A Geometrical Approach to Bell's Theorem

by

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Abstract

Bell's theorem can be proved through simple geometrical reasoning, without the need for the ψ-function, probability distributions, or calculus. The proof is based on N. David Mermin's explication of the Einstein-Podolsky-Rosen-Bohm experiment, which involves Stern-Gerlach detectors which flash red or green lights when detecting spin-up or spin-down. The statistics of local hidden variable theories for this experiment can be arranged in colored strips from which simple inequalities can be deduced. These inequalities lead to a demonstration of Bell's theorem. Moreover, all local hidden variable theories can be graphed in such a way as to enclose their statistics in a pyramid, with the quantum-mechanical result lying a finite distance beneath the base of the pyramid.

I. INTRODUCTION

Can local measurements influence distant events? In their famous paper, Einstein, Podolsky, and Rosen (EPR for short) asked this question in the context of quantum mechanics[1]. Subsequent authors have clarified the issues involved in this surprisingly subtle question, among them David Bohm. Bohm posed the EPR paradox in terms of a particular experimental arrangement[2]. The EPR-Bohm *Gedankenexperiment* in turn helped lead John S. Bell to his celebrated theorem. Bell[3-4] proved the answer to be yes; local measurements can influence distant events. He proved it by showing that hidden variables of a particular sort, namely local hidden variables, cannot explain quantum statistics.

Numerous authors since Bell have sought to reduce the theorem to its essence, including Eugene Wigner[5] and N. David Mermin[6]. Mermin gives a remarkably clear and simple explanation of what Bell's theorem is all about. And while he gives an example of local hidden variables which cannot explain quantum statistics, he does not actually give a proof of Bell's theorem.

My purpose here is to elaborate on Mermin's view a little and give a geometrical demonstration of Bell's theorem, without the use of the ψ-function, probability distributions, or calculus. In addition to geometry, only a little algebra is needed. It is to be hoped that this exposition will be intelligible to high school students; reading Mermin's article[6] first would help in this regard.

There are other geometrical approaches to Bell's theorem, such as that of Fivel[7] and of Massad and Aravind[8]. The latter base their proof for a spin 3/2 particle on the Penrose dodecahedron.

II. EPR-BOHM EXPERIMENT

Suppose (as Mermin does) we have the usual EPR-Bohm experimental arrangement: two detectors, A and B, as shown in Figure 1. Each detector has Stern-Gerlach magnets, which

measure spin-up or spin-down for a neutron. (Even though a neutron has no charge it still has a magnetic moment.) The magnets in each detector can be rotated to positions 1, 2, or 3; the three positions are 120° apart from each other. If a neutron entering detector A is deflected toward the north pole as it passes between the magnets, then the red light flashes; if deflected toward the south, the green light flashes. Detector B does the opposite: a red light flashes when the neutron is deflected toward the south pole, and green when deflected toward the north. Thus in Mermin's set-up the detectors are wired differently.

Now consider the following experiment: two neutrons in the singlet state (in which the total angular momentum is zero) fly apart; one neutron enters detector A; the other enters B. The setting of each detector and which light flashes (red or green) is recorded. The detector settings are then varied at random and two more neutrons in the singlet state fly apart and enter the detectors. The whole process is repeated over and over long enough until ideal statistics are obtained. We will write (1,2) when detector A is set to position 1 and B to position 2, and (3,1) when A is set to 3 and B to 1, etc.; so that the setting for A is written as the first number in parentheses and the setting for B the second number. Similiarly, we will write RG when the light on detector A flashes red and the light on B flashes green, and so on.

Figure 2 shows a portion of the data thus obtained. The lights flash RR half the time and GG the other half when the settings are (1,1), (2,2), and (3,3). For the other settings the lights sometime flash the same colors and sometimes opposite colors. All together the lights flash RR, GG, RG, and GR each one-fourth of the time in the EPR-Bohm experiment.

III. BELL'S CLASSICAL MODEL

Bell[3] examined the EPR-Bohm experiment from the viewpoint of a model from classical physics which can be understood in the context of Mermin's flashing lights. In his model a particle has spin angular momentum L. Upon entering detector A the light flashes G when $L \cdot H < 0$ and R when $L \cdot H > 0$; detector B does the opposite. Here H is the magnetic field vector and the

local hidden variables are the direction cosines of the spin vectors. They are local because the direction cosines apply only to each individual particle. They are not fields, for example, which pervade all of space and would therefore be nonlocal. They are hidden because we have no direct knowledge of them. And of course the direction cosines are variables because the orientation of the spins can vary.

