NASA/TM—2000-210361

R-Function Relationships for Application in the Fractional Calculus

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August 2000

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R-Function Relationships for Application in the Fractional Calculus

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Abstract

The F-function, and its generalization the R-function, are of fundamental importance in the fractional calculus. It has been shown that the solution of the fundamental linear fractional differential equation may be expressed in terms of these functions. These functions serve as generalizations of the exponential function in the solution of fractional differential equations. Because of this central role in the fractional calculus, this paper explores various intrarelationships of the R-function, which will be useful in further analysis.

Relationships of the R-function to the common exponential function, e^t, and its fractional derivatives are shown. From the relationships developed, some important approximations are observed. Further, the inverse relationships of the exponential function, e^t, in terms of the R-function are developed. Also, some approximations for the R-function are developed.

1. Introduction

The F-function [1] defined as

$$F_q(a,t) \equiv \sum_{n=0}^{\infty} \frac{a^n t^{(n+1)q-1}}{\Gamma((n+1)q)}, \tag{1.1}$$

and its generalization the R-function [2],

$$R_{q,v}(a,c,t) \equiv \sum_{n=0}^{\infty} \frac{a^n (t-c)^{(n+1)q-1-v}}{\Gamma((n+1)q-v)}, \tag{1.2}$$

are of fundamental importance in the fractional calculus. In this paper our interest will be confined to $t > c = 0$, $q \geq 0$, and $v \leq q$. Lorenzo and Hartley [2] have determined a variety of relationships associated with the R-function, including those involving relationships with the circular and hyperbolic functions as well as other advanced functions. A few more relationships to advanced functions are also presented in the Appendix of this paper. It has been shown ([1] and elsewhere) that the solution of the fundamental linear fractional differential equation

$${}_aD_t^q x(t) + a\,x(t) = f(t) \tag{1.3}$$

may be expressed in terms of these functions. As in the case of ordinary differential equations combinations and convolutions of R-functions are used to express the solutions of systems of fractional differential equations. Because of this central role in the fractional calculus, and since

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$R_{q,0}(a,0,t) = F_q(a,t)$ this paper explores various intrarelationships of the *R*-function, which will be useful in further analysis and application.

The general character of the *R*-function is shown in figure 1. Figure 1 shows the effect of variations of q with $v = 0$ and $a = \pm 1$. The exponential character of the function is readily observed (see, $q = 1$).

The Laplace transform of the *R*-function, starting at $t = c = 0$,

$$L\{R_{q,v}(a,0,t)\} = \frac{s^v}{s^q - a}, \tag{1.4}$$

is derived in reference [2]. It is also noted that

$$R_{1,0}(a,0,t) = R_{1,0}(1,0,at) = \sum_{n=0}^{\infty} \frac{(at)^n}{\Gamma(n+1)} = e^{at}. \tag{1.5}$$

These relationships will be useful in the analysis that follows. This is special, because in general

$$R_{q,v}(a,0,t) \neq R_{q,v}(1,0,at). \tag{1.6}$$

The following useful relationship however, is shown

$$R_{q,v}(1,0,at) = \sum_{n=0}^{\infty} \frac{(at)^{(n+1)q-1-v}}{\Gamma((n+1)q - v)} = a^{q-1-v} \sum_{n=0}^{\infty} \frac{(a^q)^n t^{(n+1)q-1-v}}{\Gamma((n+1)q - v)}, \tag{1.7}$$

therefore

$$R_{q,v}(1,0,at) = a^{q-1-v} R_{q,v}(a^q,0,t). \tag{1.8}$$

Alternatively this may be written as

$$R_{q,v}(a,0,t) = a^{(v+1-q)/q} R_{q,v}(1,0,a^{1/q}t). \tag{1.9}$$

In what follows in this paper, intrarelationships between *R*-functions of different arguments will be developed.

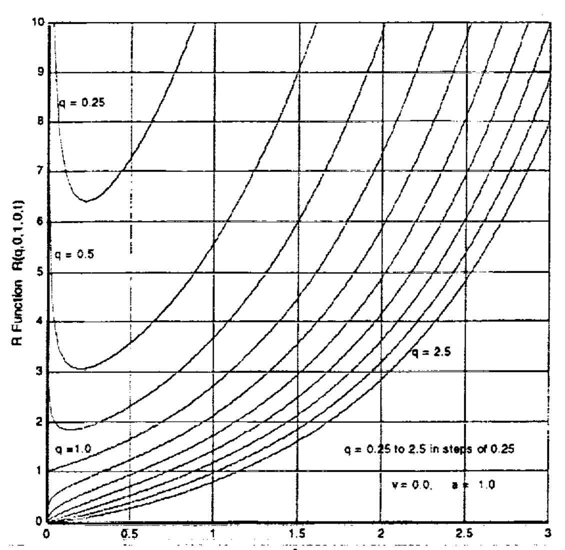

Figure 1a. Effect of q on $R_{q,0}(1,0,t)$, $v = 0.0$, $a = 1.0$

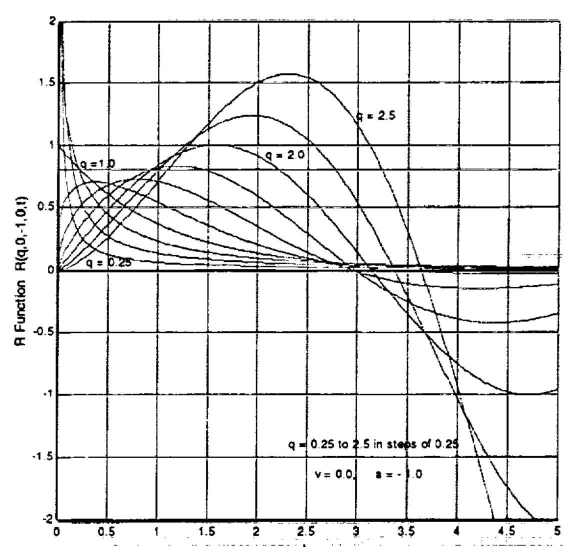

Figure 1b. Effect of q on $R_{q,0}(-1,0,t)$, $v = 0.0$, $a = -1.0$

2. Relationships for $R_{m,0}$ in Terms of $R_{1,0}$

This section will develop the relation

$$R_{m,0}(1,0,t)=f(R_{1,0}(a,0,t))=f(e^{at}) \qquad m=1,2,3\ldots \,. \tag{2.1}$$

We consider first the even cases, in particular $m=2$. We have then

$$R_{2,0}(1,0,t)=\sum_{n=0}^{\infty}\frac{t^{2n+1}}{\Gamma(2n+2)}=\frac{t}{\Gamma(2)}+\frac{t^3}{\Gamma(4)}+\frac{t^5}{\Gamma(6)}+\ldots\,. \tag{2.2}$$

Now since

$$R_{1,0}(1,0,t)=\sum_{n=0}^{\infty}\frac{t^{n}}{\Gamma(n+1)}=1+\frac{t}{\Gamma(2)}+\frac{t^2}{\Gamma(3)}+\frac{t^3}{\Gamma(4)}+\ldots \tag{2.3}$$

and

$$R_{1,0}(-1,0,t)=\sum_{n=0}^{\infty}\frac{(-1)^n t^{n}}{\Gamma(n+1)}=1-\frac{t}{\Gamma(2)}+\frac{t^2}{\Gamma(3)}-\frac{t^3}{\Gamma(4)}+\ldots \tag{2.4}$$

it is readily seen by substitution that

$$R_{2,0}(1,0,t)=\frac{1}{2}(R_{1,0}(1,0,t)-R_{1,0}(-1,0,t))=\frac{e^t-e^{-t}}{2}=Sinh(t)\,. \tag{2.5}$$

An alternative approach to this problem is available through the Laplace transform,

$$\begin{aligned} L\{R_{2,0}(1,0,t)\}&=\frac{1}{s^2-1}=\frac{1}{2}\left(\frac{1}{s-1}-\frac{1}{s+1}\right)\\ &=\frac{1}{2}\sum_{k=0}^{1}cis(2\pi\,k/2)\left(\frac{1}{s-cis(2\pi\,k/2)}\right), \end{aligned} \tag{2.6}$$

where $cis(\phi)=\cos(\phi)+i\sin(\phi)$. The inverse transform of this equation, of course, yields the equation (2.5) result.

