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Initialized Fractional Calculus

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Abstract

This paper demonstrates the need for a nonconstant initialization for the fractional calculus and establishes a basic definition set for the initialized fractional differintegral. This definition set allows the formalization of an initialized fractional calculus. Two basis calculi are considered; the Riemann-Liouville and the Grünwald fractional calculi. Two forms of initialization, terminal and side are developed.

1. Introduction

The issue of initialization has been an essentially unrecognized problem in the development of the fractional calculus. Liouville's definition ([5], p.21) for the fractional integral with lower limit of $-\infty$ and Reimann's ([5], p.21) choice of the lower limit of c, were in fact related to the issue of initialization. Ross [8],[9] recognizes that to satisfy composition of the fractional differintegral, that the integrated function and its (integer order) derivatives must be zero for times up to and including the start of fractional differintegration. Podlubny ([7] pp. 125,133) recognizes the need for initialization but carries it no further. Ross provides a history of the fractional calculus (see [6], p8.) in which he quotes A. Cayley referring to Riemann's paper "The greatest difficulty in Riemann's theory, it appears to me, is the interpretation of the complementary function...." Ross continues, "The question of the existence of a complementary function caused much confusion. Liouville and Peacock were led into error, and Riemann became inextricably entangled in his concept of a complementary function." In retrospect the difficulties of Riemann over the role of the complementary function, which has been abandoned in this mathematics, may in fact have been related to the issue of initialization. The complementary function issue is raised here because an initialization function, which accounts for the effect of history, for fractional integrals and derivatives, will appear in the definitions presented. Its form is similar to Riemann's complementary function, however, the meaning and use of this function is different (now clear).

In the solution of fractional differential equations with an assumed history, it has been implicitly inferred by many authors ([1], [5], [6], [7], [10], and others), that an initializing constant, or set of constants, representing the value(s) of the fractional differintegrals (at $t = 0$) will provide an adequate representation for the effects of the past for each differintegral. That this is **not true** will be demonstrated in this paper.

Finally, because the constant initialization of the past is insufficiently general, the widely used contemporary equation for the Laplace transform for the differintegral ([6], p.135 for

example), based on that assumption also lacks sufficient generality. An alternative generalized form is presented.

1.1 Proof of Non-Constant Initialization

Consider the following qth order fractional integrals of $f(t)$, the first starting at time $t = a$, and the second starting at time $t = c > a$

$$_a d_t^{-q} f(t) = \frac{1}{\Gamma(q)} \int_a^t (t-\tau)^{q-1} f(\tau) d\tau \qquad (1.1.1)$$

and

$$_c d_t^{-q} f(t) = \frac{1}{\Gamma(q)} \int_c^t (t-\tau)^{q-1} f(\tau) d\tau. \qquad (1.1.2)$$

Assume that $f(t)$ is zero for all $t \le a$, then the time period between $t = a$ and $t = c$ maybe considered to be the "history" of the fractional integral starting at $t = c$, namely, $_c d_t^{-q} f(t)$. Then, we should expect that when this integral, $\left(_c d_t^{-q} f(t)\right)$, is properly initialized that it should function as a continuation of the integral starting at $t = a$. To achieve this an initialization must be addended to $_c d_t^{-q} f(t)$ so that the resulting fractional integration starting at $t = c$ should be identical to the result starting at $t = a$ for $t > c$. Thus, calling ψ the unknown initialization we have that

$$_c d_t^{-q} f(t) + \psi = {}_a d_t^{-q} f(t) \qquad t > c. \qquad (1.1.3)$$

Then

$$\psi = {}_a d_t^{-q} f(t) - {}_c d_t^{-q} f(t) \qquad t > c. \qquad (1.1.4)$$

$$\psi = \frac{1}{\Gamma(q)} \int_a^c (t-\tau)^{q-1} f(\tau) d\tau \equiv {}_a d_c^{-q} f(t) \qquad t > c. \qquad (1.1.5)$$

Here ψ is seen to be a function of the independent variable t, completing the proof. We see that ψ is a generalization of the case for the ordinary integral ($q = 1$), where

$$\psi = \int_a^c f(\tau) d\tau = \text{constant.}$$

The insight for this behavior was originally obtained through the study of one-dimensional semi-infinite diffusion and wave equations [2].

Having now recognized the need for a more general initialization, it must be decided if it is prudent to proceed as is done in the ordinary (integer order) calculus. That is, to append the initialization (constant of integration or constant initialization terms of ordinary differential equations) when required in an ad hoc manner or to formalize the process.

