeneous boundary conditions. The rest of the statement concerning $\phi_R(\cdot)$ is immediate. $\square$

Remark

Let $\phi_P(t,x,z)$ denote the Possio solution. Then

$$\phi_R(t,x,z) = \phi_P(t,x,z) - \phi_M(t,x,z)$$

satisfies the Kutta-Joukowski condition and further

$$\phi_R(0,x,z) = 0 = \dot{\phi}_R(0,x,z)$$

if the downwash initial conditions are zero:

$$w_a(0,x) = 0 = \dot{w}_a(0,x), \quad |x| < b$$

and

$$\phi_R(t,x,z)$$

is thus the response solely to the forcing function on the right in (5.11), with zero boundary as well as initial conditions.

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13. ABSTRACT (Maximum 200 words)

This paper presents a new analytical treatment of Unsteady Aerodynamics – the linear theory covering the subsonic compressible (inviscid) case – drawing on some recent work in Operator Theory and Functional Analysis. The specific new results are: (a) An existence and uniqueness proof for the Laplace transform version of the Possio integral equation as well as a new closed form solution approximation thereof. (b) A new representation for the time-domain solution of the subsonic compressible aerodynamic equations emphasizing in particular the role of the initial conditions.

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