ANALYTIC STEADY-STATE ACCURACY OF A SPACECRAFT ATTITUDE ESTIMATOR

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Abstract

This paper extends Farrenkopf's analysis of a single-axis spacecraft attitude estimator using gyro and angle sensor data to include the angle output white noise of a rate-integrating gyro. Analytic expressions are derived for the steady-state pre-update and post-update angle and drift bias variances and for the state update equations. It is shown that only part of the state update resulting from the angle sensor measurement is propagated to future times

Introduction

There has long been an interest in Kalman filtering[1,2] to optimally combine star tracker and gyro data for spacecraft attitude estimation.[3,4] Farrenkopf found an analytic solution for the steady-state accuracy of a single-axis Kalman filter combining data from a gyro and an angle sensor, which has proven very useful for preliminary analysis of spacecraft attitude determination systems.[5] Farrenkopf implicitly assumed a rate gyro model, however, and many missions employ highly accurate rate-integrating gyros (RIGs). The aim of this paper is to modify Farrenkopf's result to be applicable to rate-integrating gyros. Although the intermediate results are significantly more complex than Farrenkopf's, the final expressions for the steady-state covariance are a simple modification to his.

We begin by discussing the dynamic models for the spacecraft attitude and the gyro and the models for the gyro and star tracker measurements. Then we derive the equations for gyro and star tracker covariance updates and obtain the steady-state covariance, including a numerical example. The explicit state update equations are derived next; and it is shown that the angle update due to a star tracker measurement can be divided into two parts, only one of which is propagated to future times. The conclusions are stated at the end of the paper.

Dynamics

The single-axis spacecraft dynamics are given by

$$\dot{\theta} = \omega, \tag{1}$$

where θ is the rotation angle and ω is the true angular velocity. Because of its own internal dynamics, the RIG does not measure θ exactly, but instead accumulates its own angle ϕ. The RIG dynamic equation for this angle is

$$\dot{\phi} = \omega + b + n_v, \tag{2}$$

where b is the gyro drift rate and n_v is a zero mean Gaussian white noise process. The drift rate is assumed to satisfy

$$\dot{b} = n_u, \tag{3}$$

where n_u is also a zero mean Gaussian white noise process. These processes are assumed to obey

$$E\{n_u(t)n_u(t')\} = \sigma_u^2 \delta(t - t'), \tag{4a}$$

$$E\{n_v(t)n_v(t')\} = \sigma_v^2 \delta(t - t'), \tag{4b}$$

and

$$E\{n_u(t)n_v(t')\} = 0, \tag{4c}$$

where $E\{.\}$ denotes the expectation value and $\delta(.)$ is the Dirac delta function.

Farrenkopf took $\dot{\phi}$ to be the measured gyro rate and eliminated ω between Eqs. (1) and (2). We will not make this assumption, since it does not provide a convenient procedure for including angle output white noise (also known as readout noise or electronic noise) on the RIG output.[6] The procedure we adopt requires a dynamic model for the angular rate. Since we are interested in applications that use the gyros for rate measurements rather than Eulerian dynamic models, we approximate the rate as a pure random walk,

$$\dot{\omega} = n_w,$$

(5)

where n_w is a third zero mean Gaussian white noise process, uncorrelated with n_u and n_v, and obeying

$$E\{n_w(t)n_w(t')\} = \sigma_w^2 \delta(t - t').$$

(6)

We will eventually consider the limit of very large σ_w.

The dynamic variables are the four components of the state vector

$$x = [\theta, b, \phi, \omega]^T,$$

(7)

which obeys the dynamic equation

$$\dot{x} = Ax + [0, n_u, n_v, n_w]^T,$$

(8)

with

$$A \equiv \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

(9)

This state vector contains two more components than Farrenkopf's, which only contains θ and b. With a caret denoting expectation values, Eq. (8) gives the state propagation equation

$$\hat{x}(t) = \Phi(t - t_0)\hat{x}(t_0),$$

(10)

where $\Phi(t - t_0)$ is the state transition matrix

$$\Phi(t - t_0) = \begin{bmatrix} 1 & 0 & 0 & t - t_0 \\ 0 & 1 & 0 & 0 \\ 0 & t - t_0 & 1 & t - t_0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(11)

The state covariance matrix

$$P \equiv E\{(x - \hat{x})(x - \hat{x})^T\}$$

(12)

obeys

$$P(t) = \Phi(t - t_0)P(t_0)\Phi^T(t - t_0) + \int_{t_0}^{t} \Phi(t - t')Q\Phi^T(t - t')dt',$$

(13)

where

$$Q \equiv \text{diag}([0, \sigma_u^2, \sigma_v^2, \sigma_w^2])$$

(14)

is the process noise spectral density matrix.

Measurement Models

Our estimator processes gyro measurements at a time interval τ, and star sensor measurements at a multiple of this,

$$T = n\tau. \tag{15}$$

Since we are looking for a steady-state solution, we will only consider a single star tracker measurement followed by n gyro measurements. We will require the covariance after processing all these measurements to be identical to the covariance at the beginning of the time interval of length T, which is assumed to be immediately before the star tracker measurement at time zero.

