NEW DEVELOPMENTS IN QUATERNION ESTIMATION FROM VECTOR OBSERVATIONS

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ABSTRACT

This paper contains a critical comparison of estimators minimizing Wahba's loss function. Some new results are presented for the QUaternion ESTimator (QUEST) and EStimators of the Optimal Quaternion (ESOQ and ESOQ2) to avoid the computational burden of sequential rotations in these algorithms. None of these methods is as robust in principle as Davenport's q method or the Singular Value Decomposition (SVD) method, which are significantly slower. Robustness is only an issue for measurements with widely differing accuracies, so the fastest estimators, the modified ESOQ and ESOQ2, are well suited to sensors that track multiple stars with comparable accuracies. More robust forms of ESOQ and ESOQ2 are developed that are intermediate in speed.

INTRODUCTION

In many spacecraft attitude systems, the attitude observations are naturally represented as unit vectors. Typical examples are the unit vectors giving the direction to the sun or a star and the unit vector in the direction of the Earth's magnetic field. This paper will consider algorithms for estimating spacecraft attitude from vector measurements taken at a single time, which are known as "single-frame" methods or "point" methods, in contrast to filtering methods that employ information about spacecraft dynamics. Almost all single-frame algorithms are based on a problem proposed in 1965 by Grace Wahba[1]. Wahba's problem is to find the orthogonal matrix A with determinant +1 that minimizes the loss function

$$L(A) \equiv \tfrac{1}{2}\sum_i a_i \left|\mathbf{b}_i - A\mathbf{r}_i\right|^2 . \qquad (1)$$

where $\{\mathbf{b}_i\}$ is a set of unit vectors measured in a spacecraft's body frame, $\{\mathbf{r}_i\}$ are the corresponding unit vectors in a reference frame, and $\{a_i\}$ are non-negative weights. In this paper we choose the weights to be inverse variances, $a_i = \sigma_i^{-2}$, in order to relate Wahba's problem to Maximum Likelihood Estimation[2]. This choice differs from that of Wahba and many other authors, who assumed the weights normalized to unity.

It is possible and has proven very convenient to write the loss function as

$$L(A) = \lambda_0 - \mathrm{tr}(AB^{\mathrm{T}}) \qquad (2)$$

with

$$\lambda_0 \equiv \sum_i a_i \qquad (3)$$

and

$$B \equiv \sum_i a_i \mathbf{b}_i \mathbf{r}_i^{\mathrm{T}} . \qquad (4)$$

Now it is clear that $L(A)$ is minimized when the trace, $\mathrm{tr}(AB^{\mathrm{T}})$, is maximized.

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This has a close relation to the orthogonal Procrustes problem, which is to find the orthogonal matrix A that is closest to B in the sense of the Frobenius (or Euclidean, or Schur, or Hilbert-Schmidt) norm[3]

$$\|M\|_F^2 \equiv \sum_{i,j} M_{ij}^2 = \mathrm{tr}(MM^T). \tag{5}$$

Now

$$\|A-B\|_F^2 = \|A\|_F^2 + \|B\|_F^2 - 2\,\mathrm{tr}(AB^T) = 3 + \|B\|_F^2 - 2\,\mathrm{tr}(AB^T), \tag{6}$$

so Wahba's problem is equivalent to the orthogonal Procrustes problem with the proviso that the determinant of A must be $+1$.

The purpose of this paper is to give an overview in a unified notation of algorithms for solving Wahba's problem, to provide accuracy and speed comparisons, and to present two significant enhancements of existing methods. The popular QUaternion EStimator (QUEST) and EStimators of the Optimal Quaternion (ESOQ and ESOQ2) algorithms avoid singularities by employing a rotated reference system. A method introduced in this paper uses the diagonal elements of the B matrix to determine a desirable reference system, avoiding expensive sequential computations. Also, tests show that a first-order expansion in the loss function is adequate, avoiding the need for iterative refinement of the loss function, and motivating the introduction of new first-order versions of ESOQ and ESOQ2, which are at present the fastest known first-order methods for solving Wahba's problem.

FIRST SOLUTIONS OF WAHBA'S PROBLEM

J. L. Farrell and J. C. Stuelpnagel[4], R. H. Wessner[5], J. R. Velman[6], J. E. Brock[7], R. Desjardins, and Wahba presented the first solutions of Wahba's problem. Farrell and Stuelpnagel noted that any real square matrix, including B, has the polar decomposition

$$B = WR, \tag{7}$$

where W is orthogonal and R is symmetric and positive semidefinite. Then R can be diagonalized by

$$R = VDV^T, \tag{8}$$

where V is orthogonal and D is diagonal with elements arranged in decreasing order. The optimal attitude estimate is then given by

$$A_{opt} = WV\,\mathrm{diag}[1 \;\; 1 \;\; \det W]\,V^T. \tag{9}$$

In most cases, $\det W$ is positive and $A_{opt} = W$, but this is not guaranteed. Wessner proposed the solution

$$A_{opt} = (B^T)^{-1}(B^TB)^{1/2} = B(B^TB)^{-1/2}. \tag{10}$$

This has the disadvantage of requiring B to be non-singular, which means that a minimum of three vectors must be observed, although it is well known that two vectors are sufficient to determine the attitude.

SINGULAR VALUE DECOMPOSITION (SVD) METHOD

This method has not been widely used in practice, because of its computational expense, but it yields valuable analytic insights[8,9]. The matrix B has the Singular Value Decomposition[3]:

$$B = U\Sigma V^T = U\,\mathrm{diag}[\Sigma_{11} \;\; \Sigma_{22} \;\; \Sigma_{33}]\,V^T, \tag{11}$$

where U and V are orthogonal, and the singular values obey the inequalities $\Sigma_{11} \ge \Sigma_{22} \ge \Sigma_{33} \ge 0$. Then

$$\mathrm{tr}(AB^T) = \mathrm{tr}(A\,V\,\mathrm{diag}[\Sigma_{11} \;\; \Sigma_{22} \;\; \Sigma_{33}]\,U^T) = \mathrm{tr}(U^TA\,V\,\mathrm{diag}[\Sigma_{11} \;\; \Sigma_{22} \;\; \Sigma_{33}]). \tag{12}$$

The trace is maximized, consistent with the constraint $\det A = 1$, by

$$U^TA_{opt}\,V = \mathrm{diag}[1 \;\; 1 \;\; (\det U)(\det V)], \tag{13}$$

which gives the optimal attitude matrix

$$A_{opt} = U\,\mathrm{diag}[1 \;\; 1 \;\; (\det U)(\det V)]\,V^T. \tag{14}$$

The SVD solution is completely equivalent to the original solution by Farrell and Stuelpnagel, since Eq. (14) is identical to Eq. (9) with $U = WV$. The difference is that robust SVD algorithms exist now[3,10]. In fact, computing the SVD is one of the most robust numerical algorithms.

It is convenient to define

$$s_1 \equiv \Sigma_{11},\ s_2 \equiv \Sigma_{22},\ \text{and}\ s_3 \equiv (\det U)(\det V)\,\Sigma_{33}, \tag{15}$$

so that $s_1 \geq s_2 \geq |s_3|$. We will loosely refer to s_1, s_2, and s_3 as the singular values, although the third singular value of B is actually $|s_3|$. It is clear from Eq. (11) that redefinition of the basis vectors in the reference or body frame affects V or U, respectively, but does not affect the singular values.

The covariance of the rotation angle error vector in the body frame is given by

$$P = U\,\mathrm{diag}[(s_2+s_3)^{-1}\ \ (s_3+s_1)^{-1}\ \ (s_1+s_2)^{-1}]U^{\mathrm{T}}. \tag{16}$$

DAVENPORT'S q METHOD

Davenport provided the first useful solution of Wahba's problem for spacecraft attitude determination[11,12]. He parameterized the attitude matrix by a unit quaternion[13,14]

$$q = \begin{bmatrix} \mathbf{q} \\ q_4 \end{bmatrix} = \begin{bmatrix} \mathbf{e}\sin(\phi/2) \\ \cos(\phi/2) \end{bmatrix}, \tag{17}$$

as

$$A(q) = (q_4^2 - |\mathbf{q}|^2)I + 2\mathbf{q}\mathbf{q}^{\mathrm{T}} - 2q_4[\mathbf{q}\times]\ . \tag{18}$$

The rotation axis $\mathbf{e}$ and angle ϕ will be useful later. Since $A(q)$ is a homogenous quadratic function of q, we can write

$$\mathrm{tr}(AB^{\mathrm{T}}) = q^{\mathrm{T}}Kq \tag{19}$$

where K is the symmetric traceless matrix

$$K \equiv \begin{bmatrix} S - I\,\mathrm{tr}B & \mathbf{z} \\ \mathbf{z}^{\mathrm{T}} & \mathrm{tr}B \end{bmatrix} \tag{20}$$

with

$$S \equiv B + B^{\mathrm{T}} \qquad \text{and} \qquad \mathbf{z} \equiv \begin{bmatrix} B_{23} - B_{32} \\ B_{31} - B_{13} \\ B_{12} - B_{21} \end{bmatrix} = \textstyle\sum_i a_i \mathbf{b}_i \times \mathbf{r}_i\,. \tag{21}$$

It is easy to prove that the optimal unit quaternion is the normalized eigenvector of K with the largest eigenvalue, *i.e.*, the solution of

$$Kq_{\mathrm{opt}} = \lambda_{\max} q_{\mathrm{opt}}\,. \tag{22}$$

With Eqs. (2) and (19), this gives the optimized loss function as

$$L(A_{\mathrm{opt}}) = \lambda_{\mathrm{o}} - \lambda_{\max}\,. \tag{23}$$

Very robust algorithms exist to solve the symmetric eigenvalue problem[3,10].

The eigenvalues of the K matrix, $\lambda_{\max} \equiv \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4 \equiv \lambda_{\min}$, are related to the singular values by[11]

$$\lambda_1 = s_1 + s_2 + s_3, \quad \lambda_2 = s_1 - s_2 - s_3, \quad \lambda_3 = -s_1 + s_2 - s_3, \quad \lambda_4 = -s_1 - s_2 + s_3. \tag{24}$$

The eigenvalues sum to zero because K is traceless. There is no unique solution if the two largest eigenvalues of K are equal, or $s_2 + s_3 = 0$. This is not a failure of the q method; it means that the data aren't sufficient to determine the attitude uniquely. Equation (16) shows that the covariance is infinite in this case. This is expected, since the covariance should be infinite when the attitude is unobservable.