In Bell's model the flashing lights from the settings (1,1), (2,2), and (3,3) in the EPR-Bohm experiment appear to be understandable. In the settings (1,1), for example, the lights flash either RR or GG, reflecting conservation of angular momentum: if detector A measures (say) $L \cdot H$ < 0, then detector B *has* to measure $L \cdot H$ > 0, giving a response GG. The lights can *never* flash RG or GR. This makes sense: in the singlet state the spin vector of one particle points in one direction and the spin vector of the other particle points in the opposite direction. Moreover, for (1,1) the lights flash RR half the time, and GG the other half. This also makes sense: we would expect the axis defined by the spin vectors to be randomly distributed in space, so that half the time a given detector measures $L \cdot H$ > 0 and the other half $L \cdot H$ < 0. Identical remarks hold for settings (2,2) and (3,3). Bell's model explains these aspects of the data.

With the settings (1,2) the lights in the EPR-Bohm experiment can flash anything: RR, RG, GR, or GG. This is likewise true for (1,3), (2,1), (2,3), (3,1), and (3,2). This also makes sense in Bell's classical model: with the axes of the detectors canted at 120° to each other, when $L \cdot H$ > 0 for one detector, sometimes $L \cdot H$ > 0 for the other detector as well, causing them to flash different colors.

Where Bell's classical model breaks down is in trying to explain the *statistics* of the flashing lights when all of the data are looked at: in the EPR-Bohm experiment the lights flash RR one-quarter of the time, GG a quarter, RG a quarter, and GR a quarter. It was Bell's great accomplishment to show that these statistics cannot be reproduced by his classical model -- or any other model whatsoever where the two detectors are independent of one another. By "independent" we mean no connection between the detectors. For instance, if detector A is on setting 1 and destined to flash G when after the neutrons separate, then it should not matter

whether detector B is suddenly set to 1, 2, or 3; detector A will still flash G. There is no communication between the two.

IV. LOCAL HIDDEN VARIABLES

Let us consider a local hidden variables experiment, or LHV experiment for short, just to illustrate the steps leading to the geometrical proof of Bell's theorem. Unlike the quantum-mechanical EPR-Bohm experiment, the detectors are now independent of each other. Figure 3 gives an example of various equally long runs in this hypothetical experiment when the detectors have fixed settings. The hidden variables $\{a_i\}$, $\{b_i\}$, ... can be anything or combinations of anything: spin direction, quark coordinates, or other mundane or exotic parameters. It doesn't matter, so long as they are local.

We examine the N outcomes for a particular run when the detectors are set to (1,1). The outcomes can be considered to be written on a paper tape, rather like the sales slip given to a customer at a supermarket check-out counter. As Figure 3 shows, the lights flash RR when the hidden variables have values $\{a_i\}$, GG when the hidden variables have values $\{b_i\}$, then GG again for $\{c_i\}$, and so on. Half the time the lights flash RR and the other half GG. (In the figure N = 10 for the purposes of illustration; we will actually base our proof on long runs which give ideal statistics.) The column on the right shows the product for each outcome. The product rules are simple: $R \times R = G \times G = +1$, while $R \times G = G \times R = -1$. For (1,1) each product is necessarily +1. The sum s_{11} of all the products is shown at the bottom of the product column; in this case, s_{11} = +10. Likewise for the equally long runs (2,2) and (3,3) nothing but +1's appear in the product column, so that $s_{22} = s_{33} = +10$ as well. Naturally, when these other runs are made we do not expect the hidden variables to appear in the same order as for (1,1). For run (2,2), for instance, the actual order might have been $\{f_i\}$, $\{b_i\}$, $\{d_i\}$, ..., but we can rearrange rows as we see fit and

not disturb the sums, and we choose for the moment to arrange them so that the hidden variables appear in alphabetical order.

Now consider (1,2). The most important thing to note about this run is that detector A on setting 1 still responds to $\{a_i\}$ with R and detector B on setting 2 with G. Similarly, to $\{b_i\}$ A responds with G and B with R, and so on. The responses of a given detector are determined by its own setting and the hidden variables of the neutron entering it; not on the setting of the other detector nor on the hidden variables of the other neutron, for what goes on at one detector is independent of the other, as assumed.

In (1,2) +1's and -1's appear in the product column, unlike (1,1), (2,2), and (3,3), which have only +1's. The sum s_{12} could now be a positive or negative number; in the figure, it happens to be negative: $s_{12} = -6$. In considering (2,1) it is clear that by symmetry $s_{21} = s_{12}$ always, so that $s_{21} = -6$. Likewise $s_{31} = s_{13}$ and $s_{32} = s_{23}$ by symmetry. For the figure $s_{13} = s_{31} = +2$ and $s_{23} = s_{32} = -6$. In this example s_{12} just happens to be equal to s_{23}; but other examples can be found where this is not the case.