Because of the increased complexity of the initialization relative to the integer order calculus case it is prudent to formalize the initialization, that is to include an initialization term into the definitions for the fundamental operators. The remainder of this paper will establish the definition set for an initialized fractional calculus, consider briefly the 'Ross criteria" [8],[9] for a calculus, present a generalized (corrected) form for the Laplace transform of a differintegral, and demonstrate the solution of properly initialized fractional differential equations.

2. Definitions for the Initialized Fractional Calculus

Several bases are possible for the initialized fractional calculus, these include the Riemann-Liouville, and the Grünwald formulations. The Riemann-Liouville form of the fractional calculus will be the only basis considered here. A consideration of a Grünwald based initialized fractional calculus may be found in [3]. In the development that follows, attention is restricted to real values of the order, q, of the various differintegrals.

Two types of initializations are considered, 'terminal initialization", where it is assumed that the differintegral operator can only be initialized ("charged") by effectively differintegrating prior to the "start" time, $t = c$, and "side initialization", where a fully arbitrary initialization may be applied to the differintegral operator at time $t = c$. The terminologies "terminal charging" and "side charging" have also been used to describe these initialization processes [3]. For discussion purposes, it is assumed that t (time) is the independent variable associated with the fractional differintegration and the function to be differintegrated is $f(t)$.

2.1 Initialization of Fractional Integrals

2.1.1 Terminal Initialization

Terminal initialization is considered first. It is assumed that the fractional integration of interest "starts" at $t = c$ (i.e. point of initialization). Further, $f(t) = 0$ for all $t \leq a$, and the fractional integration takes place for $t > c \geq a$. The initialization period (or space) is defined as the region $a \leq t \leq c$.

The standard (contemporary) definition of a fractional integral will be accepted only when the differintegrand $f(t) = 0$ for all $t \leq a$

Then,

$$_aD_t^{-v} f(t) \equiv \frac{1}{\Gamma(v)} \int_a^t (t-\tau)^{v-1} f(\tau)d\tau, \qquad v \geq 0, t > a, \quad (2.1.1.1)$$

subject to $f(t) = 0$ for all $t \leq a$. The following definition of fractional integration will apply generally (i.e., at any $t > c$) :

$$_cD_t^{-v} f(t) \equiv \frac{1}{\Gamma(v)} \int_c^t (t-\tau)^{v-1} f(\tau)d\tau + \psi(f,-v,a,c,t), \quad v \geq 0, t > c, \quad (2.1.1.2)$$

$c \geq a$ and $f(t) = 0 \ \forall \ t \leq a$.

The function $\psi(f,-v,a,c,t)$ is called the initialization function and will be chosen such that

$$_aD_t^{-v} f(t) = {_cD_t^{-v}} f(t) \qquad t > c. \quad (2.1.1.3)$$

Substituting the results from equations 2.1.1.1 and 2.1.1.2, then, for $t > c$ and $v \geq 0$ gives

$$\frac{1}{\Gamma(v)} \int_a^t (t-\tau)^{v-1} f(\tau)d\tau = \frac{1}{\Gamma(v)} \int_c^t (t-\tau)^{v-1} f(\tau)d\tau + \psi(f,-v,a,c,t) \quad (2.1.1.4)$$

Because $\int_a^t g(\tau)d\tau = \int_a^c g(\tau)d\tau + \int_c^t g(\tau)d\tau$,

$$\psi(f,-v,a,c,t) = {_aD_c^{-v}} f(t) = \frac{1}{\Gamma(v)} \int_a^c (t-\tau)^{v-1} f(\tau)d\tau, \qquad t > c, \quad v > 0. \quad (2.1.1.5)$$

This expression for $\psi(t)$ applies for the terminal initialization condition. Clearly, ψ brings to the definition of the fractional integral the effect of the past, namely the effect of fractionally integrating $f(t)$ from a to c. This effect will, of course, influence behavior after the time, $t = c$. The ψ function has the effect of allowing the function $f(t)$ and its derivatives to start at a value other than zero, namely the value $_aD_c^{-q}f(t)\,\big|_{t=c}$, and the ψ-function continues to contribute to the differintegral response after $t = c$. That is, a function of time is added to the uninitialized integral, not just a constant.

