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Model Refinement Using Eigensystem Assignment

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A novel approach for the refinement of finite-element-based analytical models of flexible structures is presented. The proposed approach models the possible refinements in the mass, damping, and stiffness matrices of the finite element model in the form of a constant gain feedback with acceleration, velocity, and displacement measurements, respectively. Once the free elements of the structural matrices have been defined, the problem of model refinement reduces to obtaining position, velocity, and acceleration gain matrices with appropriate sparsity that reassign a desired subset of the eigenvalues of the model, along with partial mode shapes, from their baseline values to those obtained from system identification test data. A sequential procedure is used to assign one conjugate pair of eigenvalues at each step using symmetric output feedback gain matrices, and the eigenvectors are partially assigned, while ensuring that the eigenvalues assigned in the previous steps are not disturbed. The procedure can also impose that gain matrices be dissipative to guarantee the stability of the refined model. A numerical example, involving finite element model refinement for a structural testbed at NASA Langley Research Center (Controls-Structures-Interaction Evolutionary model) is presented to demonstrate the feasibility of the proposed approach.

Introduction: Problem Statement

TYPICALLY, the spacecraft structure can be modeled as a linear, time-invariant flexible system, which in turn can be represented by the following second-order dynamical equations:

$$M\ddot{x} + D\dot{x} + Kx = Hf \tag{1}$$

where M is the positive definite mass matrix, D is the positive definite (semidefinite in the presence of rigid-body modes) damping matrix, K is the positive definite (semidefinite in the presence of rigid-body modes) stiffness matrix, H is the disturbance input influence matrix, x is a $k \times 1$ vector of displacements, and f is a $e \times 1$ vector of disturbances to the system. Usually, a finite element analysis is used to obtain these matrices analytically. However, the accuracy of the finite element model in predicting the dynamical behavior of the structure depends on a number of factors, such as proper knowledge of element and component material and geometric properties, appropriate meshing, correct joint modeling, etc. From past experience with flexible structures, the accuracy of the finite element model is limited when compared to test results from modal parameter identification. In almost every structure, the modal frequencies and amplitudes predicted using finite element models differ from those obtained from modal testing. This lack of accuracy in modal parameters can be a detriment to control system design. Control system design for flexible systems is challenging because of their special dynamic characteristics: a large number of structural modes within the controller bandwidth; low, closely spaced modal frequencies; very small inherent damping; and insufficient knowledge of the parameters.[1]

Control system design requires accurate knowledge of the plant that is to be controlled. In the case of spacecraft control systems, this means that an accurate knowledge of the parameters associated with the flexible modes of the spacecraft, such as modal frequencies, damping ratios, and mode shapes, is required. The need for accurate knowledge is particularly critical for the modal frequencies. In traditional gain-stabilized spacecraft control design, this knowledge is required to achieve nominal performance while guaranteeing stability margins in the form of phase and gain margins. In modern control system design, which may be gain or phase stabilized, this knowledge is required to achieve nominal performance as well as specific degrees of stability and performance robustness.

One approach to obtain accurate models of flexible structures is to use models that can be extracted directly from system identification (ID) or modal test data. This is a feasible approach, which has been quite successful in a number of applications. However, the usefulness of this approach is limited in that the refined model obtained applies only to the hardware configuration of the system ID. In other words, the model obtained from system ID data is only valid at the input/output channels that are used in the test setup. If the model of a component changes, additional inputs or outputs are included, or simply new elements are added, the model obtained through system ID loses its relevancy unless additional system ID tests are performed. Moreover, these models do not easily lend themselves for other required performance and reliability analyses, such as stress and strain analysis, vibration and jitter analysis, etc. To overcome the limited aspect of the system ID models, one can use an analytical model obtained through finite element analysis, provided that these models can be made to have sufficient accuracy for design and analysis. Thus, in this paper we address the problem of refining the analytical model of the flexible spacecraft using the system ID data.

To date, different techniques have been proposed for refining the finite element model of a flexible structure based on modal testing or system ID procedures.[2,3] Model refinement involves techniques that refine the finite element model by minimizing the level of disagreement between the model and test results. These techniques generally start with a set of parameters of the model (typically, physical parameters at the element level, e.g., material and geometric properties) and systematically tune those parameters to reduce or minimize some measure of disparity between the model and test data. This may be a time-related measure, for example, the difference in time history responses, or a modal-related measure, for example, difference in modal frequencies. Various optimization schemes and least-square techniques have been suggested for the refinement process.

