On Geomagnetism and Paleomagnetism I

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Abstract. A partial description of Earth's broad scale, core-source magnetic field has been developed and tested three ways. The description features an expected, or mean, spatial magnetic power spectrum that is approximately inversely proportional to horizontal wavenumber atop Earth's core (Stevenson, 1983; McLeod, 1985; 1996). This multipole spectrum describes a magnetic energy range; it is not steep enough for Gubbins' (1975) magnetic dissipation range. Temporal variations of core multipole powers about mean values are to be expected and are described statistically, via trial probability distribution functions, instead of deterministically, via trial solution of closed transport equations. The distributions considered here are closed and neither require nor prohibit magnetic isotropy. The description is therefore applicable to, and tested against, both dipole and low degree non-dipole fields.

In Part I, a physical basis for an expectation spectrum is developed and checked. The description is then combined with main field models of twentieth century satellite and surface geomagnetic field measurements to make testable predictions of the radius of Earth's core. The predicted core radius is 0.7% above the 3480 km seismologic value. Partial descriptions of other planetary dipole fields are noted.

1 INTRODUCTION & BACKGROUND

The primary physical source of the geomagnetic field is widely held to be electric current flowing in Earth's electrically conducting, ferro-metallic liquid outer core and solid inner core. Other sources include weaker current in the resistive, ferro-magnesian silicate and oxide mantle, magnetization in the colder crust, electric current in the saline hydrosphere, and external currents in the ionosphere above and magnetosphere beyond. Solenoidal magnetic induction B within the source regions can be represented mathematically via poloidal and toroidal scalars (see, e.g., Backus, 1986). On and above Earth's roughly spheroidal surface, however, the field of deep internal origin can be represented by the negative gradient of a scalar potential V that satifies Laplace's equation ($B = -\nabla V$, $\nabla^2 V = 0$). At time t and position r in geocentric spherical polar coordinates (radius r, colatitude θ, and east longitude ϕ), the spherical harmonic expansion of zero mean V is well-known to be

$$V(r,t) = a \sum_{n=1}^{\infty} (a/r)^{n+1} \sum_{m=0}^{n} [g_n^m(t)\cos m\phi + h_n^m(t)\sin m\phi]P_n^m(\cos\theta) , \qquad (1)$$

where P_n^m is the Schmidt-normalized associated Legendre function of degree n and order m, and $[g_n^m(t), h_n^m(t)]$ are Schmidt-normalized Gauss coefficients at reference radius a, here taken to be 6371.2 km.

Gauss coefficients have long been estimated by a weighted least squares fit to measured geomagnetic data (see, e.g., Langel, 1987). This usually requires truncating the sum over degrees at finite N. Accurate statistical information about Gauss coefficients could supplement such regularizing truncation, increase the accuracy of estimated coefficients, and improve the reliability of associated uncertainty estimates (Gubbins,

1983; McLeod, 1986; Backus, 1988). Lacking sufficient prior information, a statistical hypothesis may be advanced and tested against observations. The *physical* significance of such tests depends upon the physical basis of the hypothesis tested. Discrepancies between theory and observation often lead to more accurate hypotheses and measurements. So let us consider such hypotheses, develop their theoretical predictions, and test the predictions against geomagnetic and paleomagnetic observations.

Many hypotheses concern the internal spatial geomagnetic power spectrum, denoted via R_n. Lowes (1966) and others (see Cain et al., 1989) show the mean square potential field represented by spherical harmonics of degree n, averaged over a spherical shell of radius r enclosing the sources, to be

$$R_n(r,t) = (n + 1)(a/r)^{2n+4} \sum_{m=0}^{n} ([g_n^m(t)]^2 + [h_n^m(t)]^2) . \qquad (2)$$

For vacuum magnetic permeability μ_0, the magnetic energy per unit shell thickness due to an equivalent, geocentric, n^{th} degree multipole source equals $2\pi r^2 R_n/\mu_0$. The latter form a discrete spectrum with units of spatial power or force. "Multipole powers" R_n have SI units of $(Tesla)^2$ and collectively sum to the mean square field on the shell. Invariant under coordinate rotations, but not translations, the R_n offer isotropic, inhomogenous measures of an evidently anisotropic and heterogenous field. The spatial power spectrum of the secular variation, denoted $F_n(r,t)$, is obtained by replacing the Gauss coefficients in (2) with secular variation (SV) coefficients $(\partial_t g_n^m, \partial_t h_n^m)$; note $\partial_t^2 R_n \neq F_n \neq \partial_t R_n$.

Multipole powers $R_n(a,1980)$ computed from coefficients fitted to Magsat data are interpreted in terms of a predominantly core-source spectrum R_{nc} for $n \leq 12$ and a predominantly crustal-source spectrum R_{nx} for $n \geq 16$ (Langel & Estes, 1982). Corrections for lithospheric magnetization and finite mantle resistivity are arguably small only at low degrees and long periods, respectively; therefore, to ease description of a field originating in a core of mean radius c, use of (2) is here limited to $n \leq 12$, super-annual SV, and $r \geq c$. Seismologic estimates of c vary, but are denoted $c_s = 3480$ km (Dziewonski & Anderson, 1981; Kennet, Engdahl & Buland, 1995).

Several functional forms have been suggested for R_{nc}. The theoretical core-source geomagnetic spectrum of McLeod (1985; 1996 equation 20a) earns special attention, not only becasue it can be closely fitted to observational R_n, but because it was obtained from consideration of advective and diffusive effects on a core field. McLeod's rule states that the spatial magnetic power spectrum of the core-source field, $R_{nc}(a)$, is approximately proportional to $(n + 1/2)^{-1}(c/a)^{2n}$ for finite degrees $2 \leq n \leq N'$. Here we advance and test the hypothesis that this form approximates geologically long time average (over ~50 Myr), or expectation, multipole powers $\{R_{nc}\}$ throughout a finite magnetic energy range $1 \leq n \leq N_E$. Evidently,

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$N_E \geq 12$. Natural fluctuations about such mean values are to be expected from time-dependent magnetic, mass, momentum and energy transport within Earth's core. Over longer time intervals, changes in outer core boundary conditions, composition, and stratification may change $\{R_{nc}\}$ itself.

In Part I, after summarizing some previous work with the spectrum, a simple physical model of the core-mantle boundary region is used to develop an expected core multipole spectrum. The model features laterally decorrelated magnetic transport beneath a thin viscous sub-layer. This specifies the form of SV spectrum $\{F_{nc}\}$. A relation between *effective* radial wavenumbers and horizontal wavenumbers in the vertical field component is needed to specify $\{R_{nc}\}$. One such relation is advanced and tested against observational spectral ratios with satisfactory results. It yields an expected spectrum similar to McLeod's. A trial distribution function for R_{nc} is advanced and discussed, mainly to guide the paleomagnetic predictions and tests presented in Part II. Geomagnetic predictions of the mean radius of Earth's core based on McLeod's rule are then tested against seismologic values and found to be quite accurate.

2 SOME PREVIOUS SPECTRAL FORMS

Exponential Spectra. The geometric attenuation factor, $(a/r)^{2n+4}$ in (2), is a power law in radius; yet it is exponential in degree. Following Lowes (1974), $R_n(a)$ has often been approximated by the exponential $A^*(c^*/a)^{2n+4}$. Linear regression through observational values of $\log[R_n(a)]$ gives

$$\log[R_n^*(a)] = [2\log(c^*/a)]n + [\log(A^*) + 4\log(c^*/a)] . \tag{3}$$

The slope of this line implies the radius c^* at which downwardly continued spectrum $R_n^*(a)[a/r]^{2n+4}$, denoted $R_n^*(r)$, would appear independent of n. A graph of $\log[R_n^*(r)]$ as a function of n is a straight line that "levels off" at $r = c^*$. If $R_n^*(r)$ were extrapolated to arbitrarily high degree, then its sum over n would diverge at $r \leq c^*$; therefore, such extrapolation is physically invalid at $r \leq c^*$. This might be due to failure of the potential field representation, so c^* might be the minimum radius of a sphere containing the sources. Lowes' (1974) spectrum, fitted to degrees 1 through 8, levels off about 480 km below c_s. Modern dipole power R_1 exceeds R_1^* as well as higher multipole powers; it has often been excluded from subsequent regressions, which raises c^* towards c_s.