The integer order integrals under terminal initialization are of special interest. Evaluating equation 2.1.1.5, for example, for $v = 1$, indicates that $\psi(f,-1,a,c,t)=$ constant. The general case is readily shown to be

$$\psi(f,-n,a,c,t)=\sum_{i=0}^{n}c_it^i\ , \qquad\qquad n=1,2,3\cdots\ .$$

This, of course, is the same effect as seen in the integer order calculus using the "constant of integration." It is important to note that the initialization of the qth fractional integral of $f(t)$ is not unique in the following sense. That is, $f(t)$ can be considered as a composite function, for example, $f(t)=g(t)U(t-c)+h(t)\big(U(t-a)-U(t-c)\big)$, where $U(t)$ is the unit step function

$$U(t)=\begin{cases}0 & t<0 \\ 1 & t>0\ .\end{cases}$$

Then for this composite function $f(t)$, it is the function $g(t)U(t-c)$ that is being differintegrated and $h(t)\big(U(t-a)-U(t-c)\big)$ is the function on which the initialization is based. This is analogous to choosing an arbitrary constant value to initialize (the integration of) dy/dt in the solution of an ordinary differential equation.

2.1.2 Side Initialization

When side initialization is in effect as opposed to terminal initialization equation 2.1.1.2 is still taken as the operative definition and,

$$\psi = \psi(t)(\text{i.e., is arbitrary}) \qquad\qquad\qquad (2.1.2.1)$$

That is, equation 2.1.1.5 no longer (generally) applies.

2.2 Initialization of Fractional Derivatives

To extend the definition to the fractional derivatives, some issues must be addressed. The definition of the fractional derivative raises the following important questions in the context of initialization. *Do fractional derivatives require an initializing function in general? Further, do integer order derivatives in this context require initialization functions?* Clearly as we commonly think of derivatives, in the integer order calculus, the derivative is a local property and is represented geometrically as the slope of the function being considered and as such it requires no initialization. In the solution of differential equations the initialization constants which set the initial values of the derivatives really have the effect of accounting for the integration of the derivative from minus infinity to the starting time of the integration (of the differential equation).

A study of the representation of semi-infinite systems using fractional differential equations [2],[3] indicates that the fractional derivative is not a local property as appears to be the case for

integer order derivatives (in the integer order calculus). Further, to solve such fractional differential equations an initialization function *is* required to handle the effect of the distributed initialization. Also, the integer order derivative in the fractional context can be shown to require an initialization function. Therefore, the answer to both questions is clearly YES. The impact of this is to require an initialization function for the fractional (and integer order) derivatives.

Thus, a generalized **integer** order differentiation is defined as

$$_cD_t^m f(t) \equiv \frac{d^m}{dt^m} f(t) + \psi(f,m,a,c,t) \qquad t > c, \qquad (2.2.1)$$

where m is a positive integer and where $\psi(f,m,a,c,t)$ is an initialization function. This is, of course, a generalization of the definition of the derivative, and for many cases, for example, usually in the integer order calculus, ψ will be taken to be zero. It will be shown later that, for $m = 1$ with the condition of terminal initialization that $\psi(f,1,a,c,t) = 0$.

Now the **uninitialized** fractional derivative is defined as

$$_aD_t^q f(t) \equiv {}_aD_t^m {}_aD_t^{-p} f(t) \qquad q \geq 0, t > a, \text{ and } f(t) = 0 \ \forall t \leq a, \qquad (2.2.2)$$

and (for convenience) m is the least integer greater than q, and $q = m - p$. Now as in the fractional integral case $\psi(f,-p,a,a,t) = 0$. Further, since $\psi(h,m,a,a,t) = 0$, where $h(t) = {}_aD_t^{-p} f(t)$, this definition specializes to the contemporary definition of the fractional derivative.

Now the **initialized** fractional derivative is defined as

$$_cD_t^q f(t) \equiv {}_cD_t^m f(t) {}_cD_t^p f(t) \qquad q \geq 0, \ t > c \geq a, \qquad (2.2.3)$$

where (for convenience) m is the least positive integer greater than q and $q = m - p$.
The above definitions, equations (2.2.1) to (2.2.3), hold for both terminal initialization as well as side initialization.

2.2.1 Terminal Initialization

The initialization of the fractional derivative proceeds in a manner similar to the fractional integral, that is, it shall be required that;

$$_cD_t^q f(t) = {}_aD_t^q f(t) \qquad \forall t > c \geq a \text{ with } q \geq 0. \qquad (2.2.1.1)$$

Specifically, this requires compatibility of the derivatives starting at $t = a$ and $t = c$, for $t > c$. It follows then that

$$_cD_t^m {}_cD_t^{-p} f(t) = {}_aD_t^m {}_aD_t^{-p} f(t), \qquad q > 0, \ t > c \geq a. \qquad (2.2.1.2)$$

Expanding the generalized integral terms

$$_cD_t^m \left(\frac{1}{\Gamma(p)} \int_c^t (t-\tau)^{p-1} f(\tau) d\tau + \psi(f,-p,a,c,t) \right) =$$

$$_aD_t^m \left(\frac{1}{\Gamma(p)} \int_a^t (t-\tau)^{p-1} f(\tau) d\tau + \psi(f,-p,a,a,t) \right), \qquad t > c. \qquad (2.2.1.3)$$