This paper describes a novel approach for the refinement of finite element models. The approach presumes that modal analysis or system ID tests have been performed and that modal parameters, such as frequencies, damping ratios, and mode shapes (at sensor locations), have been identified for modes in the range of interest. The proposed approach models the possible refinements in the mass, damping, and stiffness matrices of the finite element model in the form of a constant gain feedback with acceleration, velocity, and displacement measurements, respectively. The freedom to change

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model parameters, as well as the relative degree of change desired in one parameter with respect to the rest, is embedded in the elements of the input and output influence matrices for the various measurements. Once the elements of the input and output influence matrices have been defined and fixed, the problem of model refinement reduces to obtaining position, velocity, and acceleration gain matrices that reassign a desired subset of the eigenvalues of the model, along with partial mode shapes, from their baseline to target values. Hence, the problem of model refinement becomes a problem of eigensystem assignment with output feedback. However, symmetry and the positive definiteness requirement of the mass matrix, and the positive definiteness (semidefiniteness if rigid-body modes are present) requirement of the stiffness and damping matrices, necessitates that gain matrices should be constrained such that the refined mass matrix remains symmetric and positive definite and the refined stiffness and damping matrices remain symmetric and positive definite (semidefinite). Moreover, the refinements in the system matrices should accommodate the sparsity of the nominal model, such that unwarranted additional connections are not introduced through refinement. In this paper, a procedure for obtaining symmetric gain matrices via eigensystem assignment is described first. To perform the required eigensystem assignment, a modified procedure to the sequential algorithm outlined in Ref. 4 is followed. The modified procedure provides the ability to use acceleration feedback, needed to refine the mass matrix, as well as the capability to assign eigenvectors partially. Second, additional constraints, in the form of quadratic inequality constraints, are outlined to render the symmetric gain matrices dissipative, thus guaranteeing the stability of the refined model. Finally, quadratic equality constraints are introduced to accommodate the sparsity requirements of the refined model. A numerical example involving model refinement of a structural testbed at NASA Langley [CSI evolutionary model (CEM) phase II] is presented to demonstrate the application of this approach.

Model Refinement

By observing of the nominal dynamical model of the system given in Eq. (1), the dynamics of the refined system may be written as

$$(M + \Delta M)\ddot{x} + (D + \Delta D)\dot{x} + (K + \Delta K)x = Hf \tag{2}$$

where ΔM is a symmetric matrix representing the refinement in the mass matrix, satisfying $(M + \Delta M) > 0$; ΔD is a symmetric matrix representing the refinement in the damping matrix, satisfying $(D + \Delta D) > 0$; and ΔK is a symmetric matrix representing the refinement in the stiffness matrix, satisfying $(K + \Delta K) > 0$. Note that the positive definiteness conditions for the refined stiffness and damping matrices reduce to positive semidefiniteness in the presence of rigid-body modes. Now, expand the refinement gain matrices as follows:

$$\Delta M = L_M G_M L_M^T, \qquad \Delta D = L_D G_D L_D^T, \qquad \Delta K = L_K G_K L_K^T \tag{3}$$

where L_M is a matrix representing the distribution of refinements that are allowed in the mass matrix. The elements of matrix L_M can vary depending on what elements in the mass matrix are chosen to vary. For example, if the chosen element of the mass matrix is the one at the ith row and column, then all of the elements of the ith row of matrix L_M may be chosen to be zeros, except one, which is set to 1. The matrix G_M represents the symmetric gain matrix associated with the mass matrix (acceleration gain matrix) that determines the extent of the refinement. The matrices L_D, G_D, L_K, and G_K are similarly defined to characterize the refinement for the damping and stiffness matrices, respectively. Note that the refinements in the mass, damping, and stiffness matrices may be viewed as symmetric, constant-gain acceleration, velocity, and position feedback. The system equations for the refined system may be rewritten as follows

$$M\ddot{x} + D\dot{x} + Kx = Hf + u$$

$$u = -\begin{bmatrix} L_K G_K L_K^T & L_D G_D L_D^T & L_M G_M L_M^T \end{bmatrix} \begin{Bmatrix} x \\ \dot{x} \\ \ddot{x} \end{Bmatrix} \tag{4}$$

Assume that a number of modes in the desired frequency range have been identified via a system ID procedure, and let the modal frequencies, damping ratios, and modal amplitudes (at the disturbance and measurement locations) be denoted by Ω_t, Z_t, and Φ_t, respectively. Here, Ω_t is an $r \times 1$ real vector of natural frequencies of the r identified modes; Z_t is an $r \times 1$ real vector of modal damping ratios; and Φ_t is an $s \times r$ complex matrix, whose r columns represent the mode shapes of these identified modes at s locations. After noting that, for real systems, complex eigenvalues occur in conjugate pairs, let the target eigenvalues and eigenvectors be defined as

$$\Lambda_t^{2i-1} = -Z_t^i\Omega_t^i + j\Omega_t^i\sqrt{1 - Z_t^{i^2}}, \qquad \Psi_t^{2i-1} = \Phi_t^i$$

$$\Lambda_t^{2i} = -Z_t^i\Omega_t^i - j\Omega_t^i\sqrt{1 - Z_t^{i^2}}, \qquad \Psi_t^{2i} = \overline{\Phi_t^i}$$