Langel & Estes (1982) interpret $R_n(a)$ from their degree 23 model MGST 10/81 of Magsat data in terms of dominant core dipole, core non-dipole ($2 \leq n \leq 12$), and crustal fields ($n \geq 16$). Their non-dipole core spectrum R_{nc}^* levels off 174 km below c_s. Their crustal spectrum R_{nx}^* levels off 83 km below a.

Voorhies (1984) used only the mean square radial field component per harmonic degree from model MGST 10/81, which is $(n+1)R_n/(2n+1)$. His non-dipole core spectrum $B_{rc}^{*2}(n)$ levels off 200 km below c_s

and was used to argue against narrow scale, intensely magnetized core spots at the core-mantle boundary. His crustal spectrum $B_{rx}^{*2}(n)$ levels off 9.9 km below a, perhaps suggesting crustal rather than deep lithospheric sources. About 10% of R_{12} was considered crustal in origin.

Cain et al. (1989) used their degree 63 numerical integration model M07AV6 of Magsat data to obtain a non-dipole core spectrum that levels off 73 km below c_s; a crustal spectrum that flattens out 14 km below the reference sphere; and a satellite altitude noise level.

Constable & Parker (1988) chose a non-dipole spectrum which is exactly flat at the core surface to guide construction of a statistical model of Gauss coefficients. Except for the manifestly anisotropic dipole, and perhaps quadrupole, normal probability distribution functions (PDFs) with m-independent variances were assigned to the coefficients. The n-dependent variances were obtained from a level spectrum at c_s fitted to observational R_n of model GSFC 9/80 (Langel, et al., 1982). The model passed a Kolmogorov-Smirnoff test for degrees 2-8.

Hulot, LeMouël & Wahr (1992) inferred a non-dipole core spectrum that levels off about 235 km below c_s. The stationary isotropic statistical model of Hulot & LeMouël (1994) also assigns normal PDFs with m-independent variances to non-dipole Gauss coefficients. Variances obtained from R_{nc}^* decrease quite rapidly with n due to the depth of c^* below c_s. They also offer impressive non-dipole Kolmogorov-Smirnoff test results, note that isotropic models give chi-squared PDFs for normalized non-dipole powers, and further suggest a PDF for the axial dipole coefficient that combines two Gaussian distributions: one for normal and one for reversed polarities (see McFadden & McElhinny, 1982).

The statistical models cited above allow for the dominant geometric attenuation of $R_{nc}(r)$ with radius and seem to offer unobjectionable, albeit different, statistical hypotheses. Yet physical reasons to expect non-interdependent Gauss coefficients, isotropy, equal degree variances at some radius c^*, and a purely exponential spectrum - all while granting special exemptions to the axial dipole - are elusive.

For example, with magnetic permeability μ and electric conductivity σ simply treated as if uniform in a spherical fluid outer core moving at velocity $v(r,t)$, analysis of the well-known induction equation,

$$\partial_t B = \nabla \times (v \times B) + (\mu\sigma)^{-1} \nabla^2 B , \tag{4}$$

indeed shows that magnetic diffusion of different spherical harmonic modes proceeds independently; yet poloidal and toroidal magnetic modes are cross-coupled by heterogeneous fluid motion – the motion needed to maintain a core-source field against resistive decay of its source currents over geologic time via dynamo action (see, e.g., Larmor, 1919; Elsasser, 1946; Gubbins & Roberts, 1987). Moreover, Earth's

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present dipole is evidently anisotropic and has power far greater than obtained by extrapolating the foregoing exponential spectra to the first degree. Furthermore, no firmly established, fully developed statistical theory of conceivably turbulent core magnetohydrodynamic transport appears to demand equal mean magnetic pressure per harmonic degree at radius c^* (level $R_n(c^*)$) instead of, say, equal total magnetic energy outside the core per harmonic degree and order (level $(n+1)(2n+1)R_n(c)$), equal mean radial gradient in magnetic pressure per harmonic degree atop the core (level $R_n(c)/(n+2)$), or some other spectral form at degrees below 13. Alternatives to a purely exponential spectrum, and to statistical presumption of isotropy, independence, and normally distributed coefficients, are available and have been used to describe geo-paleomagnetic field behavior.

Power Law Spectra by the Core-Mantle Boundary. The exponential decay of $R_n^*(r \geq c_s > c^*)$ with n is faster than needed for finite mean magnetic energy density (faster than $n^{-1-\varepsilon}$ with $\varepsilon > 0$). Indeed, the finitude of Ohmic dissipation within the core and Gubbins' (1975) famous expression for the minimum value thereof imply that, for degrees in a magnetic dissipation range defined by $n \geq N_D$,

$$R_{nc}(a) \leq K_G\, n^{-2-\delta}\, (c/a)^{2n} \ , \tag{5}$$

where K_G is a constant and $\delta > 0$. Though finite, N_D might be very much greater than 12.

Much work on idealized hydromagnetic turbulence has considered the magnetic energy spectrum $E(k)$ as a function of Cartesian wavenumber k (see, e.g., Gubbins, 1974; Pouquet, Frisch & Léorat, 1976; Moffat, 1978; Krause & Rädler, 1980). Although the proportionality $E(k) \propto k^{-3/2}$ is indicated for the inertial subrange of three-dimensional, homogeneous, isotropic, incompressible hydromagnetic turbulence, Pouquet et al. (1976) found that kinetic helicity injection leads to an inverse cascade of magnetic helicity and $E(k) \propto k^{-1}$ at low k. Citing these findings, Stevenson (1983) suggested that a power law spectrum is possible – albeit difficult to apply due to anisotropy. His arguments indicate k^{-1} and he noted some similiarity between an n^{-1} spectrum and observational R_n downwardly continued to radius 0.55a for $n \leq 8$.

Define horizontal wavenumber $k_{hn} \equiv [n(n+1)/c^2]^{1/2}$ via the surface Laplacian. If $E(k) \propto k^{-1}$, if $E(k) \propto \{R_{nc}(c)\}$, and if $k^2 \propto k_{hn}^2$, then $\{R_{nc}(c)\} \propto k^{-1} \propto [n(n+1)]^{-1/2}$. The latter form is closer to McLeod's $(n + 1/2)^{-1}$ than Stevenson's n^{-1} relation, notably at degrees 1 and 2.

Gubbins & Bloxham (1985) discuss solution norms for supplementary regularization of core-source field models. Each such norm corresponds to a downwardly continued spectrum that is inversely proportional to a polynomial of n. Norms compatible with (5) are needed for $n \geq N_D$. Solution norms may also be used to construct a smooth, or damped, core field models for $n \leq N_D$ (see also Shure, Parker &

Backus (1982); Langel (1987); and Backus (1988)). Such models have many uses; however, a desire for a smooth model ought not be mistaken for measured data. To avoid built-in assumptions about c or R_{nc}, smoothed field models are not used in this paper.

McLeod's (1985; 1994, 1996) theoretical core-source spectrum for degrees $2 \leq n \leq N'$ is written

$$R_{nc}(a) \approx R_{nc}^M(a) = K_M (n + 1/2)^{-1} (c/a)^{2n} , \tag{6}$$

where $K_M = 5 \times 10^9$ nT^2. The downwardly continued spectrum $R_{nc}^M(r)$ is proportional to $(n + 1/2)^{-1}$ at the core surface; there is no radius at which it levels off. McLeod (1996) derived (6) from functional forms of (i) the core-source SV spectrum F_{nc} and (ii) the *temporal* geomagnetic power spectrum $P_n(\omega)$ that depends upon temporal frequency ω as well as harmonic degree. The form of F_{nc} is akin to that caused by random lateral advection of magnetic field lines at the top of a high conductivity liquid core. The form of $P_n(\omega)$ is appropriate to a two time-scale model of the two processes, magnetic flux diffusion and fluid motion, that change the core field according to the induction equation (4).