QUATERNION ESTIMATOR (QUEST)

This algorithm, first applied in the MAGSAT mission in 1979, has been the most widely used algorithm for Wahba's problem[15,16]. Equation (22) is equivalent to the two equations

$$[(\lambda_{\max} + \text{tr}B)I - S]\mathbf{q} = q_4\mathbf{z} \tag{25}$$

and

$$(\lambda_{\max} - \text{tr}\,B)q_4 = \mathbf{q}^{\mathrm{T}}\mathbf{z}. \tag{26}$$

Equation (25) gives

$$\mathbf{q} = q_4[(\lambda_{\max} + \text{tr}B)I - S]^{-1}\mathbf{z} = q_4\{\text{adj}[(\lambda_{\max} + \text{tr}B)I - S]\mathbf{z}\}/\det[(\lambda_{\max} + \text{tr}B)I - S]. \tag{27}$$

The optimal quaternion is then given by

$$q_{\text{opt}} = \frac{1}{\sqrt{\gamma^2 + |\mathbf{x}|^2}}\begin{bmatrix}\mathbf{x}\\ \gamma\end{bmatrix}, \tag{28}$$

where

$$\mathbf{x} \equiv \text{adj}[(\lambda_{\max} + \text{tr}\,B)I - S]\mathbf{z} = [\alpha I + (\lambda_{\max} - \text{tr}\,B)S + S^2]\mathbf{z} \tag{29}$$

and

$$\gamma \equiv \det[(\lambda_{\max} + \text{tr}B)I - S] = \alpha(\lambda_{\max} + \text{tr}\,B) - \det S, \tag{30}$$

with

$$\alpha \equiv \lambda_{\max}^2 - (\text{tr}\,B)^2 + \text{tr}(\text{adj}S). \tag{31}$$

The second form on the right sides of Eqs. (29) and (30) follows from the Cayley-Hamilton Theorem[3,16]. These computations require knowledge of $\lambda_{\max}$, which is obtained by substituting Eqs. (28) and (29) into Eq. (26), yielding:

$$0 = \psi(\lambda_{\max}) \equiv \gamma(\lambda_{\max} - \text{tr}B) - \mathbf{z}^{\mathrm{T}}[\alpha I + (\lambda_{\max} - \text{tr}\,B)S + S^2]\mathbf{z}. \tag{32}$$

Substituting α and γ from Eqs. (30) and (31) gives a fourth-order equation in $\lambda_{\max}$, which is simply the characteristic equation $\det(\lambda_{\max}I - K) = 0$. Shuster observed that $\lambda_{\max}$ can be easily obtained by Newton-Raphson iteration of Eq. (32) starting from λ_0 as the initial estimate, since Eq. (23) shows that $\lambda_{\max}$ is very close to λ_0 if the optimized loss function is small[15,16]. In fact, a single iteration is generally sufficient. But numerical analysts know that solving the characteristic equation is one of the *worst* ways to find eigenvalues, in general, so QUEST is in principle less robust than Davenport's original q method. The analytic solution of the characteristic equation is slower and no more accurate than the iterative solution.

Shuster provided an estimate of the covariance of the rotation angle error vector in the body frame,

$$P = \left[\sum\nolimits_i a_i(I - \mathbf{b}_i\mathbf{b}_i^{\mathrm{T}})\right]^{-1}. \tag{33}$$

He also showed that $2L(A_{\text{opt}})$ obeys a χ^2 probability distribution with $2n_{\text{obs}} - 3$ degrees of freedom, to a good approximation and assuming Gaussian measurement errors, where n_{obs} is the number of vector observations[17]. This can often provide a useful data quality check.

The optimal quaternion is not defined by Eq. (28) if $\gamma^2 + |\mathbf{x}|^2 = 0$. Applying the Cayley-Hamilton theorem twice to eliminate S^4 and S^3 after substituting Eq. (29) gives, with some tedious algebra,

$$\gamma^2 + |\mathbf{x}|^2 = \gamma(\mathrm{d}\psi/\mathrm{d}\lambda), \tag{34}$$

where $\psi(\lambda)$ is the quartic function defined implicitly by Eq. (32). It follows from the discussion following Eq. (15) that $\mathrm{d}\psi/\mathrm{d}\lambda$ is invariant under rotations, since the coefficients in the polynomial $\psi(\lambda)$ depend only on the singular values of B, and $\mathrm{d}\psi/\mathrm{d}\lambda$ must be nonzero for the Newton-Raphson iteration for $\lambda_{\max}$ to be successful[20]. The singular condition $\gamma^2 + |\mathbf{x}|^2 = 0$ is thus seen to be equivalent to $\gamma = 0$, which means that $(q_{\text{opt}})_4 = 0$ and the optimal attitude represents a 180° rotation. Shuster devised the method of sequential rotations to avoid this singular case[15-18].

REFERENCE FRAME ROTATIONS

The $(q_{opt})_4 = 0$ singularity occurs because QUEST does not treat the four components of the quaternion on an equal basis. Davenport's q method treats the four components symmetrically and is nonsingular, but some other methods have singularities similar to that in QUEST. These singularities can be avoided by solving for the attitude with respect to a reference coordinate frame rotated from the original frame by 180° about the x, y, or z coordinate axis. That is, we solve for one of the quaternions

$$q^i \equiv q \otimes \begin{bmatrix} \mathbf{e}_i \\ 0 \end{bmatrix} = \begin{bmatrix} \mathbf{q} \\ q_4 \end{bmatrix} \otimes \begin{bmatrix} \mathbf{e}_i \\ 0 \end{bmatrix} = \begin{bmatrix} q_4 \mathbf{e}_i - \mathbf{q} \times \mathbf{e}_i \\ -\mathbf{q} \cdot \mathbf{e}_i \end{bmatrix} \qquad \text{for } i = 1, 2, 3, \tag{35}$$

where $\mathbf{e}_i$ is the unit vector along the i^{th} coordinate axis. We use the convention of Reference 14 for quaternion rotations, rather than the historic convention. These products are trivial to implement by merely permuting and changing signs of the of the quaternion components. For example,

$$q^1 = [q_1, q_2, q_3, q_4]^{\text{T}} \otimes [1, 0, 0, 0]^{\text{T}} = [q_4, -q_3, q_2, -q_1]^{\text{T}}. \tag{36}$$

The equations for the inverse transformations are the same, since a 180° rotation in the opposite direction has the same effect. These rotations are also easy to implement on the input data, since a rotation about axis i, for example, simply changes the signs of the jth and kth columns of the B matrix, where $\{i, j, k\}$ is a permutation of $\{1, 2, 3\}$.

The original QUEST implementation performed sequential rotations, one axis at a time, until an acceptable reference coordinate system was found. It is clearly preferable to avoid the computations required by sequential rotations by choosing a single desirable rotation as early in the computation as possible. The optimal rotated coordinate frame could be found by investigating the components of an *a priori* quaternion, which is always available in a star tracker application, since a fairly good *a priori* attitude estimate is needed to identify the stars in the tracker's field of view. If the fourth component has the largest magnitude, no rotation is performed, while a rotation about the ith axis is performed if the ith component has the largest magnitude. Then Eq. (36) or its equivalent shows that the fourth component of the rotated quaternion will have the largest magnitude. This magnitude must be at least 1/2, but may not be any greater, because it is possible for all the components of a unit quaternion to have magnitude 1/2.

The need for a priori attitude information can be avoided by substituting information from the B matrix. We can expand K in terms of its eigenvectors and eigenvalues as

$$K = \sum_{\mu=1}^{h} \lambda_\mu q_\mu q_\mu^{\text{T}} = \tfrac{1}{4} \sum_{\mu=1}^{h} \lambda_\mu (4 q_\mu q_\mu^{\text{T}} - I) = \tfrac{1}{4} \sum_{\mu=1}^{4} \lambda_\mu \begin{bmatrix} A(q_\mu) + A^{\text{T}}(q_\mu) - I \operatorname{tr} A(q_\mu) & \mathbf{w}_\mu \\ \mathbf{w}_\mu^{\text{T}} & \operatorname{tr} A(q_\mu) \end{bmatrix}, \tag{37}$$

where q_μ is the normalized eigenvector of K corresponding to the eigenvalue λ_μ and

$$\mathbf{w}_\mu \equiv \begin{bmatrix} A(q_\mu)_{23} - A(q_\mu)_{32} \\ A(q_\mu)_{31} - A(q_\mu)_{13} \\ A(q_\mu)_{12} - A(q_\mu)_{21} \end{bmatrix}. \tag{38}$$

The second step in Eq. (37) follows from the fact that the eigenvalues sum to zero, and the third step follows from Eq. (18). We use Greek indices to label different quaternions, to avoid confusion with Latin indices used to label quaternion components. Comparison of Eqs. (37) and (20) establishes the identity[19]

$$B = \tfrac{1}{4} \sum_{\mu=1}^{4} \lambda_\mu A(q_\mu). \tag{39}$$

From Eqs. (18) and (39), we have

$$\operatorname{tr} B = \sum_{\mu=1}^{4} \lambda_\mu (q_\mu)_4^2, \tag{40}$$

and

$$2B_{kk} = \sum_{\mu=1}^{4} \lambda_\mu (q_\mu)_k^2 + \operatorname{tr} B. \tag{41}$$

These relations are interesting because the discussion following Eq. (15) makes it clear that the eigenvalues are not affected by a redefinition of the basis vectors in the body or reference frames, although the quaternion is affected by such a redefinition. Suppose now that we find the maximum of $\{B_{11}, B_{22}, B_{33}, \operatorname{tr} B\}$. If tr$B$ is the maximum, and if λ_{max} is well separated from the other eigenvalues of the K matrix, Eqs. (40) and (41) suggest that the fourth component of q_{opt} has the largest magnitude, so no rotation is necessary. If B_{ii} is the maximum, the same considerations suggest that the ith component of q_{opt} has the largest magnitude, so a rotation about the ith axis is performed. This will tend to put the largest component of q_{opt} in the fourth position in the rotated frame, which means that the rotation angle in the rotated frame is small, and the 180° rotation singularity is avoided. The reference system rotation is easily "undone" by Eq. (36) or its equivalent after the optimal quaternion has been computed.