The $m=4$ case is now considered, then

$$R_{4,0}(1,0,t)=\sum_{n=0}^{\infty}\frac{t^{4n+3}}{\Gamma(4n+4)}=\frac{t^3}{\Gamma(4)}+\frac{t^7}{\Gamma(8)}+\frac{t^{11}}{\Gamma(12)}+\cdots\,. \tag{2.7}$$

Again by substitution it is readily verified that

$$R_{4,0}(1,0,t)=\frac{1}{4}(R_{1,0}(1,0,t)-R_{1,0}(-1,0,t)+iR_{1,0}(i,0,t)-iR_{1,0}(-i,0,t)) \tag{2.8}$$

$$=\frac{1}{4}\sum_{k=0}^{3}cis(2\pi\,k/4)R_{1,0}(cis(2\pi\,k/4),0,t)=\frac{1}{4}\sum_{k=0}^{3}e^{i\pi k/2}R_{1,0}(e^{i\pi k/2},0,t), \tag{2.9}$$

$$=\frac{1}{2}Sinh(t)-\frac{1}{2}Sin(t) \tag{2.10}$$

Examining the above solutions, equations (2.5) and (2.8), it is observed that the values of the coefficients and the a parameter of the $R_{1,0}$-functions lay on the unit circle of the complex plane for their respective m values. This will be validated for the $m = 3$ case. Using the Laplace transform we have

$$L\{R_{3,0}(1,0,t)\} = \frac{1}{s^3 - 1}. \tag{2.11}$$

Now the roots of $s^3 - 1 = 0$ are

$$s = 1^{1/3} = \cos\left(\frac{2\pi k}{3}\right) + i\sin\left(\frac{2\pi k}{3}\right) = cis\left(\frac{2\pi k}{3}\right) \quad k = 0,1,2. \tag{2.12}$$

Thus

$$\begin{aligned} L\{R_{3,0}(1,0,t)\} &= \frac{1}{s^3 - 1} \\ &= \frac{1}{3}\sum_{k=0}^{2} cis(2\pi k/3)\left(\frac{1}{s - cis(2\pi k/3)}\right). \end{aligned} \tag{2.13}$$

Inverse transforming yields

$$R_{3,0}(1,0,t) = \frac{1}{3}\sum_{k=0}^{2} cis(2\pi k/3)\left(R_{1,0}(cis(2\pi k/3),0,t)\right). \tag{2.14}$$

This may also be written as

$$R_{3,0}(1,0,t) = \frac{1}{3}\sum_{k=0}^{2} cis(2\pi k/3)\, e^{cis(2\pi k/3)t}. \tag{2.15}$$

The above results, equations (2.5), (2.8), and (2.13), are now generalized to give

$$R_{m,0}(1,0,t) = \frac{1}{m}\sum_{k=0}^{m-1} cis(2\pi k/m)\left(R_{1,0}(cis(2\pi k/m),0,t)\right), \qquad m = 1,2,3,\ldots, \tag{2.16}$$

also

$$R_{m,0}(1,0,t) = \frac{1}{m}\sum_{k=0}^{m-1} e^{i2\pi k/m}\, e^{cis(2\pi k/m)t}, \qquad m = 1,2,3,\ldots. \tag{2.17}$$

Thus any $R_{m,0}$ function may be written in terms of $R_{1,0}$ for $m = 1,2,3\ldots$. Consideration of the principle value $k = 0$, gives the result

$$R_{m,0}(1,0,t)_{k=0} \cong \frac{1}{m}e^{t}, \tag{2.18}$$

this is found to be a useful approximation for $t > m$.

The results of equations (2.16) and (2.17) may be generalized to arbitrary positive value for the parameter a, this more general result is given by

$$R_{m,0}(a,0,t) = \frac{1}{m}\sum_{k=0}^{m-1} a^{(1-m)/m} cis(2\pi k/m) R_{1,0}\left(a^{1/m} cis(2\pi k/m),0,t\right), \quad a > 0, \quad m = 1,2,3,\ldots, \tag{2.19}$$

and

$$R_{m,0}(a,0,t) = \frac{1}{m}\sum_{k=0}^{m-1} a^{(1-m)/m} e^{i2\pi k/m}\, e^{a^{1/m} cis(2\pi k/m)t}, \quad a > 0, \qquad m = 1,2,3,\ldots. \tag{2.20}$$

For negative values of the a parameter the following forms apply

$$R_{m,0}(-a,0,t)=-\frac{1}{m}\sum_{k=0}^{m-1}a^{(1-m)/m}cis(\alpha)R_{1,0}\left(a^{1/m}cis(\alpha),0,t\right), \qquad a>0, \quad m=1,2,3,\dots, \tag{2.21}$$

and

$$R_{m,0}(-a,0,t)=-\frac{1}{m}\sum_{k=0}^{m-1}a^{(1-m)/m}e^{i\alpha}\, e^{a^{1/m}cis(\alpha)\,t}, \qquad a>0, \qquad m=1,2,3,\dots, \tag{2.22}$$

where $\alpha=(2k+1)\pi/m$.

3. Relationships for $R_{1/m,0}$ in Terms of $R_{1,0}$

In this section we seek to express $R_{1/m,0}\left(a^{1/m},0,t\right)$in terms of $R_{1,0}(a,0,t)$, where $m=1,2,3,\dots$. The initial interest will be $1/m=1/2$. Then applying the Laplace transform

$$L\left\{R_{1/2,0}\left(a^{1/2},0,t\right)\right\}=\frac{1}{s^{1/2}-a^{1/2}}\,\frac{s^{1/2}+a^{1/2}}{s^{1/2}+a^{1/2}}=\frac{s^{1/2}}{s-a}+\frac{a}{s-a}. \tag{3.1}$$

Inverse transforming gives

$$R_{1/2,0}\left(a^{1/2},0,t\right)=R_{1,1/2}(a,0,t)+a\,R_{1,0}(a,0,t), \tag{3.2}$$

or

$$R_{1/2,0}\left(a^{1/2},0,t\right)={}_0d_t^{1/2}\,R_{1,0}(a,0,t)+a\,R_{1,0}(a,0,t). \tag{3.3}$$

It terms of the exponential function

$$R_{1/2,0}\left(a^{1/2},0,t\right)={}_0d_t^{1/2}e^{at}+a\,e^{at}. \tag{3.4}$$

The $m=4$ case is considered next. The Laplace transform is applied

$$L\left\{R_{1/4,0}\left(a^{1/4},0,t\right)\right\}=\frac{1}{s^{1/4}-a^{1/4}}\,\frac{s^{1/4}+a^{1/4}}{s^{1/4}+a^{1/4}}\,\frac{s^{1/4}-ia^{1/4}}{s^{1/4}-ia^{1/4}}\,\frac{s^{1/4}+ia^{1/4}}{s^{1/4}+ia^{1/4}}, \tag{3.5}$$

$$=\frac{s^{3/4}+a^{1/4}s^{1/2}+a^{1/2}s^{1/4}+a^{3/4}}{s-a}, \tag{3.6}$$

$$=\sum_{k=0}^{3}\frac{a^{(3-k)/4}s^{k/4}}{s-a}. \tag{3.7}$$

Inverse transforming yields the desired result

$$R_{1/4,0}\left(a^{1/4},0,t\right)=\sum_{k=0}^{3}a^{(3-k)/4}R_{1,k/4}(a,0,t), \tag{3.8}$$

or

$$R_{1/4,0}\left(a^{1/4},0,t\right)=\sum_{k=0}^{3}a^{(3-k)/4}\,{}_0d_t^{k/4}R_{1,0}(a,0,t). \tag{3.9}$$

The general results are seen to be

$$\boxed{\begin{array}{l} R_{1/m,0}\left(a^{1/m},0,t\right)=\sum_{k=0}^{m-1} a^{(m-1-k)/m} R_{1,k/m}(a,0,t), \quad a>0, \qquad m=1,2,3,\ldots \qquad (3.10) \\ \text{and} \\ R_{1/m,0}\left(a^{1/m},0,t\right)=\sum_{k=0}^{m-1} a^{(m-1-k)/m}\, {}_0d_t^{k/m} R_{1,0}(a,0,t), \quad a>0, \qquad m=1,2,3,\ldots. \qquad (3.11) \end{array}}$$

These results are now validated for the $m = 3$ case. Then

$$R_{1/3,0}\left(a^{1/3},0,t\right)=\sum_{k=0}^{2} a^{(2-k)/3} R_{1,k/3}(a,0,t) \tag{3.12}$$

Laplace transforming this equation gives

$$\frac{1}{s^{1/3}-a^{1/3}}=\frac{a^{2/3}}{s-a}+\frac{a^{1/3}s^{1/3}}{s-a}+\frac{s^{2/3}}{s-a}, \tag{3.13}$$

$$=\frac{s^{2/3}+a^{1/3}s^{1/3}+a^{2/3}}{\left(s^{1/3}-a^{1/3}\right)\left(s^{2/3}+a^{1/3}s^{1/3}+a^{2/3}\right)}=\frac{1}{s^{1/3}-a^{1/3}}. \tag{3.14}$$

