Insolation and the Precession Index

by

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Abstract

Simple nonlinear climate models yield a precession index-like term in the temperature. Despite its importance in the geologic record, the precession index $e \sin\omega$, where e is the Earth's orbital eccentricity and ω is the Sun's perigee in the geocentric frame, is not present in the insolation at the top of the atmosphere. Hence there is no one-for-one mapping of 23,000 and 19,000 year periodicities from the insolation to the paleoclimate record; a nonlinear climate model is needed to produce these periods. Two such models, a grey body and an energy balance climate model with an added quadratic term, produce $e \sin \omega$ terms in temperature. These terms, which without feedback mechanisms achieve extreme values of about ±0.48 K for the grey body and ±0.64 K for the energy balance model, simultaneously cool one hemisphere while they warm the other. Moreover, they produce long-term cooling in the northern hemisphere when the Sun's perigee is near northern solstice and long-term warming in the northern hemisphere when the perigee is near southern solstice. Thus this seemingly paradoxical mechanism works against the standard model which requires cool northern summers (Sun far from Earth in northern summer) to build up northern ice sheets, so that if the standard model is correct it may be more efficient than previously thought. Alternatively, the new mechanism could possibly be dominant and indicate southern hemisphere control of the northern ice sheets, wherein the southern oceans undergo a long-term cooling when the Sun is close to the Earth during southern summer. The cold water eventually flows north, cooling the northern hemisphere. This might explain why the northern oceans lag the southern ones when it comes to orbital forcing.

1. **Introduction**

An important component of *Milankovitch's* [1941] astronomical theory of climate change is the precession index. The precession index, along with the Earth's obliquity and orbital eccentricity, are believed to be the major controling factors of climate change in the last few million years [e.g., *Hays et al.*, 1976; *Berger et al.*, 1984; *Hinnov*, 1999]. The precession index is $e \sin \omega$, where e is the Earth's orbital eccentricity and (assuming a geocentric point of view) ω is the argument of perigee of the Sun's orbit about the Earth. The precession index spectrum has a major peak at 23 kyr and a smaller one at 19 kyr (1 kyr = 1000 years).

It is an interesting fact that the equation for the insolation at the top of the atmosphere contains no terms which look like $e \sin \omega$ (*Rubincam* [1994; 1996]; *Hinnov*, 1999; and Bruce G. Bills, private communication, 1994). Therefore the equation contains no long-period 23 kyr and 19 kyr terms. Because of its undoubted importance in the paleoclimate record, the existence of the precession index must be due to the Earth's nonlinear response to the insolation.

The standard explanation for the importance of the precession index, namely that cool northern summers are required for the growth of ice sheets [e.g., *Milankovitch*, 1941, pp. 435-435] is in fact such a nonlinear model. Assuming the Earth responds to mainly to northern summertime cooling is to manipulate the insolation into giving 23 kyr and 19 kyr frequencies in ice volume [*Rubincam*, 1996]. However, the standard "model" given by *Milankovitch* [1941] is not really a quantitative model; instead he merely correlates the snow line with summer insolation and calls the result "proof." This procedure is questionable, given all of the factors which might contribute to the position of the snowline besides just insolation.

Below I demonstrate below two quantitative nonlinear models which produce $e \sin \omega$ in temperature from the $e \sin \omega$-free insolation. The first is the simplest of all "climate" models: a grey body. In this model insolation is proportional to T^4, where T is surface temperature. I write temperature as $T = T_0 + \Delta T$ and expand T^4 to order $(\Delta T)^2$. The other is an energy balance climate model with added T^2, T^3, and T^4 terms; these are also expanded to order $(\Delta T)^2$. In both these nonlinear models cross-product of terms in the insolation produce $\Delta T_{pi} \propto -e \sin \omega \, P_1(\sin \phi)$ in the surface temperature, where $P_1(\sin \phi)$ is the Legendre polynomial of degree 1, ϕ is latitude, and the subscript "pi" on ΔT stands for "precession index". The magnitude of the temperature change can reach about ± 0.48 K for the grey body and about ± 0.64 K for the energy balance model when the eccentricity e reaches its maximum value of 0.06 and $\omega = 90°, 270°$.

Because $P_1(\sin \phi) = \sin \phi$, which changes sign when the equator is crossed, these models indicate simultaneous warming in one hemisphere and cooling in the other. Moreover, both models yield a precession index with a seemingly paradoxical sign: long-term cooling in the northern hemisphere when the Sun is near perihelion during northern summer and a long-term warming in the north when the Sun is near aphelion during northern summer. This effect thus works in opposition to the usual explanation of $e \sin \omega$, wherein cool summers (Sun far from Earth during northern summer) are required for the snow linger in order to build up into ice sheets. In this case the new mechanism warms the ground during the cool northern summers, making it harder for the standard model to work.

Alternatively, the effect presented here may dominate the standard mechanism. The new effect might argue for southern hemisphere control of the northern ice sheets, the idea

being that the southern oceans cool when the Sun is near the Earth during southern summer; the cold water flows north, eventually cooling the northern hemisphere enough to build up ice sheets. This scenario might explain why the northern oceans lag the southern in the orbital forcing.