It will be useful for future reference to note that Eqs. (18), (21), and (39) give

$$\mathbf{z} = \sum_{\mu=1}^{4} \lambda_\mu (q_\mu)_4 \mathbf{q}_\mu . \tag{42}$$

Using the orthonormality of the eigenvectors and Eq. (40), we find that

$$|\mathbf{z}|^2 = \sum_{\mu=1}^{4} \lambda_\mu^2 (q_\mu)_4^2 - (\operatorname{tr} B)^2 . \tag{43}$$

and thus that

$$|\mathbf{z}| \le \max_{\mu=1,\cdots,4} |\lambda_\mu| = \max(\lambda_{max}, -\lambda_{min}) . \tag{44}$$

FAST OPTIMAL ATTITUDE MATRIX (FOAM)

The SVD decomposition of B gives a convenient representation for adj$B$, det$B$, and $\|B\|_F^2$. These can be used to write the optimal attitude matrix as[20,21]

$$A_{opt} = (\kappa\lambda_{max} - \det B)^{-1}[(\kappa + \|B\|_F^2)B + \lambda_{max} \operatorname{adj} B^T - BB^T B], \tag{45}$$

where

$$\kappa \equiv \tfrac{1}{2}(\lambda_{max}^2 - \|B\|_F^2) . \tag{46}$$

It's important to note that all the quantities in Eqs. (45) and (46) can be computed without performing the SVD of B. In this method, λ_{max} is found from

$$\lambda_{max} = \operatorname{tr}(A_{opt} B^T) = (\kappa\lambda_{max} - \det B)^{-1}\left[\left(\kappa + \|B\|_F^2\right)\|B\|_F^2 + 3\lambda_{max} \det B - \operatorname{tr}\left(BB^T BB^T\right)\right], \tag{47}$$

or, after some matrix algebra,

$$0 = \psi(\lambda_{max}) \equiv (\lambda_{max}^2 - \|B\|_F^2)^2 - 8\lambda_{max} \det B - 4\|\operatorname{adj} B\|_F^2 . \tag{48}$$

Equations (32) and (48) for $\psi(\lambda_{max})$ would be numerically identical with infinite-precision computations, but the FOAM form of the coefficients is less subject to errors arising in finite-precision computations.

The FOAM algorithm gives the convenient form for the error covariance:

$$P = (\kappa\lambda_{max} - \det B)^{-1}(\kappa I + BB^T) . \tag{49}$$

A quaternion can be extracted from A_{opt}, with a cost of 13 MATLAB flops. This has two advantages: the four-component quaternion is often preferable to the nine-component attitude matrix, and the quaternion can be easily normalized if A_{opt} is not exactly orthogonal due to computational errors[22].

ESTIMATOR OF THE OPTIMAL QUATERNION (ESOQ or ESOQ1)

Davenport's eigenvalue equation, Eq. (22), says that the optimal quaternion is orthogonal to all the columns of the matrix

$$H \equiv K - \lambda_{\max} I, \tag{50}$$

which means that it must be orthogonal to the three-dimensional subspace spanned by the columns of H. The optimal quaternion is conveniently computed as the generalized four-dimensional cross-product of any three columns of this matrix[23-25].

Another way of seeing this result is to examine the classical adjoint of H. Representing K in terms of its eigenvalues and eigenvectors and using the orthonormality of the eigenvectors gives, for any scalar λ,

$$\operatorname{adj}(K - \lambda I) = \operatorname{adj}\left[\sum_{\mu=1}^{4}(\lambda_\mu - \lambda) q_\mu q_\mu^{\mathrm{T}}\right] = \sum_{\mu=1}^{4}(\lambda_\nu - \lambda)(\lambda_\rho - \lambda)(\lambda_\tau - \lambda) q_\mu q_\mu^{\mathrm{T}}, \tag{51}$$

where $\{\mu, \nu, \rho, \tau\}$ is a permutation of $\{1, 2, 3, 4\}$. Setting $\lambda = \lambda_{\max} \equiv \lambda_1$ causes all the terms in this sum to vanish except the first, with the result

$$\operatorname{adj} H = (\lambda_2 - \lambda_{\max})(\lambda_3 - \lambda_{\max})(\lambda_4 - \lambda_{\max}) q_{\text{opt}} q_{\text{opt}}^{\mathrm{T}} \tag{52}$$

Thus q_{opt} can be computed by normalizing any non-zero column of $\operatorname{adj} H$, which we denote by index k. Let F denote the symmetric 3×3 matrix obtained by deleting the kth row and kth column from H, and let $\mathbf{f}$ denote the three-component column vector obtained by deleting the kth element from the kth column of H. Then the kth element of the optimal quaternion is given by

$$(q_{\text{opt}})_k = -c \det F, \tag{53}$$

and the other three elements are

$$(q_{\text{opt}})_{1,\cdots,k-1,k+1,\cdots,4} = c(\operatorname{adj} F)\mathbf{f}, \tag{54}$$

where the coefficient c is determined by normalizing the quaternion. It is desirable to let k denote the column with the maximum Euclidean norm, which is the column containing the maximum diagonal element of the adjoint, owing to the symmetry of H. Computing all the diagonal elements of adj$H$, though not as burdensome as QUEST's sequential rotations, is somewhat expensive. It can be avoided by using the trace of the B matrix as in QUEST. In the ESOQ case, however, no rotation is performed; we merely choose k to be the index of the maximum element of $\{B_{11}, B_{22}, B_{33}, \operatorname{tr} B\}$.

The matrix F depends on the maximum eigenvalue $\lambda_{\max}$; but it is interesting to note that $\mathbf{f}$ does not depend on $\lambda_{\max}$, which only appears in the diagonal elements of H. The original formulation of ESOQ used the analytic solution of the characteristic equation[26]; but the analytic formula sometimes gives complex eigenvalues, which is theoretically impossible for a real symmetric matrix. These errors arise from inaccurate values of the coefficients of the quartic characteristic equation, not from the solution method. It is faster, and equally accurate, to compute $\lambda_{\max}$ by iterative solution of Eq. (48). Equation (32) would give a faster solution, but it would be less robust, and an even more efficient solution is described below.

First Order Update (ESOQ1.1)

Test results show that higher-order updates do not improve the performance of the iterative methods, providing motivation for developing a first-order approximation. The matrix H can be expanded to first order in $\Delta\lambda \equiv \lambda_0 - \lambda_{\max}$ as

$$H = H^0 + (\Delta\lambda) I, \tag{55}$$

where

$$H^0 \equiv K - \lambda_0 I. \tag{56}$$

Then

$$F = F^0 + (\Delta\lambda) I, \tag{57}$$

where F^0 is derived from H^0 in the same way that F is derived from H. Matrix identities give

$$\operatorname{adj} F = \operatorname{adj} F^0 + \Delta\lambda[(\operatorname{tr} F^0)I - F^0] \tag{58}$$

and

$$\det F = \det F^0 + (\Delta\lambda)\operatorname{tr}(\operatorname{adj} F^0), \tag{59}$$

to first order in $\Delta\lambda$. The characteristic equation can be expressed to the same order as

$$0 = \det H = (H_{kk}^0 + \Delta\lambda)\det F - \mathbf{f}^{\mathrm{T}}(\operatorname{adj} F)\mathbf{f} = H_{kk}^0 \det F^0 - \mathbf{f}^{\mathrm{T}}\mathbf{g} + \left[H_{kk}^0 \operatorname{tr}(\operatorname{adj} F^0) + \det F^0 - \mathbf{f}^{\mathrm{T}}\mathbf{h}\right]\Delta\lambda, \tag{60}$$

where

$$\mathbf{g} \equiv (\operatorname{adj} F^0)\mathbf{f} \qquad \text{and} \qquad \mathbf{h} \equiv [(\operatorname{tr} F^0)I - F^0]\mathbf{f}. \tag{61}$$

Equation (60) is easily solved for $\Delta\lambda \equiv \lambda_0 - \lambda_{\max}$, and then the first order quaternion estimate is given by

$$(q_{\text{opt}})_k = -c[\det F^0 + (\Delta\lambda)\operatorname{tr}(\operatorname{adj} F^0)] \tag{62}$$

and

$$(q_{\text{opt}})_{1,\cdots,k-1,k+1,\cdots,4} = c(\mathbf{g} + \mathbf{h}\,\Delta\lambda). \tag{63}$$

SECOND ESTIMATOR OF THE OPTIMAL QUATERNION (ESOQ2)

This algorithm uses the rotation axis/angle form of the optimal quaternion, as given in Eq. (17). Substituting these into Eqs. (25) and (26) gives

$$(\lambda_{\max} - \operatorname{tr} B)\cos(\phi/2) = \mathbf{z}^{\mathrm{T}}\mathbf{e}\sin(\phi/2) \tag{64}$$

and

$$\mathbf{z}\cos(\phi/2) = [(\lambda_{\max} + \operatorname{tr} B)I - S]\mathbf{e}\sin(\phi/2) \tag{65}$$

Multiplying Eq. (65) by $(\lambda_{\max} - \operatorname{tr} B)$ and substituting Eq. (64) gives

$$M\mathbf{e}\sin(\phi/2) = \mathbf{0}, \tag{66}$$

where

$$M \equiv (\lambda_{\max} - \operatorname{tr} B)[(\lambda_{\max} + \operatorname{tr} B)I - S] - \mathbf{z}\mathbf{z}^{\mathrm{T}} = [\mathbf{m}_1 \vdots \mathbf{m}_2 \vdots \mathbf{m}_3]. \tag{67}$$