Voorhies & Conrad (1996) questioned the existence of physical sources giving a level spectrum, the exclusion of dipole power from estimates of R_{nc}^*, and the idea that $N_D < 14$. Examination of the spectrum from the degree 60 Magsat model M102189 of Cain, Holter & Sandee (1990) confirmed that dipole power is large, and quadrupole power is small, compared with a trend at degrees 3-12. This was thought to be an artifact of geologic undersampling: values for R_1 and R_2 at 1980, being calculated from but 3 and 5 Gauss coefficients respectively, could be fairly far from multi-million year mean values. If so, perhaps a key to past dipole behavior can be found in present higher degree multipoles. We tried the power law form at c_s ($R_{nc}^\gamma(c_s) = K_\gamma n^{-\gamma}$) by fitting a linear function of $\ln(n)$ to $\ln[R_n(c_s)]$ for degrees 3-12. We found γ to be 0.94. Crustal sources may lead to a slight underestimate of γ, so we set γ to 1, estimated K_γ alone, and found the summed squared residuals per degree of freedom to be less than when γ and K_γ were coestimated. Moreover, the extrapolation to dipole power appeared much more satisfactory than for a plain exponential form. With neither theoretical nor experimental reasons to assume a level spectrum at either c_s or c^*, and seeing fair predictions for R_1 and R_2, we learned to expect a core-source spectrum of the form

$$R_{nc}(r,t) \approx \{R_{nc}^1(r)\} = K_1 n^{-1} (c/r)^{2n+4} \tag{7}$$

for $1 \leq n \leq N_E$, where the curly brackets represent expectation value and N_E is the finite, maximum degree of the magnetic energy range ($1 \leq N_E < N_D$). Whether or not a Stevenson (1983) relation like (7) describes multi-Myr means, it does not specify changes in c, γ, or K_γ due to planetary evolution over several Gyr.

If (7) accurately describes the time-averaged core-source field, then natural deviations $[R_{nc}(t) - \{R_{nc}\}]$ eventually relax to zero and perhaps change sign. Finding R_1 somewhat high, and R_2 low, at 1980 led us to predict relaxation of these deviations. We tested this against the main field and SV coefficients of undamped model GSFC 12/83 (Langel & Estes, 1985). R_1 is decreasing and R_2 is increasing as predicted. Moreover, for all orders m, $(\partial_t g_1{}^m)/g_1{}^m < 0$ and $(\partial_t g_2{}^m)/g_2{}^m > 0$ for both $g_n{}^m$ and $h_n{}^m$. The plain chance of such perfect correlation of signs is 1/8 for the dipole, 1/32 for the quadrupole, and 1/256 for all eight coefficients together. This otherwise remarkable coincidence can be viewed as efficient relaxation of the core-source field towards geologic mean values like (6) or (7).

3 STATISTICAL PHYSICS OF THE CORE-SOURCE FIELD

For simplicity, the transition from core to mantle is approximated by a sharp material interface: a jump discontinuity in molecular material properties with radius. Both hydrodynamic and macroscopic magnetic stress tensors are, in contrast, treated as continuous across the interface to ensure finite divergence of the magnetohydrodynamic stress and finite viscous and Lorentz force densities. If the fluid wets the mantle and is of approximately Newtonian rheology, then motion of the viscous fluid vanishes at the interface. The molecular kinematic shear viscosity of the liquid metal v is very small, about 3×10^{-7} m^2/s (Poirer, 1988; also see Lumb & Alldridge, 1991), so lateral motion ought not be neglected beneath a thin viscous sub-layer. A laminar sub-layer as thin as the Ekman depth d_E or the Hartmann depth d_H of about 8 cm or 20 cm, respectively, need not imply that diffusion is unimportant in the sub-adjacent region, which might be a thicker boundary layer that has not usually been distinguished from an underlying main stream.

Traditional analyses yield a thin, weak visco-magnetic boundary layer that neither absorbs appreciable normal fluid flow nor generates much electric current (Roberts & Scott, 1965; Backus, 1968; Ball, Kahle & Vestine 1968; Hide & Stewartson, 1972; Benton, 1981; Gubbins & Roberts, 1987; Benton, 1992). Advection of the magnetic field at the top of the main stream is thus mainly lateral; moreover, the jump in magnetic field across such a layer is so small as to be negligible for the radial component B_r. The radial component of (4) has thus been used to interpret SV in terms of broad scale fluid flow, and occasionally flux diffusion, at the top of the main stream. Constrained inversions of modern geomagnetic secular change indicate a main stream speed U of about 7.5 km/yr (see, e.g., Voorhies, 1995). Such inversions often damp out poorly resolved narrow scale motions. Yet the corresponding main Reynolds number Uc_s/v of about 3×10^9 is so much greater than the boundary number Ud_E/v of about 60 as to suggest some ephemeral small scale motions occur in the core. Secular change caused by such motions might be described via a pseudo-random walk of magnetic field line footpoints.

For example, Benton's (1992) lateral flow scale of 4.8 km might describe the width of fronts between broad regions of more uniform core flow, but it might also describe a seething mass of short-lived, rotationally polarized hydromagnetic eddies. Such eddies would individually induce unobservably narrow scale field variations (degrees of roughly 3,000), yet may collectively contribute to observable broad scale SV. Statistical parameterization of this contribution could be an efficient, useful, and informative complement to deterministic modeling.

There is a direct statistical approach to lateral magnetic transport at the base of the viscous sub-layer: the equivalent source approach of McLeod (1996). This yields a kinematically unbiased ensemble mean SV spectrum $\{F_n\}$ (section 3.1). Whether or not main stream dynamics cause geologic time averages to differ from such an ensemble average, observable core signal diffuses through the sub-layer and resistive mantle. To do so, magnetic change originating deeper in the core, be it induced by eddies 2 or 22,000 km across, induces electric current, hence excess field line curvature ($\nabla^2 B/|B|$), in the sub-layer (section 3.2). One hypothesis about such curvature is advanced and tested by observation and analysis of the emergent core field (section 3.3). It yields an expected spatial power spectrum *for the broad-scale core-source field* $\{R_{nc}\}$ similar to McLeod's rule. It does not specify spectral variance $\{[R_{nc} - \{R_{nc}\}]^2\}$ or higher moments. Closure is sought via trial PDFs for R_{nc} that (i) have variances determined by the mean and (ii) neither require nor prohibit isotropy (section 3.4).

3.1 Statistical Kinematics of the Secular Variation.

Following McLeod (1996), quasi-static lateral magnetic transport causes differential exterior secular change ΔB equivalent to differential dipole moments Δd_i scattered atop the main stream ($r = c^-$). To see this, recall that a single magnetic flux vector $B_0 dA$ at fixed position x_0 on the surface c^- of the source region acts as the point source of a dipole field with moment proportional to $B_0 dA$. The magnetostatic field at position x due to this *equivalent source* is well-known (see, e.g., Jackson, 1975 equation 5.64). Infinitesimal quasi-static lateral displacement Δx of this single magnetic vector, with no change in orientation and magnitude, causes a net change in the exterior field equivalent to a differential quadrupole moment at $x_0 + \Delta x/2$. More generally, there is a magnetic vector at each position on the source-surface; lateral transport can replace the vector at x_0 with an adjacent vector of slightly different orientation and magnitude; and the change in the exterior field is equivalent to that of a differential *dipole* moment Δd_0 at x_0. Given many equivalent source changes Δd_i at x_i on c^-, the total change in the exterior field ΔB at $|x|$ > c^- follows by superposition. In the continuum, differential surface magnetic moment density at x' on c^- replaces discrete Δd_i as the equivalent source of exterior secular change $\Delta B(x,t)$, which follows by

integration over **x'**. Random lateral transport of the core-source field thus produces a corresponding change in the field above.

Elements of the dyad formed by two differential dipole moments at two well separated points $[\Delta \mathbf{d}_1][\Delta \mathbf{d}_2]^T$ may be either positive or negative. The average over a kinematically unbiased ensemble of dyads gives zero cross-correlations, but non-zero auto-correlations. Then the expected spatial power spectrum *for broad-scale secular variation* is equivalent to the spectrum for laterally uncorrelated, randomly varying dipole moments on the shell of radius c^-

$$\{F_{nc}(r>c^-)\} = C\, n\,(n+1/2)\,(n+1)\,(c^-\!/r)^{2n+4} \,. \tag{8}$$

This differs slightly from McLeod (1996, equation (11)) because changes in horizontal as well as radial components of the equivalent core-source field can contribute to F_{nc}. (Consider rotation about the vertical of a horizontal equivalent source on the equator). Laterally decorrelated point sources may offer the roughest physical model of SV, yet the sum of geometrically attenuated cubic spectrum (8) over n converges on all spheres above the equivalent source layer.

When (8) is used to describe a geologic time average, natural fluctuations $[F_n(t) - \{F_n\}]$ are described as transient lateral cross-correlations between equivalent SV sources. Failure of (8) at spatial scales corresponding to $n \geq N_E$ may in part be due to persistent mesoscale cross-correlations. Moreover, description of non-stationarity via (8) is limited to changes in amplitude C and in c^- between geologic time averaging intervals. Because C is proportional to mean square differential dipole moment $\{(\Delta \mathbf{d}_i /\Delta t)^2\}$, it should, by (4), tend to increase with mean transport speed and mean field intensity.