The star tracker measurement at time zero is modeled as the spacecraft rotation angle plus a white noise term:

$$\theta_0 = \theta(0) + n_n, \tag{16}$$

where n_n is the star tracker measurement noise, which is assumed to obey

$$E\{n_n n_n\} = \sigma_n^2 \delta_{ij}. \tag{17}$$

The sensitivity matrix for the star tracker measurement is the row vector

$$H_{st} = [1, 0, 0, 0]. \tag{18}$$

At time $i\tau$, the gyro provides a measurement which is its integrated angle plus a white noise term:

$$\phi_i = \phi(i\tau) + n_i^c, \quad \text{for} \quad i = 1, 2, \dots, n, \tag{19}$$

where ϕ_i is the measured value and n_i^c is the gyro measurement noise, which is assumed to obey

$$E\{n_i^c n_j^c\} = \sigma_c^2 \delta_{ij}, \tag{20}$$

with δ_{ij} denoting the Kronecker delta. The gyro noise and the star tracker noise are assumed to be uncorrelated with each other and with the process noise. The sensitivity matrix for the gyro measurement is the row vector

$$H_g = [0, 0, 1, 0]. \tag{21}$$

These are the models we need to derive our results.

Gyro Measurement Processing

Assume that we have performed a star tracker update at time zero, so that $\hat{x}(0+)$ and $P(0+)$ denote the state expectation and covariance immediately after a star tracker update. Then $\hat{x}(i\tau-)$ and $P(i\tau-)$ for $i = 1, 2, \dots, n$ will denote the state expectation and covariance immediately before the gyro update at time $i\tau$, and $\hat{x}(i\tau+)$ and $P(i\tau+)$ will denote these quantities immediately after the gyro update. The steady-state condition is that $P(n\tau+)$, the covariance immediately before the star tracker update at time T, is equal to $P(0-)$, the covariance immediately before the star tracker update at time zero but *after* the gyro update immediately preceding it.

Denoting the components of the post-gyro-update covariance by lower case p,

$$P(i\tau+) \equiv \begin{bmatrix} p_{\theta\theta}(i) & p_{\theta b}(i) & p_{\theta\phi}(i) & p_{\theta\omega}(i) \\ p_{\theta b}(i) & p_{bb}(i) & p_{b\phi}(i) & p_{b\omega}(i) \\ p_{\theta\phi}(i) & p_{b\phi}(i) & p_{\phi\phi}(i) & p_{\phi\omega}(i) \\ p_{\theta\omega}(i) & p_{b\omega}(i) & p_{\phi\omega}(i) & p_{\omega\omega}(i) \end{bmatrix}, \tag{22}$$

Eqs. (11)–(14) give

$$P(i\tau-) = \begin{bmatrix} 1 & 0 & 0 & \tau \\ 0 & 1 & 0 & 0 \\ 0 & \tau & 1 & \tau \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_{\theta\theta}(i-1) & p_{\theta b}(i-1) & p_{\theta\phi}(i-1) & p_{\theta\omega}(i-1) \\ p_{\theta b}(i-1) & p_{bb}(i-1) & p_{b\phi}(i-1) & p_{b\omega}(i-1) \\ p_{\theta\phi}(i-1) & p_{b\phi}(i-1) & p_{\phi\phi}(i-1) & p_{\phi\omega}(i-1) \\ p_{\theta\omega}(i-1) & p_{b\omega}(i-1) & p_{\phi\omega}(i-1) & p_{\omega\omega}(i-1) \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & \tau \\ 0 & 1 & 0 & 0 \\ 0 & \tau & 1 & \tau \\ 0 & 0 & 0 & 1 \end{bmatrix}^{\mathrm{T}}$$

$$+ \begin{bmatrix} \frac{1}{3}\tau^3\sigma_w^2 & 0 & \frac{1}{3}\tau^3\sigma_w^2 & \frac{1}{2}\tau^2\sigma_w^2 \\ 0 & \tau\sigma_u^2 & \frac{1}{2}\tau^2\sigma_u^2 & 0 \\ \frac{1}{3}\tau^3\sigma_w^2 & \frac{1}{2}\tau^2\sigma_u^2 & \tau\sigma_v^2 + \frac{1}{3}\tau^3(\sigma_u^2+\sigma_w^2) & \frac{1}{2}\tau^2\sigma_w^2 \\ \frac{1}{2}\tau^2\sigma_w^2 & 0 & \frac{1}{2}\tau^2\sigma_w^2 & \tau\sigma_w^2 \end{bmatrix}.$$

(23)

The standard Kalman filter equations for the gyro measurement update give[1,2]

$$P(i\tau+) = P(i\tau-) - P(i\tau-)H_g^{\mathrm{T}}\Delta_i^{-1}H_g P(i\tau-),$$

(24)

where Δ_i is the scalar

$$\Delta_i \equiv H_g P(i\tau-)H_g^{\mathrm{T}} + \sigma_\epsilon^2$$

$$= p_{\phi\phi}(i-1) + 2\tau[p_{b\phi}(i-1) + p_{\phi\omega}(i-1)] + \tau^2[p_{bb}(i-1) + 2p_{b\omega}(i-1) + p_{\omega\omega}(i-1)] + \tau\sigma_v^2 + \frac{1}{3}\tau^3(\sigma_u^2+\sigma_w^2) + \sigma_\epsilon^2.$$

(25)

Combining Eqs. (22)–(25) gives the ten scalar equations

$$p_{\theta\theta}(i) = p_{\theta\theta}(i-1) + 2\tau p_{\theta\omega}(i-1) + \tau^2 p_{\omega\omega}(i-1) + \frac{1}{3}\tau^3\sigma_w^2$$
$$- \Delta_i^{-1}\{p_{\theta\phi}(i-1) + \tau[p_{\theta\omega}(i-1) + p_{\theta b}(i-1) + p_{\phi\omega}(i-1)] + \tau^2[p_{b\omega}(i-1) + p_{\omega\omega}(i-1)] + \frac{1}{3}\tau^3\sigma_w^2\}^2,$$