Since $\psi(f,-p,a,a,t) = 0$ and using the definition (eq.(2.2.1)) for the integer order derivative, equation (2.2.1.3) may be written as

$$\frac{d^m}{dt^m}\left\{\frac{1}{\Gamma(p)}\int_c^t(t-\tau)^{p-1}f(\tau)d\tau+\psi(f,-p,a,c,t)\right\}+\psi(h_1,m,a,c,t)=$$

$$\frac{d^m}{dt^m}\frac{1}{\Gamma(p)}\int_a^t(t-\tau)^{p-1}f(\tau)d\tau+\psi(h_2,m,a,a,t),\quad t>c,\qquad(2.2.1.4)$$

where $h_1={}_aD_t^{-p}f(t)$ and $h_2={}_aD_t^{-p}f(t)$. The integer derivative is uninitialized at $t=a$, therefore $\psi(h_2,m,a,a,t)=0$. Then combining integrals gives

$$\psi(h_1,m,a,c,t)=\frac{d^m}{dt^m}\left(\frac{1}{\Gamma(p)}\int_a^c(t-\tau)^{p-1}f(\tau)d\tau-\psi(f,-p,a,c,t)\right)\quad t>c.\qquad(2.2.1.5)$$

Under the condition of terminal initialization of the fractional integral, the argument of the derivative above is zero thus $\psi(h_1,m,a,c,t)=0$. For the case of side initialization of the fractional integral part of the fractional derivative $\psi(f,-p,a,c,t)$ is arbitrary. Thus it can be seen from the above equation that either $\psi(f,-p,a,c,t)$ or $\psi(h_1,m,a,c,t)$ can be arbitrary but not both while still satisfying the requirements of the initialization (equation (2.2.1.5)).

2.2.2 Side Initialization

The fractional derivative, side initialization case, can now be stated as

$$_cD_t^q f(t)={}_cD_t^m\left\{\frac{1}{\Gamma(p)}\int_c^t(t-\tau)^{p-1}f(\tau)d\tau+\psi(f,-p,a,c,t)\right\},\quad q\geq0,t>c,\qquad(2.2.2.1)$$

and m is the least positive integer $>q$ with $q=m-p$, or equivalently as

$$_cD_t^q f(t)=\frac{d^m}{dt^m}\frac{1}{\Gamma(p)}\int_c^t(t-\tau)^{p-1}f(\tau)d\tau+\frac{d^m}{dt^m}\psi(f,-p,a,c,t)+\psi(h,m,a,c,t),q\geq0,(2.2.2.2)$$

where m is as above, $t>c$, and $h(t)={}_aD_t^{-p}f(t)$. Here both initialization terms are arbitrary and thus may be considered as a single (arbitrary) term, namely

$$\psi(f,q,a,c,t)\equiv\frac{d^m}{dt^m}\psi(f,-p,a,c,t)+\psi(h,m,a,c,t).\qquad(2.2.2.3)$$

In the case of terminal initialization of the fractional integral part of equation (2.2.2.1) $\psi(f,-p,a,c,t)$ will be as defined by equation (2.1.1.5). It is noted, that the a and c arguments in the ψ functions in equations (2.2.2.1) to (2.2.2.3) are carried for parallelism to the previous cases and are not intended to infer an initialization period under side initialization.

2.3 A Simple Example

A simple example will be helpful. Consider the semi-integral of $f(t)=(t-a)U(t-a)$, then from ([6], pp. 63-64) the uninitialized semi-integral becomes

$$_aD_t^{-1/2}(t-a)U(t-a)=\frac{(t-a)^{3/2}}{\Gamma(2.5)},\qquad t>a.\qquad(2.3.1)$$

Now, initializing from the point $t=c$

$$_cD_t^{-1/2}(t-a)U(t-a) = \frac{1}{\Gamma(1/2)}\int_c^t (t-\tau)^{-1/2}(\tau-a)d\tau + \psi(f,-1/2,a,c,t) \quad t>c$$

$$= \frac{2}{3\Gamma(1/2)}\left((t-c)^{1/2}(2t+c-3a)\right) + \psi(f,-1/2,a,c,t) \quad t>c. \tag{2.3.2}$$

Consider now the terminal initialization,

$$\psi(f,-1/2,a,c,t) = \frac{1}{\Gamma(1/2)}\int_a^c (t-\tau)^{-1/2}(\tau-a)d\tau \quad t>c,$$

$$= \frac{2}{3\Gamma(1/2)}\left[(t-c)^{1/2}(-2t+3a-c)-(t-a)^{1/2}(-2t+2a)\right], \quad t>c. \tag{2.3.3}$$

