



A Generalized Wall Function

Tsan-Hsing Shih
Institute for Computational Mechanics in Propulsion, Cleveland, Ohio

Louis A. Povinelli, Nan-Suey Liu, and Mark G. Potapczuk
Glenn Research Center, Cleveland, Ohio

J.L. Lumley
Cornell University, Ithaca, New York

National Aeronautics and
Space Administration

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Tsan-Hsing Shih

ICOMP, NASA John H. Glenn Research Center, Cleveland, OH 44142.

Louis A. Povinelli, Nan-Suey Liu, Mark G. Potapczuk

NASA John H. Glenn Research Center, Cleveland, OH 44135.

J. L. Lumley

Cornell University, Ithaca, N.Y.

April 21, 1999

ABSTRACT

The asymptotic solutions, described by Tennekes and Lumley (1972), for surface flows in a channel, pipe or boundary layer at large Reynolds numbers are revisited. These solutions can be extended to more complex flows such as the flows with various pressure gradients, zero wall stress and rough surfaces, etc. In computational fluid dynamics (CFD), these solutions can be used as the boundary conditions to bridge the near-wall region of turbulent flows so that there is no need to have the fine grids near the wall unless the near-wall flow structures are required to resolve. These solutions are referred to as the wall functions. Furthermore, a generalized and unified law of the wall which is valid for whole surface layer (including viscous sublayer, buffer layer and inertial sublayer) is analytically constructed. The generalized law of the wall shows that the effect of both adverse and favorable pressure gradients on the surface flow is very significant. Such an unified wall function will be useful not only in deriving analytic expressions for surface flow properties but also bringing a great convenience for CFD methods to place accurate boundary conditions at any location away from the wall. The extended wall functions introduced in this paper can be used for complex flows with acceleration, deceleration, separation, recirculation and rough surfaces.

1 INTRODUCTION

An asymptotic solution for the inertial sublayer in a channel or pipe flow at large Reynolds numbers can be written as (Millikan, 1938)

$$\frac{U}{u_\tau} = \frac{1}{\kappa} \ln \left(\frac{u_\tau y}{\nu} \right) + C. \quad (1)$$

where U is the mean velocity, u_τ is the skin friction velocity defined by the wall stress τ_w as $u_\tau = \sqrt{|\tau_w|/\rho}$, y is the normal distance from the wall, ν and ρ are the viscosity and density of

the fluid. $\kappa \approx 0.41$ and $C \approx 5.0$. Eq.(1) is also theoretically valid and only valid for a flat plate boundary layer, but it has been applied to other wall bounded flows with some successes despite its formal validity. For a boundary layer with an adverse pressure gradient and zero wall stress, Tennekes and Lumley (1972) derived another asymptotic solution which reads

$$\frac{U}{u_p} = \alpha \ln \left(\frac{u_p y}{\nu} \right) + \beta. \quad (2)$$

where u_p is defined by the adverse wall pressure gradient as $u_p = [(\nu/\rho)|dP_w/dx|]^{1/3}$, and $\alpha \approx 5$, $\beta \approx 8$ according to the experimental data of Stratford (1959). Eq.(2) has not been paid much attention in computational fluid dynamics. Apparently, Eq.(1) will become erroneous for flows near separation or re-attachment points because there the wall stress, hence the skin friction velocity, is nearly zero. On the other hand, Eq.(2) will not be valid for boundary layer flows with a small or zero pressure gradient because u_p is nearly zero.

In this paper, we will briefly repeat the analyses of Tennekes and Lumley and introduce a more general asymptotic solution for the surface flow valid for both the zero or nonzero wall stress and the zero or nonzero wall pressure gradient. Therefore, the solution can be used for flows with acceleration, deceleration, separation and recirculation.

The basic idea is to assume, at large Reynolds numbers, the existence of a surface layer distinct from the outer layer in a boundary layer flow. The existence of the law of the wall in the surface layer and the existence of the velocity-defect law in the outer layer will lead to an asymptotic solution for the surface flow in the region where

$$\frac{y}{\delta} \ll 1, \quad \frac{y}{\ell_\nu} \gg 1. \quad (3)$$

where, δ represents the thickness of the boundary layer, ℓ_ν is the length scale related to the viscosity of the fluid which will be defined later. The region in which Eq.(3) holds is called the inertial sublayer. In the vicinity of the wall where y/ℓ_ν is of order one, the turbulent stress is significantly suppressed and this region is called viscous sublayer. An asymptotic solution can be obtained for both the inertial sublayer and the viscous sublayer. The region between these two sublayers is called buffer layer where the turbulent and viscous stresses are of same order. A simple model of turbulent stress leads to an expression which matches both the viscous and inertial sublayers and leads to a unified law of the wall, similar to the one proposed by Spalding (1961). We will start first with the boundary layer flow over a smooth wall. A general asymptotic solution for the surface layer will be obtained. The effect of rough surfaces on the solution will then be considered.

2 WALL BOUNDED TURBULENT FLOWS

The equations of motion for steady two-dimensional incompressible flow in a Cartesian coordinate system are

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0, \quad (4)$$

$$U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial y} = -\frac{1}{\rho} \frac{\partial P}{\partial x} - \frac{\partial \overline{uu}}{\partial x} - \frac{\partial \overline{uv}}{\partial y} + \nu \frac{\partial^2 U}{\partial x^2} + \nu \frac{\partial^2 U}{\partial y^2}, \quad (5)$$

$$U \frac{\partial V}{\partial x} + V \frac{\partial V}{\partial y} = -\frac{1}{\rho} \frac{\partial P}{\partial y} - \frac{\partial \overline{uv}}{\partial x} - \frac{\partial \overline{vv}}{\partial y} + \nu \frac{\partial^2 V}{\partial x^2} + \nu \frac{\partial^2 V}{\partial y^2}. \quad (6)$$

For simplicity, we are considering boundary layer flows in which the effect of wall curvature is neglected. The existence of the wall and the no-slip condition at the wall will create, at a sufficiently large Reynolds number, a very thin surface layer near the wall, which is distinct from the outer layer of a turbulent boundary layer. In the surface layer, the flow is largely affected by the viscosity, the governing equations (4-6) can be significantly simplified and their solutions are of certain forms called the law of the wall. On the other hand, in the outer layer of a boundary layer remote from the surface layer, the flow is less or nearly not affected by the viscosity, the solution of the above equations is of another forms called the velocity-defect law.

2.1 The law of the wall

In a coordinate system with the wall at $y = 0$, the boundary layer flow in the half-plane $y \geq 0$ is mainly in the x direction, except near the separation or re-attachment point. In general, the wall stress τ_w and the wall pressure gradient $(dP/dx)_w$ are non-zero and can be either positive or negative with respect to x direction. We may define a velocity scale u_c using these two wall parameters as follows,

$$u_c = \sqrt{\frac{1}{\rho} |\tau_w| + \left(\frac{\nu}{\rho} \left| \frac{dP_w}{dx} \right| \right)^{1/3}}. \quad (7)$$

Thus defined u_c will never become zero in any boundary layer flows with either zero wall stress or zero pressure gradient because u_τ and u_p cannot be zero at the same time. With u_c we may define a viscous length scale $\ell_\nu = \nu/u_c$. This viscous length scale is usually very small comparing with other length scales of the boundary layer, for example, the boundary layer thickness δ and the downstream length scale L . That is, $u_c \delta / \nu \gg 1$ and $u_c L / \nu \gg 1$.