(26a)

$$p_{\theta b}(i) = p_{\theta b}(i-1) + \tau p_{b\omega}(i-1) - \Delta_i^{-1}\{p_{\theta\phi}(i-1) + \tau[p_{\theta\omega}(i-1) + p_{\theta b}(i-1) + p_{\phi\omega}(i-1)]$$
$$+ \tau^2[p_{b\omega}(i-1) + p_{\omega\omega}(i-1)] + \frac{1}{3}\tau^3\sigma_w^2\}\{p_{b\phi}(i-1) + \tau[p_{bb}(i-1) + p_{b\omega}(i-1)] + \frac{1}{2}\tau^2\sigma_u^2\},$$

(26b)

$$p_{bb}(i) = p_{bb}(i-1) + \tau\sigma_u^2 - \Delta_i^{-1}\{p_{b\phi}(i-1) + \tau[p_{bb}(i-1) + p_{b\omega}(i-1)] + \frac{1}{2}\tau^2\sigma_u^2\}^2,$$

(26c)

$$p_{\theta\phi}(i) = \sigma_\epsilon^2\,\Delta_i^{-1}\{p_{\theta\phi}(i-1) + \tau[p_{\theta\omega}(i-1) + p_{\theta b}(i-1) + p_{\phi\omega}(i-1)] + \tau^2[p_{b\omega}(i-1) + p_{\omega\omega}(i-1)] + \frac{1}{3}\tau^3\sigma_w^2\},$$

(26d)

$$p_{b\phi}(i) = \sigma_\epsilon^2\,\Delta_i^{-1}\{p_{b\phi}(i-1) + \tau[p_{bb}(i-1) + p_{b\omega}(i-1)] + \frac{1}{2}\tau^2\sigma_u^2\},$$

(26e)

$$p_{\phi\phi}(i) = \sigma_\epsilon^2\,\Delta_i^{-1}(\Delta_i - \sigma_\epsilon^2),$$

(26f)

$$p_{\theta\omega}(i) = p_{\theta\omega}(i-1) + \tau p_{\omega\omega}(i-1) + \frac{1}{2}\tau^2\sigma_w^2 - \Delta_i^{-1}\{p_{\phi\omega}(i-1) + \tau[p_{b\omega}(i-1) + p_{\omega\omega}(i-1)] + \frac{1}{2}\tau^2\sigma_w^2\} \times$$
$$\{p_{\theta\phi}(i-1) + \tau[p_{\theta\omega}(i-1) + p_{\theta b}(i-1) + p_{\phi\omega}(i-1)] + \tau^2[p_{b\omega}(i-1) + p_{\omega\omega}(i-1)] + \frac{1}{3}\tau^3\sigma_w^2\},$$

(27a)

$$p_{b\omega}(i) = p_{b\omega}(i-1) - \Delta_i^{-1}\{p_{\phi\omega}(i-1) + \tau[p_{b\omega}(i-1) + p_{\omega\omega}(i-1)] + \frac{1}{2}\tau^2\sigma_w^2\} \times$$
$$\{p_{b\phi}(i-1) + \tau[p_{bb}(i-1) + p_{b\omega}(i-1)] + \frac{1}{2}\tau^2\sigma_u^2\},$$

(27b)

$$p_{\phi\omega}(i) = \sigma_\epsilon^2\Delta_i^{-1}\{p_{\phi\omega}(i-1) + \tau[p_{b\omega}(i-1) + p_{\omega\omega}(i-1)] + \frac{1}{2}\tau^2\sigma_w^2\},$$

(27c)

and

$$p_{\omega\omega}(i) = p_{\omega\omega}(i-1) + \tau\sigma_w^2 - \Delta_i^{-1}\{p_{\phi\omega}(i-1) + \tau[p_{b\omega}(i-1) + p_{\omega\omega}(i-1)] + \frac{1}{2}\tau^2\sigma_w^2\}^2.$$

(27d)

It is clear that significant simplifications are required in order to find a steady-state solution. We will consider the limit that σ_w becomes much larger than all the other measurement and process noise variances. This means that the

filter is ignorant of the spacecraft dynamics, in the sense of modeling the torques on the spacecraft and its response to these torques. In fact, we may have some knowledge of these quantities, but the filter does not make use of this knowledge, instead relying entirely on the RIG for information on the angular motion. This is the most common case in practice. Equation (27d) shows that $p_{\omega\omega}(i)$ becomes large along with σ_w, so we will take

$$p_{\omega\omega}(i) = \alpha_i \tau \sigma_w^2 + \bar{p}_{\omega\omega}(i) \tag{28}$$

where the tilde denotes the part that remains finite as σ_w becomes large. None of the other components of $P(i\tau+)$ contain terms that become large in proportion to σ_w. Now Eq. (25) gives

$$\Delta_i = \tau^3 (\alpha_{i-1} + \tfrac{1}{3}) \sigma_w^2 + \tilde{\Delta}_i , \tag{29}$$

where $\tilde{\Delta}_i$ is the finite part

$$\tilde{\Delta}_i \equiv p_{\phi\phi}(i-1) + 2\tau[p_{b\phi}(i-1) + p_{\phi\omega}(i-1)] + \tau^2[p_{bb}(i-1) + 2p_{b\omega}(i-1) + \bar{p}_{\omega\omega}(i-1)] + \tau\sigma_v^2 + \tfrac{1}{3}\tau^3\sigma_u^2 + \sigma_\epsilon^2 . \tag{30}$$