In the following we will be concerned with the sum S of all the runs:

$$S = s_{11} + s_{22} + s_{33} + s_{12} + s_{21} + s_{13} + s_{31} + s_{23} + s_{32}.$$

Using $s_{11} + s_{22} + s_{33} = 3s_{11}$ and $s_{12} = s_{21}$ etc., this becomes

$$S = 3s_{11} + 2s_{12} + 2s_{13} + 2s_{23} \tag{1}$$

in general. Our LHV experiment gives $S = 3 (+10) + 2 (-6) + 2 (+2) + 2 (-6) = 30 - 12 + 4 - 12 = +10$. On the other hand, in the ideal statistics of the quantum-mechanical EPR-Bohm experiment, the lights flash RR, GG, RG, and GR one-quarter of the time each, so that $S = 0$ for the EPR-Bohm experiment.

Question: can we find local hidden variables and construct detectors such that (i) the detectors are independent of each other, (ii) a given detector on a given setting flashes R half the time during the run and G the other half, (iii) the detectors always flash the same color for (1,1), (2,2), and (3,3), and (iv) flash opposite colors often enough for the other combination (1,2), (1,3), etc. so that S = 0? If we can, then we can reproduce the statistics of the EPR-Bohm experiment.

Bell's theorem says we can't. The proof is based only on the possible responses of the detectors.

To see this, we will rearrange the rows, so that all of the green responses are on top for (1,1) and all the red responses at the bottom, as shown in Figure 4. Naturally (1,2), (1,3), and (2,3) must also be rearranged to keep the proper responses with the proper hidden variables. This is also shown in the figure. These rearrangements do not affect the sums s_{11}, s_{12}, etc., or S. Only (1,1), (1,2), (1,3), and (2,3) are shown in Figure 4 since by (1) these are all that are needed to compute S.

We now proceed more abstractly. First, we omit the hidden variable list and the product column from the tapes as distractions. Second, we get rid of the letters G and R and the lines separating the outcomes, so that for our particular example (1,1), (1,2), etc. appear as in Figure 5. Third, we normalize the length of each tape by dividing by 9N, the total number of responses for all nine different runs, so that each tape is now 1/9 unit long. Now $s_{11} = +1/9$, $s_{12} = -1/15$, $s_{13} = +1/45$, and $s_{23} = -1/15$ in our example, so that S = +1/9, by (1).

Thus we go from discrete entries on the tapes to red and green bars. This transition from writing letters in boxes to simple geometry frees us to consider any number of hidden variables and give a very general proof of Bell's theorem.

V. BELL'S THEOREM

We will now abandon our specific example and prove Bell's theorem by considering a collection of tapes from a generic LHV experiment, as shown in Figure 6. It is assumed that the number of outcomes is so large that ideal statistics are obtained. Each tape is once again normalized in length so as to be 1/9 unit long. Center lines are drawn across each tape halfway down, to distinguish the top half from the bottom half, since we will need to consider each half in our proof.

Moving on to our proof, suppose in (1,2) x is the total length of the green area of detector B on setting 2 which is above the center line; then $0 \leq x \leq 1/18$. Further, the total length of the green area below the center line is 1/18 - x and obviously satisfies $0 \leq 1/18 - x \leq 1/18$. The sum s_{12} is easy to figure out: for the top half of the tape, an amount x will be positive and 1/18 - x will be negative, and for the bottom half an amount x will be positive and 1/18 - x will be negative, so that $s_{12} = x - (1/18 - x) + x - (1/18 - x) = 4x - 1/9$. Likewise with (1,3) if y is the total length of the green area when detector B is on setting 3, then $s_{13} = 4y - 1/9$. Clearly $s_{11} = 1/9$. Thus S, using (1), can be written

$$S = 3 (1/9) + 2 (4x - 1/9) + 2 (2y - 1/9) + 2 s_{23} = 8x + 8y - 1/9 + 2s_{23} \qquad (2)$$

We now need to consider s_{23}. Let us find its smallest value, so that S is the smallest possible number. How do we find the smallest value of s_{23}? Clearly we need to match as much green area on one side of the tape with as much red area as possible on the other. One way to accomplish this is the following. On the top half of the tape push all of the green area on the left side to the top and all of the green area against the center line, as shown in Figure 7. The area of each color is conserved in the process.