Let us start with the simplification of the governing equations (4) -(6) for the flows in the thin surface layer using an order of magnitude analysis (see Tennekes and Lumley, 1972). We use u_c to scale both the mean velocity U and turbulent velocities u, v . Let L be the downstream length scale and $\ell_\nu = \nu/u_c$ be the length scale in y direction. With $\partial U / \partial x \sim u_c / L$ and $\partial V / \partial y \sim V / \ell_\nu$, the continuity equation (4) gives $V \sim u_c \ell_\nu / L$. The left hand side of Eq.(6) is then of order $u_c^2 \ell_\nu / L^2$. The orders of magnitude of the turbulent terms in Eq.(6) are

$$\frac{\partial \overline{vv}}{\partial y} = O\left(\frac{u_c^2}{\ell_\nu}\right), \quad \frac{\partial \overline{uv}}{\partial x} = O\left(\frac{u_c^2}{L}\right); \quad (8)$$

the viscous terms in Eq.(6) are of order

$$\nu \frac{\partial^2 V}{\partial y^2} = O\left(\frac{u_c^2}{L}\right), \quad \nu \frac{\partial^2 V}{\partial x^2} = O\left(\frac{u_c^2 \ell_\nu^2}{L L^2}\right). \quad (9)$$

Because $u_c L/\nu \gg 1$ or $\ell_\nu/L \ll 1$, the major turbulence term, $\partial\overline{v\overline{v}}/\partial y$, must be balanced by the pressure term in the surface layer, that is

$$\frac{1}{\rho} \frac{\partial P}{\partial y} + \frac{\partial\overline{v\overline{v}}}{\partial y} = 0. \quad (10)$$

Integration of Eq.(10) from the wall to y (within the surface layer, i.e., $y/\delta \ll 1$) and differentiation with respect to x lead to an expression for $\partial P/\partial x$:

$$\frac{1}{\rho} \frac{\partial P}{\partial x} = \frac{1}{\rho} \frac{dP_w}{dx} - \frac{\partial\overline{v\overline{v}}}{\partial x}. \quad (11)$$

where, P_w is the pressure at the wall. With Eq.(11), the pressure term in the x -momentum equation (5) can be expressed in terms of the wall pressure.

Now we may estimate the various terms in Eq.(5) in the surface layer as follows

$$\begin{aligned} U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial y} &= O\left(\frac{u_c^2}{L}\right), \\ \frac{\partial\overline{v\overline{v}}}{\partial x} &= O\left(\frac{u_c^2}{L}\right), \quad \frac{\partial\overline{u\overline{u}}}{\partial x} = O\left(\frac{u_c^2}{L}\right), \quad \frac{\partial\overline{u\overline{v}}}{\partial y} = O\left(\frac{u_c^2}{\ell_\nu}\right), \\ \nu \frac{\partial^2 U}{\partial x^2} &= O\left(\frac{u_c^2 \ell_\nu}{L}\right), \quad \nu \frac{\partial^2 U}{\partial y^2} = O\left(\frac{u_c^2}{\ell_\nu}\right). \end{aligned} \quad (12)$$

The major terms in Eq.(12) are $\partial\overline{u\overline{v}}/\partial y$ and $\nu\partial^2 U/\partial y^2$ if $\ell_\nu/L \ll 1$. Therefore, the x -momentum equation (5) can be approximated in the surface layer as follows

$$\frac{\partial}{\partial y} \left(-\overline{u\overline{v}} + \nu \frac{\partial U}{\partial y} \right) = \frac{1}{\rho} \frac{dP_w}{dx}. \quad (13)$$

Integration of this equation from $y = 0$ to y (within the surface layer) yields

$$-\overline{u\overline{v}} + \nu \frac{\partial U}{\partial y} = \frac{\tau_w}{\rho} + \frac{y}{\rho} \frac{dP_w}{dx}, \quad (14)$$

where $\tau_w = \mu(\partial U/\partial y)_{y=0}$ is the wall stress. The no-slip condition at the wall for turbulent velocities has been imposed. This is a general equation for surface flows under the condition $u_c L/\nu \gg 1$. Eq.(14) indicates that, in general, the surface flows are affected by both the wall stress and the wall pressure gradient. The relative importance of these two terms depends on the flow situation. For example, for a boundary layer flow with a strong adverse pressure gradient, the wall stress τ_w can become very small and even vanish. In this case, the adverse wall pressure gradient controls the surface flow and the total shear stress (the left hand side of Eq.(14)) is not constant across the surface layer. On the other hand, if the pressure gradient is zero or small compared to the wall stress, then the wall stress dominates the surface flow and the total stress is constant or nearly constant across the whole surface layer.

The flows we want to consider here include acceleration, deceleration and even recirculation. Therefore, the two terms on the right hand side of Eq.(14) could be of the same order of magnitude,

or one is large than the other. Note that because Eq.(14) is linear, we may deal with these two factors separately by decompose U and $-\overline{uv}$ into two parts (Tennekes, 1968, has shown with a multivariate asymptotic technique that this is a valid procedure). Following Tennekes and Lumley, we write

$$U = U_1 + U_2, \quad (15)$$

$$-\overline{uv} = -(\overline{uv})_1 - (\overline{uv})_2. \quad (16)$$

The first part, represented by U_1 and $-(\overline{uv})_1$, is associated with the wall stress τ_w/ρ only and the second part, represented by U_2 and $-(\overline{uv})_2$, is solely related to the pressure gradient $(y/\rho)dP_w/dx$, i.e.,

$$-(\overline{uv})_1 + \nu \frac{\partial U_1}{\partial y} = \frac{\tau_w}{\rho}, \quad (17)$$

$$-(\overline{uv})_2 + \nu \frac{\partial U_2}{\partial y} = \frac{y}{\rho} \frac{dP_w}{dx}. \quad (18)$$

The first part of the flow in Eq.(17) has only one characteristic velocity u_τ and one characteristic length ν/u_τ . Similarly, the second part of flow in Eq.(18) has only one characteristic velocity u_p and one characteristic length ν/u_p . Therefore, the nondimensional form of Eq.(17) and Eq.(18) can be written as

$$\frac{-(\overline{uv})_1}{u_\tau^2} + \frac{\partial(U_1/u_\tau)}{\partial(u_\tau y/\nu)} = \frac{\tau_w}{\rho u_\tau^2}, \quad (19)$$

$$\frac{-(\overline{uv})_2}{u_p^2} + \frac{\partial(U_2/u_p)}{\partial(u_p y/\nu)} = \left(\frac{u_p y}{\nu}\right) \frac{\nu}{\rho} \frac{dP_w/dx}{u_p^3}. \quad (20)$$

There are no additional parameters in the boundary conditions on Eq.(19) and Eq.(20) because the boundary conditions are homogeneous (both the mean velocity and the turbulent stress are zero at $y = 0$). Therefore, the solution of Eq.(19) and Eq.(20) must be of the forms:

$$\frac{U_1}{u_\tau} = \frac{\tau_w}{\rho u_\tau^2} f_1 \left(\frac{u_\tau y}{\nu} \right), \quad (21)$$

$$\frac{-(\overline{uv})_1}{u_\tau^2} = \frac{\tau_w}{\rho u_\tau^2} g_1 \left(\frac{u_\tau y}{\nu} \right), \quad (22)$$

and

$$\frac{U_2}{u_p} = \frac{\nu}{\rho} \frac{dP_w/dx}{u_p^3} f_2 \left(\frac{u_p y}{\nu} \right), \quad (23)$$

$$\frac{-(\overline{uv})_2}{u_p^2} = \frac{\nu}{\rho} \frac{dP_w/dx}{u_p^3} g_2 \left(\frac{u_p y}{\nu} \right). \quad (24)$$

Eq.(21)-Eq.(24) are called the law of the wall.