These computations lose numerical significance if $(\lambda_{\max} - \operatorname{tr} B)$ and $\mathbf{z}$ are close to zero, which would be the case for zero rotation angle. We can always avoid this singular condition by using one of the sequential reference system rotations[15–18] to ensure that tr$B$ is less than or equal to zero. If we rotate the reference frame about the ith axis, then

$$(\operatorname{tr} B)_{\text{rotated}} = B_{ii} - B_{jj} - B_{kk} = 2B_{ii} - \operatorname{tr} B, \tag{68}$$

where $\{i, j, k\}$ is a permutation of $\{1, 2, 3\}$ and the components not marked "rotated" are defined with respect to the unrotated reference frame. Thus no rotation is performed if tr$B$ is the minimum of $\{B_{11}, B_{22}, B_{33}, \operatorname{tr} B\}$, while a rotation about the ith axis is performed if B_{ii} is the minimum. This will ensure the most negative value for the trace in the rotated frame. The rotation is easily "undone" by Eq. (36) or its equivalent after the quaternion has been computed. It follows from the orthogonality of the eigenvectors that

$$\lambda_{\min} \le \operatorname{tr} B \le \lambda_{\max}, \tag{69}$$

Equation (40) shows that $\operatorname{tr} B = \lambda_{\max}$ if and only if q_{opt} is the quaternion of the identity attitude, with $\cos(\phi/2) = 1$, while $\operatorname{tr} B = \lambda_{\min}$ means that the identity quaternion is an eigenvector of K with eigenvalue $\lambda_{\min}$. The latter equality requires q_{opt} to be the quaternion of a 180° rotation, with $\cos(\phi/2) = 0$, since the orthogonality of the eigenvectors means that any two of the rotation matrices $A(q_\mu)$ differ by a 180° rotation. However, q_{opt} being the quaternion of a 180° rotation does not imply $\operatorname{tr} B = \lambda_{\min}$, but only the weaker condition $\lambda_{\min} \le \operatorname{tr} B \le \lambda_2$. Thus a small value of $\operatorname{tr} B$ is a better indicator of an attitude far from the identity than a large value of $\operatorname{tr} B$ is of an attitude far from a 180° rotation, so this procedure should be even more reliable for avoiding singularities in ESOQ2 than in QUEST or ESOQ.

Equation (66) says that the rotation axis is a null vector of M. The columns of adj M are the cross products of the columns of M:

$$\operatorname{adj} M = [\mathbf{m}_2 \times \mathbf{m}_3 \,\vdots\, \mathbf{m}_3 \times \mathbf{m}_1 \,\vdots\, \mathbf{m}_1 \times \mathbf{m}_2]. \tag{70}$$

Because M is singular, all these columns are parallel, and all are parallel to the rotation axis $\mathbf{e}$. Thus we set

$$\mathbf{e} = \mathbf{y}/|\mathbf{y}|, \tag{71}$$

where $\mathbf{y}$ is the column of $\operatorname{adj} M$ (*i.e.*, the cross product) with maximum norm. Because M is symmetric, it is only necessary to find the maximum diagonal element of its adjoint to determine which column to use. The rotation angle is found from Eq. (64) or one of the components of Eq. (65). Equations (40) and (44) show that choosing the rotated reference system that provides the most negative value of tr$B$ makes Eq. (64) the best choice. With Eq. (71), this can be written

$$(\lambda_{\max} - \operatorname{tr} B)|\mathbf{y}|\cos(\phi/2) = (\mathbf{z}\cdot\mathbf{y})\sin(\phi/2), \tag{72}$$

which means that there is some scalar η for which

$$\cos(\phi/2) = \eta(\mathbf{z}\cdot\mathbf{y}) \tag{73}$$

and

$$\sin(\phi/2) = \eta(\lambda_{\max} - \operatorname{tr} B)|\mathbf{y}|. \tag{74}$$

Substituting into Eq. (17) and using Eq. (71) gives the optimal quaternion as[27,28]

$$q_{\text{opt}} = \frac{1}{\sqrt{|(\lambda_{\max} - \operatorname{tr} B)\mathbf{y}|^2 + (\mathbf{z}\cdot\mathbf{y})^2}} \begin{bmatrix} (\lambda_{\max} - \operatorname{tr} B)\mathbf{y} \\ \mathbf{z}\cdot\mathbf{y} \end{bmatrix}. \tag{75}$$

Note that it is not necessary to normalize the rotation axis. ESOQ2 does not define the rotation axis uniquely if M has rank less than two. This includes the usual case of unobservable attitude and also the case of zero rotation angle. Requiring tr$B$ to be non-positive avoids zero rotation angle singularity, however. We compute $\lambda_{\max}$ by iterative solution of Eq. (48) in the general case, as for ESOQ.

First Order Update (ESOQ2.1)

The motivation for and development of this algorithm are similar to the those of the first order update used in ESOQ1.1. The matrix M can be expanded to first order in $\Delta\lambda \equiv \lambda_0 - \lambda_{\max}$ as

$$M = M^0 + (\Delta\lambda)N, \tag{76}$$

where

$$M^0 \equiv (\lambda_0 - \operatorname{tr}B)[(\lambda_0 + \operatorname{tr}B)I - S] - \mathbf{z}\mathbf{z}^{\mathrm{T}} = [\mathbf{m}_1^0 \,\vdots\, \mathbf{m}_2^0 \,\vdots\, \mathbf{m}_3^0] \tag{77}$$

and

$$N \equiv S - 2\lambda_0 I = [\mathbf{n}_1 \,\vdots\, \mathbf{n}_2 \,\vdots\, \mathbf{n}_3]. \tag{78}$$

To this same order, we have

$$\mathbf{y} \equiv \mathbf{m}_i \times \mathbf{m}_j = (\mathbf{m}_i^0 + \mathbf{n}_i\Delta\lambda) \times (\mathbf{m}_j^0 + \mathbf{n}_j\Delta\lambda) = \mathbf{y}^0 + \mathbf{p}\,\Delta\lambda, \tag{79}$$

where

$$\mathbf{y}^0 \equiv \mathbf{m}_i^0 \times \mathbf{m}_j^0 \qquad \text{and} \qquad \mathbf{p} \equiv \mathbf{m}_i^0 \times \mathbf{n}_j + \mathbf{n}_i \times \mathbf{m}_j^0. \tag{80}$$

The maximum eigenvalue can be found from condition that M is singular; to first-order:

$$0 = \det M = (\mathbf{m}_i \times \mathbf{m}_j)\cdot\mathbf{m}_k = (\mathbf{y}^0 + \mathbf{p}\,\Delta\lambda)\cdot(\mathbf{m}_k^0 + \mathbf{n}_k\Delta\lambda) = \mathbf{y}^0\cdot\mathbf{m}_k^0 + (\mathbf{y}^0\cdot\mathbf{n}_k + \mathbf{m}_k^0\cdot\mathbf{p})\Delta\lambda, \tag{81}$$

where $\{i, j, k\}$ is a cyclic permutation of $\{1, 2, 3\}$. This is solved for $\Delta\lambda \equiv \lambda_0 - \lambda_{\max}$, and then the attitude estimate is found by substituting Eq. (79) and $\lambda_{\max} = \lambda_0 - \Delta\lambda$ into Eq. (75).

There is an interesting relation between the eigenvalue condition $\det M(\lambda) = 0$ used in ESOQ2.1 and the condition $\psi(\lambda) = 0$ used in other algorithms. Straightforward matrix algebra shows

$$\det M(\lambda) = (\lambda - \operatorname{tr} B)^2 \psi(\lambda). \tag{82}$$

Thus $\det M(\lambda)$ has the four roots of $\psi(\lambda)$, the eigenvalues of Davenport's K matrix, and an additional double root at tr$B$. Equation (40) shows that tr$B$ depends on the reference frame axes, and choosing the reference axes maximizing –tr$B$ assures that these two spurious roots are far from the desired root at λ_{max}.

TESTS

We test the accuracy and speed of MATLAB implementations of these methods, using simulated data. The q and SVD methods use the functions `eig` and `svd`, respectively; the others use the equations in this paper. MATLAB uses IEEE double-precision floating-point arithmetic, in which the numbers have approximately 16 significant decimal digits[29].

We analyze three test scenarios. In all these scenarios, the pointing of one spacecraft axis, which we take to be the spacecraft x axis, is much better determined that the rotation about this axis. This is a very common case that arises in spacecraft that point a single instrument (like an astronomical telescope) very precisely. This is also a characteristic of attitude estimates from a single narrow-field-of-view star tracker, where the rotation about the tracker boresight is much less well determined than the pointing of the boresight. The x axis error and the yz error, which is the error about an axis orthogonal to the x axis and determines the x axis pointing, are computed from an error quaternion q_{err} by writing

$$\begin{aligned} q_{err} &= \begin{bmatrix} \mathbf{q}_{err} \\ q_{err4} \end{bmatrix} = \begin{bmatrix} \mathbf{e}_x \sin(\phi_x/2) \\ \cos(\phi_x/2) \end{bmatrix} \otimes \begin{bmatrix} \mathbf{e}_{yz} \sin(\phi_{yz}/2) \\ \cos(\phi_{yz}/2) \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{e}_x \cos(\phi_{yz}/2)\sin(\phi_x/2) + \mathbf{e}_{yz}\cos(\phi_x/2)\sin(\phi_{yz}/2) - (\mathbf{e}_x \times \mathbf{e}_{yz})\sin(\phi_x/2)\sin(\phi_{yz}/2) \\ \cos(\phi_x/2)\cos(\phi_{yz}/2) \end{bmatrix}, \end{aligned} \tag{83}$$

where $\mathbf{e}_x = [1\ 0\ 0]^T$ and $\mathbf{e}_{yz}$ is a unit vector orthogonal to $\mathbf{e}_x$. We can always find ϕ_x in $[-\pi, \pi]$ and ϕ_{yz} in $[0, \pi]$ by selecting $\mathbf{e}_{yz}$ appropriately. Then, since $\mathbf{e}_{yz}$ and $\mathbf{e}_x \times \mathbf{e}_{yz}$ form an orthonormal basis for the y-z (or 2-3) plane, the error angles are given by

$$\phi_x = 2\tan^{-1}(q_{err1}/q_{err4}) \tag{84}$$

and

$$\phi_{yz} = 2\sin^{-1}\left(\sqrt{q_{err2}^2 + q_{err3}^2}\right). \tag{85}$$

Equations (84) and (85) would be unchanged if the order of the rotations about $\mathbf{e}_x$ and $\mathbf{e}_{yz}$ were reversed; only the unit vector $\mathbf{e}_{yz}$ would be different.