Substituting Eq. (28) and

$$\Delta_i^{-1} = \tau^{-3}(\alpha_{i-1} + \tfrac{1}{3})^{-1}\sigma_w^{-2}[1 - \tau^{-3}(\alpha_{i-1} + \tfrac{1}{3})^{-1}\sigma_w^{-2}\tilde{\Delta}_i + \ \dots \ ] , \tag{31}$$

into Eqs. (26) and (27) and equating terms of first order in σ_w^2 gives

$$\alpha_i = (\tfrac{1}{4} + \alpha_{i-1})/(1 + 3\alpha_{i-1}) . \tag{32}$$

Equating terms that are independent of σ_w^2 gives the ten equations

$$p_{\theta\theta}(i) = p_{\theta\theta}(i-1) + p_{\phi\phi}(i-1) - 2p_{\theta\phi}(i-1) + 2\tau[p_{b\phi}(i-1) - p_{\theta b}(i-1)] + \tau^2 p_{bb}(i-1) + \tau\sigma_v^2 + \tfrac{1}{3}\tau^3\sigma_u^2 + \sigma_\epsilon^2 , \tag{33a}$$

$$p_{\theta b}(i) = p_{\theta b}(i-1) - p_{b\phi}(i-1) - \tau p_{bb}(i-1) - \tfrac{1}{2}\tau^2\sigma_u^2 , \tag{33b}$$

$$p_{bb}(i) = p_{bb}(i-1) + \tau\sigma_u^2 , \tag{33c}$$

$$p_{\theta\phi}(i) = \sigma_\epsilon^2 , \tag{33d}$$

$$p_{b\phi}(i) = 0 , \tag{33e}$$

$$p_{\phi\phi}(i) = \sigma_\epsilon^2 , \tag{33f}$$

$$p_{\theta\omega}(i) = p_{\theta\omega}(i-1) - p_{\phi\omega}(i-1) - \tau p_{b\omega}(i-1) + k_{i-1}\tau^{-1}\{p_{\phi\phi}(i-1) - p_{\theta\phi}(i-1) \\ + \tau[p_{\phi\omega}(i-1) + 2p_{b\phi}(i-1) - p_{\theta\omega}(i-1) - p_{\theta b}(i-1)] + \tau^2[p_{bb}(i-1) + p_{b\omega}(i-1)] + \tau\sigma_v^2 + \tfrac{1}{3}\tau^3\sigma_u^2 + \sigma_\epsilon^2\} , \tag{34a}$$

$$p_{b\omega}(i) = p_{b\omega}(i-1) - k_{i-1}[p_{bb}(i-1) + p_{b\omega}(i-1) + \tau^{-1}p_{b\phi}(i-1) + \tfrac{1}{2}\tau\sigma_u^2] , \tag{34b}$$

$$p_{\phi\omega}(i) = k_{i-1}\tau^{-1}\sigma_\epsilon^2 , \tag{34c}$$

and

$$\bar{p}_{\omega\omega}(i) = (k_{i-1} - 1)^2 \bar{p}_{\omega\omega}(i-1) + 2k_{i-1}(k_{i-1} - 1)[\tau^{-1}p_{\phi\omega}(i-1) + p_{b\omega}(i-1)] \\ + k_{i-1}^2\tau^{-2}[p_{\phi\phi}(i-1) + 2\tau p_{b\phi}(i-1) + \tau^2 p_{bb}(i-1) + \tau\sigma_v^2 + \tfrac{1}{3}\tau^3\sigma_u^2 + \sigma_\epsilon^2] , \tag{34d}$$

where

$$k_i \equiv (\tfrac{1}{2} + \alpha_i)/(\tfrac{1}{3} + \alpha_i) . \tag{35}$$

Terms that go to zero as σ_w becomes infinitely large are ignored.

We will see below that the star tracker update doesn't affect α_i. Thus we can assume that the coefficient α has attained a steady-state value resulting from repeated iterations of Eq. (32). This is the positive root of

$$\alpha = (\tfrac{1}{4} + \alpha)/(1 + 3\alpha), \tag{36}$$

or

$$\alpha = 1/\sqrt{12}. \tag{37}$$

The corresponding steady-state value of k is, from Eq. (35),

$$k = 3 - \sqrt{3}. \tag{38}$$

We now want to consider the overall effect on the covariance of the n gyro updates. Since $p_{\theta\omega}$, $p_{b\omega}$, $p_{\phi\omega}$, and $p_{\omega\omega}$ do not appear on the right side of Eqs. (33), these equations can be combined in the 3×3 matrix form

$$
\begin{bmatrix}
p_{\theta\theta}(i) & p_{\theta b}(i) & p_{\theta\phi}(i) \\
p_{\theta b}(i) & p_{bb}(i) & p_{b\phi}(i) \\
p_{\theta\phi}(i) & p_{b\phi}(i) & p_{\phi\phi}(i)
\end{bmatrix}
=
\begin{bmatrix}
1 & -\tau & -1 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
p_{\theta\theta}(i-1) & p_{\theta b}(i-1) & p_{\theta\phi}(i-1) \\
p_{\theta b}(i-1) & p_{bb}(i-1) & p_{b\phi}(i-1) \\
p_{\theta\phi}(i-1) & p_{b\phi}(i-1) & p_{\phi\phi}(i-1)
\end{bmatrix}
\begin{bmatrix}
1 & -\tau & -1 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{bmatrix}^{T}
$$
$$
+
\begin{bmatrix}
\tau\sigma_v^2 + \tfrac{1}{3}\tau^3\sigma_u^2 + \sigma_\epsilon^2 & -\tfrac{1}{2}\tau^2\sigma_u^2 & \sigma_\epsilon^2 \\
-\tfrac{1}{2}\tau^2\sigma_u^2 & \tau\sigma_u^2 & 0 \\
\sigma_\epsilon^2 & 0 & \sigma_\epsilon^2
\end{bmatrix}. \tag{39}
$$