Now suppose $x + y \leq 1/18$. The green parts will contribute -x - y to s_{23}, while the red spaces opposite each other in the middle contribute 1/18 - x - y, so that the total contribution of the top half of the tape to s_{23} is 1/18 - 2x - 2y. Likewise, the bottom half of the tape also contributes 1/18 - 2x - 2y, giving a total of 1/9 - 4x - 4y as the lower limit on s_{23}. Thus

$$s_{23} \geq 1/9 - 4x - 4y \qquad (3)$$

Plugging (3) into (2) gives $S \geq +1/9$.

By similar reasoning, it is easy to show that if $x + y > 1/18$, then $S \geq 16x + 16y - 7/9$. So it must be that

$$S \geq +1/9 \qquad (4)$$

for any x and y. Hence the smallest possible value is $S = +1/9$ in a local hidden variables theory. However, in the EPR-Bohm experiment (wherein $x = y = 1/72$) we have $S = 0$. Therefore the statistics of the EPR-Bohm experiment cannot be explained by any local local hidden variables theory at all. Thus quantum mechanics cannot be a local hidden variables theory. This is Bell's theorem.

Suppose the actual statistics of the EPR-Bohm experiment are found to differ from quantum mechanics by $S = \varepsilon$ (ε being a small number), indicating the need for a new theory beyond quantum mechanics. Can the new theory be local? The answer is no, because of the finite difference between ε and $+1/9$. This unbridgable gap is also part of Bell's theorem. Any new theory must of necessity be nonlocal to explain the data.

Martin Gardner[9] invented a parlor game based on the EPR-Bohm experiment. The object was to reproduce the EPR-Bohm statistics. This game cannot be played with the desired result.

It can also be shown that with local hidden variables the lights must flash the same colors at least 5/9 of the time, in agreement with Mermin[6]. The expression (4) is somewhat reminiscent of the Clauser-Horne-Holt-Shimony inequality[10].

VI. A LAZY PYRAMID

We found above the minimum possible value value of S for any x and y in a local theory. What about the maximum value of S in a local theory?

The maximum value of S can be found with the same geometrical technique we used to find s_{23}, only this time we match as much green with green and as much red with red as possible. The answer turns out to be S = 16x + 1/9 for x $\leq$ y and S = 16y + 1/9 for x $\geq$ y. The graph of all possible values of S *versus* x and y, wherein the lights flash equally RR and GG for settings (1,1), (2,2) and (3,3), is given in Figure 8. It shows that all local theories are entombed in the lazy, three-sided pyramid whose base levitates 1/9 unit above the x-y plane. The LHV experiment of Figures 3 - 5 just happened to have the minimum value of S. For the EPR-Bohm experiment, S is located in the x-y plane at x = y = 1/72, beneath the pyramid and unreachable by any local theory.

All possible theories, local or nonlocal, for which the lights flash red-red half the time and green-green the other half when the detectors are set to (1,1), (2,2), or (3,3) can also be graphed, as shown in Figure 9. The bounds are determined by the fact that the most s_{23} can be is +1/9 and the least -1/9. The two figures apply only to the experimental arrangement where the detector settings are 120°. The shapes will alter if the detectors are set for other angles.

VII. BELL'S PREFERENCE

Despite occasional "proofs" of their impossibility, hidden variable theories which reproduce quantum statistics (or close to it) do exist[11]. They are of course nonlocal.

If the nonlocal hidden variables do exist, then they have curious properties. For instance, they do not cause the correlations to decay with distance. This is just like local hidden variables. Moreover, the influence of one detector on the other apparently must travel faster than light, but in

such a way that quantum mechanics cannot be used to communicate information at superluminal speeds[12-13].

John S. Bell himself was partial to Louis de Broglie's pilot wave theory, which was independently rediscovered and developed by David Bohm[14].

Again and again in his book *Speakable and Unspeakable in Quantum Mechanics* Bell[11] proffered the de Broglie-Bohm theory for consideration by the physics community, which he felt ignored pilot waves too much. On page 160, for instance, he asked the following rhetorical questions:

> But why then had Born not told me [in his book *Natural Philosophy of Cause and Chance*] of this 'pilot wave'? If only to point out what was wrong with it? Why did von Neumann not consider it? More extraordinarily, why did people go on producing 'impossibility proofs', after 1952, and as recently as 1978? When even Pauli, Rosenfeld, and Heisenberg could produce no more devastating criticism of Bohm's version than to brand it as 'metaphysical' and 'ideological'? Why is the 'pilot wave' picture ignored in textbooks? Should it not be taught, not as the only way, but as an antidote to the prevailing complacency? To show that vagueness, subjectivity, and indeterminism, are not forced on us by experimental facts, but by deliberate theoretical choice?