The numerical evaluations of these equations for specific numerical values, $a=-1$, $c=1$, are shown in the graphs of figure 1.
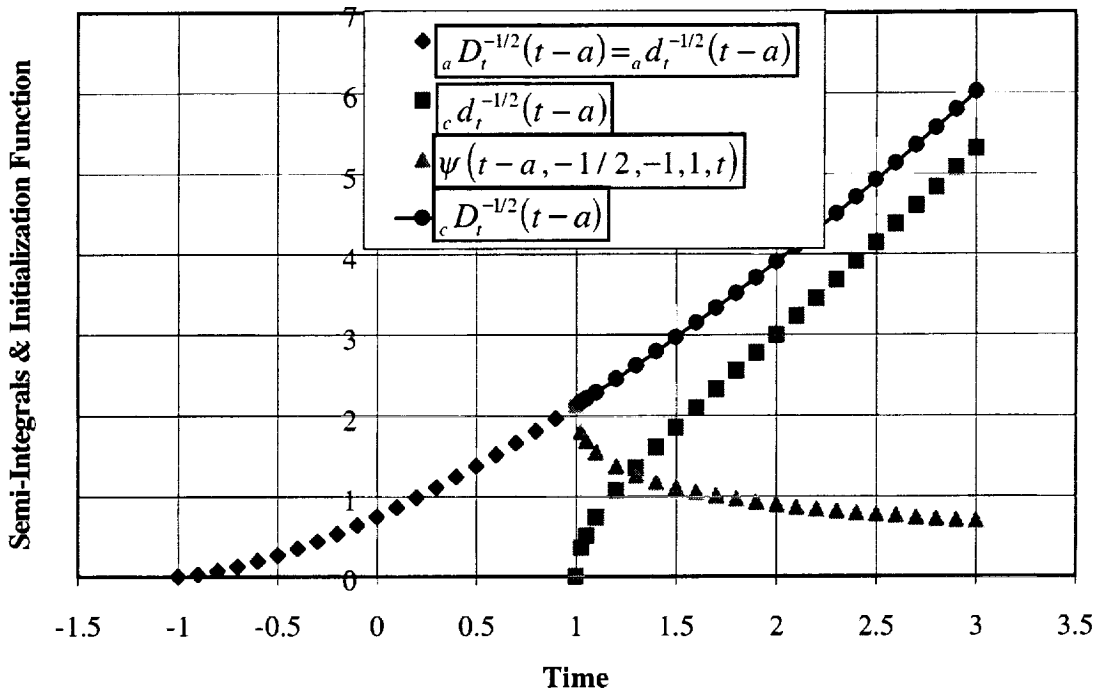

**Figure 1 Sample Problem---Semi-Integral of (t-a) vs Time with
a= -1, c=1**

3. Criteria for A Fractional Calculus

Ross [9] provides a set of criteria for a fractional calculus. The criteria include the following properties: backward compatibility with the normal (integer order) calculus, the zero property, namely $_cD_t^0 f(t) = f(t)$, linearity of the operators, and that the index law should hold, that is, $_cD_t^u {}_cD_t^v f(t) = {}_cD_t^v {}_cD_t^u f(t) = {}_cD_t^{u+v} f(t)$. Under the conditions of terminal initialization the above properties are each shown to hold ([3], pp. 26-44). This provides credibility to the

initialized fractional calculus, also completion of the proofs yields constraints on the allowable initialization functions. For example, linearity of the fractional integral

$$_cD_t^{-v}\left(b\,f(t)+k\,g(t)\right)=b\ _cD_t^{-v}f(t)+k\ _cD_t^{-v}g(t)\quad t>c,\tag{3.1}$$

holds if and only if

$$\psi\left(bf+kg,-v,a,c,t\right)=b\psi\left(f,-v,a,c,t\right)+k\psi\left(g,-v,a,c,t\right)\quad t>c.\tag{3.2}$$

4. Laplace Transform of Fractional Differintegrals

The Laplace transform of the differintegral is an important tool in the solution of fractional differential equations. The following form for the Laplace transform of the fractional derivative is given by many authors (see for example, ([7], p.105), ([6], p.134), or an equivalent form in ([5], p.123))

$$L\left\{_0D_t^p f(t)\right\}=s^p F(s)-\sum_{k=0}^{n-1}s^k\ _0D_t^{p-k-1}f(t)\Big|_{t=0}\qquad n-1\le p<n.\tag{4.1}$$