This equation represents the combined effect of propagation over a time step and a gyro update. It is easy to show by induction that for $i > 0$

$$
\begin{bmatrix}
p_{\theta\theta}(i) & p_{\theta b}(i) & p_{\theta\phi}(i) \\
p_{\theta b}(i) & p_{bb}(i) & p_{b\phi}(i) \\
p_{\theta\phi}(i) & p_{b\phi}(i) & p_{\phi\phi}(i)
\end{bmatrix}
=
\begin{bmatrix}
1 & -i\tau & -1 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
p_{\theta\theta}(0) & p_{\theta b}(0) & p_{\theta\phi}(0) \\
p_{\theta b}(0) & p_{bb}(0) & p_{b\phi}(0) \\
p_{\theta\phi}(0) & p_{b\phi}(0) & p_{\phi\phi}(0)
\end{bmatrix}
\begin{bmatrix}
1 & -i\tau & -1 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{bmatrix}^{T}
$$
$$
+
\begin{bmatrix}
i\tau\sigma_v^2 + \tfrac{1}{3}(i\tau)^3\sigma_u^2 + \sigma_\epsilon^2 & -\tfrac{1}{2}(i\tau)^2\sigma_u^2 & \sigma_\epsilon^2 \\
-\tfrac{1}{2}(i\tau)^2\sigma_u^2 & i\tau\sigma_u^2 & 0 \\
\sigma_\epsilon^2 & 0 & \sigma_\epsilon^2
\end{bmatrix}. \tag{40}
$$

Star Tracker Measurement Processing

Using the notation and steady-state relation introduced at the beginning of the previous section, Eq. (40) gives

$$
\begin{aligned}
P_{\theta\theta}(0-) &= P_{\theta\theta}(n\tau+) \\
&= P_{\theta\theta}(0+) + P_{\phi\phi}(0+) - 2P_{\theta\phi}(0+) + 2T\,[P_{b\phi}(0+) - P_{\theta b}(0+)] + T^2 P_{bb}(0+) + T\sigma_v^2 + \tfrac{1}{3}T^3\sigma_u^2 + \sigma_\epsilon^2,
\end{aligned} \tag{41a}
$$

$$P_{\theta b}(0-) = P_{\theta b}(n\tau+) = P_{\theta b}(0+) - P_{b\phi}(0+) - T\,P_{bb}(0+) - \tfrac{1}{2}T^2\sigma_u^2, \tag{41b}$$

$$P_{bb}(0-) = P_{bb}(n\tau+) = P_{bb}(0+) + T\sigma_u^2, \tag{41c}$$

$$P_{\theta\phi}(0-) = P_{\theta\phi}(n\tau+) = \sigma_\epsilon^2, \tag{41d}$$

$$P_{b\phi}(0-) = P_{b\phi}(n\tau+) = 0, \tag{41e}$$

and

$$P_{\phi\phi}(0-) = P_{\phi\phi}(n\tau+) = \sigma_\epsilon^2. \tag{41f}$$

The star tracker measurement update at time zero gives

$$P(0+) = P(0-) - P(0-)H_{st}^T[H_{st}P(0-)H_{st}^T + \sigma_n^2]^{-1}H_{st}P(0-). \tag{42}$$

Inserting this into Eqs. (41a–c) gives the steady-state conditions

$$[\sigma_e^2 + \sigma_n^2 + TP_{\theta b}(0-)]^2 = [P_{\theta\theta}(0-) + \sigma_n^2][T^2P_{bb}(0-) - P_{\theta\theta}(0-) + T\sigma_v^2 + \tfrac{1}{3}T^3\sigma_u^2 + 2\sigma_e^2 + \sigma_n^2], \tag{43a}$$

$$[\sigma_e^2 - P_{\theta\theta}(0-) + TP_{\theta b}(0-)]P_{\theta b}(0-) = [P_{\theta\theta}(0-) + \sigma_n^2][TP_{bb}(0-) + \tfrac{1}{2}T^2\sigma_u^2], \tag{43b}$$

and

$$P_{\theta b}^2(0-) = [P_{\theta\theta}(0-) + \sigma_n^2]T\sigma_u^2. \tag{43c}$$

The steady-state values of $P_{b\omega}$, $P_{\phi\omega}$, and $P_{\omega\omega}$ are not of interest. We note, however, that Eq. (42) does not change the component of $P_{\omega\omega}$ proportional to σ_w^2, so α is left unaltered by the star tracker measurement, as promised.

Defining a dimensionless variable ζ by the relation

$$P_{\theta b}(0-) = -T^{1/2}\sigma_u\sigma_n\zeta, \tag{44}$$

Eq. (43c) gives

$$P_{\theta\theta}(0-) = (\zeta^2 - 1)\sigma_n^2, \tag{45a}$$

and it follows from Eq. (42) that

$$P_{\theta\theta}(0+) = (1 - \zeta^{-2})\sigma_n^2. \tag{45b}$$

Equations (43b) and (42) give

$$P_{bb}(0\mp) = [\zeta - (1 + S_e^2)\zeta^{-1}]T^{-1/2}\sigma_u\sigma_n \pm \tfrac{1}{2}T\sigma_u^2, \tag{46}$$

where

$$S_e \equiv \sigma_e/\sigma_n. \tag{47a}$$

We similarly define, following Farrenkopf,

$$S_v \equiv T^{1/2}\sigma_v/\sigma_n \tag{47b}$$

and

$$S_u \equiv T^{3/2}\sigma_u/\sigma_n. \tag{47c}$$

The remaining task is to find ζ by solving Eq. (43a), which can be written

$$[\zeta^2 - 2(\tfrac{1}{4}S_u + \gamma)\zeta + 1 + S_e^2][\zeta^2 - 2(\tfrac{1}{4}S_u - \gamma)\zeta + 1 + S_e^2] = 0, \tag{48}$$

with

$$\gamma \equiv (1 + S_e^2 + \tfrac{1}{4}S_v^2 + \tfrac{1}{48}S_u^2)^{1/2}. \tag{49}$$