Remarks. The quasi-magnetostatic derivation of (8) is so fundamental that it does not require use of Faraday's induction equation. Magnetic transport equation (4) obtains when (i) Galilean invariance and Ohm's law, hence transport of non-uniform field $\mathbf{B}(r,t)$ by non-uniform fluid velocity $\mathbf{v}(r,t)$, replace transport of equivalent sources; (ii) Ampere currents of density $\mathbf{J}$ are the true core-sources; (iii) Maxwell displacement currents are filtered out via the quasi-steady approximation ($\nabla \cdot \mathbf{J} = 0$); and (iv) μ and σ are treated as if uniform scalars. For stationary reference mass density $\rho_0(r)$, the anelastic approximation of Gubbins & Roberts (1987) is replaced with $\nabla \cdot \rho_0 \mathbf{v} = 0$. Then expansion of a rising fluid parcel ($\nabla \cdot \mathbf{v} = -\mathbf{v} \cdot \nabla \ln \rho_0 > 0$) tends to reduce its magnetic flux density and non-stationarity is discussed separately. At the base of the sub-layer, omitting terms proportional to small v_r (hence $\nabla \cdot \mathbf{v}$), the radial component of (4) is

$$\partial_t B_r + \mathbf{v} \cdot \nabla_s B_r = B_r \partial_r v_r + (\mu \sigma c)^{-1} \nabla^2_r B_r \,, \tag{9}$$

where ∇_s denotes the surface gradient operator.

Observable changes in the core-source field caused by advection $(-v \bullet \nabla_s B_r)$ and by downwelling $(B_r \partial_r v_r)$ are *indistinguishable* from quasi-static lateral transport and confluence of equivalent sources. The broad scale part of these changes are here described via (8). Ohmic resistance to Ampere currents appears as magnetic diffusion in (9). Observable changes in the field due to diffusion are indistinguishable from changes in equivalent source moments. Whether or not magnetic diffusion alters (8), as seems likely for n $\geq N_D$ and during laterally correlated free decay, it is essential in the sub-layer where v falls to zero.

The lateral decorrelation used to obtain (8) by no means prohibits turbulent dynamo action at depth; indeed, the more precisely correlated fluctuations balance diffusion, the better (8) should describe any residual SV. Although changes in horizontal field components also contribute to (8), the normal field component on the two boundaries $(r = c, r \rightarrow \infty)$ still determines the exterior potential field. So long as B_r remains effectively continuous across the sub-layer, so does $\partial_t B_r$; then jumps in $\nabla_s \bullet B_r v$ and $(\mu \sigma r)^{-1} \nabla^2 r B_r$ across the sub-layer are the same by (9). The statistical description of broad-scale SV (8) is, by continuity, therefore applied atop the core-mantle interface.

3.2 Apparent Scale Heights and Characteristic Time-Constants

At the top of the viscous sub-layer, v vanishes by the no-slip boundary condition. There (4) reduces to the magnetic diffusion equation with radial component

$$\partial_t B_r = (\mu \sigma c)^{-1} \nabla^2 r B_r = (\mu \sigma)^{-1} \partial_r^2 B_r^*. \tag{10a}$$

Here $\partial_r^2 B_r^*$ denotes the jump in $\partial_r^2 B_r$ across the core-mantle interface. Clearly B_r^* is the core-source correction $(B_r + \partial_r V)$ for $r < c$. It vanishes at c by continuity. From (10a) alone, B_r^* would seem comparable in magnitude to $B_r(c)$ at a depth below c, denoted k_r^{-1}, of about $[<[B_r(c)]^2>/<[\mu \sigma \partial_t B_r(c)]^2>]^{1/2}$, where the angle brackets denote the average over the sphere. This is an *apparent* scale height because (10a) does not hold in moving fluid below c. At Magsat epoch 1980, root mean square (rms) values for the downwardly continued broad scale radial main field and SV are about 3 gauss and 3 μT/yr; with $\mu = \mu_o$, assuming $\sigma \approx 5 \times 10^5$ S/m would return an apparent scale height of about 70 km.

Denote real, Schmidt-normalized spherical harmonic coefficients of B_r^* within the core by $[G_n^m(r,t), H_n^m(r,t)]$. Omitting corrections for mantle conductivity and crustal sources, (10a) implies

$$(n + 1)(a/c)^{n+2} \partial_t g_n^m = (\mu \sigma)^{-1} \partial_r^2 G_n^m$$
$$(n + 1)(a/c)^{n+2} \partial_t h_n^m = (\mu \sigma)^{-1} \partial_r^2 H_n^m \tag{10b}$$

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(This plausibility argument neither suggests nor requires a weak field deep in the core). More generally, when the physical dependencies of $[k_{rn}(t)]^4$ on underlying hydromagnetic interactions, and on magnetic modes contributing to many other multipole powers $R_{jc}(c,t)$ for $j \neq n$, dominate its dependence on $R_{nc}(c,t)$, then (13) can be a useful approximation.

Statistically, for any set of PDFs assigned to (g_n^m, h_n^m) and for any set of PDFs assigned to $[\partial_r^2 G_n^m, \partial_r^2 H_n^m]$ at r = c, define the ratio of expectation values $\{B_{rc}^2(n)\}/\{[\partial_r^2 B_r^*(n)]^2\}$ to be k_{rn}^{-4}; k_{rn} is the *effective* radial wavenumber for the radial field configured in harmonics of degree n. As in (11a-d), one obtains

$$\{F_{nc}\} = (\mu\sigma k_{rn}^2)^2 \{R_{nc}\} . \tag{14a}$$

To the extent that expectation values correspond to geologic time averages, (14a) corresponds to (12) rather than (13); indeed, only during intervals when correlated fluctuations are negligible compared with the product of means could carefully averaged apparent scale heights $\{[k_{rn}(t)]^4\}^{-1/4}$ be approximated by reciprocal effective radial wavenumbers k_{rn}^{-1}. This might be so when excess curvature at low degree n is driven by narrower scale fluid motions, perhaps an ensemble of ephemeral eddies underlying the laminar sub-layer, that depend but weakly on multipole power at degee n.

Similarly, for any set of PDFs assigned to (g_n^m, h_n^m) and for any set of PDFs assigned to $[\partial_t g_n^m, \partial_t h_n^m]$, define the ratio of expectation values $\{R_{nc}\}/\{F_{nc}\}$ to be τ_n^2. The characteristic time-constants

$$\tau_n \equiv (\{R_{nc}\}/\{F_{nc}\})^{1/2} = \mu\sigma(k_{rn})^{-2} , \tag{14b}$$

remain independent of radius in a source-free exterior. For example, if PDFs assigned to (g_n^m, h_n^m) were independent of m, and, in marked contrast to (8), SV were attributed to uniform westward drift at azimuthal velocity $-w_0\sin\theta$ alone, then $\{F_{nc}\}$ would be $(nw_0/c)^2\{R_{nc}\}/3$ and τ_n^2 would be $3c^2(nw_0)^{-2}$; however, all the R_n, F_n, and g_n^0 would be steady.

3.3 Constant Aspect Ratio Hypothesis

Effective radial wavenumbers k_{rn} for the core-source field can be compared with known horizontal wavenumbers k_{hn}. To do so, introduce the square aspect ratio $A_n \equiv k_{rn}^2/k_{hn}^2$ and rewrite (14b) as

$$\tau_n \equiv (\{R_{nc}\}/\{F_{nc}\})^{1/2} = \mu\sigma c^2[A_n \, n(n+1)]^{-1} . \tag{15}$$

With no kinematical reason to prefer aspect ratios that either increase or decrease with n, constant $A_n = A$ seems the natural hypothesis to be tested.

Some insight into this hypothesis is obtained by comparing radial diffusivity $(\tau_n k_m^2)^{-1}$, which is molecular $(\mu\sigma)^{-1}$ in the anisotropic laminar sub-layer, with lateral empirical diffusivities $(\tau_n k_{hn}^2)^{-1} = A_n/\mu\sigma$. (These are "empirical" only in that they might eventually be determined from the ratio of geologic time-averaged observational spectra τ_n). The A_n are amplification factors converting molecular diffusivity into lateral empirical diffusivities. They are also indicies of anisotropy. If there were but one lateral empirical diffusivity equal to a single lateral eddy diffusivity UL, then the single squared aspect ratio A would equal the magnetic Reynolds number $\mu\sigma$UL. This describes a scale invariant diffusive anisotropy. In contrast, the uniform westward drift example above has but one lateral eddy, yet many lateral empirical diffusivities, and would have A_n be $\mu\sigma w_0 c(3)^{-1/2}(n+1)^{-1}$.