The results of equations 3.10 and 3.11 are extended to the case of a negative a parameter by use of the following expressions

$$\boxed{\begin{array}{l} R_{1/m,0}\left(-a^{1/m},0,t\right)=\sum_{k=0}^{m-1}(-1)^{m-1-k} a^{(m-1-k)/m} R_{1,k/m}\left((-1)^m a,0,t\right), \quad a>0, \quad m=1,2,3\ldots \qquad (3.15) \\ \text{and} \\ R_{1/m,0}\left(-a^{1/m},0,t\right)=\sum_{k=0}^{m-1}(-1)^{m-1-k} a^{(m-1-k)/m}\, {}_0d_t^{k/m} R_{1,0}\left((-1)^m a,0,t\right), \quad a>0, \quad m=1,2,3\ldots \qquad (3.16) \end{array}}$$

4. Relationships for the Rational Form $R_{m/p,0}$ in Terms of $R_{1/p,0}$

In this part we wish to relate $R_{m/p,0}(1,0,t)$ to $R_{1/p,0}(a,0,t)$, where m and p are positive integers. This is a generalization of the preceding parts. We start with the example $R_{3/2}(1,0,t)$, applying the Laplace transform gives

$$L\{R_{3/2,0}(1,0,t)\}=\frac{1}{s^{3/2}-1}. \tag{4.1}$$

Now the roots of $s^{3/2}-1=0$ are $s=1^{2/3}=cis\left(2\pi k\,\tfrac{2}{3}\right)$ however because of the periodicity of the roots equivalent results are obtained from $cis(2\pi k/3)$. Thus equation 4.1 may be written as

$$L\{R_{3/2,0}(1,0,t)\}=\frac{1}{s^{3/2}-1}=\frac{1}{\left(s^{1/2}-c_0\right)\left(s^{1/2}-c_1\right)\left(s^{1/2}-c_2\right)}$$

$$=\frac{A_0}{s^{1/2}-c_0}+\frac{A_1}{s^{1/2}-c_1}+\frac{A_2}{s^{1/2}-c_2}, \tag{4.2}$$

where $c_k = cis(2\pi k/3)$. The values for this problem are

$$c_0 = 1, \qquad c_1 = -0.5 + 0.866i, \qquad c_2 = -0.5 - 0.866i.$$

The A_k are determined from partial fraction expansion to be

$$A_0 = \frac{1}{3} \qquad A_1 = \frac{1}{3}\left(\frac{1}{-0.5-0.866i}\right) \quad A_2 = \frac{1}{3}\left(\frac{1}{-0.5+0.866i}\right).$$

The A_k are recognized as $A_k = c_k/3$. This gives the following for the transform

$$\begin{aligned} L\{R_{3/2,0}(1,0,t)\} &= \frac{1}{s^{3/2}-1} \\ &= \frac{cis(0)}{3(s^{1/2}-cis(0))} + \frac{cis(2\pi/3)}{3(s^{1/2}-cis(2\pi/3))} + \frac{cis(4\pi/3)}{3(s^{1/2}-cis(4\pi/3))}. \end{aligned} \tag{4.3}$$

The inverse transform then is given as

$$R_{3/2,0}(1,0,t) = \frac{1}{3}\sum_{k=0}^{2} cis(2\pi k/3) R_{1/2,0}(cis(2\pi k/3),0,t). \tag{4.4}$$

As in the previous parts these results are generalized to the following

$$R_{m/p,0}(1,0,t) = \frac{1}{m}\sum_{k=0}^{m-1} cis(2\pi k/m) R_{1/p,0}(cis(2\pi k/m),0,t) \;\; m = 1,2,3\ldots \;\; p = 1,2,3\ldots. \tag{4.5}$$

Since $e^{i\alpha} = cis(\alpha)$ this may be written as

$$R_{m/p,0}(1,0,t) = \frac{1}{m}\sum_{k=0}^{m-1} e^{i2\pi k/m} R_{1/p,0}\left(e^{i2\pi k/m},0,t\right) \;\; m = 1,2,3\ldots \;\; p = 1,2,3\ldots. \tag{4.6}$$

These results also may be generalized to include a nonunity value for the *a* parameter. The general form is given by

$$R_{m/p,0}(a,0,t) = \frac{1}{m}\sum_{k=0}^{m-1} a^{(1-m)/m} cis(2\pi k/m) R_{1/p,0}\left(a^{1/m} cis(2\pi k/m),0,t\right), \quad a > 0, \;\; m = 1,2,3\ldots, \;\; p = 1,2,3\ldots, \tag{4.7}$$

and

$$R_{m/p,0}(a,0,t) = \frac{1}{m}\sum_{k=0}^{m-1} a^{(1-m)/m} e^{i2\pi k/m} R_{1/p,0}\left(a^{1/m} e^{i2\pi k/m},0,t\right), \quad a > 0, \;\; m = 1,2,3\ldots, p = 1,2,3\ldots. \tag{4.8}$$

The above equations (4.7) and (4.8) allow any rational based, $q = m/p$, $R_{m/p,0}$-function to be expressed in terms of its basis $R_{1/p,0}$-functions.

The results of equations (4.7) and (4.8) are extended to the case of negative a parameter by the following equations

$$R_{m/p,0}(-a,0,t)=-\frac{1}{m}\sum_{k=0}^{m-1}a^{(1-m)/m}cis((2k+1)\pi/m)R_{1/p,0}\left(a^{1/m}cis((2k+1)\pi/m),0,t\right),$$
$$a>0,\qquad m=1,2,3\ldots,\; p=1,2,3\ldots, \tag{4.9}$$

and

$$R_{m/p,0}(-a,0,t)=-\frac{1}{m}\sum_{k=0}^{m-1}a^{(1-m)/m}e^{i((2k+1)\pi/m)}R_{1/p,0}\left(a^{1/m}e^{i((2k+1)\pi/m)},0,t\right),$$
$$a>0,\; m=1,2,3\ldots,p=1,2,3\ldots. \tag{4.10}$$

These results, equations (4.7) to (4.10), are the most general (direct) relationships to the basis function presented in this paper. The $R_{1/p,0}$ are seen as basis functions for any $R_{m/p,0}$.

5. Relationships for $R_{1/p,0}$ in Terms of $R_{m/p,0}$

This section develops the reciprocal relation to that formed in the previous section. This form will be useful in developing the inverse relationships, which follow in later sections. Consider the case for $p=2,\ m=1$, then the Laplace transform is

$$L\{R_{1/2,0}(a^{1/2},0,t)\}=\frac{1}{s^{1/2}-a^{1/2}}\frac{s^{1/2}+a^{1/2}}{s^{1/2}+a^{1/2}}=\frac{s^{1/2}+a^{1/2}}{s-a}, \tag{5.1}$$

$$=\frac{s^{1/2}}{s-a}+\frac{a^{1/2}}{s-a}, \tag{5.2}$$

Inverse Laplace transforming gives

$$R_{1/2,0}(a^{1/2},0,t)=R_{1,1/2}(a,0,t)+a^{1/2}R_{1,0}(a,0,t), \tag{5.3}$$

or

$$R_{1/2,0}(a^{1/2},0,t)={}_0d_t^{1/2}\,R_{1,0}(a,0,t)+a^{1/2}R_{1,0}(a,0,t). \tag{5.4}$$

The case $p=2,m=4$, is now considered. The Laplace transform is given by

$$L\{R_{1/2,0}(a^{1/4},0,t)\}=\frac{1}{s^{1/2}-a^{1/4}}\frac{s^{1/2}+a^{1/4}}{s^{1/2}+a^{1/4}}\frac{s^{1/2}-ia^{1/4}}{s^{1/2}-ia^{1/4}}\frac{s^{1/2}+ia^{1/4}}{s^{1/2}+ia^{1/4}}, \tag{5.5}$$

$$=\frac{s^{3/2}+a^{1/4}s^{1}+a^{1/2}s^{1/2}+a^{3/4}}{s^{2}-a}=\sum_{k=0}^{3}\frac{a^{(3-k)/4}s^{k/2}}{s^{2}-a}. \tag{5.6}$$

Inverse transforming yields

$$R_{1/2,0}(a^{1/4},0,t)=\sum_{k=0}^{3}a^{(3-k)/4}R_{2,k/2}(a,0,t), \tag{5.7}$$

or

$$R_{1/2,0}(a^{1/4},0,t)=\sum_{k=0}^{3}a^{(3-k)/4}\,{}_0d_t^{k/2}R_{2,0}(a,0,t). \tag{5.8}$$