The physically significant root of Eq. (48) is the largest root,

$$\zeta = \gamma + \tfrac{1}{4}S_u + \tfrac{1}{2}(2\gamma S_u + S_v^2 + \tfrac{1}{3}S_u^2)^{1/2}, \tag{50}$$

where γ is assumed to be positive. Equation (50) gives

$$(1 + S_e^2)\zeta^{-1} = \gamma + \tfrac{1}{4}S_u - \tfrac{1}{2}(2\gamma S_u + S_v^2 + \tfrac{1}{3}S_u^2)^{1/2}, \tag{51}$$

which can be used to rewrite Eq. (49) in the more convenient form

$$P_{bb}(0\mp) = (2\gamma\,T^{1/2}\sigma_u\sigma_n + \sigma_v^2 + \tfrac{1}{3}T^2\sigma_u^2)^{1/2}\sigma_u \pm \tfrac{1}{2}T\sigma_u^2 \,. \tag{52}$$

The principal results of this paper are given by Eqs. (45), (49), (50) and (52). Although the derivation is much more involved than Farrenkopf's and the notation is somewhat different, the final results only differ by the S_e^2 term in Eq. (49). For $\sigma_e = 0$, these equations are completely equivalent to Farrenkopf's. It is remarkable that these results do not depend in any way on the gyro update interval τ.

Approximate Forms

Approximate expressions valid for rapid updates are obtained by retaining only the lowest order terms in S_v and S_u. Equations (49) and (50) give

$$\zeta \approx (1 + S_e^2)^{1/2} + \tfrac{1}{2}\left[2(1 + S_e^2)^{1/2}S_u + S_v^2\right]^{1/2} \tag{53}$$

which gives the approximate forms for the angle and drift bias variances

$$P_{\theta\theta}(0-) = (\sigma_n^2 + \sigma_e^2)^{1/2}\left[2T^{3/2}\sigma_u(\sigma_n^2 + \sigma_e^2)^{1/2} + T\sigma_v^2\right]^{1/2} + \sigma_e^2, \tag{54a}$$

$$P_{\theta\theta}(0+) = \frac{\sigma_n^4}{(\sigma_n^2 + \sigma_e^2)^{3/2}}\left[2\,T^{3/2}\sigma_u(\sigma_n^2 + \sigma_e^2)^{1/2} + T\sigma_v^2\right]^{1/2} + \frac{\sigma_n^2\sigma_e^2}{\sigma_n^2 + \sigma_e^2}, \tag{54b}$$

and

$$P_{bb}(0\mp) \approx [2\,T^{1/2}\sigma_u(\sigma_n^2 + \sigma_e^2)^{1/2} + \sigma_v^2]^{1/2}\sigma_u. \tag{55}$$

The difference between the pre-update and post-update drift bias variance is of order σ_u^2, which is negligible in this approximation. When σ_e is negligible, the post-update and pre-update angle variances are equal in the approximation of rapid updates, also.

Numerical Example

Consider a rate-integrating gyro with

$$\sigma_v = 0.025\,\mathrm{deg}/\sqrt{\mathrm{hour}} = 7.27\,\mu\mathrm{rad}/\sqrt{\mathrm{sec}}, \tag{56a}$$

$$\sigma_u = 3.7 \times 10^{-3}\,\mathrm{deg}/\mathrm{hour}^{3/2} = 3 \times 10^{-4}\,\mu\mathrm{rad}/\mathrm{sec}^{3/2}, \tag{56b}$$

and

$$\sigma_e = 15\,\mu\mathrm{rad}. \tag{56c}$$

These numbers are characteristic of a ring-laser gyro with very low drift but with significant angle white noise. Assume that the star tracker measurement noise is

$$\sigma_n = 15\,\mu\mathrm{rad}. \tag{56d}$$

Figure 1 shows the steady-state pre-update and post-update angle standard deviations, which are the square roots of $P_{\theta\theta}(0\mp)$, for star tracker update times between 0.01 sec and 100 sec. The solid curves are the exact values given by Eqs. (45), the dashed curves are the result of ignoring σ_e in these exact equations, and the dot-dash curves are rapid-update approximations of Eq. (54). For each pair of curves, the upper curve is the pre-update value, and the lower curve is the post-update value. The figure shows that the rapid-update approximations always lie between the pre-update and post-update exact values, and that these approximations are reasonably accurate for update times of less than one second with this set of gyro parameters.

The steady-state drift bias standard deviations are not plotted, since they always lie between 0.0467 μradians/sec and 0.0468 μradians/sec for these gyro parameters, whether or not the σ_e terms are included. The lower value is just equal to $\sqrt{\sigma_u \sigma_v}$, the common limit of Eqs. (52) and (55) for small σ_u.