More generally, many superimposed eddies of many lateral scales k_{hi}^{-1}, speeds $U_i(t)$, and circulation times $|U_i(t)k_{hi}|^{-1}$ can reconfigure flux from main field degrees j to other degrees n via (9). From a purely kinematic viewpoint, the many lateral eddy diffusivities $|U_i(t)/k_{hi}|$ do not explictly depend upon this SV degree n. Even an implicit dependence via selection rules (e.g., $n \sim |i \pm j|$) becomes obscured over time as eddies mix and remix all magnetic modes via interactions with intermediating modes. Moreover, an ensemble mean lateral eddy diffusivity, averaged over a kinematically unbiased ensemble of ephemeral eddies, cannot depend on the SV the eddies happen to induce. When, on average, neither radial molecular nor lateral eddy diffusivities $\{|U_i(t)/k_{hi}|\}$ depend on SV degree, they cannot be used to construct lateral empirical diffusivities that do. The hypothesis of one such quantity, hence constant aspect ratio, remains as the seemingly natural null-hypothesis. Of course, detailed kinematical assumption (e.g., pure westward drift) or dynamical argumentation (e.g., strong Lorentz feedback) as to which particular eddies and magnetic modes might eventually dominate can suggest other hypotheses.

This hypothesis is necessarily restricted to horizontal scales exceeding the thickness of the sub-layer itself ($k_{hn}^{-1} \gg d_E$). It is clearly intended for magnetic scales ($n \le N_E$, $n \ll N_D$) broader than those dominated by molecular diffusion, hence time scales long compared with sub-layer delay time (e.g., $\tau_n \gg \mu\sigma d_H^2$). Moreover, so long as fluctuations in R_n and F_n are about the same magnitude as the means, a single epoch value for a single spectral ratio (e.g., $R_6(t_1)/F_6(t_1)$) could easily differ from τ_n by a factor of two or more. Such natural variability ought not be mistaken for a multiplicity of aspect ratios.

Observational Check. Reliable, undamped geomagnetic field models through degree and order 12 or more are available, albeit not for the geologically long time intervals needed to determine $\{F_n(t)\}/\{R_n(t)\}$. Observational values of $F_n(a,t)/R_n(a,t)$ were, however, calculated for epochs 1960, 1970,

and 1980 from undamped Magsat model GSFC 9/80 (Langel et al., 1982) fifteen years ago in hopes of deducing a dispersion relation for SV. The constant aspect ratio hypothesis predicts

$$\tau_n^{-2} = \{F_{nc}\}/\{R_{nc}\} = (A/\mu\sigma c^2)^2[n(n+1)]^2 \; ; \tag{16}$$

therefore, at each epoch we fitted $\ln A' + \beta \ln[n(n+1)]$ by least squares to the observational $\ln(F_n/R_n)$ for degrees 3 through 12. The three resulting estimates of β average to 1.957 ± 0.156 (1σ). This agrees with the constant aspect ratio prediction $\beta = 2$. It does not agree with the uniform westward drift example above.

The three estimates of $\ln A'$ indicate $(A')^{-1/2}$ is 2,639 years to within a factor of 1.94 (2σ); therefore, very recent field behavior suggests $\tau_n \approx 2,640/[n(n+1)]$ years. The extrapolation of this τ_n relation to the dipole, $\tau_1 \approx 1320$ yr, agrees with the observational values $[R_1(t)/F_1(t)]^{1/2}$ from GSFC 9/80, which average to 1275 yr. The extrapolated range for the quadrupole, $\tau_2 \approx 220\text{-}880$ yr, exceeds observational $[R_2(t)/F_2(t)]^{1/2}$, which average to 126 yr, perhaps due to a rapid rebound from diminished R_2.

With this τ_n relation and core radius $c = c_m = 3.5$ Mm (from section 4), the lateral empirical diffusivity is about 150 m^2/s. With (14b), and supposing $k_{r1} \geq \pi/c$ to allow forced dipole decay, this τ_n relation also offers an upper bound on core magnetic diffusivity. The bound, $(\mu\sigma)^{-1} \leq 30$ m^2/s, is uncertain by a factor of two, as is the lower bound on core conductivity, $\sigma \geq 2.7 \times 10^4$ S/m, obtained by assuming $\mu = \mu_0$. The bound is consistent with a high conductivity liquid metallic outer core.

When emerging SV is traced to underlying fluid motion, the bound may be interpreted as a turbulent diffusivity. To reckon molecular magnetic diffusivity, divide 30 m^2/s by the *effective* magnetic Reynolds number $R_m^* = 2A/\pi^2$. If a molecular conductivity is 5×10^5 S/m is assumed, then R_m^* is 19 and square aspect ratio A is about 92, each to within a factor of two.

Generalized Stevenson-McLeod Relations. On the constant aspect ratio hypothesis, equations (16) and (8) at the top of the sub-layer are combined to yield

$$
\begin{aligned}
\{R_{nc}(r)\} = \tau_n^2\{F_{nc}(r)\} &= [\mu\sigma c^2/An(n+1)]^2[Cn(n+1/2)(n+1)](c/r)^{2n+4} \\
&= K(n+1/2)[n(n+1)]^{-1}(c/r)^{2n+4} \tag{17a} \\
&\approx K[n(n+1)]^{-1/2}(c/r)^{2n+4} \tag{17b} \\
&\approx K(n+1/2)^{-1}(c/r)^{2n+4} \tag{17c}
\end{aligned}
$$

for $1 \leq n \leq N_E$ and $r \geq c$. With K set to $K_M(a/c)^4$ and r set to a, (17c) reduces to McLeod's rule (6); however, (17) is obtained from constant aspect ratio, instead of a form for $P_n(\omega)$, and does not exclude

dipole power. This allows more extensive testing. Failure of (17) at narrow spatial scales ($n \approx N_E$) may in part be due to failure of (16) as well as (8).

Particular spectral forms (7) and (17a-c) are of general form

$$\{R_{nc}(r)\} = K\, q(n)\, (c/r)^{2n+4}. \tag{18}$$

The various $q(n)$ approach Stevenson's (1983) relation at high n; they are approximately $2/(2n+1)$ and are thus but minor variations on McLeod's rule when compared with extending it to the dipole and to geologically long time intervals. The latter correspond to the hypothesis that, over such intervals, the core-source dipole field *at the top of the core* might be exceptional compared with other multipole fields in its nearly axial orientation, but not in its magnitude.

3.4 Statistical Core Field Hypothesis

With the expectation value of core multipole power $\{R_{nc}\}$ given by (17), the statistical core field hypothesis advanced here is that, over geologic time intervals, normalized core magnetic multipole power $(2n+1)R_{nc}/\{R_{nc}\}$ at radii $r \geq c$ is distributed as chi-squared (χ^2) with $2n+1$ degrees of freedom. These particular trial probability distribution functions (PDFs) for the R_{nc} are

$$\wp[(2n+1)R_{nc}/\{R_{nc}\}] = \wp_{2n+1}(\chi^2) = [2^{n+1/2}\,\Gamma(n+1/2)]^{-1}\,(\chi^2)^{n-1/2}\,\exp[-\chi^2/2], \tag{19}$$

where Γ is the gamma function, the $\{R_{nc}\}$ are from (17), and $1 \leq n \leq N_E$. By (19), standard tables giving the probability of obtaining $\chi^2 \leq X^2$ also give the probability of finding $R_{nc}/\{R_{nc}\} \leq X^2/(2n+1)$. Note that (19) does not postulate PDFs for each individual Gauss coefficient; moreover, $\{R_{nc}\}$ from (17) is not a plain exponential and does not level off at any depth.

Distribution (19) neither requires nor prohibits equipartitioning of multipole power among the various orders m within degree n. It holds if Gauss coefficients of degree n are random samples of a population with a zero mean Gaussian PDF of variance $\{R_{nc}(a)\}/[(n+1)(2n+1)]$. It can also hold when these coefficients are not normally distributed. Indeed, there are an infinite number of PDFs for individual coefficients of any particular degree that yield (19) (see Appendix A). Distribution (19) may thus summarize core processes that cause fluctuations about a geologic mean magnetic energy density of $\{R_{nc}(c)\}/2\mu$ per degree, but distribute such energy abnormally among the orders within that degree due to geometric effects. Earth rotation is an obvious source of such anisotropy.