The above results are generalized in the following forms

$$R_{1/p,0}\left(a^{1/m},0,t\right)=\sum_{k=0}^{m-1} a^{(m-1-k)/m} R_{m/p,k/p}(a,0,t), \qquad a>0, \quad p=1,2,3\ldots, \; m=1,2,3\ldots \tag{5.9}$$

and

$$R_{1/p,0}\left(a^{1/m},0,t\right)=\sum_{k=0}^{m-1} a^{(m-1-k)/m}\, {}_0d_t^{k/p} R_{m/p,0}(a,0,t), \; a>0, \; p=1,2,3\ldots, \; m=1,2,3\ldots \, . \tag{5.10}$$

These results will now be tested on the following case $p=2$, $m=3$. Then

$$R_{1/2,0}\left(a^{1/3},0,t\right)=\sum_{k=0}^{2} a^{(2-k)/3} R_{3/2,k/2}(a,0,t), \tag{5.11}$$

$$=a^{2/3}R_{3/2,0}(a,0,t)+a^{1/3}R_{3/2,1/2}(a,0,t)+R_{3/2,1}(a,0,t). \tag{5.12}$$

Applying the Laplace transform gives

$$\frac{1}{s^{1/2}-a^{1/3}}=\frac{a^{2/3}}{s^{3/2}-a}+\frac{a^{1/3}s^{1/2}}{s^{3/2}-a}+\frac{s}{s^{3/2}-a}, \tag{5.13}$$

$$=\frac{s+a^{1/3}s^{1/2}+a^{2/3}}{\left(s^{1/2}-a^{1/3}\right)\left(s+a^{1/3}s^{1/2}+a^{2/3}\right)}, \tag{5.14}$$

$$=\frac{1}{s^{1/2}-a^{1/3}}, \tag{5.15}$$

providing a validation point for the general form (equation 5.9).

The results of equations (5.9) and (5.10) may also be extended to the case for a negative a parameter. These results are given as

$$R_{1/p,0}\left(-a^{1/m},0,t\right)=\sum_{k=0}^{m-1}(-1)^{m-1-k} a^{(m-1-k)/m} R_{m/p,k/p}\left((-1)^m a,0,t\right),$$
$$a>0, \quad p=1,2,3\ldots, \; m=1,2,3\ldots, \tag{5.16}$$

and

$$R_{1/p,0}\left(-a^{1/m},0,t\right)=\sum_{k=0}^{m-1}(-1)^{m-1-k} a^{(m-1-k)/m}\, {}_0d_t^{k/p} R_{m/p,0}\left(\left(-1\right)^m a,0,t\right),$$
$$a>0, \quad p=1,2,3\ldots, \quad m=1,2,3\ldots \, . \tag{5.17}$$

6. Relating $R_{m/p,0}$ to the Exponential Function $R_{1,0}(b,0,t)=e^{bt}$

Using the results of the previous sections, it is now possible to express any $R_{q,0}(a,0,t)$ in terms of $R_{1,0}(b,0,t)=e^{bt}$, for $q=m/p$ (rational). Two results are required, equations (3.11) and (4.8). For clarity of discussion we rewrite equation (3.11) in the following terms

$$R_{1/p,0}(b,0,t)=\sum_{j=0}^{p-1} b^{p-1-j}\,{}_0d_t^{j/p}\,R_{1,0}\left(b^p,0,t\right)$$

$$=\sum_{j=0}^{p-1} b^{p-1-j}\,{}_0d_t^{j/p}e^{b^p t}\,,\quad b\geq 0,\quad p=1,2,3\ldots. \tag{6.1}$$

Now this result may be directly substituted into equation (4.8) to give

$$R_{m/p,0}(a,0,t)=\frac{1}{m}\sum_{k=0}^{m-1} a^{(1-m)/m}e^{i2\pi k/m}\sum_{j=0}^{p-1}\left(a^{1/m}e^{i2\pi k/m}\right)^{p-1-j}{}_0d_t^{j/p}R_{1,0}\left(\left(a^{1/m}e^{i2\pi k/m}\right)^p,0,t\right),$$

$$m=1,2,3\ldots,p=1,2,3\ldots. \tag{6.2}$$

This now may be written as

$$R_{m/p,0}(a,0,t)=\frac{1}{m}\sum_{k=0}^{m-1}\sum_{j=0}^{p-1} a^{(p-j-m)/m}e^{i2\pi k(p-j)/m}\,{}_0d_t^{j/p}R_{1,0}\left(\left(a^{p/m}e^{i2\pi kp/m}\right),0,t\right),$$

$$a\geq 0,\quad m=1,2,3\ldots,p=1,2,3\ldots. \tag{6.3}$$

$$R_{m/p,0}(a,0,t)=\frac{1}{m}\sum_{k=0}^{m-1}\sum_{j=0}^{p-1} a^{(p-j-m)/m}e^{i2\pi k(p-j)/m}\,R_{1,j/p}\left(\left(a^{p/m}e^{i2\pi kp/m}\right),0,t\right),$$

$$a\geq 0,\qquad m=1,2,3\ldots,p=1,2,3\ldots. \tag{6.4}$$

Thus, from equation (6.3) the generalized exponential function $R_{m/p,0}$ may now be expressed as a function of fractional derivatives of the common exponential function

$$R_{m/p,0}(a,0,t)=\frac{1}{m}\sum_{k=0}^{m-1}\sum_{j=0}^{p-1} a^{(p-j-m)/m}e^{i2\pi k(p-j)/m}\,{}_0d_t^{j/p}\exp\left(\left(a^{p/m}e^{i2\pi kp/m}\right)t\right),$$

$$a\geq 0\,,\ m=1,2,3\ldots,\ p=1,2,3\ldots. \tag{6.5}$$

These results, equations (6.3) to (6.5), contain the results of equations (2.19), (2.20), (3.10), and (3.20).

The case for negative a parameter follows a similar development as above. Equation (4.10) is written as

$$R_{m/p,0}(-a,0,t)=-\frac{1}{m}\sum_{k=0}^{m-1} a^{(1-m)/m}e^{i\alpha}R_{1/p,0}\left(a^{1/m}e^{i\alpha},0,t\right),$$

$$a>0,\quad m=1,2,3\ldots,p=1,2,3\ldots, \tag{6.6}$$

where $\alpha = (2k+1)\pi/m$. Now equation (6.1) may be substituted into equation (6.5) to give

$$\boxed{\begin{aligned} &R_{m/p,0}(-a,0,t) = -\frac{1}{m}\sum_{k=0}^{m-1}\sum_{j=0}^{p-1} e^{i\alpha(p-j)} a^{(p-j-m)/m} {}_0d_t^{j/p} R_{1,0}\left(a^{p/m}e^{i\alpha p},0,t\right), \\ &\qquad\qquad a>0,\ \ m=1,2,3\ldots,\ p=1,2,3\ldots, \\ &\text{or} \\ &R_{m/p,0}(-a,0,t) = -\frac{1}{m}\sum_{k=0}^{m-1}\sum_{j=0}^{p-1} e^{i\alpha(p-j)} a^{(p-j-m)/m} R_{1,j/p}\left(a^{p/m}e^{i\alpha p},0,t\right), \\ &\qquad\qquad a>0,\ \ m=1,2,3\ldots,\ p=1,2,3\ldots, \\ &\text{or} \\ &R_{m/p,0}(-a,0,t) = -\frac{1}{m}\sum_{k=0}^{m-1}\sum_{j=0}^{p-1} e^{i\alpha(p-j)} a^{(p-j-m)/m} {}_0d_t^{j/p} \exp\left(\left(a^{p/m}e^{i\alpha p}\right)t\right), \\ &\qquad\qquad a>0,\ \ m=1,2,3\ldots,\ p=1,2,3\ldots. \end{aligned}}$$

(6.7), (6.8), (6.9)

These results, equations (6.3) to (6.5), and (6.7) to (6.9) are the most general (direct) expressions for the R-function in terms of the common exponential function presented in this paper.