The Laplace transform of the fractional integral is given ([7], p.104), and ([6], p.134) as,

$$L\left\{_0D_t^{-p} f(t)\right\}=s^{-p} F(s)\qquad\qquad p>0.\tag{4.2}$$

The form of the summation in equation (4.1) infers that the p- th order derivative is being decomposed into a fractional integral (order < 1) and (n-1) order 1 derivatives. Further inferred is that each derivative is initialized by an impulse at $t=0$. This situation is a residual from the integer order calculus and lacks sufficient generality for a properly initialized fractional calculus. It should be noted that in equations (4.1) and (4.2) above D^p and D^{-p} refer to the contemporary uninitialized fractional derivative and integral respectively.

For the initialized fractional calculus the Laplace transform of the initialized fractional differintegral ([3], p.61) is given by

$$L\left\{_0D_t^q f(t)\right\}=s^q L\left\{f(t)\right\}+L\left\{\psi(f,q,a,0,t)\right\},\qquad\forall q,\tag{4.3}$$

where ψ depends on; the assumed past history of $_0D_t^q f(t)$, and the assumed decomposition of $_0D_t^q f(t)$ (as required in the problem definition). It should be noted that for $q>0$ in the most general case (side initialization) that the equivalent form for ψ must be used. That is

$$\psi(f,q,a,0,t)=\frac{d^m}{d\,t^m}\psi(f,-p,a,0,t)+\psi(h,m,a,0,t)\quad q>0,\quad\text{where }h(t)=_a d_t^{-p}f(t).\tag{4.4}$$

More powerful forms than equation (4.3) have been derived that account for all possible decompositions of $_0D_t^q f(t)$. For a detailed explanation of the decomposition issue the reader is referred to ([3], p.46-58).

5. Fractional Differential Equations

Proper initialization is crucial in the solution and understanding of fractional differential equations. The application of the initialized fractional calculus to the solution of initialized fractional differential equations will be illustrated with the following examples.

5.1 Example 1

Podlubny ([7], p. 138) and Oldham and Spanier ([6], p. 157) consider the following fractional differential equation

$$_0D_t^{1/2}f(t)+b\,f(t)=0, \quad t>0; \qquad _0D_t^{-1/2}f(t)\big|_{t=0}=C.$$ (5.1.1)

The notation $_0D_t^{1/2}f(t)$ is that of Podlubny, and of course, refers to the uninitialized derivative. Equation (4.1) above, is used by Podluny, to obtain the Laplace transform of equation (5.1.1) as

$$F(s)=\frac{C}{s^{1/2}+b}.$$ (5.1.2)

The inverse transform is then given in terms of a two-parameter Mittag-Leffler series expansion

$$f(t)=Ct^{-1/2}E_{\frac{1}{2},\frac{1}{2}}\left(-b\sqrt{t}\right).$$ (5.1.3)

For $b=1$, then this is determined to be

$$f(t)=C\left(\frac{1}{\sqrt{\pi t}}-e^t\,erfc\left(\sqrt{t}\right)\right),$$ (5.1.4)

which agrees with the result of Oldham and Spanier [6]. This is contrasted with the following approach using the results from the initialized fractional calculus.

We now solve equation (5.1.1) again, but now $_0D_t^{1/2}f(t)$ is interpreted as an initialized fractional derivative as defined in part 2.2 above. Thus, we have

$$_0D_t^{1/2}f(t)+bf(t)=0, \quad t>0, \qquad \psi(f,1/2,a,0,t) \text{ is arbitrary.}$$ (5.1.5)

This may be rewritten as

$$_0d_t^{1/2}f(t)+\psi(f,1/2,a,0,t)+bf(t)=0, \quad t>0, \qquad \psi(f,1/2,a,0,t)\text{ is arbitrary.}$$ (5.1.6)

The Laplace transform of equation (5.1.5) using equation (4.3) is

$$F(s)=\frac{-\psi(f,1/2,a,0,s)}{s^{1/2}+b}=\frac{-\psi(s)}{s^{1/2}+b}.$$ (5.1.7)

This equation should be contrasted to equation (5.1.2) above, they are only the same when $\psi(t)=-C\delta(t)$, that is, when an impulse at $t=0$ is used to initialize the fractional differential equation!