State Update Equations

The steady-state Kalman gains and state update equations are also of interest. The steady-state gains for the gyro update are given by

$$K_g = P(i\tau+)H_g^T\sigma_e^{-2} = [1, 0, 1, k/\tau]^T. \tag{57}$$

It is remarkable that these gains are constant, independent of the number of gyro updates since the previous star tracker update. The state update equation is

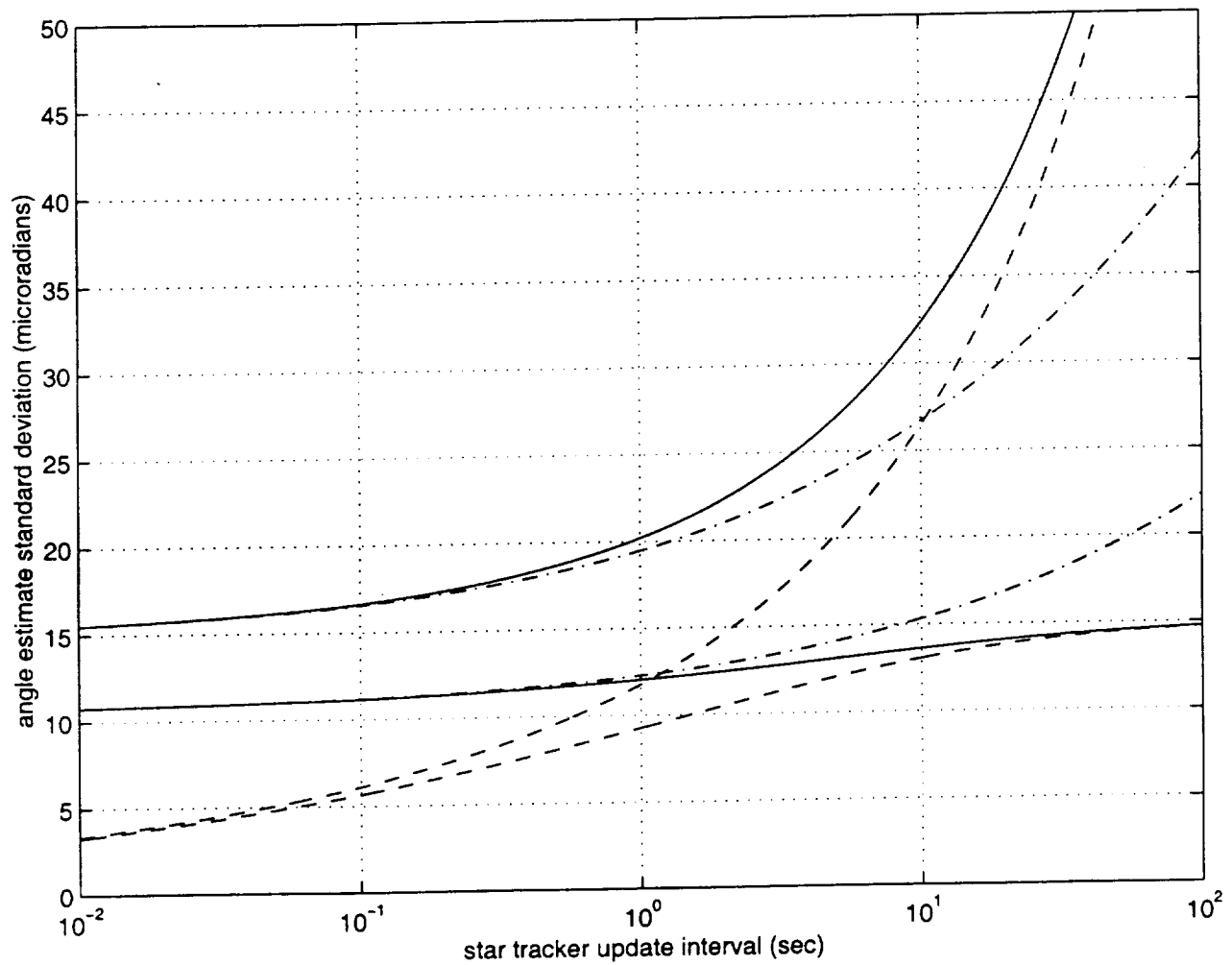$$\hat{x}(i\tau+) = \hat{x}(i\tau-) + K_g[\phi_i - \phi(i\tau-)]. \tag{58}$$

Figure 1. Steady-state pre-update and post-update angle standard deviations

Solid curves from Eqs. (45), dashed curves from Eqs. (45) with $\sigma_e = 0$, dot-dash curves from Eq. (54)

This gives, for the individual components:

$$\hat{\theta}(i\tau+) = \hat{\theta}(i\tau-) + \phi_i - \hat{\phi}(i\tau-) = \hat{\theta}((i-1)\tau+) + \phi_i - \hat{\phi}((i-1)\tau+) - \tau\hat{b}((i-1)\tau+),$$ (59a)

$$\hat{b}(i\tau+) = \hat{b}(i\tau-) = \hat{b}((i-1)\tau+),$$ (59b)

$$\hat{\phi}(i\tau+) = \hat{\phi}(i\tau-) + \phi_i - \hat{\phi}(i\tau-) = \phi_i,$$ (59c)

and

$$\hat{\omega}(i\tau+) = \hat{\omega}(i\tau-) + k\tau^{-1}[\phi_i - \hat{\phi}(i\tau-)] = (1-k)\hat{\omega}((i-1)\tau+) + k\tau^{-1}[\phi_i - \hat{\phi}((i-1)\tau+) - \tau\hat{b}((i-1)\tau+)],$$ (59d)

where we have used Eqs. (10) and (11) to include the state propagation from the previous gyro update.

The first three of Eqs. (59) have a straightforward interpretation for $i \geq 2$. In this case $\hat{\phi}((i-1)\tau+) = \phi_{i-1}$ and the estimate of the spacecraft rotation angle is the previous estimate plus the rotation sensed by the gyro, corrected for the estimated drift bias. The estimate of the bias is not refined by the gyro measurement, as expected; and the estimate of the gyro-sensed angle is equal to the gyro measurement itself. We will return to the $i = 1$ case after considering the star tracker update.