2.   Insolation

The insolation is given by

$$F_S = QS$$

where

$$4Q = F_s^0\left(\frac{r_0}{a}\right)^2$$

with F_s^0 being 1371 W m^{-2} when the reference distance r_0 is $r_0 = a = 1$ AU [*Hickey et al.*, 1988], where a is the semimajor axis of the Earth's orbit, and

$$S = 4\sum_{\ell=0}^{\infty} d_\ell \sum_{m=0}^{\ell} (2-\delta_{0m})\frac{(\ell-m)!}{(\ell+m)!}P_{\ell m}(\sin\phi)$$

$$\bullet \sum_{p=0}^{\ell}\sum_{q=-\infty}^{+\infty} F_{\ell m p}(\varepsilon)\, W_{\ell-2p,q}(e) \tag{1}$$

$$\bullet \begin{bmatrix}\cos\\\sin\end{bmatrix}_{\ell-m\,odd}^{\ell-m\,even} [(\ell-2p)\omega + (\ell-2p+q)M + m(\Omega-\eta-\lambda)]$$

[*Rubincam*, 1994]. Here e is orbital eccentricity, ω is the argument of perigee, M is the mean anomaly of the Sun's orbit about the Earth (I assume a geocentric point of view), and Ω is the position of the line of nodes of the Sun's orbit measured from the equinox; $\Omega = 0$. The angle ε is the Earth's obliquity (currently about 23.44°), and ϕ is latitude and λ is east longitude on the Earth, while η is the hour angle. The $P_{\ell m}(\sin\phi)$ are the associated Legendre polynomials of degree ℓ and order m . The $F_{\ell m p}(\varepsilon)$ are the inclination functions from celestial mechanics; *Rubincam* [1994, Table 2] gives them for degree 1, while *Kaula* [1966] and *Caputo* [1967] list them for degrees 2 through 4. The $W_{\ell\text{-}2p,q}(e)$ are not the eccentricity functions from celestial mechanics; rather, they are special eccentricity functions associated solely with the insolation and are tabulated in *Rubincam* [1994]. (There is a typographical error in Table 3 of that paper; the entry for $\ell - 2p = \pm 1$, $q = \pm 1$ should read $2e - 3e^3/2$ instead of $2e - 3e^4/2$. Also, the $(2^\ell - 2s)!$ factor in the equation following (7) in the same paper should read $(2\ell - 2s)!$)

The zonal insolation can be found from (1) by setting $m = 0$. The zonal insolation given by equation (8) in *Rubincam* [1994] contains some errors. The corrected equation for the zonal insolation at the top of the atmosphere is

$$S \ =$$

$$P_0(\sin \phi) \bullet \left[1 \ + \ \frac{e^2}{2} \ + \ 2e \cos M \ + \ \frac{5}{2}e^2 \cos 2M \right]$$

$$+ \ 2 P_1(\sin \phi)$$

$$\bullet \left[(1 - \frac{e^2}{2}) \sin \varepsilon \sin (\omega + M) \ + \ 2e \sin \varepsilon \sin (\omega + 2M) \right.$$

$$\left. + \ \frac{e^2}{8} \sin \varepsilon \sin (\omega - M) \ + \ \frac{27}{8}e^2 \sin \varepsilon \sin (\omega + 3M) \right]$$

$$+ \ \frac{5}{4} P_2(\sin \phi)$$

(2)

$$\bullet \left[(1 + \frac{e^2}{2})(\frac{3}{4}\sin^2 \varepsilon - \frac{1}{2}) \ - \ \frac{3}{4}(1 - \frac{7}{2}e^2) \sin^2 \varepsilon \cos (2\omega + 2M) \right.$$

$$+ \ 2e (\frac{3}{4}\sin^2 \varepsilon - \frac{1}{2}) \cos M \ + \ \frac{3}{4}e \sin^2 \varepsilon \cos (2\omega + M)$$

$$- \ \frac{9}{4}e \sin^2 \varepsilon \cos (2\omega + 3M) \ + \ \frac{5}{2}e^2 (\frac{3}{4}\sin^2 \varepsilon - \frac{1}{2}) \cos 2M$$

$$\left. - \ \frac{39}{8}e^2 \sin^2 \varepsilon \cos (2\omega + 4M) \right]$$

$$+ \ ...$$

The $P_i(sin \ \phi)$ are the usual Legendre polynomials

$$P_0(\sin \phi) \ = \ 1 \ ,$$

$$P_1(\sin \phi) \;=\; \sin \phi \quad ,$$

$$P_2(\sin \phi) \;=\; \frac{3}{2}\sin^2 \phi \;-\; \frac{1}{2} \quad ,$$

$$P_3(\sin \phi) \;=\; \frac{5}{2}\sin^3 \phi \;-\; \frac{3}{2}\sin \phi \quad ,$$

etc. Being just the zonal (longitude-independent) insolation, the diurnal terms are not given in (2).

The important thing to notice about (1) and (2) is that they do *not* contain a term of the form

$$S \;=\; \ldots \;+\; coefficient \times e \sin \omega \;+\; \ldots$$

In other words, there is no term in the insolation which looks like the precession index. Hence there is no term in the insolation which has periods of 23 kyr and 19 kyr. Bruce G. Bills, in an independent analysis (unpublished), arrived at the same conclusion. This lack of a precession index was in fact demonstrated early on by *Humphreys* [1964, pp. 85-87] in his classic book on meteorology.

The only thing which even remotely looks like the precession index is the $2e \sin \varepsilon$ $\sin (\omega + 2M)$ term which multiplies $P_1 (\sin \phi)$. However, this term has a high frequency due to the presence of the $2M$. Its period is about half a year, not 19 kyr or 23 kyr.

This result has caused some confusion [*Berger*, 1996; *Rubincam*, 1996] because 23 kyr and 19 kyr periods are strongly indicated in the paleoclimate records [e.g., *Hays et al.*,

1976; *Berger et al.*, 1984]. If the Milankovitch theory of climate is correct, how can there be an $e \sin \omega$ signal in the paleoclimate records when it does not exist in the insolation?