Now the R-function and its Laplace transform [4], are given by

$$R_{q,v}(\alpha,c,t)\equiv\sum_{n=0}^{\infty}\frac{(\alpha)^n(t-c)^{(n+1)q-v-1}}{\Gamma((n+1)q-v)}\Leftrightarrow\frac{s^v}{s^q-\alpha}, \quad \text{Re}(q-v)>0,\ \text{Re}(s)>0.$$ (5.1.8)

The general inverse for equation (5.1.7), is obtained by applying Laplace convolution integral,

$$f(t)=-\int_0^t R_{1/2,0}(-b,0,t-\tau)\psi(\tau)d\tau \quad t>0.$$ (5.1.9)

Thus, with arbitrary $\psi(t)$ this provides the *most* general solution to equation (5.1.1), or (5.1.5). If we take $\psi(t)=-C\delta(t)$ in equation (5.1.9), the result is

$$f(t)=CR_{1/2,0}(-b,0,t),$$ (5.1.10)

which is identical with the result of equation (5.1.2). In the context of the initialized fractional calculus, this might be considered as a pathological result of little interest. A more useful result

would be obtained under the assumption of terminal initialization, that is apply equation (4.4) with $q = m - p$, $m = 1$, $p = 1/2$, and $\psi(h,m,a,0,t) = 0$. Then

$$\psi(f,1/2,a,0,t) = \frac{d}{dt}\psi(f,-1/2,a,0,t) = \frac{d}{dt}\frac{1}{\Gamma(1/2)}\int_a^0 (t-\tau)^{-1/2} f(\tau)d\tau. \qquad (5.1.11)$$

In this evaluation ($f(t)$, $t < 0$) need not be identical to ($f(t)$, $t > 0$), if $f(t)$ is considered to be a composite function.

5.2 Example 2

A second fractional differential equation is considered by Podlubny ([7], p. 139), and has a special case considered by Oldham and Spanier ([6], p.159 $Q = 1$, $q = 1/2$). The general case, of which it is said "encounters great difficulties except when $q - Q$ is integer or half-integer," is given by

$$_0D_t^Q f(t) + {}_0D_t^q f(t) = h(t), \qquad C = \left[{}_0D_t^{Q-1} f(t) + {}_0D_t^{q-1} f(t)\right]_{t=0}, \qquad (5.2.1)$$

Again, the notation $_0D_t^Q f(t)$ is that of Podlubny, and refers to the uninitialized derivative. Here, we shall assume the notation is that of this paper, that is represents the initialized fractional derivative, and we shall relax the requirement that $q - Q$ is integer or half-integer. The initialization of equation (5.2.1) will be replaced by two separate ones to identify the most general solution. Then the Laplace transform of equation (5.2.1) gives

$$s^Q F(s) + s^q F(s) = H(s) - \psi_1(f,Q,a,0,s) - \psi_2(f,q,a,0,s), \qquad (5.2.2)$$

where the ψ's are subscripted for convenience. This may be written, assuming that $Q > q$, as

$$F(s) = \left(\frac{s^{-q}}{s^{Q-q}+1}\right)\left(H(s) - \psi_1(f,Q,a,0,s) - \psi_2(f,q,a,0,s)\right). \qquad (5.2.3)$$

The solution is easily written using the R-function as,

$$f(t) = \int_0^t R_{Q-q,-q}(-1,0,t-\tau)\left(h(\tau) - \psi_1(f,Q,a,0,\tau) - \psi_2(f,q,a,0,\tau)\right)d\tau. \qquad (5.2.4)$$

This solution may be specialized to that of Podlubny by taking $\psi_1 = -C_1\delta(t)$, $\psi_2 = -C_2\delta(t)$ and $C = C_1 + C_2$, namely

$$f(t) = -CR_{Q-q,-q}(-1,0,t) + \int_0^t R_{Q-q,-q}(-1,0,t-\tau)\left(h(\tau)\right)d\tau. \qquad (5.2.5)$$

The important issue here is that the form of equation (5.2.5) does not allow the effect of continuing the past as does equation (5.2.4).

5.3 Example 3

In this example, an approach will be demonstrated in which the entire fractional differential equation is initialized as opposed to the above examples where the individual derivatives were initialized. We now generalize equation (5.2.1) as follows

$$_cD_t^Q f(t) + {}_cD_t^q f(t) = h(t). \qquad (5.3.1)$$

This may be rewritten as

$$_cd_t^Q f(t) + {}_cd_t^q f(t) = h(t) - \psi_1(f,Q,a,c,t) - \psi_2(f,q,a,c,t) = h(t) - \psi_{eqiv}(t). \qquad (5.3.2)$$

Now the Laplace transform of the uninitialized fractional derivative is ([3], p.60)

$$L\{{}_bd_t^q f(t)\} = L\{{}_0d_t^q(u(t-b)f(t))\} = e^{-bs}s^q L\{f(t+b)\}. \qquad (5.3.3)$$

Thus, the Laplace transform of equation (5.3.2), for $Q > q$, is given by

$$L\{f(t+c)\} = \frac{h(s) - \psi_{eqiv}(s)}{e^{-cs}s^q(s^{Q-q}+1)} = e^{cs}G(s)(h(s) - \psi_{eqiv}(s)) \qquad (5.3.4)$$

Clearly the behavior of the fractional differential equation (or the system it represents) is captured by the function $G(s) = 1/(s^q(s^{Q-q}+1))$. We now consider two time domain segments, for domain (1) we take $a_1 = c_1 = 0$, and for domain (2) $a_2 = 0$, $c_2 = 1$.