In a dynamo, kinetic energy maintains magnetic energy against Ohmic dissipation. The Coriolis force does no work; it does not change the kinetic energy of a fluid parcel. Rotational polarization of (geologically) turbulent core flow might thus produce magnetic anisotropy without much deviation from (17). Anisotropy due to motional inductive effects resembling the Coriolis vorticity effect should distinguish axial from equatorial dipoles. For example, consider surficially geostrophic flow in the uppermost main stream: $\hat{r} \cdot \nabla \times (\Omega \times v) \approx 0$ for bulk angular velocity Ω. Then fluid downwelling implies poleward flow. By (9), fluid downwelling draws in or attracts magnetic field line footpoints to form regions of strong radial field (core spots); the accompanying poleward flow implies poleward drifting core spots. This provides a flux partitioning mechanism for axial dipole formation, growth, and fluctuation (Voorhies 1991; 1992).

This mechanism for inducing planetary magnetic anisotropy relies on the kinematic boundary condition indicated by a rigid mantle; it might also operate where a stably stratified fluid bounds a convecting conductor. Whether or not it indicates deviations from (17), such as persistent quadrupole diminution (Stevenson, 1983) or a shift of magnetic energy from quadrupolar to dipolar configurations, remains to be seen. Other mechanisms, perhaps involving a solid inner core or a thermo-chemically heterogeneous mantle, might also partition core surface magnetic energy unequally among the harmonic orders within a particular degree for geologic time intervals. Hypothesis (16), expectations (17), and distributions (19) are compatible with such anisotropy.

Earth's squared dipole moment $\mathbf{m} \cdot \mathbf{m}$ is $(4\pi a^3/\mu_0)^2 [R_1(a,t)/2]$. Granting it is mainly from the core, (19) predicts that $3\mathbf{m} \cdot \mathbf{m}/\{\mathbf{m} \cdot \mathbf{m}\}$, which equals $3R_{1c}(a)/\{R_{1c}(a)\}$, will be distributed as χ^2 with three degrees of freedom. The probability $\wp(|\chi|)d|\chi|$ of finding $|\chi|$ in the interval $[|\chi|, |\chi|+d|\chi|]$ is $\wp(\chi^2)[d\chi^2/d|\chi|]d|\chi|$, so (19) predicts a Maxwellian distribution of absolute dipole moments $|\mathbf{m}|$. The predicted the root mean square dipole moment is

$$\text{RMSDM} \equiv \{\mathbf{m} \cdot \mathbf{m}\}^{1/2} = 4\pi a^3 \{R_{1c}(a)/2\}^{1/2}/\mu_0 ; \tag{20a}$$

the predicted mean absolute dipole moment is

$$|\text{DM}| \equiv \{|\mathbf{m}|\} = 4\pi a^3 \{[R_{1c}(a)/2]^{1/2}\}/\mu_0$$
$$= 4\pi a^3 [4\{R_{1c}(a)\}/3\pi]^{1/2}/\mu_0 = (8/3\pi)^{1/2}\text{RMSDM} ; \tag{20b}$$

and the most probable absolute dipole moment is $(2/3)^{1/2}\text{RMSDM}$. The Maxwellian distribution for normalized absolute geomagnetic dipole is quite compatible with an anisotropic, dominantly axial, dipole.

In following sections, one or two parameters of spectral forms $\{R_{nc}\}$ are estimated by least squares fittings to elementary functions of observational R_n. By (19), normally distributed residuals to the R_n fitted ought not be expected; moreover, the magnitude of these residuals should be large compared with the uncertainty in R_n indicated by geomagnetic field model covariance. This is because the natural variability of the core-source field allowed by (19) turns out to be much larger than formal uncertainty estimates in the low degree R_n derived from Magsat data. The modest decrease in the natural variance of the $R_{nc}(c)$ from (19) as n increases from 1 to 12, together with the Gaussian factor in (19), suggest least squares estimates closely approximate maximum likelihood estimates. Nonetheless, each estimate does not yield rigorous covariance for parameters (K,c); stated uncertainties summarize variances caused by geomagnetic field model and truncation selection.

4 MAGNETO-LOCATION OF EARTH'S CORE

If (17) is correct, then the two parameter fit of $\{R_{nc}(a)\}$ to the spectrum calculated from spherical harmonic models of Magsat data should give accurate magnetic estimates of the core radius, denoted c_m, and amplitude K. Such estimates were computed by least squares fits of spectral form (18) to observational $R_n(a)$ data. Specifically, $\ln[\{R_{nc}(a)\}/q(n)]$, which equals $[n\ln(c_m/a)^2 + \ln K + 4\ln(c_m/a)]$, was fitted to $\ln[R_n(a)/q(n)]$. The $R_n(a)$ data fitted were calculated from the degree 13 model GSFC 12/83 (Langel & Estes 1985) and, to reduce aliasing of crustal fields, the degree 60 model M102189 (Cain et al. 1990).

The top three rows of Table I.1 give the c_m computed using the q(n) listed in the first column. The second and third columns list c_m computed from $R_n(a)$ for degrees 3 through 12; the fourth and fifth columns list c_m computed from $R_n(a)$ for degrees 1 through 12. The fourth row gives leveling radii c* obtained using plain exponential form (3); the c* computed from degrees 2 through 12 are shown in parentheses. Such omission of dipole power raises c* by about 102 km.

The top row of Table I.1 shows that core radii determined by McLeod's rule agree very well with the seismologic core radius of 3480 km. The closest agreement comes from the fit of (17c) to the $R_n(a)$ values from M102189 for degree 3 through 12. This is illustrated in Figure I.1, which graphs $R_n(a)$ (diamonds connected by dashed line segments), the fit (solid curve with $c_m = 3484.5$ km and $K_M(a/c_m)^4 = 5.3649 \times 10^{10}$ nT^2), and the extrapolation to degrees 2 and 1 (fine dashed curve). The vertical bars attached to the fit show the 80% likelihood range deduced from χ^2 per 2n+1 degrees of freedom (19). (These are *not* errorbars attached to the data, which would be very small). As expected, two of the ten points fitted lie just outside the 80% likelihood range (R_8 and R_9). The low degree extrapolation shows R_1 is greater, and R_2 less, than expected; yet both are within the 80% likelihood range.

Table I.1 shows McLeod's rule (17c) as most accurate (top row) and variation (17b) as almost as accurate (second row); even form (7) gives core radii at most 1.8% above c_s (third row). The first two values in the top row average to 3488.2 $\pm$ 10.3 km (2σ). The four values in the upper left quadrant average to 3489.9 $\pm$ 8.6 km (2σ). The last two columns show that inclusion of dipole and quadrupole powers raises c_m by at most 22.3 km. That the effect on c_m is so small ($\leq 0.64\%$) is perhaps no more surprising than the efficient relaxation of R_1 and R_2 towards expectation values. The effect of the somewhat lighter weights PDFs (19) would assign to lower degree R_n can be approximated by averaging results from degrees 1-12 with those from degree 3-12.

The fourth row of Table I.1 shows that c* from plain exponential form (3) systematically underestimate c_s by about 200 km. It is doubtful that discrepancies this large result from crustal contributions to the $R_n(a)$ fitted. Indeed, coestimation of parameters describing recently advanced power law forms for the crustal spectrum $R_{nx}(a)$ would slightly decrease, not increase, c* and c_m. Perhaps geophysically important deficiencies of (3) have been overlooked due to omission of R_1 from previous fits.

These spectral core magneto-locations are more accurate and precise than those obtained using the frozen-flux approximation alone (Hide & Malin 1981; Voorhies & Benton 1982; Voorhies 1984). The latter methods rely heavily upon uncertain phase information in harmonic orders m and, more importantly, upon uncertain secular change information from either SV models or main field models at different epochs. The spectral method allows for magnetic diffusion as well as motional induction and relies only upon the R_n spectrum at a single epoch. These advantages reduce scatter. Some 44 frozen-flux core locations obtained from diverse field models at various truncations of expansion (1) average to 3506.2 $\pm$ 300.9 km (1σ, Voorhies 1984, equation (3.21)). The twelve estimates of c_m in Table I.1 average to 3504.8 $\pm$ 37.0 km (2σ). This is but 0.71% more than the seismic value.