7. Inverse Relationships—Relationships for $R_{1,0}$ in Terms of $R_{m,k}$

In this and the following sections inverse relationships expressing the exponential function in terms of various R-functions will be developed. Consider

$$L\{e^{at}\} = \frac{1}{s-a} = \frac{s+a}{(s-a)(s+a)}, \tag{7.1}$$

$$= \frac{s}{s^2-a^2} + \frac{a}{s^2-a^2}. \tag{7.2}$$

Upon inverse transforming we have

$$e^{at} = R_{1,0}(a,0,t) = R_{2,1}(a^2,0,t) + a R_{2,0}(a^2,0,t). \tag{7.3}$$

In similar fashion for $m=4$, we have

$$L\{e^{at}\} = \frac{1}{s-a} = \frac{(s+a)(s-ia)(s+ia)}{(s-a)(s+a)(s-ia)(s+ia)}, \tag{7.4}$$

$$= \frac{s^3 + as^2 + a^2 s + a^3}{s^4 - a^4} \tag{7.5}$$

$$= \sum_{k=0}^{3} \frac{a^{3-k} s^k}{s^4 - a^4}. \tag{7.6}$$

Inverse transforming this result yields

$$e^{at} = R_{1,0}(a,0,t) = \sum_{k=0}^{3} a^{3-k}\, R_{4,k}\left(a^4,0,t\right) = \sum_{k=0}^{3} a^{3-k}\ {}_0d_t^k\, R_{4,0}\left(a^4,0,t\right). \qquad (7.7)$$

These results generalize to

$$\boxed{\begin{aligned} &e^{at} = R_{1,0}(a,0,t) = \sum_{k=0}^{m-1} a^{m-1-k}\, R_{m,k}\left(a^m,0,t\right), \qquad a>0, \quad m=1,2,3,.., \qquad (7.8)\\ &\text{and}\\ &e^{at} = R_{1,0}(a,0,t) = \sum_{k=0}^{m-1} a^{m-1-k}\ {}_0d_t^k\, R_{m,0}\left(a^m,0,t\right), \qquad a>0, \quad m=1,2,3\ldots. \qquad (7.9)\end{aligned}}$$

These results are now used to test the $m=3$ case. Then

$$e^{at} = R_{1,0}(a,0,t) = \sum_{k=0}^{2} a^{2-k}\, R_{3,k}\left(a^3,0,t\right). \qquad (7.10)$$

Thus the Laplace transform is

$$\frac{1}{s-a} = \sum_{k=0}^{2} \frac{a^{2-k} s^k}{s^3 - a^3} = \frac{s^2 + as + a^2}{s^3 - a^3}, \qquad (7.11)$$

$$= \frac{s^2 + as + a^2}{(s-a)\left(s^2 + as + a^2\right)} = \frac{1}{s-a}. \qquad (7.12)$$

For negative values of the a parameter the following forms apply

$$\boxed{\begin{aligned} &e^{-at} = R_{1,0}(-a,0,t) = \sum_{k=0}^{m-1} (-1)^{m-1-k} a^{m-1-k}\, R_{m,k}\left((-1)^m a^m,0,t\right), \quad a>0,\ m=1,2,3,..., \qquad (7.13)\\ &\text{and}\\ &e^{-at} = R_{1,0}(-a,0,t) = \sum_{k=0}^{m-1} (-1)^{m-1-k} a^{m-1-k}\ {}_0d_t^k\, R_{m,0}\left((-1)^m a^m,0,t\right), \quad a>0,\ m=1,2,3\ldots \qquad (7.14)\end{aligned}}$$

8. Inverse Relationships—Relationships for $R_{1,0}$ in Terms of $R_{1/m,0}$

In this section we seek to express $R_{1,0}(1,0,t)$ in terms of $R_{1/m,0}$, where $m=1,2,3\ldots$. The initial interest will be $1/m = 1/2$. Then

$$R_{1/2,0}(1,0,t) = \sum_{k=0}^{\infty} \frac{t^{k/2-1/2}}{\Gamma(k/2+1/2)} = \frac{t^{-1/2}}{\Gamma(1/2)} + \frac{t^0}{\Gamma(1)} + \frac{t^{1/2}}{\Gamma(3/2)} + \frac{t^1}{\Gamma(2)} + \frac{t^{3/2}}{\Gamma(5/2)} + \ldots \qquad (8.1)$$

and

$$R_{1/2,0}(-1,0,t) = \sum_{k=0}^{\infty} \frac{(-1)^k t^{k/2-1/2}}{\Gamma(k/2+1/2)} = \frac{t^{-1/2}}{\Gamma(1/2)} - \frac{t^0}{\Gamma(1)} + \frac{t^{1/2}}{\Gamma(3/2)} - \frac{t^1}{\Gamma(2)} + \frac{t^{3/2}}{\Gamma(5/2)} - \ldots. \qquad (8.2)$$

Therefore it is easily seen that

$$R_{1,0}(1,0,t)=\frac{1}{2}\left(R_{1/2,0}(1,0,t)-R_{1/2,0}(-1,0,t)\right). \tag{8.3}$$

We now consider the case $1/m=1/4$. Then

$$\begin{aligned} R_{1/4,0}(1,0,t) &= \sum_{k=0}^{\infty}\frac{t^{k/4-3/4}}{\Gamma(k/4+1/4)} \\ &= \frac{t^{-3/4}}{\Gamma(1/4)}+\frac{t^{-1/2}}{\Gamma(2/4)}+\frac{t^{-1/4}}{\Gamma(3/4)}+\frac{t^{0}}{\Gamma(4/4)}+\frac{t^{1/4}}{\Gamma(5/4)}+\frac{t^{1/2}}{\Gamma(6/4)}+\dots . \end{aligned} \tag{8.4}$$

Now it may be shown by substitution that

$$R_{1,0}(1,0,t)=\frac{1}{4}\left(R_{1/4}(1,0,t)-R_{1/4,0}(-1,0,t)+iR_{1/4,0}(i,0,t)-iR_{1/4,0}(-i,0,t)\right). \tag{8.5}$$

As in the previous section this may be generalized as

$$R_{1,0}(1,0,t)=\frac{1}{m}\sum_{k=0}^{m-1}cis(2\pi\, k/m)R_{1/m,0}(cis(2\pi\, k/m),0,t) \qquad m=1,2,3\dots . \tag{8.6}$$

Remembering that $R_{1,0}(1,0,t)=e^{t}$, this equation (8.6) is recognized as a decomposition (of the m-th order $m=1,2,3\dots$) of the exponential function. That is, each of these functions is more basic than the exponential function in that the exponential function may readily be expressed in terms of the "fractional exponential $\left(\text{i.e., } R_{1/m,0}(a,0,t)\right)$" in closed form (without differintegrating).

The results of equation (8.6) may be generalized to arbitrary value for the parameter a, this more general result is given by

$$\boxed{\begin{aligned} R_{1,0}(1,0,at) &= R_{1,0}(a,0,t)=e^{at} \\ &= \frac{1}{m}\sum_{k=0}^{m-1}a^{(1-m)/m}cis(2\pi\, k/m)R_{1/m,0}\left(a^{1/m}cis(2\pi\, k/m),0,t\right),\ a>0,\ \ m=1,2,3,\dots \\ &= \frac{1}{m}\sum_{k=0}^{m-1}a^{(1-m)/m}e^{i2\pi k/m}R_{1/m,0}\left(a^{1/m}e^{i2\pi k/m},0,t\right),\ a>0,\ \ m=1,2,3,\dots . \end{aligned}} \tag{8.7}$$

The approach to the solution for a negative a parameter will now be demonstrated for the case $1/m=1/4$. Applying the Laplace transform gives

$$\begin{aligned} L\{R_{1,0}(-a,0,t)\} &= L\{e^{-at}\} \\ &= \frac{1}{s+a}=\frac{1}{\left(s^{1/4}-c_0a^{1/4}\right)\left(s^{1/4}-c_1a^{1/4}\right)\left(s^{1/4}-c_2a^{1/4}\right)\left(s^{1/4}-c_3a^{1/4}\right)} \end{aligned} \tag{8.8}$$

where $c_k=cis((2k+1)\pi/4)$. This may be written as

$$\frac{1}{s+a}=\frac{A_0}{s^{1/4}-c_0a^{1/4}}+\frac{A_1}{s^{1/4}-c_1a^{1/4}}+\frac{A_2}{s^{1/4}-c_2a^{1/4}}+\frac{A_3}{s^{1/4}-c_3a^{1/4}}. \tag{8.9}$$

The A_k are found by partial fractions to be $A_k = -c_k / \left(4a^{3/4}\right)$. The general form is then validated and given as

$$\boxed{\begin{aligned} R_{1,0}(1,0,-at) &= R_{1,0}(-a,0,t) = e^{-at} \\ &= \frac{1}{m}\sum_{k=0}^{m-1} -a^{(1-m)/m} cis(\alpha) R_{1/m,0}\left(a^{1/m} cis(\alpha),0,t\right), \quad a>0, \quad m=1,2,3,..., \\ &= \frac{1}{m}\sum_{k=0}^{m-1} -a^{(1-m)/m} e^{i\alpha} R_{1/m,0}\left(a^{1/m} e^{i\alpha},0,t\right), \quad a>0, \quad m=1,2,3,... \ , \end{aligned}} \quad (8.10)$$

where $\alpha = (2k+1)\pi / m$.