Doubtless one reason for the widespread rejection of the de Broglie-Bohm theory was that it must have seemed like a step backward. The quantum-mechanical equations and their

Copenhagen interpretation were hard-won, obtained after years of struggle. But then Bohm came along and jettisoned the wave-particle duality, complementarity, and conscious observers of the Copenhagen interpretation in favor of objectivity, determinism, and nonlocal fields (which explained experiments of the EPR-Bohm ilk). It must have seemed as though the revolution were called off.

The Copenhagen interpretation still holds sway. One can only speculate on what today's interpretation of quantum mechanics would be if de Broglie had pushed his ideas to their logical conclusion before the Copenhagen interpretation became the prevailing view.

ACKNOWLEDGMENT

NOTES

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Black and white figure captions

Figure 1

N. David Mermin's version of the EPR-Bohm experiment. Two neutrons in the singlet state separate: one enters detector A, passing between Stern-Gerlach magnets; the other neutron enters detector B. A red light flashes on detector A when the neutron is deflected toward the north pole, and a green light flashes when deflected toward the south. Detector B does the opposite: red for south and green for north. The magnets of a given detector can be rotated to positions 1, 2, or 3. The angle between positions is 120°.

Figure 2

Quantum data produced by the detectors of Figure 1. In the first outcome detector A was set to position 1 and its red light flashed; detector B was set to position 2 and its green light flashed. The detectors flash red-red half the time and green-green half the time when they are set to the same positions (1,1) (2,2), or (3,3). These ideal statistics were manufactured by literally mixing up outcomes and drawing them out of a hat.

Figure 3

Data for given detector settings in an experiment involving local hidden variables. There are 10 outcomes in each run. The local hidden variables are shown in the braces to the left of the colors, while the product column is shown to the right. Same colors give +1 in the product column, while different colors give -1. The sum of the +1's and -1's is shown at the bottom of the product column.

Figure 4

The data of Figure 3, rearranged so that all the green responses are at the top when the detectors are set to positions (1,1).

Figure 5

The data of Figure 4, normalized so that each run is 1/9 unit long. Only (1,1), (1,2), (1,3), and (2,3) are shown, since these are all that are needed to compute S. The grey areas correspond

Figure 6

The tapes of a generic local hidden variables experiment, where N is so large as to give ideal statistics. The central lines divide each tape in half. The grey areas correspond to red and the white areas to green.

Figure 7

On the left is the (2,3) run of the previous figure. The right shows the way to minimize s_{23}. In the top half of the tape: on the left side, slide all of the green area up to the top, while on the right side slide all of the green area against the central line. In the bottom half of the tape: on the left side, slide all of the green area up against the central line, while on the right side slide all of the green to the bottom. The area of each color is conserved. The grey areas correspond to red and the white areas to green.