The success of Hide's method nontheless suggests that secular change induced by main-stream fluid motion emerges largely unaltered by a visco-magnetic boundary layer. The success of the spectral method further suggests pseudo-random lateral magnetic transport by the main stream below, and diffusion of SV signal out the top of, this thin laminar sub-layer. Taken together, these inductive geomagnetic data analyses indicate a low viscosity, electrically conducting fluid core of 3.5 Mm radius without requiring seismic, tide and nutation, gravity, or astro-geochemical data to infer core composition, liquidity, and radius.

APPENDIX A: CHI-SQUARED FROM ABNORMAL DISTRIBUTIONS

If 2n+1 independent variables x_i are drawn at random from identical, zero mean Gaussian probability distribution functions of unit variance, then the probability distribution function (PDF) for the sum of their squares, $\Sigma_i(x_i)^2$, is well-known to be chi-squared with 2n+1 degrees of freedom. It is evidently less well known that the reverse is not always true. For example, consider three independent real variables (X, Y, Z) on the open interval $(-\infty,\infty)$ with PDFs

$$\wp_M(X) = (2\pi)^{-1/2} X^2 \exp(-X^2/2) \tag{A1a}$$

$$\wp_D(Y) = \delta(Y) \tag{A1b}$$

$$\wp_D(Z) = \delta(Z) \tag{A1c}$$

where $\wp_M$ is the bi-Maxwellian and δ is the Dirac delta function. There is no chance of Y and Z being anything but zero, X can be either positive or negative, and $X^2 + Y^2 + Z^2 = \chi^2$ is distributed as

$$\wp(\chi^2)d(\chi^2) = \wp(X^2)d(X^2) = 2\wp_M(X)\,|\frac{dX}{d(X^2)}|\ d(X^2)$$

$$= (2\pi)^{-1/2} (\chi^2)^{1/2} \exp(-\chi^2/2)\, d(\chi^2)$$

$$= [2^{3/2}\,\Gamma(3/2)]^{-1} (\chi^2)^{1/2} \exp(-\chi^2/2)\, d(\chi^2) \tag{A1d}$$

where Γ is the gamma function. PDF (A1d) is chi-squared with three degrees of freedom.

The difference between example (A1) and the usual case of three normal distributions with equal variance is important. With suitably normalized dipole coefficients $(X \to g_1^0/D, Y \to 0, Z \to 0)$, (A1) describes a dipole with a zero mean bi-modally distributed axial component, variance $\{(g_1^0)^2\} = 3D^2$, and negligible tilt. This approximates a planetary magnetic field dominated by an axial dipole. The usual case (with $(X, Y, Z) \to (g_1^0/\sigma, g_1^1/\sigma, h_1^1/\sigma))$ describes a dipole with no preferred direction at all and seems even less relevant to Earth than (A1). The isotropic case, with typical dipole tilts of order $\tan^{-1}([2]^{1/2}) = 54.7°$, may be of interest for Uranus and Neptune; case (A1) may be of interest for Saturn (Connerney, Ness & Acuna, 1982); intermediate distributions are potentially more Earth-like and may be of interest for Jupiter (Connerney & Acuna, 1982).

There are an infinite number of sets of three PDFs for three independent variables which give the distribution χ^2 with three degrees of freedom. To see this, consider the possibly singular distributions for independent variables (x, y, z) on the interval $(-\infty, +\infty)$

$$\wp_1(x) = A_x |x|^{-p_x} e^{-x^2/2} \tag{A2a}$$

$$\wp_2(y) = A_y |y|^{-p_y} e^{-y^2/2} \tag{A2b}$$

$$\wp_3(z) = A_z |z|^{-p_z} e^{-z^2/2} \tag{A2c}$$

where (A_x, A_y, A_z) are normalization constants and (p_x, p_y, p_z) are power law indicies which are less than one and sum to zero. Clearly

$$\wp_1{}'(x^2)d(x^2) = 2\wp(x)\left|\frac{dx}{d(x^2)}\right| d(x^2)$$

$$= A_x{}' |x^2|^{-[(1+p_x)/2]} e^{-x^2/2} d(x^2), \tag{A2d}$$

while similar PDFs for y^2 or z^2 follow by replacing x in (A2d) with y or z, respectively. The PDF for $\xi^2 = x^2 + y^2 + z^2$ is obtained in the usual way via

$$\wp(\xi^2) = \int_0^\infty \int_0^\infty \int_0^\infty \wp_1{}'(x^2) \wp_2{}'(y^2) \wp_3{}'(x^2) \delta[\xi^2 - (x^2+y^2+z^2)]d(x^2)d(y^2)d(z^2) \tag{A3a}$$

and the fact that the offset delta-function is the inverse Lapalace transform (denoted La^{-1}) of the exponential of its offset

$$\delta[\xi^2 - (x^2+y^2+z^2)] = La^{-1}[e^{\xi^2 - (x^2+y^2+z)^2}] = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{s\xi^2} e^{-s(x^2+y^2+z^2)} ds \tag{A3b}$$

where s is the Laplace transform domain variable. Substituting (A2d) and (A3b) into (A3a), and introducing $u = x^2$, $v = y^2$ and $w = z^2$, yields

$$\wp(\xi^2) = \frac{A_x{}'A_y{}'A_z{}'}{2\pi i} \int_{-i\infty}^{i\infty} e^{s\xi^2} La[u^{-(1+p_x)/2} e^{-u/2}]$$

$$La[v^{-(1+p_y)/2} e^{-v/2}] La[w^{-(1+p_z)/2} e^{-w/2}]ds. \tag{A4a}$$

With $k_x \equiv (1 - p_x)/2$ and

$$La[u^{-(1+p_x)/2} e^{-u/2}] = La(u^{k_x-1} e^{-u/2}) = \Gamma(k_x)[s + 1/2]^{-k_x} \tag{A4b}$$

it follows that

$$\wp(\xi^2) = \frac{A_x{}'A_y{}'A_z{}'}{2\pi i} \int_{-i\infty}^{i\infty} e^{s\xi^2} \frac{\Gamma(k_x)\,\Gamma(k_y)\,\Gamma(k_z)}{(s+1/2)^{k_x+k_y+k_z}}\,ds\,. \tag{A4c}$$

Provided k_x, k_y, and k_z are all positive definite and $k_x + k_y + k_z = 3/2$,

$$\wp(\xi^2) = A_x{}'A_y{}'A_z{}'\,\Gamma(k_x)\,\Gamma(k_y)\,\Gamma(k_z)\,La^{-1}[(s+1/2)^{-3/2}]$$

$$= A_x{}'A_y{}'A_z{}'\,\frac{\Gamma(k_x)\,\Gamma(k_y)\,\Gamma(k_z)}{\Gamma(3/2)}\,(\xi^2)^{1/2}\,e^{-\xi^2/2}$$

$$= [2^{3/2}\,\Gamma(3/2)]^{-1}\,(\xi^2)^{1/2}\,e^{-\xi^2/2}\,. \tag{A4d}$$

Comparison of (A4d) with (A1d) shows that ξ^2 is in fact distributed as χ^2 with three degrees of freedom. The conditions that (k_x, k_y, k_z) are all positive definite and sum to 3/2 are equivalent to the condition that (p_x, p_y, p_z) are all less than 1 and sum to zero. There are an infinite number of triplets for (p_x, p_y, p_z) that satisfy these conditions. The form of PDFs (A2a-c) is $A|\chi|^{-p}\exp(-\chi^2)]$ and is, in retrospect, one half the chi-squared distribution with typically fractional degree of freedom $(1 - p)$ reflected about a zero mean; for $p = (1 - 2k)$, the corresponding A is $1/[2^k\Gamma(k)]$.

It has been demonstrated that assigning the distribution chi-squared with 2n+1 degrees of freedom to some variable ξ^2 does not necessarily imply that ξ^2 is the sum of 2n+1 normally distributed variables of equal variance. It particular, there are an infinite number of sets of three PDFs for the three degree 1 Gauss coefficients that imply the distribution of chi-squared with three degrees of freedom for normalized dipole power $3R_{1c}/\{R_{1c}\}$. Explict contact between (A2a-c) and PDFs for suitably normalized Gauss coefficients is omitted for brevity. Variances of PDFs (A2a-c) are restricted by the selection of power law indicies and are $2^{1/2}\Gamma(k + 1/2)/\Gamma(k)$. PDFs (A2a-c) also have singularities for $1 > p_y = p_z > 0$. Such PDFs may be of interest in wave-particle physics, but are not the intermediate cases sought for describing planetary magnetism.