The answer is that if $e \sin \omega$ is important for climate, it must be due to the way the Earth responds to the insolation [*Rubincam*, 1994, 1996]. In other words, the Earth's climate system does something nonlinear to the astronomical signal, thereby manufacturing the 23 kyr and 19 kyr periods. *Rubincam* [1994, p. 201] produced a model which in fact gave a precession index-like term in the radiation reaching the ground. That radiation is the insolation at the the top of the atmosphere multiplied by the Earth's coalbedo. Short-period terms in the albedo can multiply short-period terms in the insolation, eliciting $e \sin \omega$. But this "precession index" was extremely weak. Below I exhibit two simple models which which produce $e \sin \omega$ in the Earth's temperature and have extreme values of about ± 0.48 K and ± 0.64 K when the Earth's orbital eccentricity is at its maximum value of $e = 0.06$ and $\omega = 90°, 270°$.

3. Grey body

The first is the simplest of all "climate" models: the Earth as a grey body with albedo A and emissivity ε_{ss}. In this case

$$\varepsilon_{ss} \, \sigma T^4 \;=\; (1 - A) \, Q S \ . \tag{3}$$

where σ is the Stefan-Boltzmann constant. Writing

$$T = T_0 + \Delta T \tag{4}$$

where T is the absolute temperature yields

$$\varepsilon_{ss}\, \sigma T_0^4 \left[1 + 4\left(\frac{\Delta T}{T_0}\right) + 6\left(\frac{\Delta T}{T_0}\right)^2 + 4\left(\frac{\Delta T}{T_0}\right)^3 + \left(\frac{\Delta T}{T_0}\right)^4 \right]$$

$$= (1-A)QS = (1-A)Q + (1-A)Q\Delta S$$

(5)

where T_0 is a constant and

$$\Delta S = 2P_0(\sin\phi)\bullet\left[e\cos M\right]$$

$$+ 2P_1(\sin\phi)\bullet\left[(\sin\varepsilon\sin(\omega+M) + 2e\sin\varepsilon\sin(\omega+2M)\right]$$

$$+ \frac{5}{4}P_2(\sin\phi)\left(\frac{3}{4}\sin^2\varepsilon - \frac{1}{2}\right)\bullet\left[1 + 2e\cos M\right]$$

(6)

to the first power in e in (2). Setting

$$\varepsilon_{ss}\,\sigma T_0^4 = (1-A)Q \tag{7}$$

$$x = (\Delta T/T_0)$$

$$y = (1-A)Q\Delta S$$

in (5) makes that equation become

$$y = 4x + 6x^2 + 4x^3 + x^4.$$

This equation has the form

$$y = a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \dots \quad . \tag{8}$$

Assuming

$$\Delta T \ll T_0$$

(8) can be solved for x using *reversion of series* [*Selby*, 1974, p. 470]:

$$x = A_1 y + A_2 y^2 + A_3 y^3 + A_4 y^4 + \dots \tag{9}$$

where

$$A_1 = 1/a_1, \quad A_2 = -a_2/a_1^3, \quad \dots \quad .$$

Hence the solution of (5) becomes

$$\frac{\Delta T}{T_0} \approx \frac{(1-A)Q\Delta S}{\varepsilon_{ss}\, \sigma T_0^4} - \frac{3}{32}\left[\frac{(1-A)Q\Delta S}{\varepsilon_{ss}\, \sigma T_0^4}\right]^2 + \dots$$

Squaring ΔS as given by (6) in the above equation yields cross-product terms of the form

$$e \sin \varepsilon \sin \omega \left[\frac{1}{4} P_1(\sin \phi) \ - \ \frac{5}{32} P_1(\sin \phi) P_2(\sin \phi) \right]$$

to first power in $\sin \varepsilon$ and are mainly from the $e \cos M$ multiplying $\sin \varepsilon \sin (\omega + M) P_1(\sin \phi)$. This is easily seen from the identity

$$\sin \alpha \cos \beta \ = \ \frac{\sin (\alpha - \beta)}{2} \ + \ \frac{\sin (\alpha + \beta)}{2}$$

where $\alpha = \omega + M$ and $\beta = M$, giving a low-frequency $\sin \omega$ and a high-frequency $\sin (\omega + 2M)$. The high-frequency terms will be ignored here. Using

$$P_1(\sin \phi) P_2(\sin \phi) \ = \ \frac{2}{5} P_1(\sin \phi) \ + \ \frac{3}{5} P_3(\sin \phi) \tag{10}$$

ultimately gives

$$\frac{\Delta T}{T_0} \ = \ -\frac{9}{32} \left[\frac{(1-A)Q}{\varepsilon_{ss} \, \sigma \, T_0^4} \right]^2$$

$$\bullet \, e \sin \varepsilon \sin \omega \left[P_1(\sin \phi) \ - \ \frac{1}{2} P_3(\sin \phi) \right]$$

$$+ \quad \textit{other terms}$$

Hence the "precession index" ΔT_{pi} in temperature is

$$\Delta T_{pi} = -\frac{9\,T_0}{32}\left[\frac{(1-A)Q}{\varepsilon_{ss}\,\sigma\,T_0^4}\right]^2$$

(11)

$$\bullet\, e\sin\varepsilon\sin\omega\left[P_1(\sin\phi) - \frac{1}{2}P_3(\sin\phi)\right]$$

for a grey body for zonal insolation only. For the Earth the variations in ε are small and can be considered constant to a first approximation, so that this term behaves like $e\sin\omega$. Using $\sigma = 5.67\times 10^{-8}$ W m^{-2} K^{-4}, together with some typical values $A = 0.3$, $r_0 = a = 1$ AU, $F_s^0 = 1371$ W m^{-2}, $\varepsilon = 23.44°$, plus setting $\varepsilon_{ss} = 0.89$ with the emissivity ε_{ss} being chosen to reproduce the observed average surface temperature of the Earth $T_0 = 288.1$ K [North et al., 1981, Table 1] give

$$\Delta T_{pi} = -5.6\bullet e\sin\omega\left[P_1(\sin\phi) - \frac{1}{2}P_3(\sin\phi)\right]\ \text{K}$$

(12)

At $e = 0.06$, the maximum obliquity of the Earth's orbit, and at latitude $\phi = 43.1°$, this reaches a minimum value of $\Delta T_{pi} = -0.27$ K when $\omega = 90°$ (perihelion at northern solstice). At the same time at $\phi = -43.1°$ it reaches a maximum of $\Delta T_{pi} = +0.27$ K. The curve marked "GREY (Z)" in Figure 1 shows the temperature as a function of latitude for $e = 0.06$ and $\omega = 90°$ for a grey body using only the zonal insolation.