9. Inverse Relationships—Relationships for $e^{at} = R_{1,0}(a,0,t)$ in Terms of $R_{m/p,0}$

Using the results of the previous sections, it is now possible to express $e^{at} = R_{1,0}(a,0,t)$ in terms of any $R_{q,0}(b,0,t)$, for $q = m/p$ (rational). Two results are required, equations (8.7) and (5.10). For clarity of discussion we rewrite equation (5.10) in the following terms

$$R_{1/p,0}(b,0,t) = \sum_{j=0}^{r-1} b^{(r-1-j)} {}_0d_t^{j/p} R_{r/p,0}\left(b^r,0,t\right), \ b>0, \ p=1,2,3..., \ m=1,2,3... . \quad (9.1)$$

Substituting this equation into equation (8.7) gives

$$R_{1,0}(a,0,t) = e^{at} = \frac{1}{m}\sum_{k=0}^{m-1} a^{(1-m)/m} cis(\lambda) \sum_{j=0}^{r-1} \left(a^{1/m} cis(\lambda)\right)^{r-1-j} {}_0d_t^{j/m} R_{r/m,0}\left(\left(a^{1/m} cis(\lambda)\right)^r,0,t\right),$$
$$a>0, \quad m=1,2,3..., \quad (9.2)$$

$$\boxed{\begin{aligned} R_{1,0}(a,0,t) = e^{at} &= \frac{1}{m}\sum_{k=0}^{m-1}\sum_{j=0}^{r-1} a^{(r-j-m)/m} cis(\lambda(r-j)) R_{r/m,\,j/m}\left(a^{r/m} cis(\lambda r),0,t\right), \\ &\qquad a>0, \ r=1,2,3..., \ m=1,2,3..., \end{aligned}} \quad (9.3)$$

where $\lambda = 2\pi k / m$.

Alternatively

$$\boxed{\begin{aligned} R_{1,0}(a,0,t) = e^{at} &= \frac{1}{m}\sum_{k=0}^{m-1}\sum_{j=0}^{r-1} a^{(r-j-m)/m} e^{i(r-j)\lambda} R_{r/m,\,j/m}\left(\left(a^{r/m} e^{ir\lambda}\right),0,t\right), \\ &\qquad a>0, \ r=1,2,3..., \quad m=1,2,3.... \end{aligned}} \quad (9.4)$$

For the case of a negative a parameter, we substitute equation (9.1) into equation (8.10) to give

$$\begin{aligned} e^{-at} &= R_{1,0}(1,0,-at) = R_{1,0}(-a,0,t) \\ &= -\frac{1}{m a^{1-(1/m)}}\sum_{k=0}^{m-1} cis(\alpha) \sum_{j=0}^{r-1} \left(a^{1/m} cis(\alpha)\right)^{r-1-j} {}_0d_t^{j/m} R_{r/m,0}\left(\left(a^{1/m} cis(\alpha)\right)^r,0,t\right) \end{aligned}$$
$$a>0, \ r=1,2,3..., \ m=1,2,3,... \ , \quad (9.5)$$

where $\alpha = (2k+1)\pi / m$. Thus we have

$$R_{1,0}(1,0,-at) = R_{1,0}(-a,0,t) = e^{-at}$$
$$= -\frac{1}{m}\sum_{k=0}^{m-1}\sum_{j=0}^{r-1} a^{(r-j-m)/m}\, cis(\alpha(r-j))\, R_{r/m,\,j/m}\left(a^{r/m}\, cis(\alpha r), 0, t\right)$$
$$a>0,\ r=1,2,3\ldots,\ m=1,2,3,\ldots\ , \qquad (9.6)$$

or

$$R_{1,0}(1,0,-at) = e^{-at} = -\frac{1}{m}\sum_{k=0}^{m-1}\sum_{j=0}^{r-1} a^{(r-j-m)/m} e^{i\alpha(r-j)}\, R_{r/m,\,j/m}\left(a^{r/m} e^{i\alpha r}, 0, t\right)$$
$$a>0,\ r=1,2,3\ldots,\ m=1,2,3,\ldots\ . \qquad (9.7)$$

The expressions (9.3), (9.4), (9.6), and (9.7) are the most general expressions for the exponential function in terms of the general (rational) *R*-function $R_{r/p,0}$ (and its fractional derivatives) presented in this paper.

Tables 1 and 2 summarize the key *R*-function relationships developed in this paper in a common form. Table 1 presents the relationships for positive *b* parameter in the left-hand side of $R_{u,v}(b,0,t)$ function, while Table 2 presents the relationships for a negative *b* parameter.

Table 1. *R*-Function Relationships—Positive Parameter

	Eq. Nos.
$R_{m,0}(a,0,t) = \frac{1}{m}\sum_{k=0}^{m-1} a^{(1-m)/m} cis(\lambda) R_{1,0}\left(a^{1/m} cis(\lambda), 0, t\right)$	(2.19)
$R_{1/m,0}\left(a^{1/m},0,t\right) = \sum_{k=0}^{m-1} a^{(m-1-k)/m} R_{1,k/m}(a,0,t)$	(3.10)
$R_{m/p,0}(a,0,t) = \frac{1}{m}\sum_{k=0}^{m-1} a^{(1-m)/m} cis(\lambda) R_{1/p,0}\left(a^{1/m} cis(\lambda), 0, t\right)$	(4.7)
$R_{1/p,0}\left(a^{1/m},0,t\right) = \sum_{k=0}^{m-1} a^{(m-1-k)/m} R_{m/p,k/p}(a,0,t)$	(5.9)
$R_{m/p,0}(a,0,t) = \frac{1}{m}\sum_{k=0}^{m-1}\sum_{j=0}^{p-1} a^{(p-j-m)/m}\, cis((p-j)\lambda) R_{1,j/p}\left(\left(a^{p/m} cis(p\lambda)\right), 0, t\right)$	(6.4)
$R_{1,0}(a,0,t) = e^{at} = \sum_{k=0}^{m-1} a^{m-1-k}\, R_{m,k}\left(a^m, 0, t\right)$	(7.8)
$R_{1,0}(1,0,at) = R_{1,0}(a,0,t) = e^{at}$ $= \frac{1}{m}\sum_{k=0}^{m-1} a^{(1-m)/m} cis(\lambda) R_{1/m,0}\left(a^{1/m} cis(\lambda), 0, t\right)$	(8.7)
$R_{1,0}(a,0,t) = e^{at} = \frac{1}{m}\sum_{k=0}^{m-1}\sum_{j=0}^{r-1} a^{(r-j-m)/m}\left(cis(\lambda)\right)^{r-j} R_{r/m,\,j/m}\left(a^{r/m}\left(cis(\lambda)\right)^{r}, 0, t\right)$	(9.3)

For this table $a>0,\quad m=1,2,3\ldots,\quad p=1,2,3\ldots,\quad r=1,2,3\ldots,\quad \lambda = 2\pi k/m$

Table 2. *R*-Function Relationships—Negative Parameter

	Eq. Nos.
$R_{m,0}(-a,0,t) = -\frac{1}{m}\sum_{k=0}^{m-1} a^{(1-m)/m} cis(\alpha) R_{1,0}\left(a^{1/m} cis(\alpha),0,t\right)$	(2.21)
$R_{1/m,0}\left(-a^{1/m},0,t\right) = \sum_{k=0}^{m-1}(-1)^{m-1-k} a^{(m-1-k)/m} R_{1,k/m}\left((-1)^m a,0,t\right)$	(3.15)
$R_{m/p,0}(-a,0,t) = -\frac{1}{m}\sum_{k=0}^{m-1} a^{(1-m)/m} cis(\alpha) R_{1/p,0}\left(a^{1/m} cis(\alpha),0,t\right)$	(4.9)
$R_{1/p,0}\left(-a^{1/m},0,t\right) = \sum_{k=0}^{m-1}(-1)^{m-1-k} a^{(m-1-k)/m} R_{m/p,k/p}\left((-1)^m a,0,t\right)$	(5.16)
$R_{m/p,0}(-a,0,t) = -\frac{1}{m}\sum_{k=0}^{m-1}\sum_{j=0}^{p-1} a^{(p-j-m)/m} cis(\alpha(p-j)) R_{1,j/p}\left(a^{p/m} cis(\alpha\, p),0,t\right)$	(6.8)
$R_{1,0}(-a,0,t) = e^{-at} = \sum_{k=0}^{m-1}(-1)^{m-1-k} a^{m-1-k} R_{m,k}\left((-1)^m a^m,0,t\right)$	(7.13)
$R_{1,0}(1,0,-a\,t) = R_{1,0}(-a,0,t) = e^{-at}$ $= -\frac{1}{m}\sum_{k=0}^{m-1} a^{(1-m)/m} cis(\alpha) R_{1/m,0}\left(a^{1/m} cis(\alpha),0,t\right)$	(8.10)
$R_{1,0}(1,0,-a\,t) = R_{1,0}(-a,0,t) = e^{-at}$ $= -\frac{1}{m}\sum_{k=0}^{m-1}\sum_{j=0}^{r-1} a^{(r-j-m)/m}\, cis(\alpha(r-j)) R_{r/m,j/m}\left(a^{r/m}\, cis(\alpha\, r),0,t\right)$	(9.6)

For this table $a > 0$, $m = 1,2,3\ldots$, $p = 1,2,3\ldots$, $r = 1,2,3\ldots$, $\alpha = (2k+1)\pi/m$,

10. Approximation of the *R*-Function

Various approximations may be developed for the *R*-function. A few such approximations will be developed here. As suggested in section 2, the principle value in equation (2.17) provides the basis of such an approximation. The result, equation (2.18), is generalized to

$$R_{q,0}(1,0,t) \approx \tfrac{1}{q} e^{t}, \qquad t > q > 0.5\,. \tag{10.1}$$

The approximation is shown in figure 2.