Figure 8

All local hidden variable theories, in which the lights flash red-red half the time and green-green the other half when the detectors have the same settings, are entombed in the lazy, three-sided grey pyramid. The base of the pyramid levitates at +1/9 unit, forever out of reach of the quantum-mechanical result in the x-y plane (at x = +1/72 and y = +1/72 and marked QM).
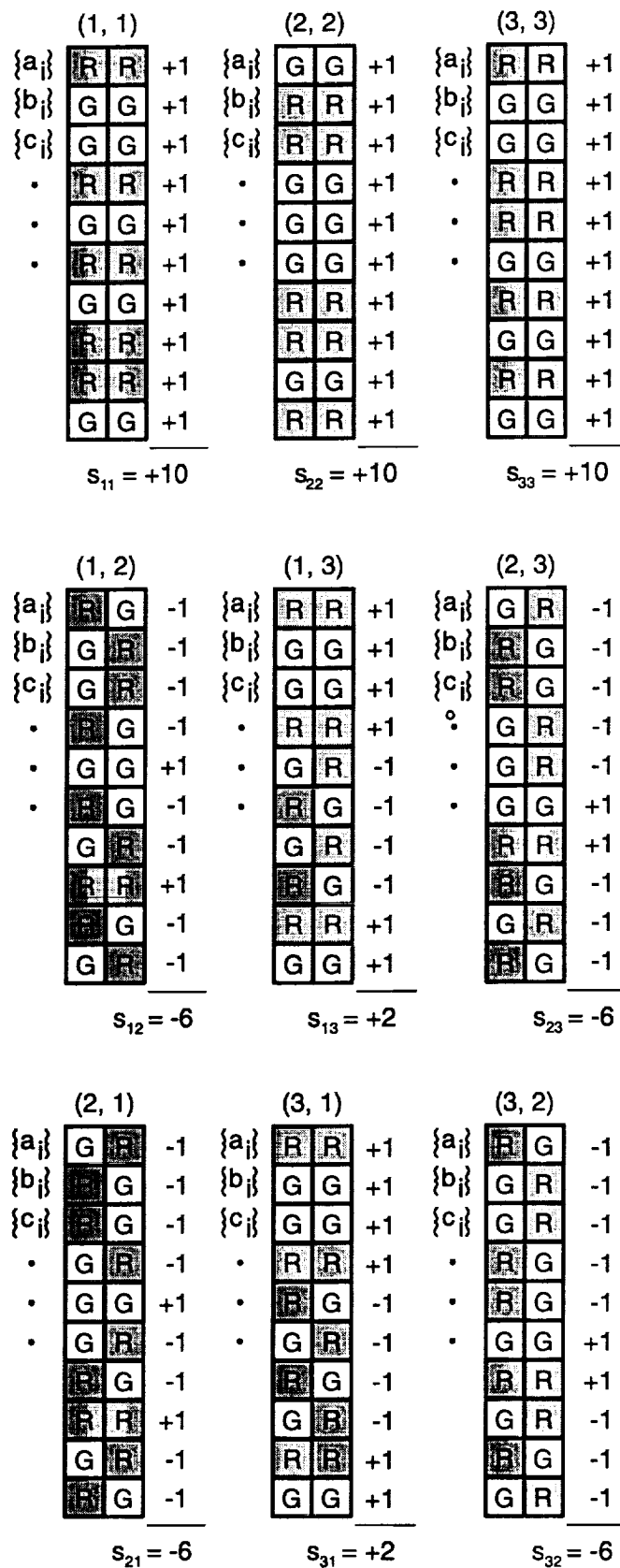
Figure 9

All local and nonlocal theories for which the lights flash red-red half the time and green-green the other half when the detectors are set on (1,1), (2,2), or (3,3) are trapped between the diamonds. The diamonds encase the pyramid. The pyramid, in contrast to Figure 8, is shown here as a solid and greatly shortened due to the compression of the vertical scale.
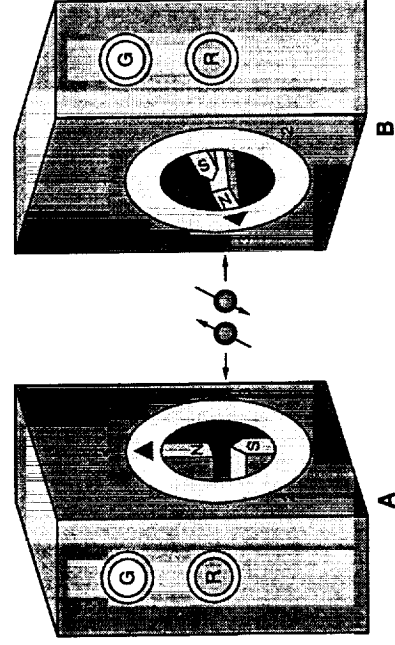
Figure 1
D. Rubincam
Am. J. Physics

| (1, 2)	R G		(2, 3)	G R
(2, 2)	R R		(1, 1)	G G
(2, 3)	G R		(3, 1)	R G
(2, 1)	G G		(1, 2)	G R
(2, 2)	G G		(2, 2)	R R
(1, 3)	R G		(3, 1)	G G
(1, 1)	R R		(1, 2)	R R
(3, 3)	G G		(2, 2)	G G
(3, 2)	R G		(2, 1)	G R
(3, 3)	G G		(2, 2)	R R
(2, 1)	R G		(1, 1)	R R
(3, 3)	G G		(1, 3)	G R
(3, 1)	R G		(1, 2)	R G
(3, 2)	G R		(3, 2)	G R
(1, 3)	G R		(2, 3)	R R
(2, 2)	G G		(1, 1)	R R
(3, 2)	R G		(2, 3)	R G
(1, 1)	G G		(3, 3)	R R
(2, 1)	R G		(2, 1)	R R
(1, 3)	G G		(2, 2)	G G
(3, 2)	G G		(3, 1)	G R
(1, 2)	G R		(2, 1)	R G
(2, 3)	G R		(1, 1)	G G
(2, 1)	G R		(2, 1)	G R
(2, 3)	R G		(1, 2)	G R
(1, 3)	R R		(2, 3)	R G
(3, 3)	R R		(3, 3)	R R
(1, 1)	R R		(1, 3)	R G
(3, 1)	G R		(3, 1)	G R
(1, 3)	G R		(1, 2)	R G
(1, 1)	G G		(3, 1)	R G
(3, 2)	R G		(3, 1)	R R
(1, 3)	R G		(3, 2)	R R
(1, 2)	G G		(2, 2)	R R
(3, 2)	G R		(3, 3)	R R
(3, 3)	G G		(2, 3)	G G