Equation (59d) is more difficult to understand. For k equal to unity, it would just say that the best estimate for the angular velocity is the gyro-sensed rate over the last measurement interval, corrected for drift. For k less than one, the updated angular velocity estimate would be some average of the previous estimate and the gyro-sensed rate over the last measurement interval. The steady-state k is greater than one, however, which means that the previous estimate of the angular velocity is given a negative weight in this average. It is worth noting that the variance of the angular velocity estimate is very large in our model, as shown by Eq. (28).

The steady-state gains for the star tracker update are given by

$$K_{st} = P(0+)H_{st}^T\sigma_n^{-2} = \begin{bmatrix} 1-\zeta^{-2} \\ -(\zeta T)^{-1}S_u \\ (S_e/\zeta)^2 \\ \sigma_n^{-2}P_{\theta\omega}(0+) \end{bmatrix},$$ (60)

where we have used

$$P_{\theta b}(0+) = \zeta^{-2}P_{\theta b}(0-) = -\zeta^{-1}T^{1/2}\sigma_u\sigma_n$$ (61a)

and

$$P_{\theta\phi}(0+) = \zeta^{-2}P_{\theta\phi}(0-) = \zeta^{-2}\sigma_e^2.$$ (61b)

The state update equation is

$$\hat{x}(0+) = \hat{x}(0-) + K_{st}[\theta_0 - \hat{\theta}(0-)].$$ (62)

Star tracker updates of $\hat{\omega}$, which depend on the steady-state value of $P_{\theta\omega}(0+)$, are not generally performed in practice. The other three components of the state vector are updated by

$$\hat{\theta}(0+) = \hat{\theta}(0-) + (1 - \zeta^{-2})[\theta_0 - \hat{\theta}(0-)],$$ (63a)

$$\hat{b}(0+) = \hat{b}(0-) - (\zeta T)^{-1}S_u[\theta_0 - \hat{\theta}(0-)],$$ (63b)

and

$$\hat{\phi}(0+) = \hat{\phi}(0-) + (S_e/\zeta)^2[\theta_0 - \hat{\theta}(0-)].$$ (63c)

It is convenient to define a modified angle estimate $\bar{\theta}$ by

$$\bar{\theta}(i\tau+) \equiv \hat{\theta}(i\tau+) \tag{64a}$$

for $i = 1, 2, \ldots, n$, and

$$\bar{\theta}(0+) \equiv \hat{\theta}(0+) + \phi_0 - \hat{\phi}(0+), \tag{64b}$$

where ϕ_0 is understood to denote the gyro measurement immediately preceding the star tracker measurement at time zero. Thus $\bar{\theta}$ is the optimal angle estimate following a gyro update, but differs from the optimal estimate after a star tracker update. Now Eq. (59a) can be written

$$\bar{\theta}(i\tau+) = \bar{\theta}((i-1)\tau+) + \phi_i - \phi_{i-1} - \tau\hat{b}((i-1)\tau+) \tag{65}$$

for all gyro updates. With this restructuring of the update equations, the need to estimate $\hat{\phi}$ vanishes.

Since $\hat{\theta}(0-)$ is the angle estimate following the gyro update immediately preceding the star tracker measurement, it is consistent with our labeling conventions to denote this quantity by $\bar{\theta}(0-)$ on the right side of Eqs. (63). This and Eqs. (50), (51), (63), and (64) enable us to write the star tracker update of the optimal angle estimate as a two-step process

$$\bar{\theta}(0+) = \bar{\theta}(0-) + \zeta^{-1}(2\gamma S_u + S_v^2 + \tfrac{1}{3}S_u^2)^{1/2}[\theta_0 - \bar{\theta}(0-)], \tag{66a}$$

followed by

$$\hat{\theta}(0+) = (\sigma_n^2 + \sigma_\epsilon^2)^{-1}[\sigma_n^2\,\bar{\theta}(0+) + \sigma_\epsilon^2\,\theta_0]. \tag{66b}$$

Equations (63) and (65) show that the update of Eq. (66a), which vanishes if $\sigma_u = \sigma_v = 0$, is propagated into future estimates of the angle and the drift bias. In contrast to this, the update of Eq. (66b), which does not depend on σ_u or σ_v and vanishes if $\sigma_\epsilon = 0$, is only effective at the time of the star tracker measurement. The appearance of Eq. (66b) as the optimal combination of independent quantities $\bar{\theta}(0+)$ and θ_0 with standard deviations σ_ϵ and σ_n is very misleading, since these quantities are correlated by Eq. (66a).

Conclusions

Analytic expressions for the steady-state accuracy of a single-axis Kalman filter combining data from a gyro and an angle sensor have proven very useful in tailoring attitude sensor specifications to mission requirements. This paper provides a very simple modification to Farrenkopf's analytic expressions to include the effects of angle white noise on the output of a rate-integrating gyro. The resulting equations depend on the star tracker update interval, but are independent of the gyro update interval. The effect of the angle white noise is significant for a specific set of gyro parameters analyzed. A rapid-update approximation to the steady-state accuracy is reasonably accurate for star tracker update rates of once per second or faster for these gyro parameters.

The model developed in this paper leads to an effective two-component filter for the attitude angle and gyro drift bias, much like Farrenkopf's. In fact, the filter is identical with Farrenkopf's if gyro angle output noise is absent. In the presence of gyro angle output white noise, the update of the angle estimate resulting from a star tracker measurement can be broken into two parts. The first part, which vanishes when the gyro drift parameters are zero, is propagated forward into future angle estimates. The second part of the update, which is independent of the gyro drift parameters and vanishes if the gyro readout white noise is zero, is only effective at the time of the star tracker measurement.

Acknowledgements

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