The following is an improved approximation when $t < 1$ and $0 < q \le 1$.

$$R_{q,0}(1,0,t) \approx \tfrac{1}{q} e^{t}\left(\frac{(1-q)^{2.5-q}}{t^{1-q}}\right), \qquad t > 0,\ \ 0 < q \le 1\,. \tag{10.2}$$

This approximation is shown graphically in Figure 3.

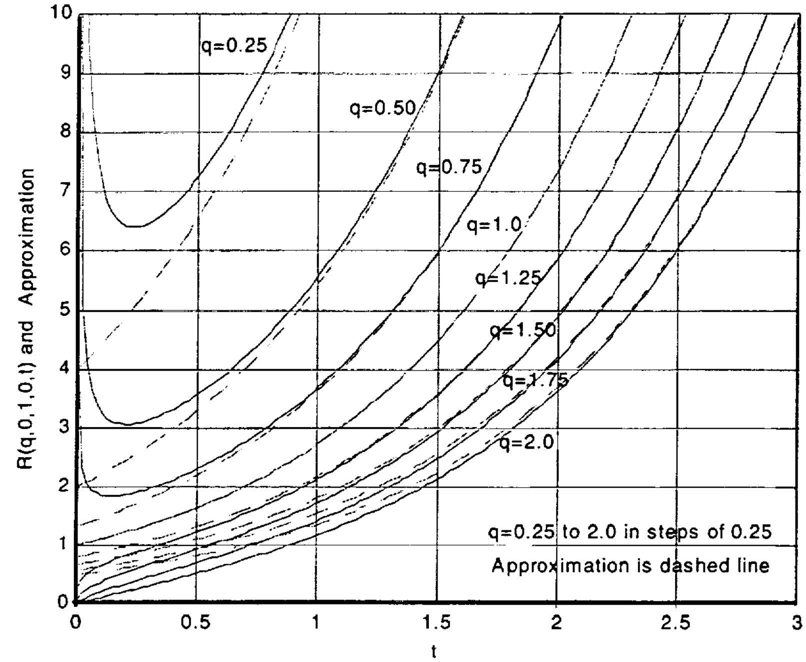

Figure 2. Approximation of $R_{q,0}(1,0,t)$ by $\frac{1}{q}e^{t}$

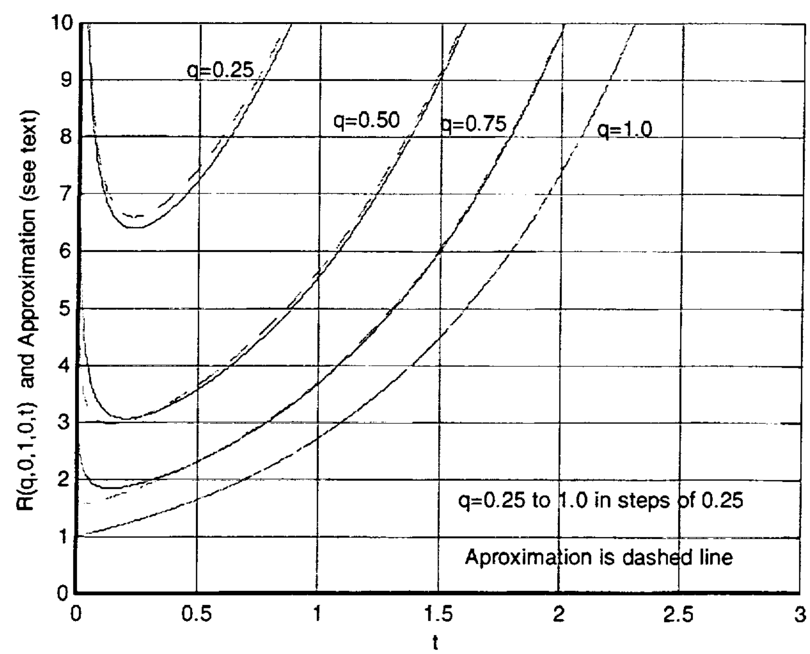

Figure 3. Approximation of $R_{q,0}(1,0,t)$ by Eq. (10.2)

The above approximations, equations (10.1) and (10.2) may be extended to include $R_{q,0}(a,0,t)$ in the following manner. By a simple replacement of the t variable with at we may rewrite equation (10.1) as

$$R_{q,0}(1,0,at) \approx \tfrac{1}{q} e^{at}, \quad a > 0, 1 \ge q > 0, at > 1 \tag{10.3}$$

Now using equation (1.9) along with equation (10.3) we infer the approximation

$$R_{q,0}(a,0,t) \approx \tfrac{1}{q} a^{(1-q)/q} e^{a^{1/q} t}, \quad a > 0, q > 0, at > 1 \tag{10.4}$$

When the a parameter of the R-function becomes negative, a different set of approximations is required. The following approximation works well for $1 \le q \le 2$

$$R_{q,0}(-1,0,t) \approx e^{-(2-q)^{1.25} t} \cos(t - \pi/2) \qquad 1 \le q \le 2, \quad t > \pi/2, \tag{10.5}$$

This approximation is shown in figure 4.

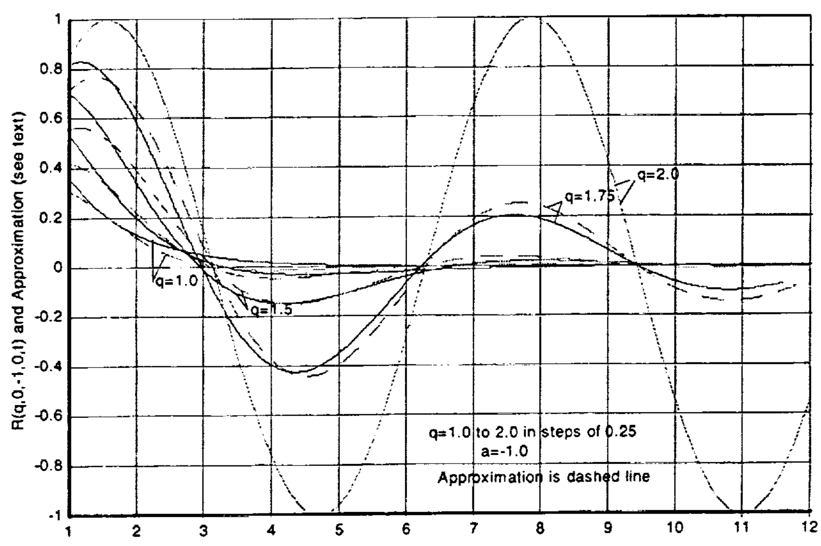

Figure 4. Approximation of $R_{q,0}(-1,0,t)$ by Eq. (10.5)

Improved approximations may be determined for particular values of q by optimizing the $A, B, C,$ and D constants in the following equation

$$R_{q,0}(-1,0,t) \approx A e^{-Bt} \cos(C(Dt - \pi/2)) \qquad 1 \le q \le 2 \; . \tag{10.6}$$

When the a parameter in equation (10.5) takes on values other than -1 the following approximation works reasonably well for values of a not too large

$$R_{q,0}(-a,0,t) \approx a^{-1/q} e^{-(2-q)^{1.25} a^{1/q} t} \cos(a^{1/q} t - \pi/2), \quad t > 1, \; 1 \le q \le 2. \tag{10.7}$$

Figure 5 graphically shows this approximation for $a = -4$.