4. Grey body by iteration

It will instructive to solve for ΔT_{pi} again through an iterative process, since it will be the method used to solve the energy balance equation below. Reversion of series cannot be used to solve a differential equation like the energy balance equation.

In this case the first-iteration temperature will be written

$$T \;=\; T_0 \;+\; \Delta T \;.$$

To first order (5) is simply

$$4\varepsilon_{ss}\,\sigma\,T_0^4\left(\frac{\Delta T}{T_0}\right) \;=\; (1-A)\,Q\,\Delta S$$

after using (7) to alias out the constant terms. To second-order

$$T \;=\; T_0 \;+\; \Delta T \;+\; \delta T$$

so that (5) will now be approximately

$$\varepsilon_{ss}\,\sigma\,T_0^4\left[4\left(\frac{\Delta T + \delta T}{T_0}\right) \;+\; 6\left(\frac{\Delta T}{T_0}\right)^2\right] \;=\; (1-A)\,Q\,\Delta S \tag{13}$$

The $(\Delta T)^2$ part will yield

$$(\Delta T)^2 \;=\; \frac{3}{16}\left[\frac{(1-A)Q}{\varepsilon_{ss}\,\sigma\,T_0^3}\right]^2$$

$$\bullet\, e \sin\varepsilon \sin\omega\left[P_1(\sin\phi) \;-\; \frac{1}{2}\,P_3(\sin\phi)\right]$$

plus other terms. Since the terms in the above equation not exist on the right side of (12), they must be cancelled by similar terms in δT; i.e.,

$$4\left(\frac{\delta T}{T_0}\right) \;+\; 6\left(\frac{\Delta T}{T_0}\right)^2 \;=\; 0$$

for the $e\,\sin\omega$ terms, thus giving

$$\delta T_{pi} \;=\; -\frac{9T_0}{32}\left[\frac{(1-A)Q}{\varepsilon_{ss}\,\sigma\,T_0^4}\right]^2$$

$$\bullet\, e \sin\varepsilon \sin\omega\left[P_1(\sin\phi) \;-\; \frac{1}{2}P_3(\sin\phi)\right]$$

which is the same as (11). This iterative approach will be used in the next section.

The above equations for the grey body uses only the zonal insolation (2). However, because the grey body has no thermal inertia, the diurnal terms in (1) will be of the same order of magnitude as the zonal terms. Including the $\ell = 1,2\; m = 1$ terms in the above approach yields the curve marked "GREY (Z + D)" in Figure 1. The diurnal terms increase ther size of the effect to extremes of ± 0.48 K.

5. Energy balance climate model

The zonal nonlinear energy balance climate model is [e.g., *North et al.*, 1981, 1983]:

$$C\frac{\partial T}{\partial t} \; - \; \frac{\partial\left[D\,(1-\mu^2)\frac{\partial T}{\partial \mu}\right]}{\partial \mu} \; + \; A_0 \; + \; \sum_{i=1}^{4}B_iT^i \; = \; (1-A)QS \tag{14}$$

where $\mu = sin\ \phi$ and the B_iT^i terms for $i \geq 2$ are the added nonlinear terms, and where now T is temperature in degrees Centigrade instead of the absolute temperature. Only the zonal terms will be needed, since the short-period diurnal terms will be damped out by the heat storage term which was not present for the grey body.

The first term $C\partial T/\partial t$ in the equation above represents heat storage, while the second term represents zonal diffusion as produced by winds and ocean currents and has a diffusion constant D. The infrared radiation leaving the Earth in the above equation is given by

$$I \; = \; A_0 \; + \; B_1T \; + \; B_2T^2 \; + \; B_3T^3 \; + \; B_4T^4 \tag{15}$$

The values for A_0, and the B_i can be estimated from the data of *Graves et al.* [1993] (see Figure 2); these values are $A_0 = 195.0$, $B_1 = 1.4158$, $B_2 = 0.02289$, $B_3 = 0.001148$, and $B_4 = 0.00002089$. These numbers do not come from a least-squares fit; rather, they were chosen to bisect the envelope of the data.

The linearized version of (14) will be solved for first. Once again writing

$$T = T_0 + \Delta T$$

gives by (15)

$$I = H_0 + \sum_{i=1}^{4} H_i (\Delta T)^i$$

where

$$H_0 = A_0 + B_1 T_0 + B_2 T_0^2 + B_3 T_0^3 + B_4 T_0^4$$

$$H_1 = B_1 + 2 B_2 T_0 + 3 B_3 T_0^2 + 4 B_4 T_0^3$$

$$H_2 = B_2 + 3 B_3 T_0 + 6 B_4 T_0^2$$

$$H_3 = B_3 + 4 B_4 T_0$$

$$H_4 = B_4$$

and now $T_0 = 14.9°$ C. [*North et al.*, 1981, Table 1], with $H_0 = 226.005$, $H_1 = 3.1389$, $H_2 = 0.1020$, $H_3 = 0.00235$, and $H_4 = 0.00002089$. The linearized, first-order equation is then

$$C\frac{\partial(T_0 + \Delta T)}{\partial t} - \frac{\partial\left[D(1-\mu^2)\frac{\partial(T_0 + \Delta T)}{\partial\mu}\right]}{\partial\mu}$$

$$+ \ H_0 + \ H_1\Delta T \ \cong \ (1-A)Q[1+\Delta S]$$

Setting

$$H_0 \ = \ (1-A)Q$$

allows the albedo A to be solved for, giving $A = 0.344$, in reasonable agreement with the observed value of 0.30 [*Stephens et al.*, 1981], and leaving

$$C\frac{\partial(\Delta T)}{\partial t} - \frac{\partial\left[D(1-\mu^2)\frac{\partial(\Delta T)}{\partial\mu}\right]}{\partial\mu} + \ H_1\Delta T \ \cong \ (1-A)Q\Delta S \ . \quad (16)$$