Figure 2
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Figure 3
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Figure 4
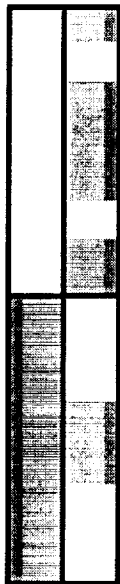D. Rubincam
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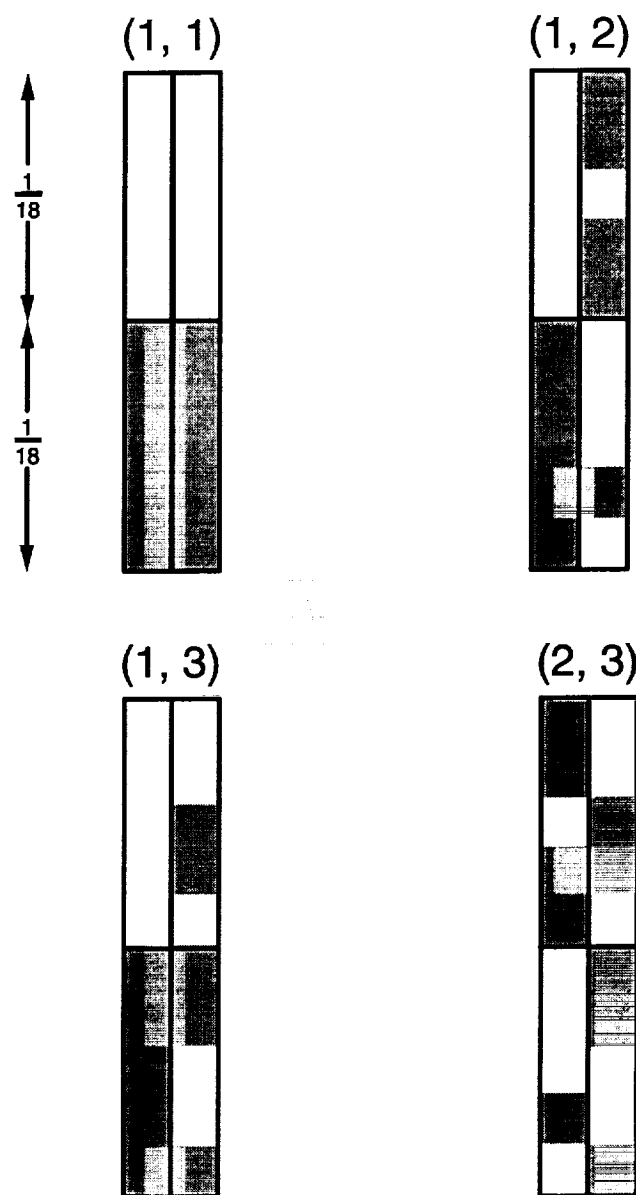
Figure 5
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(1, 1)
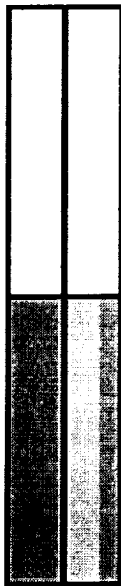
(1, 2)