2.2 The velocity-defect law

In the outer layer of a boundary layer, the boundary layer thickness δ is the only appropriate length scale. If u_τ is the only characteristic velocity for the first part of the flow, then the first part of the velocity-defect $(U_1 - U_0)/(\tau_w/\rho u_\tau^2)$ is a function of y, δ, u_τ only, where U_0 is the velocity at the edge of the boundary. Therefore, we have a system of four quantities with two dimensions. From Π theory of dimensional analysis, only two independent normalized quantities can be formed and we may write this system as

$$F \left\{ \frac{(U_1 - U_0)/u_\tau}{\tau_w/\rho u_\tau^2}, \frac{y}{\delta} \right\} = 0,$$

or

$$\frac{U_1 - U_0}{u_\tau} = \frac{\tau_w}{\rho u_\tau^2} F_1 \left(\frac{y}{\delta} \right). \quad (25)$$

For the second part of flow U_2 , we may use the above same argument to obtain a relation for U_2

$$\frac{U_2 - U_0}{u_p} = \frac{\nu}{\rho} \frac{dP_w/dx}{u_p^3} F_2 \left(\frac{y}{\delta} \right). \quad (26)$$

The relations (25) and (26) are called the velocity-defect law.

3 ASYMPTOTIC SOLUTIONS FOR SURFACE LAYER

3.1 The inertial sublayer

If the Reynolds number, $Re = u_c \delta / \nu$, is large enough, an overlapping layer between the surface layer and the outer layer may be developed. This overlapping layer is called inertial sublayer, where $y/\delta \ll 1$ and $u_c y / \nu \gg 1$. The existence of the inertial sublayer can be easily seen from the following relation

$$\frac{y}{\delta} = \frac{u_c y}{\nu} \frac{\nu}{u_c \delta}. \quad (27)$$

For example, if $u_c \delta / \nu \geq 10^4$, there will exist a region where $u_c y / \nu \geq 10^2$ and $y/\delta \leq 10^{-2}$. In the inertial sublayer, the law of the wall in the surface layer should match the velocity-defect law in the outer layer. Following Tennekes and Lumley to equate the mean velocity gradient $\partial U_1 / \partial y$ calculated from Eq.(21) and Eq.(25), we obtain

$$\left(\frac{u_\tau y}{\nu} \right) \frac{df_1}{d(u_\tau y / \nu)} = \left(\frac{y}{\delta} \right) \frac{dF_1}{d(y/\delta)}. \quad (28)$$

Eq.(28) indicates that the both sides must be equal to a same constant, $1/\kappa$, say. And this leads to a logarithmic velocity profile in the inertial sublayer:

$$\frac{U_1}{u_\tau} = \frac{\tau_w}{\rho u_\tau^2} \left[\frac{1}{\kappa} \ln \left(\frac{u_\tau y}{\nu} \right) + C \right]. \quad (29)$$

To match the case with zero pressure gradient boundary layer, $\kappa = 0.41$, $C = 5.0$. Eq.(29) can be rearranged using u_c as

$$\frac{U_1}{u_c} = \frac{\tau_w}{\rho u_\tau^2} \left[\frac{1}{\kappa} \frac{u_\tau}{u_c} \ln \left(\frac{u_c y}{\nu} \right) + C_1 \right], \quad (30)$$

where

$$C_1 = \frac{u_\tau}{u_c} \left[\frac{1}{\kappa} \ln \left(\frac{u_\tau}{u_c} \right) + C \right] \quad (31)$$

C_1 is shown in Figure 1, it goes to zero as u_τ vanishes ($u_p/u_c \rightarrow 1.0$).

Using the above same argument for the second part of the flow U_2 , based on the law of the wall (23) and the velocity-defect law (26), we may obtain the solution:

$$\frac{U_2}{u_p} = \frac{\nu}{\rho} \frac{dP_w/dx}{u_p^3} \left[\alpha \ln \left(\frac{u_p y}{\nu} \right) + \beta \right]. \quad (32)$$

To match the case with zero wall stress boundary layer, $\alpha = 5.0$ and $\beta = 8.0$. Eq.(32) can be rearranged as

$$\frac{U_2}{u_c} = \frac{\nu}{\rho} \frac{dP_w/dx}{u_p^3} \left[\alpha \frac{u_p}{u_c} \ln \left(\frac{u_c y}{\nu} \right) + C_2 \right], \quad (33)$$

where

$$C_2 = \frac{u_p}{u_c} \left[\alpha \ln \left(\frac{u_p}{u_c} \right) + \beta \right] \quad (34)$$

C_2 is shown in Figure 1, it goes to zero as u_p vanishes.

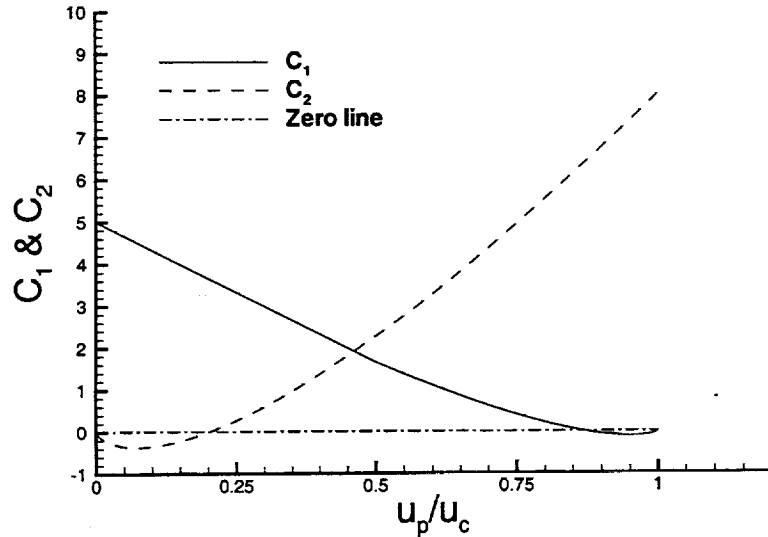


Figure 1: Coefficients C_1 and C_2 in the law of the wall

Finally, the total velocity $U = U_1 + U_2$ can be written as

$$\frac{U}{u_c} = \frac{\tau_w}{\rho u_\tau^2} \left[\frac{1}{\kappa} \frac{u_\tau}{u_c} \ln \left(\frac{u_c y}{\nu} \right) + C_1 \right] + \frac{\nu}{\rho} \frac{dP_w/dx}{u_p^3} \left[\alpha \frac{u_p}{u_c} \ln \left(\frac{u_c y}{\nu} \right) + C_2 \right]. \quad (35)$$

