EULERIAN TIME-DOMAIN FILTERING FOR SPATIAL LES

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Abstract. Eulerian time-domain filtering seems to be appropriate for LES of flows whose large coherent structures convect approximately at a common characteristic velocity; e.g., mixing layers, jets, and wakes. For these flows, we develop an approach to LES based on an explicit second-order digital Butterworth filter, which is applied in the time domain in an Eulerian context. The approach is validated through *a priori* and *a posteriori* analyses of the simulated flow of a heated, subsonic, axisymmetric jet.

1. Introduction

Historically, large-eddy simulation (LES) has relied upon spatial rather than time-domain filters. Conceptually, however, filtering in time would seem to enjoy certain advantages. First, the DNS-LES-RANS spectrum of numerical approaches would be self-consistent if time-domain filtering were exploited for LES as it is for RANS. Second, as observed by Frisch (1995) "Most experimental data on fully developed turbulence are obtained in the time domain and then recast into the space domain via the Taylor hypothesis." If time-domain analysis is natural for experiments, one wonders why spatial re-interpretation is necessary or desirable. Third, differentiation-operator/filter-operator commutation error is problematic for spatial filtering on finite domains (Blaisdell, 1997, and Vasilyev et al., 1998). Fourth, according to Moin and Jimenez (1993): "In LES, it is highly desirable for the filter width to be significantly larger than the computational mesh to separate the numerical and modeling errors. Practical considerations, however, usually require the filter width and mesh to be of the same order. In this case, there does not appear to be a necessity for higher than second order numerical methods for LES." In contrast, for the present temporally is, $\lim_{t\to-\infty} G(t,\Delta) = 0$), then partial differentiation and filtering commute. As an example of a kernel that satisfies these constraints, consider $G(t,\Delta) = H(t+\Delta)/\Delta$, where H is the Heaviside function, and whereby $\overline{\phi}(t) = \int_{t-\Delta}^{t} \phi(\tau)d\tau$. Because bounded support is the norm for time-domain filters, but not for spatial filters (Blaisdell, 1997), time-domain filtering enjoys a natural advantage with respect to commutation error.

2.2. DISCRETE CAUSAL FILTERS

The discrete analog of Eq. 1 is $\overline{\phi}_i = \sum_{j=0}^{m} p_j \phi_{i-j}$, where $\phi_i = \phi(i\Delta t, \mathbf{x})$ and Δt is a (fixed) time increment. In general, the coefficients p_j depend on the quadrature rule used to approximate the integral of Eq. 1, the specific kernel G, and the width Δ. Following Press et al. (1986), a more versatile digital filter–of *recursive* type–is given by

$$\overline{\phi}_i = \sum_{j=0}^{m} p_j \phi_{i-j} + \sum_{k=1}^{n} q_k \overline{\phi}_{i-k} \tag{3}$$

whereby $\overline{\phi}_i$ is a linear combination of previous unfiltered and filtered values. From Press et al. (1986), the frequency response of recursive filters of the form of Eq. 3 is

$$H(\Omega) = \frac{\sum_{j=0}^{m} p_j e^{-\iota j\Omega}}{1 - \sum_{k=1}^{n} q_k e^{-\iota k\Omega}} \tag{4}$$

where $\iota = \sqrt{-1}$, $\Omega = \omega^* \Delta t^*$ is the dimensionless frequency, and f^* and $\omega^* = 2\pi f^*$ are the dimensional physical and circular frequencies, respectively. (Throughout the paper, we denote dimensional quantities by asterisks.)

A class of recursive filters well-suited to LES is the "Butterworth" class. Low-pass digital Butterworth filters have the trait that p_0 vanishes in Eq. 3, which renders them fully explicit in time. The design of Butterworth filters of various order properties is discussed in Strum and Kirk (1988), to which the reader is referred. As our prototype, we adopt a second-order Butterworth filter, whose frequency response is shown in Fig. 1. The prototype, with a nominal cutoff frequency of $\Omega_c' \approx 1$, is rendered tunable by the introduction of a cutoff parameter, $R_c = \Omega_c/\Omega_c' = \Delta t/\Delta$, defined as the ratio of the actual cutoff to that of the prototype. The action of the filter on a harmonically rich signal is shown in Fig. 2 for selected values of R_c.