(1, 3)
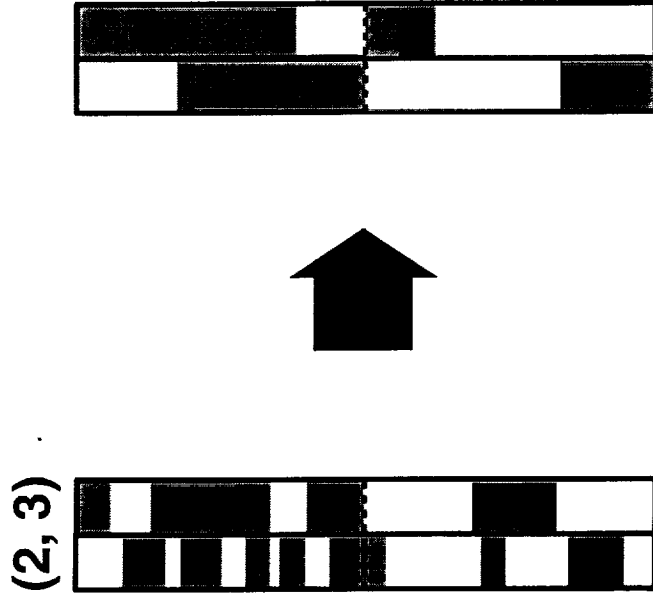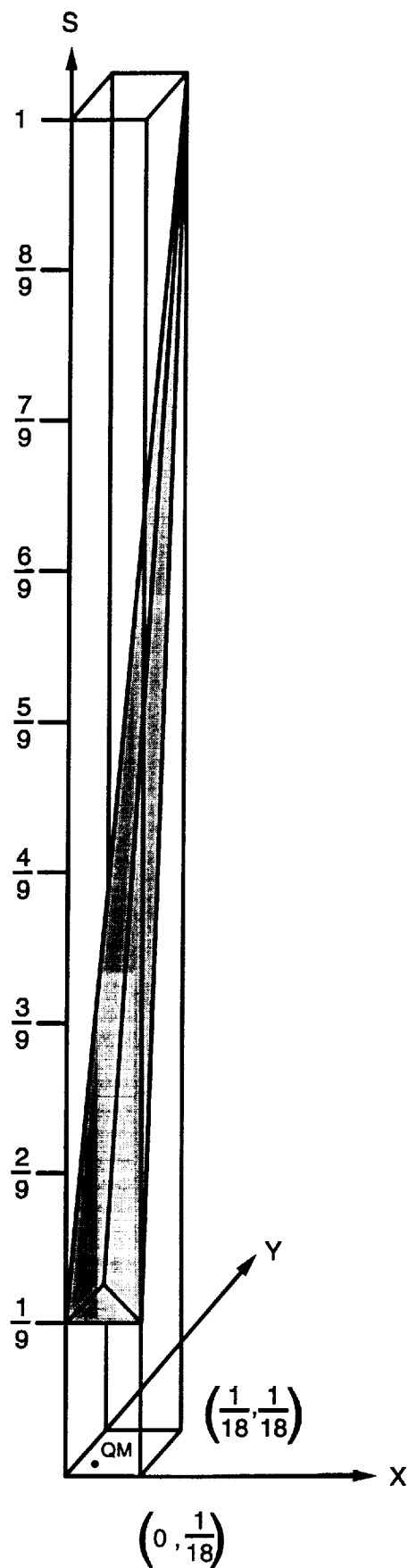
(2, 3)
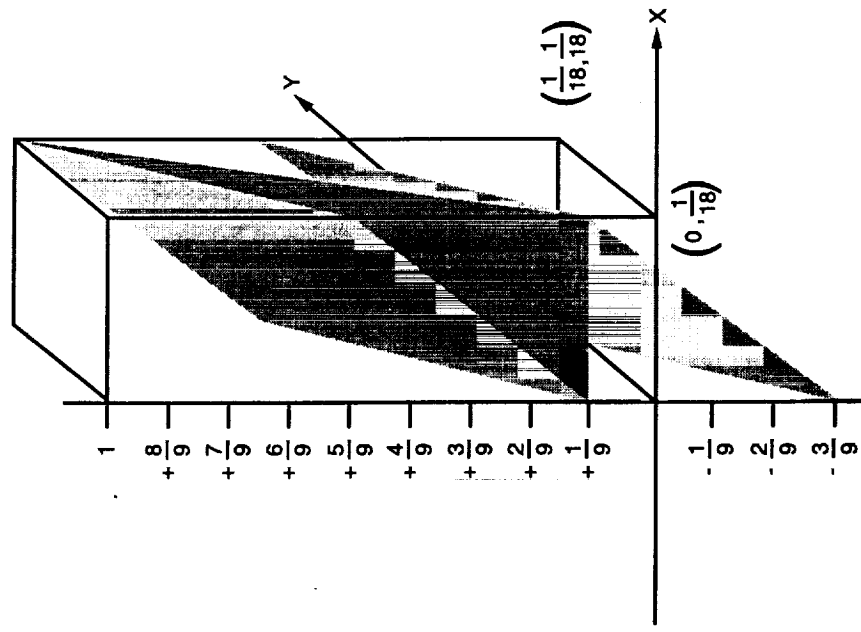
Figure 6
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(2, 3)

Figure 7
D. Rubincam
Am. J. Physics

Figure 8
D. Rubincam
Am. J. Physics

Figure 9
D. Rubincam
Am. J. Physics