With Eq.(35) and Eq.(14), it can be shown that, in the inertial sublayer ($u_c y/\nu \gg 1$), the total turbulent stress is

$$-\overline{uv} = \frac{\tau_w}{\rho} + \frac{y}{\rho} \frac{dP_w}{dx}. \quad (36)$$

or

$$\frac{-\overline{uv}}{u_c^2} = \frac{\tau_w}{\rho u_\tau^2} \left(\frac{u_\tau}{u_c} \right)^2 + \frac{dP_w/dx}{|dP_w/dx|} \left(\frac{u_p}{u_c} \right)^3 \frac{u_c y}{\nu}. \quad (37)$$

Equations (35) and (36) are the asymptotic solutions for the surface flow in the inertial sublayer, where $u_c y/\nu \gg 1$ and $y/\delta \ll 1$. These equations are called wall functions in CFD, because these relations can be used as the near-wall boundary conditions for CFD calculations. The effect of pressure gradients on the mean velocity profile described by Eq. (35) is shown in Figure 2. The left of Figure 2 shows the strong effect of adverse pressure gradients on the mean velocity profile as the ratio of u_p/u_c varies from zero to one (which corresponds respectively to the zero pressure gradient boundary layer and the boundary layer about to separate). In the boundary layers with favorable pressure gradients, the skin friction velocity u_τ increases with the pressure gradient so that $u_p/u_c = u_p/(u_\tau + u_p)$ is always less than one and it is usually a small value. For example, in a fully developed channel flow, it can be shown that $u_p/u_c = 1/(1 + R_{eh}^{1/3})$, where the Reynolds number is defined as $R_{eh} = U_{max}h/\nu$, h and U_{max} are the half width of the channel and the center line mean velocity, respectively. Therefore, if $R_{eh} \geq 10^4$, then $u_p/u_c \leq 0.044$, hence, the effect of pressure gradients on channel or pipe flows will not be very significant if the Reynolds number R_{eh} is sufficiently large. However, for accelerating boundary layer flows, the value of u_p/u_c could be much larger than 0.05 before the turbulence has been suppressed by the acceleration. The right figure of Figure 2 shows the effect of favorable pressure gradients on the mean flow up to $u_p/u_c = 0.3$, which corresponds to the flows with extremely large favorable pressure gradients. The effect is significant. The effect of pressure gradients on the turbulent stress, described by Eq.

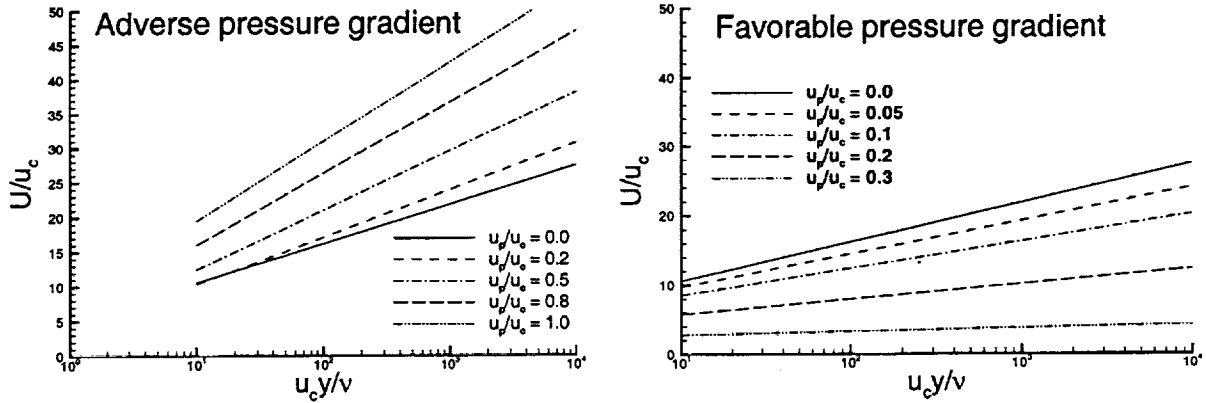


Figure 2: Effect of pressure gradients on the mean velocity

(36) or Eq. (37), is shown in Figure 3. Again, both the adverse and favorable pressure gradients have the strong effect on the turbulent shear stress and there is no constant shear stress layer for boundary layer flows with large pressure gradients. The constant shear stress layer only exists for flows with zero pressure gradients, or $u_p/u_c \ll 1$. In addition, we see $-\overline{uv}$ changes its sign for some large favorable pressure gradients in the right of Figure 3, which may indicate the limitation of Eq. (36) or Eq. (37).

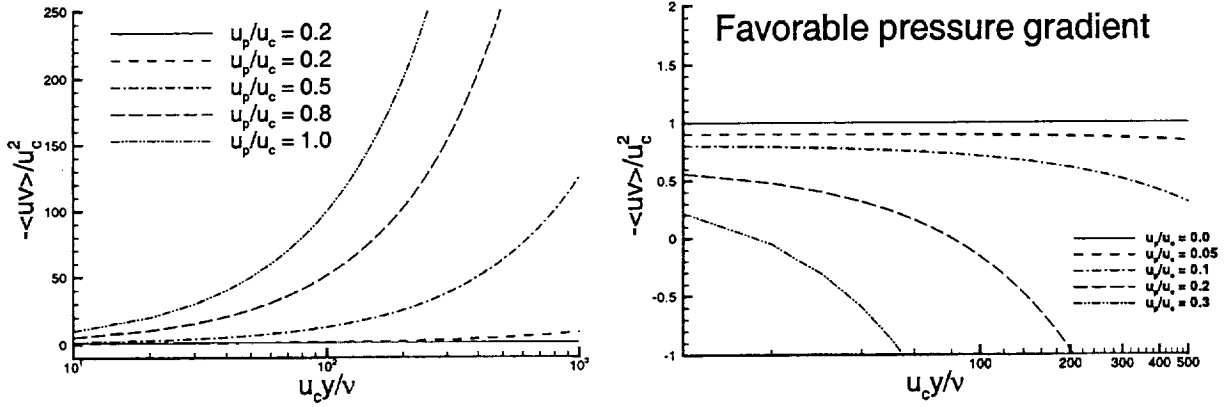


Figure 3: Effect of pressure gradients on the turbulent stress

3.2 The viscous sublayer

In the vicinity of the wall, for example, $u_c y / \nu \leq 5$, the viscous effect dominates the flow. The turbulent stress $-\overline{uv}$ is vanishingly small compared to the viscous stress $\nu \partial U / \partial y$ even though the flow is still quite disturbed and is not laminar. In fact, the ratio of the turbulent rms velocity u' to the mean velocity U at the wall is finite, i.e., $(u'/U)_{y \rightarrow 0} = const..$ In this so-called viscous sublayer, the turbulent stress can be neglected and the velocity profile can be obtained from Eq.(14) as follows

$$\frac{U}{u_c} = \frac{\tau_w}{\rho u_c^2} \left(\frac{u_c y}{\nu} \right) + \frac{1}{2} \frac{\nu}{\rho} \frac{dP_w/dx}{u_c^3} \left(\frac{u_c y}{\nu} \right)^2, \quad (38)$$

or,

$$U = \frac{\tau_w}{\rho} \frac{y}{\nu} + \frac{1}{2} \frac{\nu}{\rho} \frac{dP_w}{dx} \left(\frac{y}{\nu} \right)^2. \quad (39)$$

The effect of pressure gradients on the flow in the viscous sublayer is shown in Figure 4. Again, the effect of pressure gradients is significant.