In this equation ΔT is assumed to have the same form as the zonal ΔS as given by (6):

$$\Delta T \ = \ \tau_1\left[2e\cos(M-\psi)\right.$$

$$+ \ 2P_1(\sin\phi)\sin\varepsilon\sin(\omega+M-\psi)$$

$$\left. - \ \frac{5}{4}eP_2(\sin\phi)\cos(M-\psi)\right]$$

$$+ \ \frac{5}{4}\tau_2 P_2(\sin\phi)\left[\frac{3}{4}\sin^2\varepsilon - \frac{1}{2}\right]$$

(17)

where the observed values are $\tau_1 = 19.5$ °C., $\tau_2 = 58.7$ °C., and $\psi = 31.5°$ [*North et al.*, 1981, Table 1]. (The amplitude τ_1 and lag angle ψ were actually found by North *et al.* only for the $2P_1(sin\ \varphi)$ sin ε term; but they are assumed here to apply to the terms proportional to e as well, since they have very nearly the same frequency.)

The diffusion constant D can be found by equating the time-invariant parts of the $P_2(sin\ \phi)$ coefficients in (16), so that

$$D = \frac{(1-A)Q - H_1\,\tau_2}{6\,\tau_2} = 0.1185 \tag{18}$$

Finding C in (16) from the time-variable $P_1(sin\ \phi)$ terms is more complicated. Writing the $P_1(sin\ \phi)$ part of ΔT in the usual complex notation

$$\tilde{\tau}_1 \left[2\,P_1(\sin\ \phi)\sin\ \varepsilon\ e^{2\pi i t} \right]$$

and substituting in (16) yields

$$2\pi i C\tilde{\tau}_1 + 2D\tilde{\tau}_1 + H_1\tilde{\tau}_1 = (1-A)Q$$

where $\dot{\omega} + \dot{M} \approx 2\pi$ (with time t measured in years), so that

$$\tilde{\tau}_1 = \tau_1\,e^{-i\psi}$$

where

$$\tau_1 = \frac{(1-A)Q}{\sqrt{(2D+H_1)^2+(2\pi C)^2}}$$

and

$$\tan \psi = \frac{2\pi C}{2D+H_1}$$

Thus there are actually two equations involving C. Solving each equation for C using the observed values of H_1, τ_1, and ψ and the value for D found above yields inconsistent results: in one case $C = 1.76$ and $C = 0.33$ in the other. Fortunately the value for C is not needed in what follows below, and I will simply adopt the observed values of τ_1 and ψ in what comes next.

Proceeding to the second-order solution, once again I write

$$T = T_0 + \Delta T + \delta T$$

where δT is the second-order part of the temperature. Substituting this in (16) and eliding the first-order solution leaves

$$C\frac{\partial(\delta T)}{\partial t} \quad - \quad \frac{\partial\left[D\,(1-\mu^2)\frac{\partial(\delta T)}{\partial\mu}\right]}{\partial\mu}$$

$$+ \quad H_1\,\delta T \quad + \quad H_2\,(\Delta T)^2$$

$$\cong \quad (1-A)\,Q\,\Delta S$$

(19)

as the second-order equation. Now the $H_2\,(\Delta T)^2$ term produces cross-product terms which can be written

$$3H_2\,\tau_1^2\,(e\sin\varepsilon\sin\omega)\,[P_1(\sin\phi) \quad - \quad \frac{1}{2}P_3(\sin\phi)]$$

(20)

plus other terms. Since terms of the sort (20) do not appear on the right side of (19), they must be cancelled by similar terms in δT in the other parts of (19), just as in solving the grey body by iteration. The expression (20) above contains the long periods of $e\sin\omega$, so that $C\partial(\delta T)/\partial T$ changes slowly in (19) and can be neglected. To see this, assume δT can be written in the form $e^{2\pi i f t}$, where f is the frequency associated with the 23 kyr period. Differentiating δT with respect to time brings the f out in front of the term. Since f is small, the time derivative is small and can be ignored. Hence the precession index part of the temperature can be solved for from

$$\frac{\partial \left[D\,(1-\mu^2)\dfrac{\partial(\delta T_{pi})}{\partial\mu} \right]}{\partial\mu} \;+\; H_1\,\delta T_{pi}$$

$$\approx\;\; -3H_2\,\tau_1^2\,(e\sin\varepsilon\sin\omega)\,[P_1(\sin\phi)\;-\;\frac{1}{2}P_3(\sin\phi)]$$

producing

$$\delta T_{pi} \;\cong\; -e\,\sin\varepsilon\,\sin\omega\left(\frac{3H_2\,\tau_1^2}{2D+H_1}\right)\Bigg\{ P_1(\sin\phi)$$

$$-\;\frac{(2D+H_1)}{2(12D+H_1)}P_3(\sin\phi)]\Bigg\}$$

(21)

Substituting numerical values in (20) yields

$$\delta T_{pi} \;\cong\; -13.7\,e\,\sin\omega\,P_1(\sin\phi)$$

$$+\;\; 5.1\,e\,\sin\omega\,P_3(\sin\phi)\;\;°C.$$

(22)

For e equal to the maximum value 0.06, δT_{pi} achieves extreme values of $\delta T_{pi} = \pm\,0.64°$.

when $\omega = 90°$, 270° and $\varphi = \pm\,48.5°$. Thus this effect is larger than for the grey body

(12). The results for the energy balance climate model are labeled "EBM" in Figure 1.