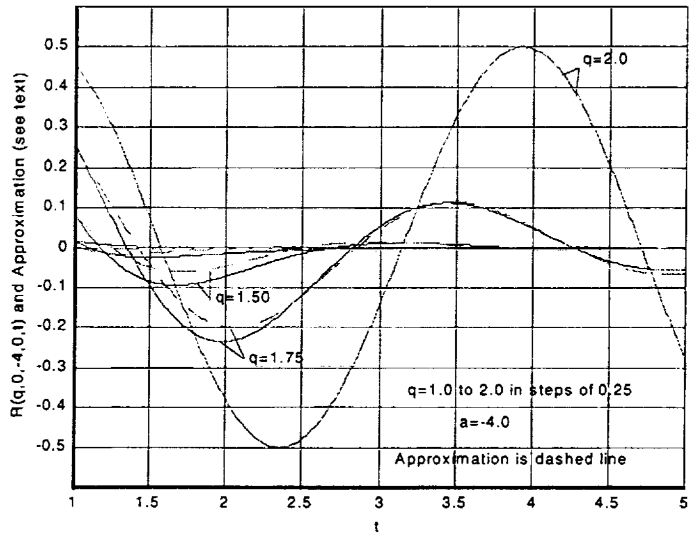

Figure 5. Approximation of $R_{q,0}(-4,0,t)$ by Eq. (10.7)

Approximations for values of $q>2$ with parameter a negative, require positive values for the argument of the real exponential in the approximation. For example for $2 \le q \le 3$ the following approximation

$$R_{q,0}(-1,0,t) \approx e^{(0.2q^2+1.45q-2.1)t} \cos((-.2q+1.4)t - \pi/2), \qquad 2 \le q \le 3, \quad t>1 \qquad (10.8)$$

is presented in figure 6.

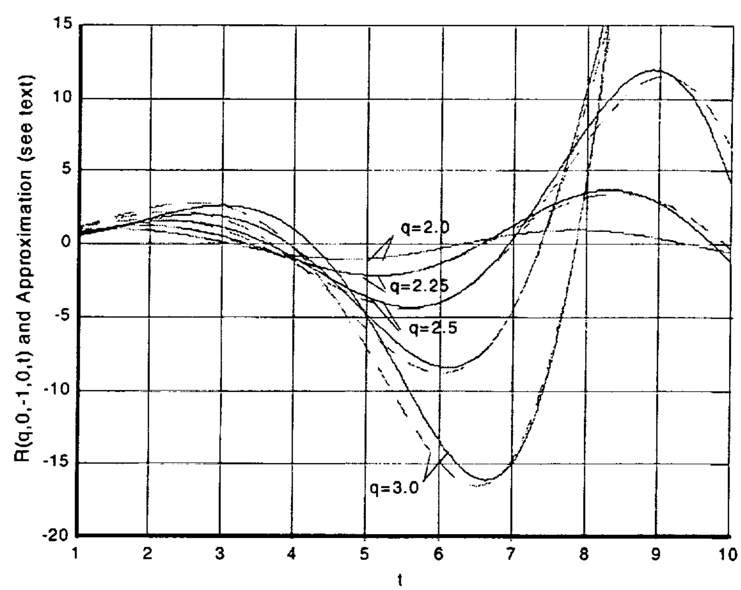

Figure 6. Approximation of $R_{q,0}(-1,0,t)$ by Eq. (10.8)

11. Discussion

This paper has presented a variety of relationships relating various R-functions. A key result was that $R_{q,0}$, with $q = m/p$ and positive rational, may be written in terms of basis R-functions $R_{1/p,0}$, equations (4.7) to (4.10). Also, reciprocal relationships have been developed expressing $R_{1/p,0}$ in terms of $R_{q,0}$, with $q = m/p$ and positive rational, (equations (5.9), (5.10), (5.16), and (5.17)).

It was also determined that $R_{q,0}$, with $q = m/p$ and positive rational, may be written in terms of fractional derivatives of $R_{1,0}$-functions (i.e., exponential functions), equations (6.3) to (6.9).

Further, the $R_{1,0}$ (exponential) functions may in turn be written as a function of basis functions $R_{1/p,0}$, (equations (7.8), (7.9), (7.13), and (7.14)). These results have allowed very general relationships to be written relating $R_{1,0}$ to $R_{q,0}$ and its fractional derivatives, with $q = m/p$ and positive rational, (equations (9.3), (9.4), (9.6), and (9.7)).

It is expected that the results presented here should be analytically very useful since the $R_{q,v}$ - function is the solution or solution basis of many fractional differential equations. It is also observed that all of the above relationships are expressed as **finite** series, the lengths of which depend on m and p.

Finally, various approximations of R-functions with both positive and negative arguments have been developed. Clearly these approximations only hint at the possibilities, and much more is possible.

Appendix

The following relationships of the R-function with advanced functions are an extension of those presented in reference [2]. The expansions used for the defined functions are all taken from reference [3].

The product of the exponential function and the complementary error function is given by

$$\exp(x)\, erfc\left(\pm\sqrt{x}\right)=\sum_{n=0}^{\infty}\frac{\left(\mp\sqrt{x}\right)^n}{\Gamma(1+n/2)}=R_{1/2,-1/2}(\mp 1,0,x). \qquad (A-1)$$

The error function as given by

$$erf(x)=\exp\left(-x^2\right)\sum_{n=0}^{\infty}\frac{x^{2n+1}}{\Gamma(n+3/2)}=R_{1,0}\left(-1,0,x^2\right)R_{1,-1/2}\left(1,0,x^2\right). \qquad (A-2)$$

The expansion for Dawson's integral becomes

$$daw\left(\sqrt{x}\right)=\frac{\sqrt{\pi\, x}}{2}\sum_{n=0}^{\infty}\frac{(-x)^n}{\Gamma(n+3/2)}=\frac{\sqrt{\pi}}{2}R_{1,-1/2}(-1,0,x). \qquad (A-3)$$

Many distributions may be expressed in terms of exponential of powers of x (see [3], p.260). Since $R_{1,0}(a,0,t)=R_{1,0}(1,0,at)=e^{at}$ these distributions may also be expressed as R-functions.

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[2] Lorenzo, C.F. and Hartley, T.T., Generalized Functions for the Fractional Calculus, NASA/TP—1999-209424/REV1, October 1999.

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REPORT DOCUMENTATION PAGE

Form Approved
OMB No. 0704-0188

1. AGENCY USE ONLY *(Leave blank)*

2. REPORT DATE
August 2000

3. REPORT TYPE AND DATES COVERED
Technical Memorandum

4. TITLE AND SUBTITLE
R-Function Relationships for Application in the Fractional Calculus

5. FUNDING NUMBERS
WU–519–30–53–00

6. AUTHOR(S)
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7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)
National Aeronautics and Space Administration
John H. Glenn Research Center at Lewis Field
Cleveland, Ohio 44135–3191

8. PERFORMING ORGANIZATION REPORT NUMBER
E–12410

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)
National Aeronautics and Space Administration
Washington, DC 20546–0001

10. SPONSORING/MONITORING AGENCY REPORT NUMBER
NASA TM—2000-210361

11. SUPPLEMENTARY NOTES
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12a. DISTRIBUTION/AVAILABILITY STATEMENT
Unclassified - Unlimited
Subject Categories: 61, 66, and 31 Distribution: Standard

This publication is available from the NASA Center for AeroSpace Information, (301) 621–0390.

12b. DISTRIBUTION CODE

13. ABSTRACT ***(Maximum 200 words)***
The *F*-function, and its generalization the *R*-function, are of fundamental importance in the fractional calculus. It has been shown that the solution of the fundamental linear fractional differential equation may be expressed in terms of these functions. These functions serve as generalizations of the exponential function in the solution of fractional differential equations. Because of this central role in the fractional calculus, this paper explores various intrarelationships of the *R*-function, which will be useful in further analysis. Relationships of the *R*-function to the common exponential function, e^t, and its fractional derivatives are shown. From the relationships developed, some important approximations are observed. Further, the inverse relationships of the exponential function, e^t, in terms of the *R*-function are developed. Also, some approximations for the *R*-function are developed.

14. SUBJECT TERMS
R-function; *F*-function; Fractional calculus; Fractional differential equations; Infinite series; Basis *R*-function

15. NUMBER OF PAGES
28

16. PRICE CODE
A03

17. SECURITY CLASSIFICATION OF REPORT	18. SECURITY CLASSIFICATION OF THIS PAGE	19. SECURITY CLASSIFICATION OF ABSTRACT	20. LIMITATION OF ABSTRACT
Unclassified	Unclassified	Unclassified	

NSN 7540-01-280-5500

Standard Form 298 (Rev. 2-89)
Prescribed by ANSI Std. Z39-18
298-102