Instead consider normal distributions for $g_1{}^1$ and $h_1{}^1$. Then (A1a-c) makes it intuitively obvious that the PDF for $g_1{}^0$ yielding chi-squared with three degrees of freedom for the distribution of normalized dipole power will be a linear combination of Gaussian and bi-Maxwellian distributions. This is in fact the case, as shown below.

2.1

Denote $[(a/c)^3 g_1{}^0, (a/c)^3 g_1{}^1, (a/c)^3 h_1{}^1]$ by $[x, y, z]$ and the variances of (x, y, z) by $(\sigma_x{}^2, \sigma_y{}^2, \sigma_z{}^2)$. The axial dipole hypothesis would have the means of y and z be zero, while isotropy of the equatorial dipole would have σ_y and σ_z be equal. So replace (A2a-c) with distributions

$$\wp_x(x)dx = (3/2\pi V^2)^{1/2}(3/V^2)[\frac{\sigma_x{}^2 - \sigma_y{}^2}{V^2} x^2 + \sigma_y{}^2]\ \exp(-3x^2/2V^2)\ dx \tag{A5a}$$

$$\wp_y(y)dy = (2\pi\sigma_y{}^2)^{-1/2}\ \exp(-y^2/2\sigma_y{}^2)dy \tag{A5b}$$

$$\wp_z(z)dz = (2\pi\sigma_z{}^2)^{-1/2}\ \exp(-z^2/2\sigma_z{}^2)dz \tag{A5c}$$

where $V^2 \equiv \sigma_x{}^2 + \sigma_y{}^2 + \sigma_z{}^2$. With $\xi^2 \equiv x^2 + y^2 + z^2 = V^2\chi^2/3$, the problem is to derive (A5a) for $\wp_x(x)$ given normal distributions (A5b-c) and

$$\wp(\xi^2)d(\xi^2) = (2^{3/2}\Gamma(3/2))^{-1}[3\xi^2/V^2]^{1/2}\exp[-3\xi^2/2V^2](3/V^2)d(\xi^2) \tag{A5d}$$

subject to the constraint that the dipole is usually mainly axial: $\sigma_x{}^2 >> \sigma_y{}^2 \approx \sigma_z{}^2$. For statstically independent (x, y, z)

$$\wp(\xi^2) = \int_0^\infty \int_0^\infty \int_0^\infty \wp_x{}'(x^2)\ \wp_y{}'(y^2)\ \wp_z{}'(z^2)\ \delta[\xi^2 - (x^2+y^2+z^2)]d(x^2)d(y^2)d(z^2) \tag{A6a}$$

$$= (2\pi\sigma_y\sigma_z)^{-1}\int_0^\infty \int_0^\infty \int_0^\infty \wp_x{}'(x^2)(y^2z^2)^{-1/2}\ \exp(-y^2/2\sigma_y{}^2 - z^2/\sigma_z{}^2)$$

$$\delta[\xi^2 - (x^2+y^2+z^2)]d(x^2)d(y^2)d(z^2) \tag{A6b}$$

or, using (A3b) and resetting $u = x^2, v = y^2, w = z^2$,

$$\wp(\xi^2) = (4\pi^2\sigma_y\sigma_z i)^{-1}\int_{-i\infty}^{i\infty}\int_0^\infty \int_0^\infty \int_0^\infty \wp_x{}'(u)\exp(-su)(vw)^{-1/2}\exp(-sv - v/2\sigma_y{}^2)$$

$$\exp(-su - w/\sigma_z{}^2)\exp(s\xi^2)\ dudvdwds\ . \tag{A6c}$$

Recognizing and evaluating the Laplace transforms gives

$$\wp(\xi^2) = (4\pi^2\sigma_y\sigma_z i)^{-1}\int_{-i\infty}^{i\infty} La[\wp_x{}'(u)]\ La[v^{-1/2}\exp(-v/2\sigma_y{}^2)]$$

$$La[w^{-1/2}\exp(-w/2\sigma_z{}^2)]\ \exp(s\xi^2)ds$$

$$= (2\pi\sigma_y\sigma_z)^{-1}\ La^{-1}[La[\wp_x{}'(u)]\ [\Gamma(1/2)]^2[(s + 1/2\sigma_y{}^2)(s + 1/2\sigma_z{}^2)]^{-1/2}\ . \tag{A6d}$$

The Laplace transforms of (A6d) and (A5d) imply

$$La[\wp_x'(u)] = \sigma_y\sigma_z(3/2V^2)^{1/2} \left[\frac{(s + 1/2\sigma_y^2)(s + 1/2\sigma_z^2)}{(s + 3/2V^2)^3}\right]^{1/2}(3/V^2) \qquad (A7a)$$

In the special case $\sigma_y = \sigma_z$,

$$\wp_x'(x^2) = \sigma_y^2(3/2V^2)^{1/2} \, La^{-1}\left[\frac{s + 1/2\sigma_y^2}{(s + 3/2V^2)^{3/2}}\right](3/V^2) \qquad (A7b)$$

and the inverse transform gives

$$\wp_x'(x^2) = \frac{\sigma_y^2(3/2V^2)^{1/2}}{2\Gamma(3/2)}(3/V^2)\left[(x^2)^{-1/2} + \frac{V^2 - 3\sigma_y^2}{\sigma_y^2V^2}(x^2)^{-1/2}\right]\exp(-3x^2/2V^2) . \qquad (A7c)$$

Because x can be of either sign, $\wp_x(x)dx$ is but half $\wp_x'(x^2)2xdx$ and (A7c) reduces to the linear combination of a zero mean bi-Maxwellian and a zero mean Gaussian (A5a).

Although (A5a-c) are but one class of PDFs consistent with the distribution chi-squared with three degrees of freedom advanced for normalized dipole power in the text, they may be of some use in planetary and perhaps stellar magnetism. In (A5a) note that $V^2 - 3\sigma_y^2 = \sigma_x^2 - \sigma_y^2 > 0$; with $\sigma_x^2 >> \sigma_y^2$, the distribution enjoys the two peaks on either side of the mean (at $g_1^0 = 0$) that correspond to normal and reversed axial dipole polarity, respectively. A suitable index of anisotropy, or 'tilt control parameter', for PDFs (A5a-c) is $\varepsilon \equiv [(\sigma_x^2 - \sigma_y^2)/V^2]$, which is 1 for a purely axial dipole, zero for randomly oriented dipoles, and -1/2 for a purely equatorial dipole. For a purely axial dipole with $\sigma_y = \sigma_z = 0$, (A5a) would vanish at x = 0; then g_1^0 could not pass through zero and reversals would be prohibited. So $\varepsilon \approx 1$ describes inhibition of reversals.

It is conjectured that PDFs (A5) describe the terrestrial dipole with fair accuracy. Very slow evolution of the variances ($\sigma_x^2, \sigma_y^2 \approx \sigma_z^2$) over geologic time is thought to reflect planetary evolution of the boundary conditions on the outer core (and vice-versa). PDFs (A5) may also describe other planetary dipoles (e.g., ε is apparently about 1 for Saturn, somewhat less than 1 for Jupiter, and near zero for Uranus and Neptune).

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Table I.1: Estimates of Earth's Core Radius in km from McLeod's Rule and Other Forms

$q(n)$	GSFC 12/83	M102189	GSFC 12/83	M102189
	$3 \leq n \leq 12$	$3 \leq n \leq 12$	$1 \leq n \leq 12$	$1 \leq n \leq 12$
$[n + 1/2]^{-1}$	3491.8	3484.5	3501.2	3496.0
$[n(n+1)]^{-1/2}$	3493.3	3486.0	3506.8	3501.6
n^{-1}	3511.4	3504.1	3543.1	3537.8
1	3261.4	3254.6	3208.6	3203.8
			(3311.8)	(3306.0)

Figure Caption

Figure I.1 shows the multipole powers $R_n(a)$ from model M102189 (Cain et al., 1990) for degrees $1 \leq n \leq 12$ (diamonds connected by dashed line segments), the two parameter fit of McLeod's rule (16c) (solid curve with $c_M = 3484.5$ km and $K_M(a/c_M)^4 = 5.3649 \times 10^{10}$ nT^2) to degrees 3 through 12, and the extrapolation to degrees 2 and 1 (fine dashed curve). Vertical bars show the 80% likelihood range from χ^2 with $2n+1$ degrees of freedom and are attached to the theory (solid curve), not the data. As expected, two of ten powers fitted lie outside this range. R_1 is greater, and R_2 less, than predicted, but both are within their 80% likelihood ranges.

Figure I.1

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