6. Comparison to obliquity changes

This result for the energy balance model can be compared to the ΔT_{obl} expected from obliquity changes. From (17) the second harmonic in temperature is

$$\frac{5}{4}\,\tau_2 P_2(\sin\,\phi)\left[\frac{3}{4}\sin^2\varepsilon - \frac{1}{2}\right]$$

This will give changes in temperature as the obliquity changes with the 41 kyr period of magnitude

$$\Delta T_{obl} = \frac{5}{4}\tau_2\left(\frac{3}{2}\sin\varepsilon\cos\varepsilon\right)\Delta\varepsilon$$

where $\Delta\varepsilon$ is the change in obliquity. Because $\Delta\varepsilon \approx \pm 1°$ over the obliquity cycle,

$$\Delta T_{obl} \approx 40.2\,\Delta\varepsilon \approx \pm 0.7\ \mathrm{K},$$

so that the precession index is about 90% the size of the obliquity variation for the energy balance model. These calculations of course assume no feedback mechanisms, such as ice-albedo feedback.

7. Diffusion constant D

The value derived here for the diffusion constant, $D = 0.1185$, is much lower than the 0.649 found by *North et al.* [1981, p. 96] for their energy balance model; thus the present nonlinear model indicates that the overall heat transport is countered by other factors when averaged around the whole globe. The grey body gives much the same result; the linearized grey body model yields from (5) and (6)

$$\Delta T \;=\; -\frac{5}{16}\left[\frac{(1-A)Q}{\varepsilon_{ss}\,\sigma\,T_0^3}\right]\left[\frac{3}{4}\sin^2\varepsilon - \frac{1}{2}\right]P_2(\sin\phi) \;=\; -24.7\,P_2(\sin\phi)\quad K$$

for the second degree harmonic temperature variation. There will also be a second-order term from squaring the $P_1(sin\ \varphi)\sin\ (\omega + M)$ in (6). That yields

$$\delta T \;=\; -\frac{1}{8T_0}\left[\frac{(1-A)Q}{\varepsilon_{ss}\,\sigma\,T_0^3}\right]^2\sin^2\varepsilon\,P_2(\sin\phi) \;=\; -2.7\,P_2(\sin\phi)\quad K$$

Adding these together gives the grey body's coefficient as - 27.4 K, which is not far from the observed value of - 28.0 K [*North et al.*, 1981, p. 100.]. Since the grey body has no latitudinal heat transport at all, this lends support to the idea that diffusion is balanced by something not covered by this simple model, yielding a small D.

On the other hand, the larger *North et al.* [1981] value for D indicates that latitudinal heat flow is substantial. And clearly diffusion has importance for the real Earth: thanks to the Gulf Stream London has an temperate climate, even though it is farther north than Winnipeg. Why are the results so different?

The resolution to the problem presumably lies in continentality, albedo, and the way the infrared radiation is handled in the linear energy balance models. While the Gulf Stream may heat Europe, continental interiors get cold; also, the albedo increases with latitude [e.g., *Stephens et al.*, 1981]. These factors, which are not included in the nonlinear models presented here, apparently average out the effect of diffusion, so that overall the Earth responds to insolation much like a grey body with a uniform albedo. The discepancy with the linear energy balance model may come from the linear model's fitting an inherently curved set of data to the form $A_0 + BT$, resulting in a large diffusion coefficient D, thus giving diffusion a spurious importance.

8. Discussion

The precession index in temperature reveals itself as the product of short-period terms in these T^2 models. This shown in Figure 3 in a highly schematic diagram of the insolation spectrum and the temperature spectrum. The Earth's nonlinear climate system creates the precession index line from the insolation. It is chiefly the $e \cos M\ P_0(\sin \varphi)$ term in S multiplying the strong annual term $\sin \varepsilon\ \sin (\omega + M)\ P_1(\sin \varphi)$ which produces $e \sin \omega P_1(\sin \varphi)$, giving the long periods of 23 kyr and 19 kyr. It is extremely interesting to note that as far as long periods are concerned, the only cross-product terms of any significance are these $e \sin \omega$ terms: there are no $e \sin 2\omega$ or pure e terms, for instance.

The magnitude of the effect is fairly large: for the energy balance model it is $\pm 0.64°$ C. at maximum eccentricity and without feedback. For comparison, the difference in the global temperature between the present and the last glacial maximum was about -4° C. [*Crowley*, 1983, p. 868].

The energy balance model gives a larger temperature change than for the grey body. This is because the data relating infrared radiation to temperature (Figure 2) shows a greater curvature than for the standard T^4 behavior of a black body.

The sign of the new mechanism appears paradoxical: when $\omega = 90°$ in (12) and (22), the Sun is close to the Earth at northern solstice, producing a long-term cooling in the northern hemisphere with simultaneous warming in the southern. Similarly, when $\omega = 270°$, the Sun is far from the northern hemisphere during northern summer and this effect produces a long-term warming there, while at the same time the southern hemisphere cools.

This is counter to intuition, which says that when the Sun is close to the northern hemisphere in northern summer that hemisphere ought to warm, not cool. However, it is the short period term $\sin \varepsilon \ \sin (\omega + M) \ P_l(\sin \varphi)$ which does the warming in accordance with intuition, as can be seen from (2). The peculiar sign of the mechanism presented here is in fact correct and necessary to achieve energy balance in the nonlinear models.

The sign is also counter to that of the standard explanation of why the precession index is important. The standard model relies on the short period terms: it calls for cool northern summers when ω is near 270° and the Sun is far from the Earth, so that snow lingers through the summers, ultimately building up into an ice sheet [e.g., *Milankovitch*, 1941, pp. 435-436; *Pisias and Imbrie*, 1986, p. 45]. By warming the northern cool summers and heating the cool winters, the effect found here presumably makes it harder for the standard model to operate. If the standard explanation is responsible for the waxing and waning of ice sheets at the 23 kyr and 19 kyr periods, then it must be more efficient than previously thought to overcome the present mechanism with its opposing sign.