Time-domain filters suffer some disadvantages relative to spatial filters. First, they require storage of past information; the higher the order, the more storage. In particular, our second-order filter requires four fields of storage for each field filtered. However, relative to DNS, the net storage savings of temporally filtered LES remains substantial. Moreover, time-domain filtering, which is one-dimensional, results in less computational where $D = \frac{\partial \tilde{u}_k}{\partial x_k}$ is the resolved-scale dilatation, $\tilde{S}_{ij} = 2(\tilde{e}_{ij} - \frac{1}{3} D\delta_{ij})$, δ_{ij} is the Kronecker delta, $\tilde{e}_{ij} = \frac{1}{2}\left[\frac{\partial \tilde{u}_i}{\partial x_j} + \frac{\partial \tilde{u}_j}{\partial x_i}\right]$ is the resolved-scale strain-rate tensor, and Φ is the dissipation function. For brevity, the physical viscosity and thermal conductivity are denoted, respectively, as $\mu_v = \tilde{\mu}/Re$ and $\kappa_v = \tilde{\mu}/(M^2 RePr)$, where Re, Pr, and M are the Reynolds, Prandtl, and Mach numbers, respectively. Similarly, the eddy viscosity and the eddy thermal conductivity are given, respectively, by $\mu_T = C_r l^2 \bar{\rho} \Pi^{1/2}$ and $\kappa_T = \mu_T/(\gamma M^2 Pr_T)$, where l is a length scale, Pr_T is the turbulent Prandtl number, γ is the ratio of specific heats, and $\Pi = \tilde{S}_{ij}\tilde{S}_{ij}$.

Of mixed type, the SEZHu model incorporates both scale-similarity and eddy-viscosity terms, which are underlined on the left and right sides of the equations above, respectively. In the momentum equation, for example, the underlined terms together model $\tau_{ij} = \bar{\rho}(\tilde{u}_i\tilde{u}_j - \widetilde{u_i u_j})$, the exact residual-stress tensor. The CNSE are recovered whenever the underlined terms are turned off, which renders the equations valid for either DNS or LES. For LES, the SGS model requires values for three constants. Following Erlebacher et al. (1992), we use $Pr_T = 0.5$, $C_r = 0.012$, and $l = 2\Delta x$.

For the axisymmetric jet-flow application, we exploit a cylindrical coordinate system, for which x (u_1) and z (u_1) are the axial and radial coordinates (velocities), respectively. Because of axisymmetry, the azimuthal coordinate (θ) does not come into play.

4. Galilean Invariance

Speziale (1987) raises issues regarding Galilean invariance and Eulerian time-domain filters. Ultimately, one can circumvent the issue by implementing temporal filtering in a Lagrangian frame of reference, as has been proposed by Meneveau et al. (1996). However, Lagrangian time-filtered approaches suffer at least one drawback, namely the introduction of additional closure equations, which renders SGS models potentially as computationally cumbersome as Reynolds-stress models. Thus, Eulerian time-domain filtering would be preferable whenever it is appropriate, which is the subject of this section.

4.1. DOPPLER EFFECT

Speziale (1985) shows that the spatially filtered part of a Galilean-invariant function is itself Galilean-invariant. Subsequently (Speziale, 1987), he implies that the same is not true of time-domain filters, which we have verified. In our words, although the governing equations themselves are Galilean invariant for time-domain filters, the individual terms of those equations are not. In essence, Eulerian temporally filtered quantities experience a Doppler jet temperature $T_j^* = 600$F (on which Mach number is based), ambient temperature $T_a^* = 70$F, nominal jet radius $R_j^* = 0.5$ in., ambient pressure $p_a^* = 216$ psf., and $Re = 10153$ (based on the jet conditions and the jet radius). The jet is assumed to be axisymmetric and fully expanded, in which case, in the absence of disturbances, the pressure is constant both radially and axially.

In the Results section to follow, lengths have been normalized by R_j^*, and the velocities, temperature, and density, have been normalized by U_j^*, T_j^*, and ρ_j^*, respectively.