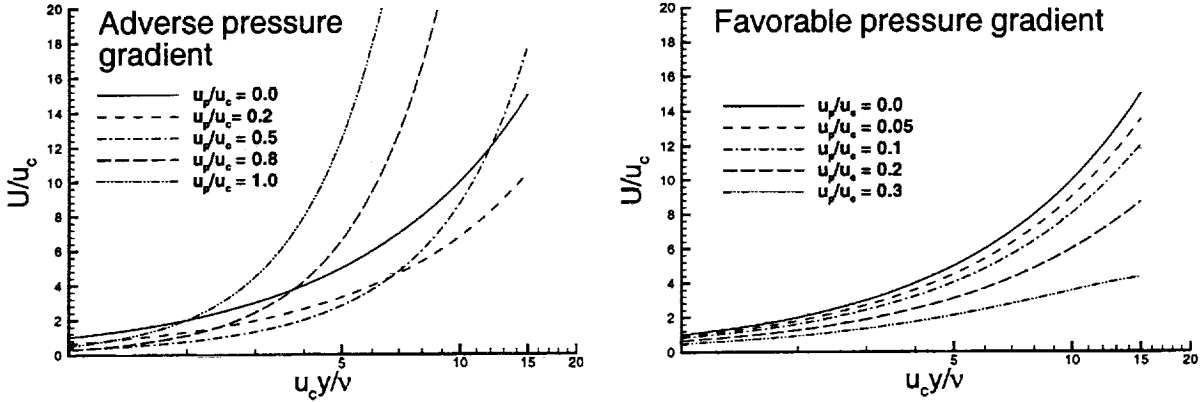


Figure 4: Effect of adverse pressure gradients on the flow in the viscous sublayer.

4 EFFECT OF ROUGH SURFACES

Now let us consider the effect of rough surfaces on the surface flow. Let h denote an rms roughness height. If the ratio h/δ is small, then the roughness will not affect the velocity-defect law. However, the law of the wall in the surface layer may need modification. Let us define $y = 0$ as the average vertical position over the rough surface. Apparently, we are interested in the flow only at $y \geq h$, because at $y = 0$ the flow field is not even defined. Now, the surface layer over a rough surface has another length scale h , in addition to ν/u_c . The ratio of the two is the roughness Reynolds number $R_h = u_c h/\nu$. If R_h is of order one, $R_h \leq 5$ say, one may expect that the roughness will have no effect on the surface flow because the roughness elements are submerged in the viscous sublayer where no turbulent stress can be generated, even though much the flow is disturbed. It can be shown that Eq.(14) is still valid for the surface flow with rough surfaces as long as $h/\delta \ll 1$ and $u_c L/\nu \gg 1$, $h/L \ll 1$. Following Tennekes and Lumley's argument^[1], the law of the wall can be written as

$$\frac{U_1}{u_\tau} = \frac{\tau_w}{\rho u_\tau^2} f_1 \left(\frac{u_\tau y}{\nu}, R_h \right), \quad (40)$$

$$\frac{U_2}{u_p} = \frac{\nu}{\rho} \frac{dP_w/dx}{u_p^3} f_2 \left(\frac{u_p y}{\nu}, R_h \right), \quad (41)$$

or,

$$\frac{U_1}{u_\tau} = \frac{\tau_w}{\rho u_\tau^2} f_1 \left(\frac{y}{h}, R_h \right), \quad (42)$$

$$\frac{U_2}{u_p} = \frac{\nu}{\rho} \frac{dP_w/dx}{u_p^3} f_2 \left(\frac{y}{h}, R_h \right). \quad (43)$$

The velocity-defect law in the outer layer of a boundary, Eq.(25) and Eq.(26), will be independent of the surface roughness as long as $h/\delta \ll 1$. The matching of the velocity derivative in the inertial

sublayer will lead to the logarithmic velocity profile with an additive function depending on R_h :

$$\frac{U}{u_c} = \frac{\tau_w}{\rho u_\tau^2} \left[\frac{1}{\kappa} \frac{u_\tau}{u_c} \ln \left(\frac{u_c y}{\nu} \right) + C_1(R_h) \right] + \frac{\nu}{\rho} \frac{dP_w/dx}{u_p^3} \left[\alpha \frac{u_p}{u_c} \ln \left(\frac{u_c y}{\nu} \right) + C_2(R_h) \right], \quad (44)$$

or

$$\frac{U}{u_c} = \frac{\tau_w}{\rho u_\tau^2} \left[\frac{1}{\kappa} \frac{u_\tau}{u_c} \ln \left(\frac{y}{h} \right) + C'_1(R_h) \right] + \frac{\nu}{\rho} \frac{dP_w/dx}{u_p^3} \left[\alpha \frac{u_p}{u_c} \ln \left(\frac{y}{h} \right) + C'_2(R_h) \right]. \quad (45)$$

For $R_h \leq 5$, as mentioned earlier, the surface roughness is too small to affect the surface flow. Hence, in Eq.(44), $C_1(R_h) \rightarrow C_1$, $C_2(R_h) \rightarrow C_2$ when $R_h \leq 5$. For $R_h \geq 5$, $C_1(R_h)$ and $C_2(R_h)$ may depend on the roughness Reynolds number R_h . The detailed relations must be determined by experiments.

However, for large values of R_h (> 30 , say), Tennekes and Lumley^[1] have shown that the coefficients $C'_1(R_h)$ and $C'_2(R_h)$ in Eq.(45) should be independent of R_h . If these coefficients are absorbed in the definition of h , then Eq.(45) may be written as

$$\frac{U}{u_c} = \frac{\tau_w}{\rho u_\tau^2} \left[\frac{1}{\kappa} \frac{u_\tau}{u_c} \ln \left(\frac{y}{h} \right) \right] + \frac{\nu}{\rho} \frac{dP_w/dx}{u_p^3} \left[\alpha \frac{u_p}{u_c} \ln \left(\frac{y}{h} \right) \right]. \quad (46)$$

With Eq.(44) or Eq.(46) and Eq.(14), it can be shown that the turbulent stress in the inertial sublayer ($u_c y/\nu \gg 1$) is

$$-\overline{uv} = \frac{\tau_w}{\rho} + \frac{y}{\rho} \frac{dP_w}{dx}. \quad (47)$$

Now let us consider whether there exists a viscous sublayer on a rough surface. Apparently, for the case of large R_h , the roughness elements are well above the viscous sublayer defined by $u_c y/\nu \leq 5$. These elements generate turbulent wakes and are responsible for essentially inviscid drag on the surface. Therefore, no viscous sublayer exists and Eq.(46) may be used down to the "wall", where $y = h$, $U = 0$.

On the other hand, however, for small $R_h \leq 5$, the viscous sublayer may exist. It can be shown that Eq.(39) is valid with the following modification:

$$U = \frac{\tau_w}{\rho \nu} (y - h) + \frac{1}{2} \frac{dP_w/dx}{\rho \nu} (y - h)^2. \quad (48)$$

where $U = 0$ at $y = h$. Eq.(46) and Eq.(48) indicate that the "effective wall" for rough surfaces is at $y = h$.

5 Unified wall function

We have discussed the asymptotic solutions for surface flows in the inertia sublayer and the viscous sublayer. The region between these two sublayers, i.e., $5 \leq u_c y/\nu \leq 30$, is called buffer layer. In this region, the viscous and turbulent stresses are of same order. No theoretical asymptotic

solution can be obtained for this layer. However, by using a simple model for the turbulent stress in the buffer layer and a proper matching procedure, we are able to obtain a single analytic function, unified wall function, for the whole surface layer which includes viscous sublayer, buffer layer and inertia sublayer. Such an unified wall function (without considering the effect of pressure gradients) was first constructed by Spalding (1961). Here, we will construct an unified wall function with the effect of pressure gradients so that it can be used for flows with acceleration, deceleration, separation and recirculation.