It may be the present mechanism which in fact predominates. While *Imbrie et al.* [1988] argue for northern hemisphere control of the northern ice sheets, the effect presented here perhaps argues for southern hemisphere control of the northern ice sheets. The idea is that during the times when ω is near 270°, the southern hemisphere undergoes a long-term cooling, especially the Antarctic ice cap and the southern oceans. The cold water makes its way north, cooling the whole Earth and eventually producing the northern ice sheets.

This could perhaps explain the phase shift seen in the sea surface temperatures for the precession index, as discussed by *Imbrie et al.* [1988, pp. 144-148]. They note that the northern oceans significantly lag the southern oceans with respect to the orbital forcing. This is what would be expected from the southern oceans cooling first, and then the northern as the cold waters spread north. Because the turn-over time of the oceans is

thousands of years, the northern hemisphere should lag the southern by this amount. This is what is observed. There is some support for this point of view in the obliquity forcing as well. Here also the northern oceans lag the southern in the obliquity forcing [*Imbrie et al.*,1988, pp. 147-148], again arguing for southern ocean control of the ice ages.

The slow ocean currents are of course not present in the model, nor are continents and ice-albedo feedback [*North et al.*, 1981, 1983; *Graves et al.*, 1993; *Short et al.*, 1991]. The continents and feedback presumably amplify the effect locally. Cross-product diurnal terms are not present either, which could possibly give rise to significant long-period terms in a model better than the energy balance model. These all represent avenues for future research.

The δT_{pi} found here may not be "the" precession index. Some other nonlinear mechanism rather than the one discussed above may the reason for the importance of $e \sin \omega$. The point to be made here is that there is no one-for-one mapping of the 23 kyr and 19 kyr cycles from insolation to the paleoclimate record. The 23 kyr and 19 kyr cycles do not exist in the insolation. They must be manufactured from the astronomical signal by the Earth's nonlinear climate system. The T^2 models are one way of doing it. There may be others.

Acknowledgment

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Figure 1

ΔT_{pi} for the grey body with the zonal insolation alone (dotted line), the grey body with the zonal and diurnal terms (dashed line), and δT_{pi} for the energy balance model (solid line) as a function of latitude φ for $e = 0.06$ and $\omega = 90°$. These values give the maximum changes in temperature.

Figure 2

The outgoing infrared radiation as a function of temperature, based on Figure 1 of *Graves et al.* [1993]. The curve, which is a polynomial in T, is chosen to bisect the envelope. It is the curvature of the data which gives rise to $e \sin \omega$ in temperature.