The total error, which is the principal angle of the rotation represented by the error quaternion, is given by

$$\cos(\phi_{total}/2) = q_{err4} = \cos(\phi_x/2)\cos(\phi_{yz}/2)\ . \tag{86}$$

Thus $\phi_{total}/2$ is the hypotenuse of a right spherical triangle with sides $\phi_x/2$ and $\phi_{yz}/2$. This is the spherical trigonometry equivalent of taking two orthogonal components of an error vector.

First Test Scenario

The first scenario simulates a single star tracker with a narrow field of view and boresight at $[1, 0, 0]^T$. This is an application for which the QUEST algorithm has been widely used. We assume that the tracker is tracking five stars at

$$\mathbf{b}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},\quad \mathbf{b}_2 = \begin{bmatrix} 0.99712 \\ 0.07584 \\ 0 \end{bmatrix},\quad \mathbf{b}_3 = \begin{bmatrix} 0.99712 \\ -0.07584 \\ 0 \end{bmatrix},\quad \mathbf{b}_4 = \begin{bmatrix} 0.99712 \\ 0 \\ 0.07584 \end{bmatrix},\quad \text{and}\quad \mathbf{b}_5 = \begin{bmatrix} 0.99712 \\ 0 \\ -0.07584 \end{bmatrix}. \tag{87}$$

We simulate 1000 test cases with uniformly distributed random attitude matrices, which we use to map the five observation vectors to the reference frame. We add Gaussian random noise with equal standard deviations of 6 arcseconds per axis to the reference vectors rather than the observation vectors, so that Eq. (87) will remain valid in the presence of noise, and then normalize the reference vectors.

The loss function is computed with measurement variances in (radians)^2, since this results in $2L(A_{opt})$ approximately obeying a χ^2 distribution. The minimum and maximum values of the loss function in the 1000 test runs are 0.23 and 12.1, respectively. The probability distribution of the loss function is plotted as the solid line in Figure 1, and several values of $P(\chi^2 \mid \nu)$ for $\chi^2 = 2L(A_{opt})$ and $\nu = 2n_{obs} - 3 = 7$ are plotted as circles[26]. The agreement is seen to be excellent.

The estimation errors in arcseconds for the star tracker scenario are presented in Table 1, as both the RSS (outside of parentheses) and the maximum (in parentheses) over the 1000 cases. The q method and the SVD method should both give the truly optimal solution, since they are based on robust matrix analysis algorithms[3,10]. The q method is taken as optimal by definition, so no estimated-to-optimal differences are presented for that algorithm, and the differences between the SVD and q methods provide an estimate of the computational errors of both methods. In particular, the loss function is computed exactly by both methods, in principle, which means in practice that it is computed to about one part in 10^5. Equation (3) gives $\lambda_0 = 5.9 \times 10^9$ rad^{-2} for this scenario, so this is the expected accuracy in double-precision machine computations. No estimate of the loss function is provided when no update of λ_{max} is performed, accounting for the lack of entries in the loss function column in the tables for these cases.

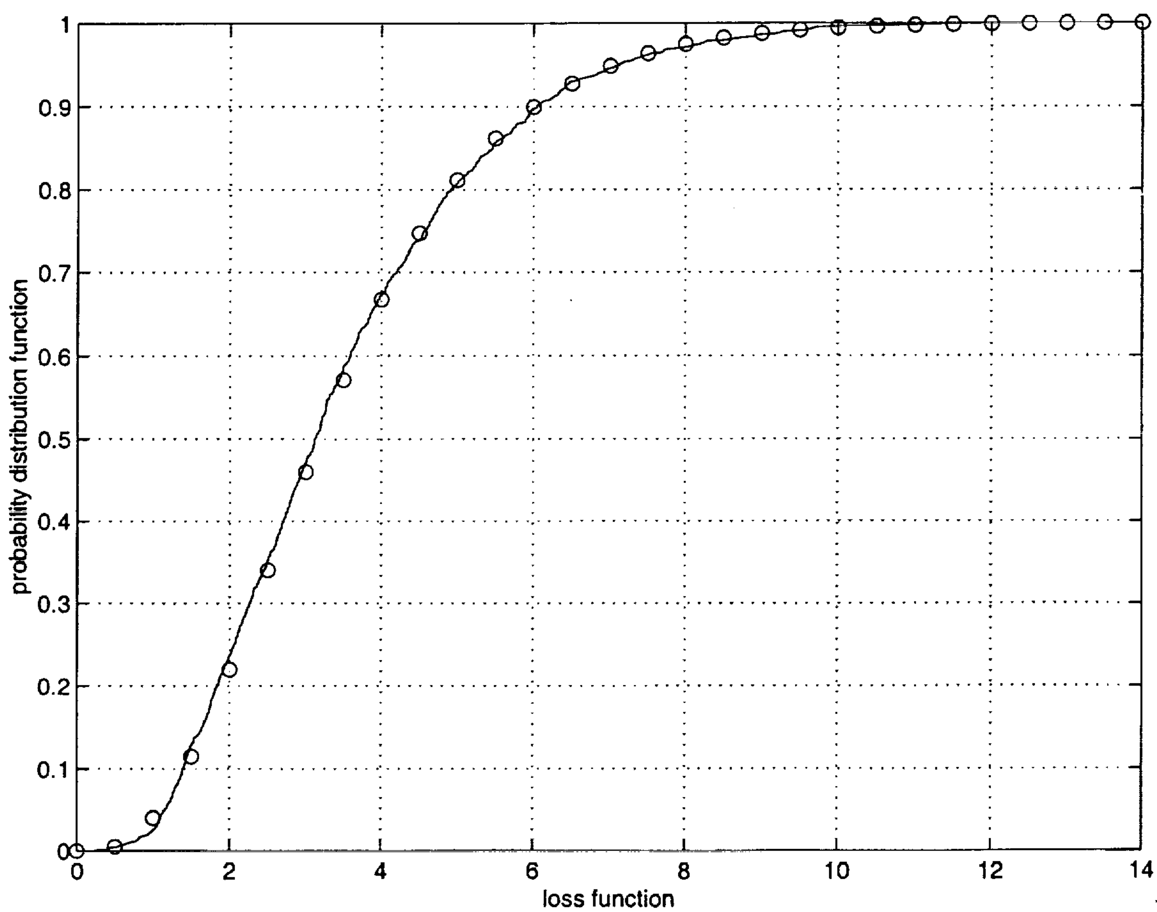

Figure 1: Empirical (solid line) and Theoretical (dots) Loss Function Distribution for the Seven-Degree-of-Freedom Star Tracker Scenario

Not all the decimal places exhibited are significant, since the results of 1000 different random cases would not agree with these to more than two decimal places. The extra decimal places are shown to emphasize the fact that although the different algorithms give results that are closer or farther from the optimal estimate, all the algorithms provide estimates that are equally close to the true attitude. The differences between the estimated and optimal values further show that one Newton-Raphson iteration for λ_{max} is always sufficient; a second iteration provides no practical improvement in the estimate for this scenario.

Equation (33) gives the covariance for the star tracker scenario as

$$P = (6\,\text{arcsec})^2[5I - \sum_{i=1}^{5}\mathbf{b}_i\mathbf{b}_i^T]^{-1} = \text{diag}[1565, 7.2, 7.2]\,\text{arcsec}^2, \tag{88}$$

which gives the standard deviations of the attitude estimation errors as

$$\sigma_x = \sqrt{1565}\,\text{arcsec} = 40\,\text{arcsec} \quad \text{and} \quad \sigma_{yz} = \sqrt{7.2+7.2}\,\text{arcsec} = 3.8\,\text{arcsec}. \tag{89}$$

It is apparent that this covariance estimate is quite accurate.

Second Test Scenario

The second scenario uses three observations with widely varying accuracies to provide a difficult test case for the algorithms under consideration. The three observation vectors are

$$\mathbf{b}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix},\quad \mathbf{b}_2 = \begin{bmatrix}-0.99712\\0.07584\\0\end{bmatrix},\quad \text{and} \quad \mathbf{b}_3 = \begin{bmatrix}-0.99712\\-0.07584\\0\end{bmatrix}. \tag{90}$$

Table 1: Estimation Errors for Star Tracker Scenario

Algorithm		RSS (max) estimated–to–optimal			RSS (max) estimated–to–true	
λ_{max} iterations		loss function	x (arcsec)	yz (arcsec)	x (arcsec)	yz (arcsec)
q	–	—	—	—	38.41 (122.5)	3.829 (9.252)
SVD	–	$0.4\ (1.8) \times 10^{-5}$	$1.4\ (5.6) \times 10^{-8}$	$0.8\ (2.9) \times 10^{-10}$	38.41 (122.5)	3.829 (9.252)
FOAM	0	—	0.014 (0.078)	$9.7\ (36) \times 10^{-5}$	38.41 (122.5)	3.829 (9.252)
	1	$0.4\ (1.6) \times 10^{-5}$	$1.5\ (5.6) \times 10^{-8}$	$26\ (104) \times 10^{-10}$	38.41 (122.5)	3.829 (9.252)
	2	$0.4\ (1.7) \times 10^{-5}$	$1.5\ (5.6) \times 10^{-8}$	$26\ (88) \times 10^{-10}$	38.41 (122.5)	3.829 (9.252)
QUEST	0	—	0.015 (0.078)	$9.6\ (36) \times 10^{-5}$	38.41 (122.5)	3.829 (9.252)
	1	$2.5\ (7.2) \times 10^{-5}$	$10.1\ (46) \times 10^{-8}$	$6.1\ (26) \times 10^{-10}$	38.41 (122.5)	3.829 (9.252)
	2	$2.9\ (8.4) \times 10^{-5}$	$11.1\ (50) \times 10^{-8}$	$7.2\ (25) \times 10^{-10}$	38.41 (122.5)	3.829 (9.252)
ESOQ2	0	—	0.014 (0.078)	$9.7\ (36) \times 10^{-5}$	38.41 (122.5)	3.829 (9.252)
	1	$0.4\ (1.7) \times 10^{-5}$	$1.5\ (6.1) \times 10^{-8}$	$2.0\ (10) \times 10^{-10}$	38.41 (122.5)	3.829 (9.252)
	2	$0.4\ (1.8) \times 10^{-5}$	$1.5\ (6.1) \times 10^{-8}$	$2.1\ (11) \times 10^{-10}$	38.41 (122.5)	3.829 (9.252)
ESOQ2.1	1	$0.4\ (1.6) \times 10^{-5}$	$1.5\ (5.9) \times 10^{-8}$	$1.9\ (12) \times 10^{-10}$	38.41 (122.5)	3.829 (9.252)
ESOQ	0	—	0.015 (0.078)	$9.6\ (36) \times 10^{-5}$	38.41 (122.5)	3.829 (9.252)
	1	$0.4\ (1.7) \times 10^{-5}$	$1.5\ (6.2) \times 10^{-8}$	$9.6\ (39) \times 10^{-10}$	38.41 (122.5)	3.829 (9.252)
	2	$0.4\ (1.8) \times 10^{-5}$	$1.5\ (6.2) \times 10^{-8}$	$9.6\ (39) \times 10^{-10}$	38.41 (122.5)	3.829 (9.252)
ESOQ1.1	1	$1.0\ (5.3) \times 10^{-5}$	$4.1\ (24) \times 10^{-8}$	$7.0\ (29) \times 10^{-10}$	38.41 (122.5)	3.829 (9.252)