5.1 Unified law of the wall

Let us start with Eqs. (17) and (18). In the buffer layer, $-(\overline{uv})_1$ and $\nu \frac{\partial U_1}{\partial y}$ are assumed of same order. The same is true for $-(\overline{uv})_2$ and $\nu \frac{\partial U_2}{\partial y}$. The buffer layer is considered very close to the wall. Therefore, we may assume that the following near-wall behaviors are valid throughout the buffer layer,

$$-(\overline{uv})_1 \sim y^3 + O(y^4), \quad U_1 \sim y + O(y^2), \quad \text{hence} \quad -(\overline{uv})_1 \sim U_1^3 + O(U_1^4) \quad (49)$$

$$-(\overline{uv})_2 \sim y^3 + O(y^4), \quad U_2 \sim y^2 + O(y^3), \quad \text{hence} \quad -(\overline{uv})_2 \sim U_2^{3/2} + O(U_2^{5/2}) \quad (50)$$

If we further model $-(\overline{uv})_1$ and $-(\overline{uv})_2$ with an eddy viscosity concept that they are proportional to the mean velocity gradient, and note that $\partial U_1/\partial y$ is order of one and $\partial U_2/\partial y$ is of order y or $U_2^{1/2}$, then we may write

$$-(\overline{uv})_1 \propto \nu \left[\frac{U_1^3}{u_\tau^3} + O\left(\frac{U_1^4}{u_\tau^4}\right) \right] \frac{\partial U_1}{\partial y}, \quad -(\overline{uv})_2 \propto \nu \left[\frac{U_2}{u_p} + O\left(\frac{U_2^2}{u_p^2}\right) \right] \frac{\partial U_2}{\partial y} \quad (51)$$

With theses models, Eqs. (17) and (18), in the region from the wall to the buffer layer, can be written as

$$\nu \left[1 + C' \frac{U_1^3}{u_\tau^3} + O\left(\frac{U_1^4}{u_\tau^4}\right) \right] dU_1 = \frac{\tau_w}{\rho} dy \quad (52)$$

$$\nu \left[1 + C'' \frac{U_2}{u_p} + O\left(\frac{U_2^2}{u_p^2}\right) \right] dU_2 = \frac{y}{\rho} \frac{dP_w}{dx} dy \quad (53)$$

Integrate the above equations, we obtain

$$Y_\tau^+ = U_1^+ + C' U_1^{+4} + O(U_1^{+5}), \quad (54)$$

$$(Y_p^+)^2 = U_2^+ + C'' U_2^{+2} + O(U_2^{+3}) \quad (55)$$

where

$$Y_\tau^+ = \frac{u_\tau y}{\nu}, \quad Y_p^+ = \frac{u_p y}{\nu}, \quad U_1^+ = \frac{\rho u_\tau^2 U_1}{\tau_w u_\tau}, \quad U_2^+ = 2 \frac{|dp_w/dx| U_2}{dp_w/dx u_p} \quad (56)$$

Equations (54) and (55) describe the behavior of U_1 and U_2 in the buffer layer which matches their behaviors in the viscous sublayer. The coefficients in the higher order terms, C', C'', \dots , are

unknown. However, they will automatically be determined by matching Equations (54) and (55) with the asymptotic solutions in the inertial sublayer. In the inertia sublayer, the asymptotic solutions (29) and (32) can be written as

$$Y_\tau^+ = \exp(-\kappa C) \exp(\kappa U_1^+) \quad (57)$$

$$(Y_p^+)^2 = \exp(-2\alpha/\beta) \exp(U_2^+/\alpha) \quad (58)$$

Their series expansions in terms of U_1^+ and U_2^+ are

$$Y_\tau^+ = \exp(-\kappa C) \left[1 + \kappa U_1^+ + \frac{1}{2} (\kappa U_1^+)^2 + \frac{1}{6} (\kappa U_1^+)^3 + \dots \right]$$

$$(Y_p^+)^2 = \exp(-2\beta/\alpha) \left[1 + U_2^+/\alpha + \dots \right]$$

Now, in order to have an expression for U_1^+ which behaves like Eq. (57) when $u_\tau y/\nu$ is large (say, > 30), otherwise, Eq. (54) when $u_\tau y/\nu \leq 30$, the following form is the simplest:

$$Y_\tau^+ = U_1^+ + \exp(-\kappa C) \left[\exp(\kappa U_1^+) - 1 - \kappa U_1^+ - \frac{1}{2} (\kappa U_1^+)^2 - \frac{1}{6} (\kappa U_1^+)^3 \right] \quad (59)$$

This analytical expression has been supported by many experimental data and is referred to as the Spalding's law of the wall. Similarly, by matching Eqs. (58) and (55) we obtain another analytical expression, the law of the wall for flows with zero wall-stress:

$$(Y_p^+)^2 = U_2^+ + \exp(-2\beta/\alpha) \left[\exp(U_2^+/\alpha) - 1 - U_2^+/\alpha \right] \quad (60)$$

The unified laws of the wall for zero pressure gradient and zero wall stress, (59) and (60), are shown in Figure 5.

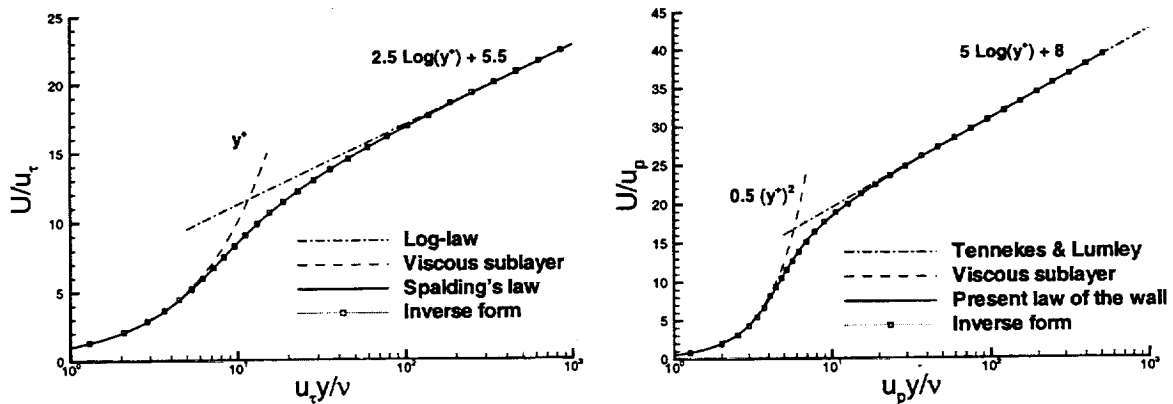


Figure 5: Unified laws of the wall

From the application point of view, we need the inverse form of Eqs. (59) and (60). Unfortunately, it is impossible to obtain a single analytical inverse form for both (59) and (60). Therefore, we

suggest the following piecemeal functions:

$$\frac{U_1}{u_\tau} = \frac{\tau_w}{\rho u_\tau^2} f_1(Y_\tau^+) \quad (61)$$

$$\frac{U_2}{u_p} = \frac{dp_w/dx}{|dp_w/dx|} f_2(Y_p^+) \quad (62)$$

where f_1 and f_2 are the piecemeal fitting functions defined as follows

$$f_1(Y_\tau^+) = \begin{cases} a_1 Y_\tau^+ + a_2 (Y_\tau^+)^2 + a_3 (Y_\tau^+)^3 & \text{if } Y_\tau^+ \leq 5; \\ b_0 + b_1 Y_\tau^+ + b_2 (Y_\tau^+)^2 + b_3 (Y_\tau^+)^3 + b_4 (Y_\tau^+)^4 & \text{if } 5 \leq Y_\tau^+ \leq 30; \\ c_0 + c_1 Y_\tau^+ + c_2 (Y_\tau^+)^2 + c_3 (Y_\tau^+)^3 + c_4 (Y_\tau^+)^4 & \text{if } 30 \leq Y_\tau^+ \leq 140 \\ \frac{1}{\kappa} \log(Y_\tau^+) + C & \text{if } 140 < Y_\tau^+ \end{cases} \quad (63)$$

$$f_2(Y_p^+) = \begin{cases} a_2 (Y_p^+)^2 + a_3 (Y_p^+)^3 & \text{if } Y_p^+ \leq 4; \\ b_0 + b_1 Y_p^+ + b_2 (Y_p^+)^2 + b_3 (Y_p^+)^3 + b_4 (Y_p^+)^4 & \text{if } 4 \leq Y_p^+ \leq 15; \\ c_0 + c_1 Y_p^+ + c_2 (Y_p^+)^2 + c_3 (Y_p^+)^3 + c_4 (Y_p^+)^4 & \text{if } 15 \leq Y_p^+ \leq 30; \\ \alpha \log(Y_p^+) + \beta & \text{if } 30 < Y_p^+ \end{cases} \quad (64)$$

The coefficients in f_1

a_1	a_2	a_3		
1.0	$1.0E-02$	$-2.9E-03$		
b_0	b_1	b_2	b_3	b_4
-0.872	1.465	$-7.02E-02$	$1.66E-03$	$-1.495E-05$
c_0	c_1	c_2	c_3	c_4
8.6	0.1864	$-2.006E-03$	$1.144E-05$	$-2.551E-08$

The coefficients in f_2

a_2	a_3			
0.5	$-7.31E-03$			
b_0	b_1	b_2	b_3	b_4
-15.138	8.4688	-0.81976	$3.7292E-02$	$-6.3866E-04$
c_0	c_1	c_2	c_3	c_4
11.925	0.93400	$-2.7805E-02$	$4.6262E-04$	$-3.1442E-06$

The inverse forms of Eq. (63) and Eq. (64), are also shown in Figure 5.

5.2 Unified wall function

Now, we may construct the unified wall function for a general surface flow using the results described in the previous section. Since

$$\frac{U}{u_c} = \frac{U_1 + U_2}{u_c} = \frac{u_\tau U_1}{u_c u_\tau} + \frac{u_p U_2}{u_c u_p}$$

we may write

$$\frac{U}{u_c} = \frac{\tau_w}{\rho u_\tau^2} \frac{u_\tau}{u_c} f_1 \left(y^+ \frac{u_\tau}{u_c} \right) + \frac{dp_w/dx}{|dp_w/dx|} \frac{u_p}{u_c} f_2 \left(y^+ \frac{u_p}{u_c} \right) \quad (65)$$

where

$$y^+ = \frac{u_c y}{\nu}$$

The left and right figures in Figure 6 show respectively the effect of adverse and favorable pressure gradients on the surface flow. As the adverse pressure gradient increases, the ratio of u_p/u_c increase from zero to one and u_τ/u_c decreases from one to zero since $u_c = u_\tau + u_p$. As the result, the mean velocity profile U/u_c is significantly affected by the adverse pressure gradient (see the left of Figure 6). For flows with favorable pressure gradients, as we have discussed before that the ratio u_p/u_c for boundary layer flows could be larger than the value for channel or pipe flows, 0.044, hence, the effect of favorable pressure gradients could be also significant. This is shown in the right of Figure 6.