6. Numerical Methodology

We view spatial DNS and LES as three-step processes. First, an unperturbed time-independent base state is obtained by boundary-layer techniques (Pruett, 1996). Second, the base state is subjected to temporally periodic perturbations; here, these are imposed through the streamwise velocity at the computational inflow boundary, as per Mankbadi et al., (1994). Numerical experimentation reveals most rapid development of the jet for a Strouhal number ($St = f_j^* R_j^* / U_j^*$) of 0.5, where f_j^* is the fundamental forcing frequency. An out-of-phase subharmonic is also included to enhance the pairing of adjacent vortices. The forcing amplitude is small–half a percent of U_j^* for the fundamental–and the forcing is is ramped up slowly to minimize temporal transients. Third, the spatial evolution of the propagating disturbances is computed by numerical solution of the unfiltered (DNS) or filtered (LES) CNSE, as discussed below.

For both DNS and LES, we adapt the high-order numerical scheme of Pruett et al. (1995), to which the reader is referred for details. Briefly, this algorithm exploits fully explicit time advancement, high-order compact-difference methods (Lele, 1992) for aperiodic spatial dimensions, and spectral collocation methods for periodic spatial dimensions. Specifically, for the present axisymmetric-jet application, we use fourth- and sixth-order compact-difference schemes in the axial and radial dimensions, respectively. (The azimuthal dimension does not come into play because of the axisymmetry assumption.) The original method of Pruett et al. (1995) used a variable step for time advancement in the context of a third-order Runge-Kutta (RK3) scheme. However, the present LES application, which involves temporal filtering, requires a constant time step. Consequently, the original RK temporal integration has been replaced by a fixed-length, third-order Adams-Bashforth (AB3) technique.

Regarding boundary conditions, for both DNS and LES, symmetry conditions are imposed along the jet axis ($z = 0$). At the inflow boundary, we specify v, w, T, and the incoming Riemann invariants. At the far-field

$R_c = 0.015625$ as cutoffs for the grid and test filters, respectively; hence, $r = 2$. (The grid-filter cutoff value is consistent with $R_c = 0.125$ for the LES calculation below, of coarser time resolution.) When adjusted for the phase lag of the test filter, correlations of 0.90 are obtained.

Following Liu et al. (1994), we also present results for $r = 1$. Whereas $r = 1$ is disallowed by conventional dynamic SGS models, Taylor-series analysis (Pruett, 1997) suggests that $r = 1$ is *optimal* for second-order filters in that the leading-order error term vanishes in the approximation of τ_{ij} by $\mathcal{L}_{ij}$. In this case, the correlation coefficients exceed 0.998 when corrected for the phase lag of the test filter. Moreover, coefficients on the order of 0.7 are obtained when the resolved and residual stresses are correlated at the same instant in time, without correction.

Present *a priori* tests suggests that strong correlations exist between the τ_{ij} and $\mathcal{L}_{ij}$, as observed also for spatial filters. We note that for $r = 1$, the stress-similarity model of Liu et al. (1994) is equivalent to the SEZHu model of Section 3 with its dissipative term turned off. It is well known, however, the similarity models alone are insufficiently dissipative for practical applications to LES. Hence, we consider an *a posteriori* test of the full SEZHu model.

7.2. LES

The SEZHu SGS model was implemented with no changes other than the incorporation of time-domain filtering in lieu of spatial filtering for the similarity term, which we evaluate in real time. Some numerical experimentation was necessary to find an appropriate level of dissipation. If the SGS model is insufficiently dissipative, the computation blows up. On the other hand, if the model is excessively dissipative, the instabilities that result in vortex shedding and pairing are suppressed. Because most model parameters were set for consistency with Erlebacher et al. (1992), dissipation was controlled by experimenting with grid resolution and with the test-filter cutoff R_c. Figure 5 presents instantaneous contours of constant density at $t_p = 18$ obtained from an LES computation of 432×192 spatial resolution, $R_c = 0.125$, and $r = 2$. Because fully explicit numerical schemes are typically over-resolved in time, it is natural that $\Delta >> \Delta t$ ($R_c << 1$). Whereas the DNS calculation required 40 CPU hours, the coarser LES calculation required but two hours. Relative to the DNS results of Fig. 3, the shear-layer roll-up and pairing events of the LES computation are retarded but not prevented. Consequently, we believe that moderately resolved LES could serve as a computational platform for investigations of jet noise, and that a SGS model built on Eulerian time-domain filtering is well-suited for this task. To this end, we extract the compressible dilatation from the LES

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