We simulate 1000 test cases as in the star tracker scenario, but with Gaussian noise of one arcsecond per axis on the first observation, and 1° per axis on the other two. This models the case that the first observation is from an onboard astronomical telescope, and the other two observations are from a coarse sun sensor and a magnetometer, for example. A very accurate estimate of the orientation of the x axis is required in such an application, but the rotation about this axis expected to be fairly poorly determined.

The minimum and maximum values of the loss function computed by the q method in the 1000 test runs for the second scenario are 0.003 and 8.5, respectively. The probability distribution of the loss function is plotted as the solid line in Figure 2, and several values of the χ^2 distribution with three degrees of freedom are plotted as circles. The agreement is almost as good as the seven-degree-of-freedom case.

The estimation errors for this scenario are presented in Table 2, which is similar to Table 1 except that the rotation errors about the x axis are given in degrees. The agreement of the q and SVD methods is virtually identical to their agreement in the star tracker scenario, but the other algorithms show varying performance. Equation (3) gives $\lambda_0 = 8.5 \times 10^{10}$ rad^{-2} for this scenario, so the expected accuracy of the loss function in double-precision machine computations is on the order of 10^{-5}, which is the level of agreement between the values computed by the q and SVD methods. None of the other methods computes the loss function nearly as accurately. This differs from the first scenario, where all the algorithms came close to achieving the maximum precision available in double-precision arithmetic.

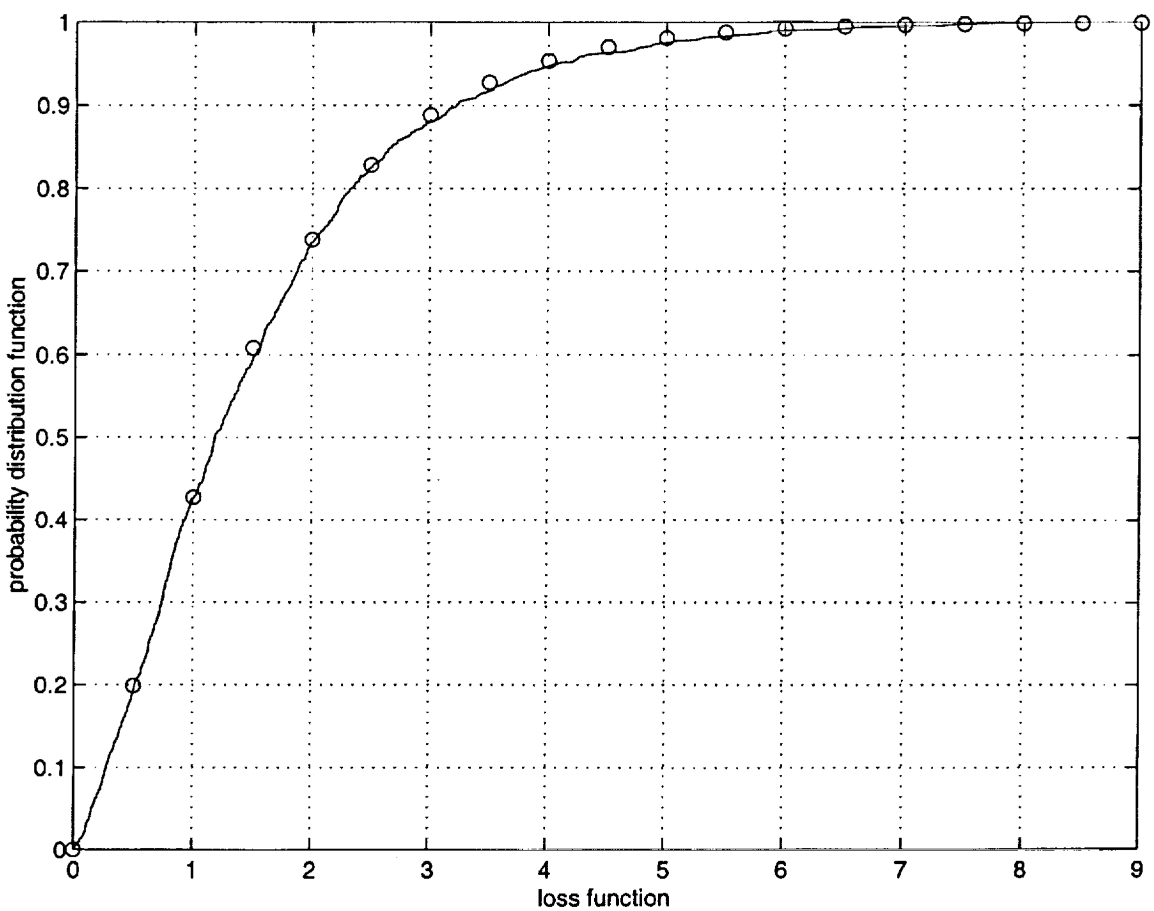

Figure 2: Empirical (solid line) and Theoretical (dots) Loss Function Distribution for the Three-Degree-of-Freedom Unequal Measurement Weight Scenario

The iterative computation of λ_{max} in QUEST, ESOQ1.1, and ESOQ2.1 is poor, but this has surprisingly little effect on the determination of the x axis pointing. The determination of the rotation about the x axis is adversely affected by an inaccurate computation of λ_{max}, however, with maximum deviations from the optimal estimate of almost 180°. The only useful results of QUEST are obtained by not performing any iterations for λ_{max}. The iterative computation of λ_{max} by Eq. (48) in FOAM, ESOQ, and ESOQ2 improves the agreement with the optimal estimate, but does not result in better agreement with the true attitude. It seems probable that Eq. (48) provides a better estimate of λ_{max} because it deals with B directly, while the other algorithms use the symmetric and skew parts S and $\mathbf{z}$ instead.

The predicted covariance in this scenario is, to a very good approximation,

$$P = \mathrm{diag}[\tfrac{1}{2}(1-0.99712^2)^{-1}\,\mathrm{deg}^2, 1\,\mathrm{arcsec}^2, 1\,\mathrm{arcsec}^2], \tag{91}$$

which gives

$$\sigma_x = 9.3\,\mathrm{deg} \quad \text{and} \quad \sigma_{yz} = 1.4\,\mathrm{arcsec}, \tag{92}$$

in agreement with the best results in Table 2.

Third Test Scenario

The third scenario investigates the effect of measurement noise mismodeling, illustrating problems that first appeared in analyzing data from the Upper Atmosphere Research Satellite[30]. Of course, no one would intentionally use erroneous models, but it can be very difficult to determine an accurate noise model for real data, and the assumption of any level of white noise is often a poor approximation to real measurement errors. This scenario uses the same three observation vectors as the second scenario, given by Eq. (90). We again simulate 1000 test cases, but with Gaussian white noise of 1° per axis on the first observation and 0.1° per axis on the other two. The estimator, however, incorrectly assumes measurement errors of 0.1° per axis on all three observations, so it weights the measurements equally.

Table 2: Estimation Errors for Unequal Measurement Weight Scenario

Algorithm	λ_{max} iterations	RSS (max) estimated–to–optimal: loss function	RSS (max) estimated–to–optimal: x (deg)	RSS (max) estimated–to–optimal: yz (arcsec)	RSS (max) estimated–to–true: x (deg)	RSS (max) estimated–to–true: yz (arcsec)
q	–	—	—	—	9.5 (34)	1.42 (3.57)
SVD	–	$1.6\ (6.9) \times 10^{-5}$	$1.4\ (8.0) \times 10^{-5}$	$7.7\ (24) \times 10^{-11}$	9.5 (34)	1.42 (3.57)
FOAM	0	—	1.5 (9.9)	$7.9\ (29) \times 10^{-3}$	9.5 (34)	1.42 (3.57)
	1	0.09 (0.7)	0.09 (1.1)	$8.0\ (26) \times 10^{-3}$	9.5 (34)	1.42 (3.57)
	2	0.0007 (0.012)	0.0008 (0.013)	$7.8\ (29) \times 10^{-3}$	9.5 (34)	1.42 (3.57)
QUEST	0	—	1.9 (12)	$0.4\ (3.8) \times 10^{-3}$	9.6 (34)	1.42 (3.57)
	1	768 (2329)	60 (170)	$2.3\ (9.0) \times 10^{-3}$	48 (90)	1.42 (3.57)
	2	1796 (38501)	62 (175)	$4.8\ (95) \times 10^{-3}$	48 (91)	1.42 (3.57)
ESOQ2	0	—	1.5 (9.9)	$1.1\ (6.8) \times 10^{-3}$	9.5 (34)	1.42 (3.57)
	1	0.09 (0.7)	0.09 (1.1)	$1.3\ (9.4) \times 10^{-3}$	9.5 (34)	1.42 (3.57)
	2	0.0007 (0.012)	0.0008 (0.013)	$1.1\ (7.1) \times 10^{-3}$	9.5 (34)	1.42 (3.57)
ESOQ2.1	1	59 (370)	39 (178)	$1.5\ (14) \times 10^{-3}$	29 (91)	1.42 (3.57)
ESOQ	0	—	1.9 (12)	$4.8\ (23) \times 10^{-3}$	9.6 (34)	1.42 (3.57)
	1	0.09 (0.7)	0.10 (1.1)	$5.3\ (28) \times 10^{-3}$	9.5 (34)	1.42 (3.57)
	2	0.0007 (0.012)	0.0008 (0.013)	$5.2\ (24) \times 10^{-3}$	9.5 (34)	1.42 (3.57)
ESOQ1.1	1	327 (1727)	60 (177)	$2.6\ (34) \times 10^{-3}$	43 (90)	1.42 (3.57)