Figure 3

Schematic spectrum of insolation and temperature. This diagram shows how the Earth's nonlinear climate system manufactures a new spectral line in temperature from existing lines in insolation. The precssion line is shown only as a single peak, and many other spectral lines are omitted for clarity.
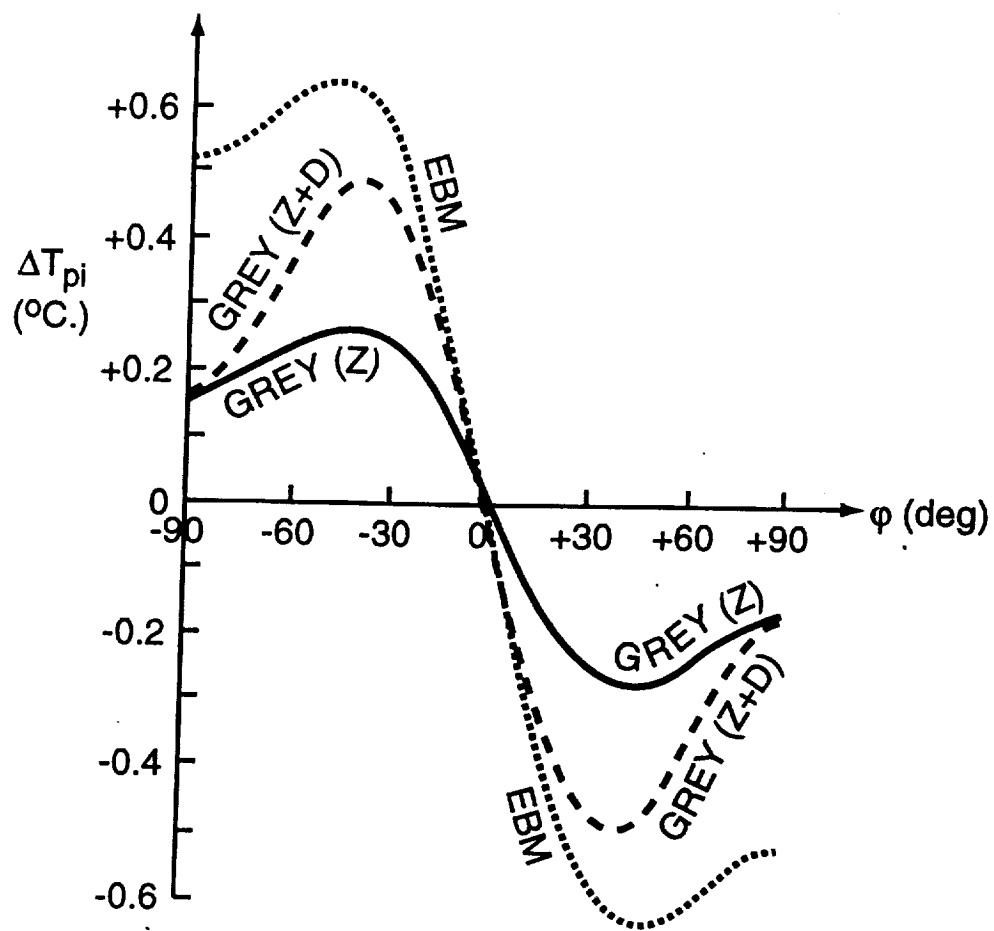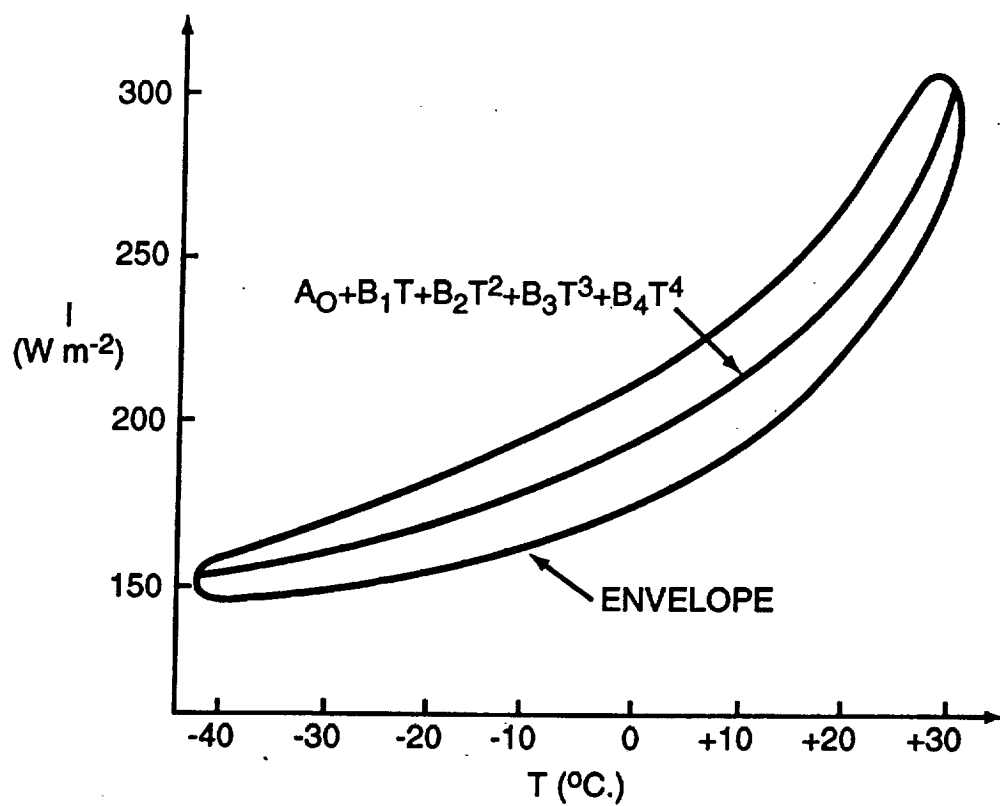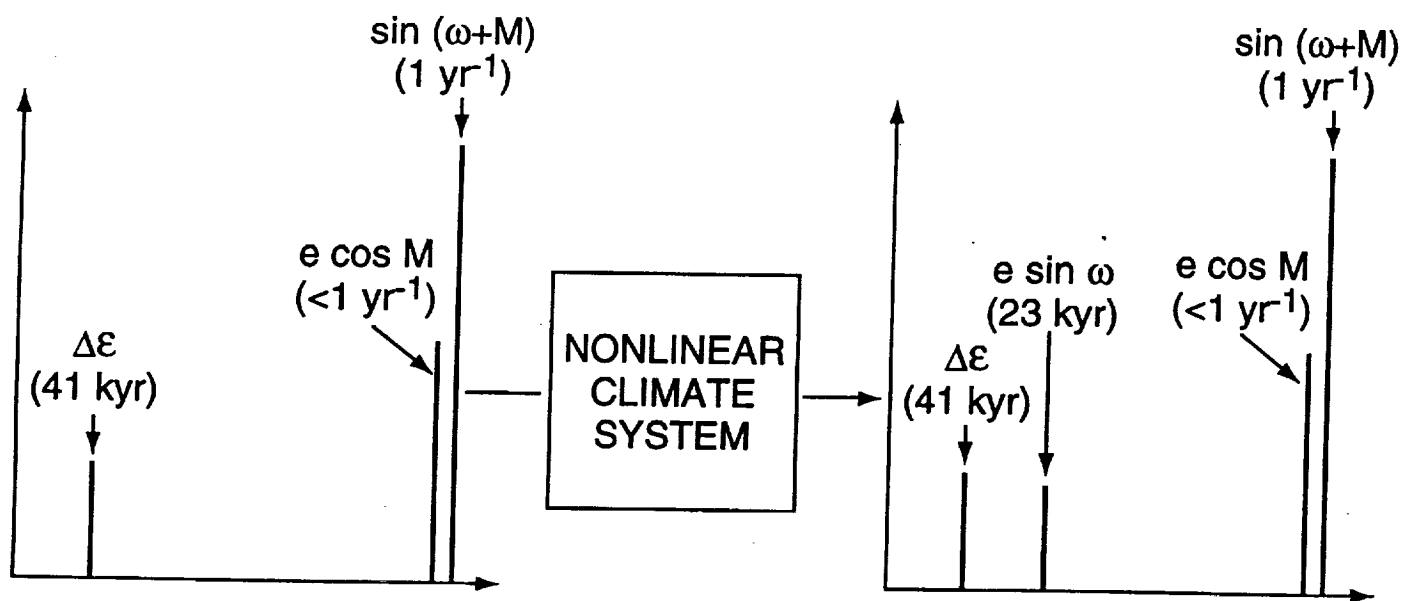
Figure 1

$A_O + B_1T + B_2T^2 + B_3T^3 + B_4T^4$

ENVELOPE

Figure 2

Figure 3