Domain (1)

Now domain (1) will be used as the initializing period for domain (2). To do this let $h_1(t) = h^*(t)(u(t) - u(t-1))$, for simplicity here we take $h^*(t) = 1$. We also take $f(t) = 0$, $\forall\ t < 0$ thereby inferring $\psi_1(f, Q, 0, 0, t) = \psi_2(f, q, 0, 0, t) = 0$. The domain (1) solution then is given by

$$L\{f_1(t)\} = \frac{(1 - e^{-s})}{s^{q+1}(s^{Q-q}+1)} = \left(\frac{1 - e^{-s}}{s}\right)G(s), \qquad (5.3.5)$$

and

$$f_1(t) = R_{Q-q,-q-1}(-1, 0, t) - u(t-1)R_{Q-q,-q-1}(-1, 0, t-1) \qquad t > 0. \qquad (5.3.6)$$

For $0 < t < 1$, then $f_1(t) = R_{Q-q,-q-1}(-1, 0, t)$.

Domain (2)

Now for domain (2), $t > 1$, we consider the initialization period to be, domain (1) above, $0 < t \le 1$, therefore, $a_2 = 0$, $c_2 = 1$, and again we take $f(t) = 0$, $\forall\ t < 0$. Further, for clarity, we only consider the unforced problem, that is we take $h_2(t) = 0$, therefore from equation (5.3.2) we have

$$_1d_t^Q f_2(t) + _1d_t^q f_2(t) = -\psi_1(f_2, Q, 0, 1, t) - \psi_2(f_2, q, 0, 1, t) = -\psi_{eqiv}(t). \qquad (5.3.7)$$

The Laplace transform, using equation (5.3.3), is given by

$$L\{f_2(t+1)\} = \frac{-\psi_{eqiv}(s)}{e^{-s}s^q(s^{Q-q}+1)} . \qquad (5.3.8)$$

The initialization $\psi_{eqiv}(t)$ is now chosen based on $h_1(t)$, $(0 < t \le 1)$, that is the historic forcing function, thus

$$\psi_{eqiv}(t) = -h_1(t) = -(u(t) - u(t-1)), \qquad (5.3.9)$$

hence

$$\psi_{eqiv}(s) = -\left(\frac{1 - e^{-s}}{s}\right). \qquad (5.3.10)$$

Substituting this result into equation (5.3.8)

$$e^{-s}L\{f_2(t+1)\} = \frac{1 - e^{-s}}{s^{q+1}(s^{Q-q}+1)} = \frac{1 - e^{-s}}{s}G(s) = L\{f_1(t)\}. \qquad (5.3.11)$$

Applying the Laplace shifting theorem, we have

$$L\{f_2(t)u(t-1)\} = L\{f_1(t)\}. \qquad (5.3.12)$$

Thus for $t > 1$ we have the important result $f_2(t) = f_1(t)$ $t > 1$, as would be expected from a proper initialization theory. It is not possible to obtain such results from the contemporary theory referenced earlier.

Summary

This paper proves the need for an initialized fractional calculus. The paper presents the definition sets required for initialized fractional calculi. Two underlying bases have been used, the Riemann-Liouville based fractional calculus and the Grünwald based fractional calculus (by reference).

The significant result is that when fractionally differintegrating with respect to t that a function of t is required as an initialization as opposed to the constant initialization used in the integer order calculus.

Two types of initialization are introduced "terminal initialization" and "side initialization". Proofs of Ross' criteria and the initialization constraints that ensue from the criteria have been referenced. Corrected forms for the Laplace transforms for fractional differintegals that properly account for the initialization function have been presented. Commonly studied fractional differential equations have been solved to demonstrate the various aspects of initialization.

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This paper demonstrates the need for a nonconstant initialization for the fractional calculus and establishes a basic definition set for the initialized fractional differintegral. This definition set allows the formalization of an initialized fractional calculus. Two basis calculi are considered; the Riemann-Liouville and the Grünwald fractional calculi. Two forms of initialization, terminal and side are developed.

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