The minimum and maximum values of the loss function computed by the q method in the 1000 test runs for the third scenario are 0.07 and 453, respectively. The probability distribution of the loss function is plotted in Figure 3. The theoretical three-degree-of-freedom distribution is not plotted, since it would be a very poor fit to the data. More than 95% of the values of $L(A_{\mathrm{opt}})$ are theoretically expected to lie below 4, according to the χ^2 distribution plotted in Figure 2, but almost half of the values of the loss function plotted in Figure 3 have values greater than 50. Shuster has emphasized that large values of the loss function are an excellent indication of measurement mismodeling or simply of bad data.

The estimation errors for this scenario are presented in Table 3, which is similar to Tables 1 and 2 except that all the angular errors are given in degrees. The truly optimal q and SVD methods agree even more closely than in the other scenarios. Equation (3) gives $\lambda_0 = 5\times10^5$ rad^{-2} for this scenario, so the expected accuracy of the loss function in double-precision machine computations is on the order of 10^{-10}, the level of agreement between the q and SVD methods. As in the second scenario, none of the other methods computes the loss function nearly as accurately. In the third scenario, though, the iterative computation of $\lambda_{\max}$ works well for all the algorithms, and both iterations improve the agreement of the loss function and attitude estimates with the optimal values. The first order refinement is reflected in a reduction of the attitude errors, particularly in determining the rotation about the x axis, but no algorithm is aided significantly by a second-order update. As in the first scenario, all the algorithms with the first order update to $\lambda_{\max}$ perform as well as the q and SVD methods.

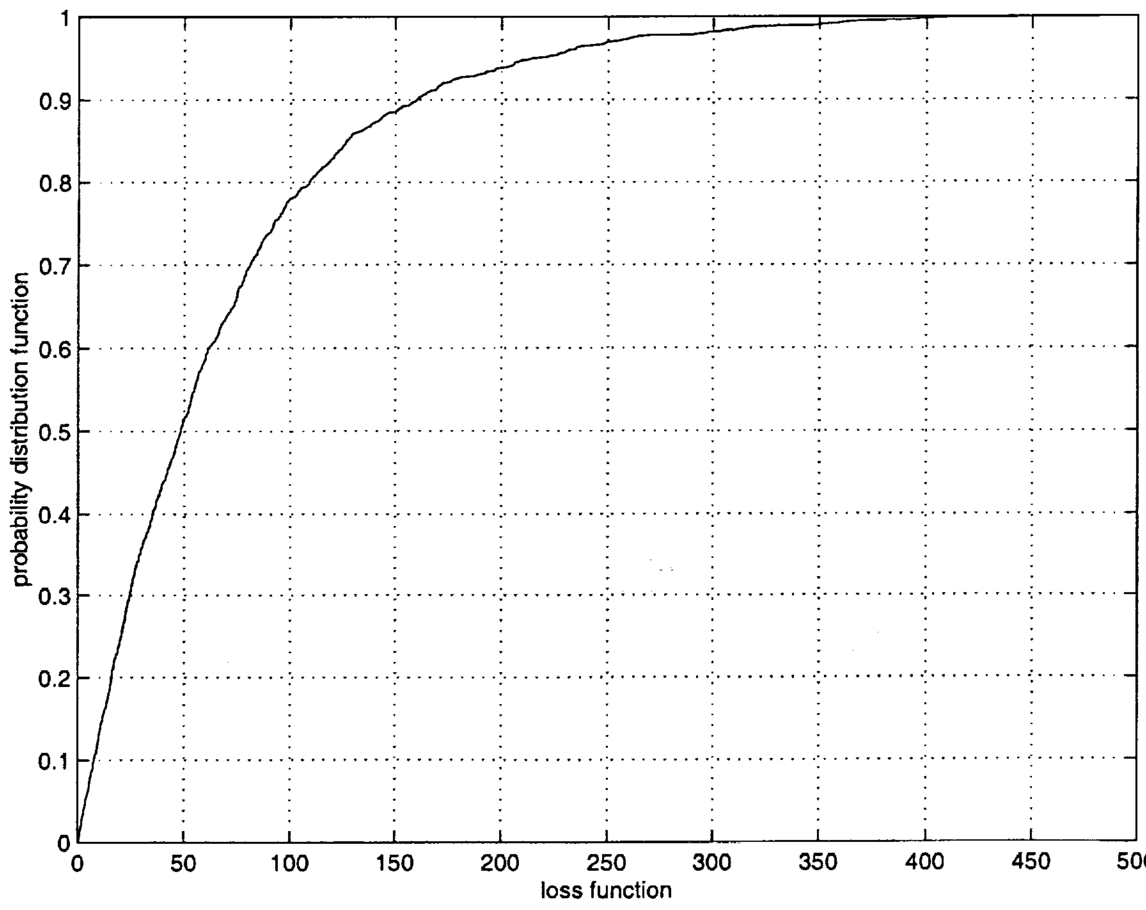

Figure 3: Empirical Loss Function Distribution for the Mismodeled Measurement Weight Scenario

Speed

There are two caveats to make with regard to timing comparisons. First, absolute speed numbers are not very important for ground computations, since the actual estimation algorithm is only a part of the overall attitude determination data processing effort. Absolute speed was more important in the past, when thousands of attitude solutions had to be computed by slower machines, which is why QUEST was so important for the MAGSAT mission. Second, the longest computation time is more important than the average time for a real-time computer in a spacecraft attitude control system or a star tracker, which must finish all its required tasks in a limited time. This penalizes QUEST for real-time applications, unless we use an *a priori* attitude estimate or information from the B matrix to eliminate the need for sequential rotations, as described above.

Figures 4 and 5 show the maximum number of MATLAB floating-point operations (flops) to compute an attitude using two to six reference vectors; the times to process more than six vectors follow the trends seen in the figure. The inputs for the timing tests are the n_{obs} normalized reference and observation vector pairs and their n_{obs} weights. One thousand test cases with random attitudes and random reference vectors with Gaussian measurement noise were simulated for each number of reference vectors.

Figure 4 plots the times of the more robust methods. The break in the plots for FOAM, ESOQ, and ESOQ2 at $n_{obs} = 3$ results from using an exact solution of the characteristic equation in the two-observation case, when $\det B = 0$ and Eq. (48) shows that $\psi(\lambda_{max})$ is a quadratic function of λ^2_{max}. For three or more observations, these algorithms are timed for a first-order update to λ_{max} using Eq. (48). Additional iterations for λ_{max} are not expensive, however, costing only 11 flops each. It is clear that the q method and the SVD method require significantly more computational effort than the other algorithms, as expected. The q method is more efficient than the SVD method, except in the least interesting two-observation case. The other three algorithms are much faster, with the fastest, ESOQ and ESOQ2, being nearly equal in speed.

Table 3: Estimation Errors for Mismodeled Measurement Weight Scenario

Algorithm	λ_{max} iterations	RSS (max) estimated–to–optimal loss function	RSS (max) estimated–to–optimal x (deg)	RSS (max) estimated–to–optimal yz (deg)	RSS (max) estimated–to–true x (deg)	RSS (max) estimated–to–true yz (deg)
q	–	—	—	—	0.96 (3.62)	0.49 (1.17)
SVD	–	$4.1\ (22) \times 10^{-10}$	$3.8\ (17) \times 10^{-12}$	$2.3\ (7.3) \times 10^{-14}$	0.96 (3.62)	0.49 (1.17)
FOAM	0	—	0.7 (5.9)	$4.0\ (21) \times 10^{-3}$	1.18 (5.42)	0.49 (1.16)
	1	2.6 (24)	0.020 (0.33)	$1.0\ (11) \times 10^{-4}$	0.96 (3.60)	0.49 (1.17)
	2	0.004 (0.07)	$0.4\ (10) \times 10^{-4}$	$1.7\ (35) \times 10^{-7}$	0.96 (3.62)	0.49 (1.17)
QUEST	0	—	0.9 (7.6)	$4.3\ (29) \times 10^{-3}$	1.27 (7.66)	0.49 (1.16)
	1	2.6 (24)	0.023 (0.33)	$1.1\ (11) \times 10^{-4}$	0.96 (3.60)	0.49 (1.17)
	2	0.004 (0.07)	$0.4\ (10) \times 10^{-4}$	$1.7\ (35) \times 10^{-7}$	0.96 (3.62)	0.49 (1.17)
ESOQ2	0	—	0.7 (5.9)	$4.0\ (21) \times 10^{-3}$	1.18 (5.42)	0.49 (1.16)
	1	2.6 (24)	0.020 (0.33)	$1.0\ (11) \times 10^{-4}$	0.96 (3.60)	0.49 (1.17)
	2	0.004 (0.07)	$0.4\ (10) \times 10^{-4}$	$1.7\ (35) \times 10^{-7}$	0.96 (3.62)	0.49 (1.17)
ESOQ2.1	1	2.6 (24)	0.020 (0.33)	$0.6\ (5.8) \times 10^{-4}$	0.96 (3.60)	0.49 (1.17)
ESOQ	0	—	0.9 (7.6)	$4.3\ (29) \times 10^{-3}$	1.27 (7.66)	0.49 (1.16)
	1	2.6 (24)	0.023 (0.33)	$1.1\ (11) \times 10^{-4}$	0.96 (3.60)	0.49 (1.17)
	2	0.004 (0.07)	$0.4\ (10) \times 10^{-4}$	$1.7\ (35) \times 10^{-7}$	0.96 (3.62)	0.49 (1.17)
ESOQ1.1	1	2.6 (24)	0.023 (0.33)	$1.3\ (27) \times 10^{-6}$	0.96 (3.60)	0.49 (1.17)

Figure 5 compares the timing of the fastest methods, which generally use zeroth order and first order approximations for λ_{max}. Both QUEST(1) and ESOQ2.1 use the exact quadratic solution for λ_{max} in the two-observation case, but ESOQ1.1 uses its faster first order approximation for any number of observations. It is clear that ESOQ and ESOQ2 are the fastest algorithms using the zeroth order approximation for λ_{max}, and ESOQ1.1 is the fastest of the first order methods.