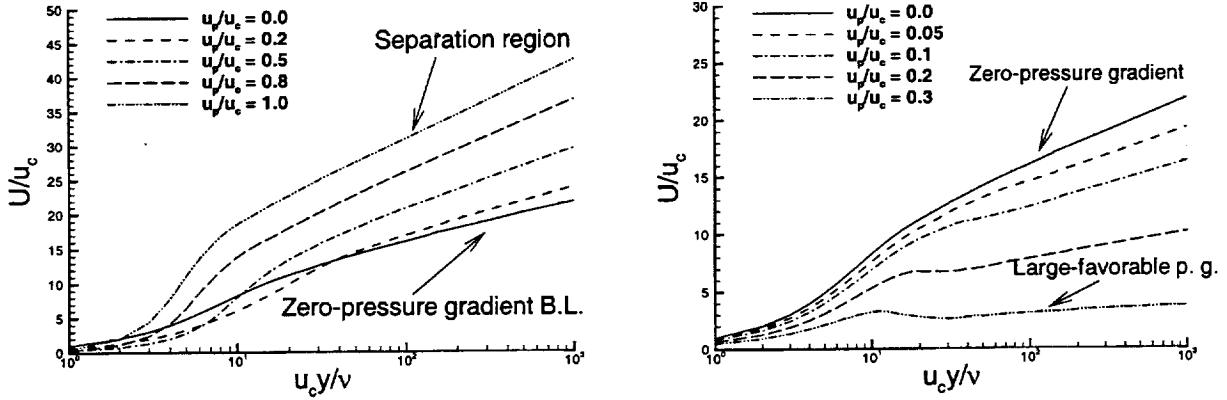


Figure 6: Effect of pressure gradients on the surface flow

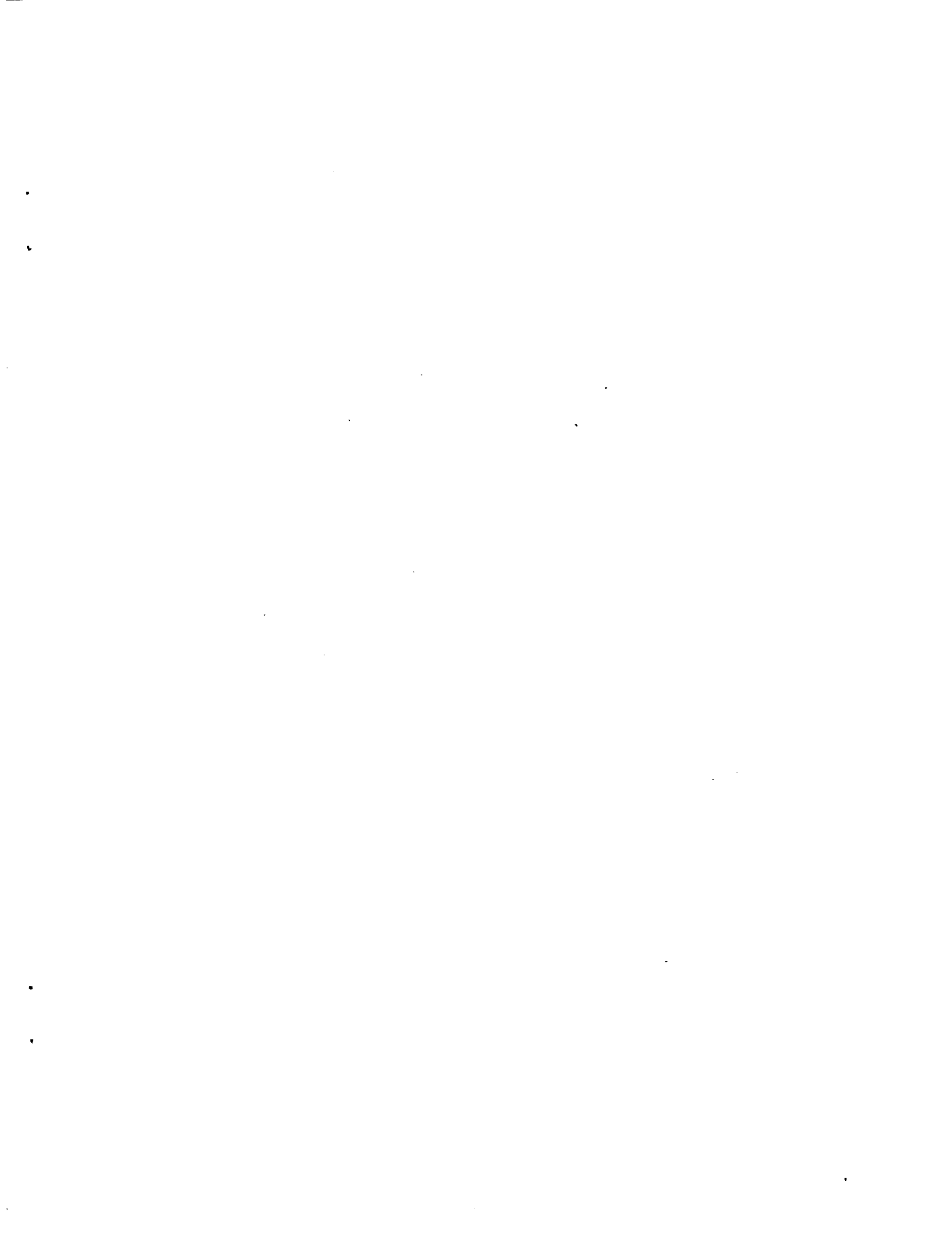
6 Conclusion

A generalized wall function has been derived, which accounts for the effect of various adverse and favorable pressure gradients and is valid from the solid wall up all the way to the inertial sublayer. We have demonstrated that the effect of pressure gradients on surface flows are very significant,

especially for the flows with adverse pressure gradients. The traditional wall function must be replaced by the new generalized wall function for flows with considerable pressure gradients. We expect that this generalized wall function will be useful for analytical studies of turbulent surface flow properties as well as for CFD applications in providing appropriate boundary conditions for high Reynolds number turbulent complex flows.

REFERENCES

1. Millikan, C.B.A., 1938, "A critical discussion of turbulent flows in channels and circular tubes." Pages 386-369 of: *Proceedings of the Fifth International Congress of Applied Mechanics*.
2. Spalding, D.B., 1961, "A single formula for the law of the wall." *Transactions of th ASME, Series E: Journal of Applied Mechanics*, 28, 455-458.
3. Stratford, B.S., 1959, "An experimental flow with zero skin friction throughout its region of pressure rise." *Journal of Fluid Mechanics*, 5, 17.
4. Tennekes, H. and Lumley, J.L., *A First Course in Turbulence*, 1972, by The Massachusetts Institute of Technology.
5. Tennekes, H., "Outline of a second order theory for turbulent pipe flow." *AIAA Journal*, vol. 6, 1968, P.1735.



REPORT DOCUMENTATION PAGE

Form Approved
OMB No. 0704-0188

Public reporting burden for this collection of information is estimated to average 1 hour per response, including the time for reviewing instructions, searching existing data sources, gathering and maintaining the data needed, and completing and reviewing the collection of information. Send comments regarding this burden estimate or any other aspect of this collection of information, including suggestions for reducing this burden, to Washington Headquarters Services, Directorate for Information Operations and Reports, 1215 Jefferson Davis Highway, Suite 1204, Arlington, VA 22202-4302, and to the Office of Management and Budget, Paperwork Reduction Project (0704-0188), Washington, DC 20503.

1. AGENCY USE ONLY (<i>Leave blank</i>)		2. REPORT DATE July 1999	3. REPORT TYPE AND DATES COVERED Technical Memorandum	
4. TITLE AND SUBTITLE A Generalized Wall Function			5. FUNDING NUMBERS WU-522-31-23-00	
6. AUTHOR(S) Tsan-Hsing Shih, Louis A. Povinelli, Nan-Suey Liu, Mark G. Potapczuk, and J.L. Lumley				
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) National Aeronautics and Space Administration John H. Glenn Research Center at Lewis Field Cleveland, Ohio 44135-3191			8. PERFORMING ORGANIZATION REPORT NUMBER E-11834	
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES) National Aeronautics and Space Administration Washington, DC 20546-0001			10. SPONSORING/MONITORING AGENCY REPORT NUMBER NASA TM-1999-209398 ICOMP-99-08	
11. SUPPLEMENTARY NOTES Tsan-Hsing Shih, Institute for Computational Mechanics in Propulsion, NASA Glenn Research Center, Cleveland, Ohio 44135; Louis A. Povinelli, Nan-Suey Liu, and Mark G. Potapczuk, NASA Glenn Research Center; J.L. Lumley, Cornell University, Ithaca, New York. ICOMP Program Director, Lou Povinelli, organization code 5880, (216) 433-5818.				
12a. DISTRIBUTION/AVAILABILITY STATEMENT Unclassified - Unlimited Subject Category: 34 This publication is available from the NASA Center for AeroSpace Information, (301) 621-0390.			12b. DISTRIBUTION CODE	
13. ABSTRACT (<i>Maximum 200 words</i>) The asymptotic solutions, described by Tennekes and Lumley (1972), for surface flows in a channel, pipe or boundary layer at large Reynolds numbers are revisited. These solutions can be extended to more complex flows such as the flows with various pressure gradients, zero wall stress and rough surfaces, etc. In computational fluid dynamics (CFD), these solutions can be used as the boundary conditions to bridge the near-wall region of turbulent flows so that there is no need to have the fine grids near the wall unless the near-wall flow structures are required to resolve. These solutions are referred to as the wall functions. Furthermore, a generalized and unified law of the wall which is valid for whole surface layer (including viscous sublayer, buffer layer and inertial sublayer) is analytically constructed. The generalized law of the wall shows that the effect of both adverse and favorable pressure gradients on the surface flow is very significant. Such as unified wall function will be useful not only in deriving analytic expressions for surface flow properties but also bringing a great convenience for CFD methods to place accurate boundary conditions at any location away from the wall. The extended wall functions introduced in this paper can be used for complex flows with acceleration, deceleration, separation, recirculation and rough surfaces.				
14. SUBJECT TERMS Turbulent flows; Wall function; Law of wall			15. NUMBER OF PAGES 22	
			16. PRICE CODE A03	
17. SECURITY CLASSIFICATION OF REPORT Unclassified	18. SECURITY CLASSIFICATION OF THIS PAGE Unclassified	19. SECURITY CLASSIFICATION OF ABSTRACT Unclassified	20. LIMITATION OF ABSTRACT	