CONCLUSIONS

This paper has examined the most useful algorithms for estimating spacecraft attitude from vector measurements based on minimizing Wahba's loss function. These were tested in three scenarios, which show that the most robust, reliable, and accurate estimators are Davenport's q method and the Singular Value Decomposition (SVD) method. This is not surprising, since these methods are based on robust and well-tested general-purpose matrix algorithms. The q method, which computes the optimal quaternion as the eigenvector of a symmetric 4×4 matrix with the largest eigenvalue, is the faster of these two.

Several algorithms are significantly less burdensome computationally than the q and SVD methods. These methods are less robust in principle, since they solve the quartic characteristic polynomial equation for the maximum eigenvalue, a procedure that is potentially numerically unreliable. Algorithms that use the form of the characteristic polynomial from the Fast Optimal Attitude Matrix (FOAM) algorithm performed as well as the q and SVD methods in practice, however. The fastest of these algorithms are the EStimators of the Optimal Quaternion, ESOQ and ESOQ2.

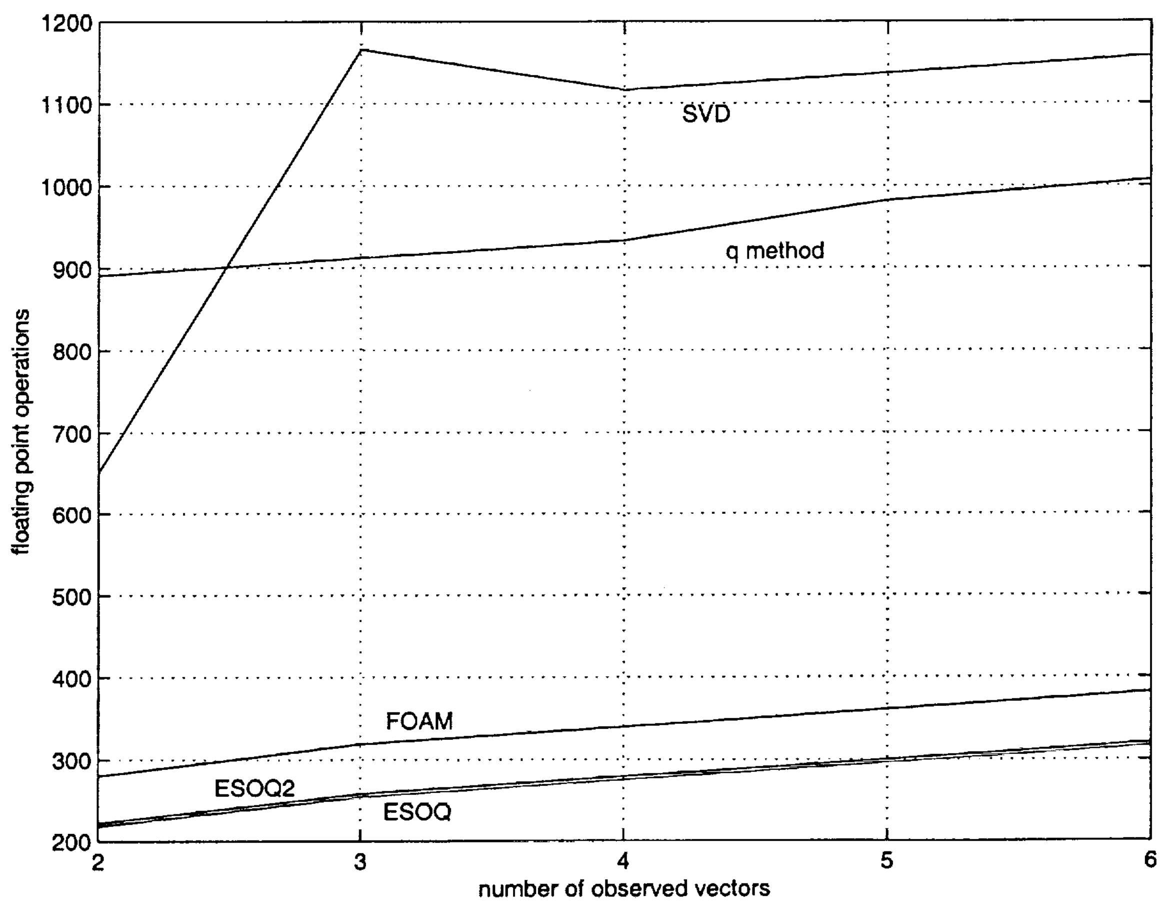

Figure 4: Execution Times for Robust Estimation Algorithms
FOAM, ESOQ, and ESOQ2 use first order approximation for λ_{max}.

Part of the speed advantage of the methods tested in this paper over previous studies results from using the information contained in the diagonal elements of B to eliminate sequential rotations in QUEST and extra computations in ESOQ and ESOQ2. It would be equally fast, and more accurate in principle, to use an *a priori* quaternion to find the best rotated reference frame. Simulations comparing the methods using the diagonal elements of B with methods using information contained in an *a priori* quaternion showed equally accurate estimates with the two approaches, however, and the former method is preferable for its generality.

All the algorithms tested perform as well as the more robust algorithms in cases where measurement weights do not vary too widely and are reasonably well modeled. These include most of the cases for which vector observations are used to compute spacecraft attitude, in particular the case of an attitude solution from multiple stars. If the measurement uncertainties are not well represented by white noise, however, an update is required, while this update can be disastrous if the measurement weights span a wide range.

The examples in the paper show that these robustness concerns are not an issue for the processing multiple star observations with comparable accuracies, the most common application of Wahba's loss function. Thus the fastest algorithms, the zeroth-order ESOQ and ESOQ2 and the first-order ESOQ1.1, are well suited to star tracker attitude determination applications. In general-purpose applications where measurement weights may vary greatly, one of the more robust algorithms may be preferred.

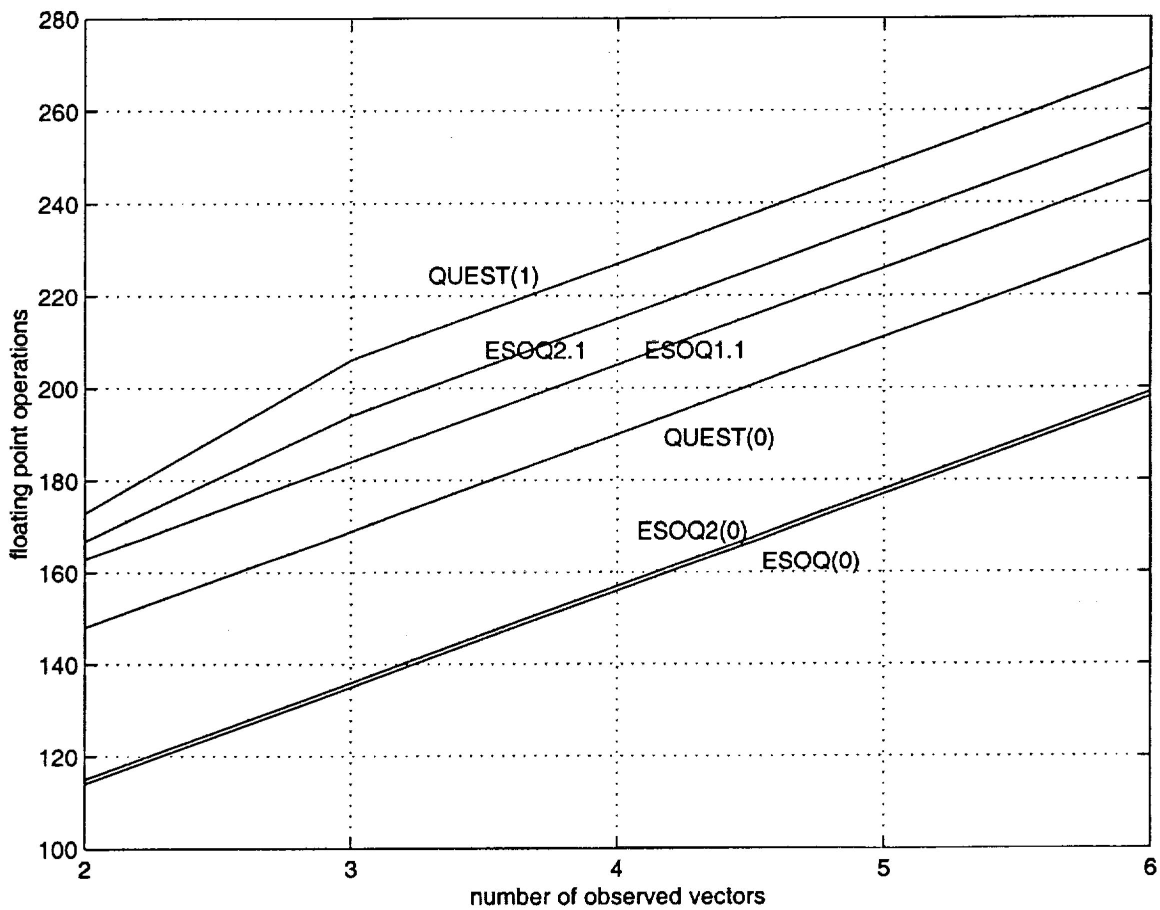

Figure 5: Execution Times for Fast Estimation Algorithms
Numbers in parentheses denote order of λ_{max} approximation.

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