$$i = 1, \ldots, r \tag{5}$$

The overbar in the expressions in this section refer to complex conjugation of the elements of the corresponding vector (or matrix) only, as opposed to the Hermitian operator, which involves transposition and complex conjugation. Now the problem of model refinement may be expressed as the problem of finding symmetric acceleration, velocity, and position gain matrices (G_M, G_D, and G_K) such that the $2r$ eigenvalues of the system

$$\left(M + L_M G_M L_M^T\right)\ddot{x} + \left(D + L_D G_D L_D^T\right)\dot{x} + \left(K + L_K G_K L_K^T\right)x = Hf \tag{6}$$

are assigned to Λ_t^i, $i = 1, 2, \ldots, 2r$, and the s elements of corresponding eigenvectors are assigned to Ψ_t^i, $i = 1, 2, \ldots, 2r$, subject to the condition that the refined mass, damping, and stiffness matrices are positive definite. The partial assignment of eigenvectors may be defined as

$$\bar{R}\Phi = \Psi_t \tag{7}$$

where matrix $\bar{R}$ represents the influence coefficient matrix for the system ID sensor locations. The procedure developed and followed to compute the gain matrices is described in the next section.

Refinements in Damping Matrix via Eigensystem Assignment

The task of assigning the eigenvalues and partial eigenvectors of the system in Eq. (6) with symmetric output feedback gain matrices is accomplished using a sequential algorithm described by Ref. 4. The algorithm is modified here to accommodate the partial eigenvector assignment and acceleration terms to include refinements in the mass matrix. For the simplicity of presentation, the procedure is described for refinements in the damping matrix alone and then for an all inclusive model refinement.

In each step of the sequential procedure, one conjugate pair of eigenvalues is assigned to desired values while making sure that the previously assigned eigenvalues are not disturbed. The procedure uses a first-order descriptor representation of the system, obtained from Eq. (6),

$$\begin{bmatrix} I & 0 \\ 0 & M \end{bmatrix} \begin{Bmatrix} \dot{x} \\ \ddot{x} \end{Bmatrix} = \begin{bmatrix} 0 & I \\ -K & -D \end{bmatrix} \begin{Bmatrix} x \\ \dot{x} \end{Bmatrix} + \begin{bmatrix} 0 \\ L_D \end{bmatrix} u + \begin{bmatrix} 0 \\ H \end{bmatrix} f$$

$$u = -G_D L_D^T \dot{x} = -G_D \begin{bmatrix} 0 & L_D^T \end{bmatrix} \begin{Bmatrix} x \\ \dot{x} \end{Bmatrix} \equiv -G_D C_D \begin{Bmatrix} x \\ \dot{x} \end{Bmatrix} \tag{8}$$

that is, the descriptor form

$$E\dot{z} = Az + Bu + Pf, \qquad u = -G_D C_D z \tag{9}$$

where

$$z = \begin{Bmatrix} x \\ \dot{x} \end{Bmatrix}$$

represents the state in the first-order descriptor form, with $C_D = B^T$. A brief description of the sequential procedure is given next. The reader is referred to Ref. 4 for a thorough description of the procedure.

1) The procedure employs the generalized ordered real Schur transformations of the system matrices, E and A.

2) Orthogonal transformations are used to move previously assigned eigenvalue pairs to the top left block of the pair $(\tilde{E}, \tilde{A})$, where $(\tilde{E}, \tilde{A})$ are in ordered real Schur form,[5] and the structure of the new gain matrix is prescribed such that it only affects the eigenvalues in the lower bottom partition of the system matrices. For example, assume that $(k-1)$ conjugate pairs of the eigenvalues have been placed in the previous steps and that they are in the top left block of $(\tilde{E}, \tilde{A})$. Let $\tilde{N}_{D_k}$ denote a matrix whose columns form an orthogonal basis for the left null space of $\tilde{C}_{D_{k1}}$, that is, $\tilde{N}_{D_k}$ is a matrix with orthogonal columns such that $\tilde{N}_{D_k}^T \tilde{C}_{D_{k1}} = 0$. Here, $\tilde{C}_{D_{k1}}$ denotes the first $2(k-1)$ column partition of the output influence matrix in transformed coordinates. If the gain matrix in the transformed coordinates $\tilde{G}_{D_k}$ is constructed as

$$\tilde{G}_{D_k} = \tilde{N}_{D_k} \hat{G}_{D_k} \tilde{N}_{D_k}^T \tag{10}$$

where $\hat{G}_{D_k}$ may be an arbitrary matrix, then output feedback with the gain matrix will not affect the $(k-1)$ eigenvalue pairs assigned in the earlier steps.

3) At each step, an intermediate gain matrix is computed to assign a pair of eigenvalues to desired values in lower bottom partition of the system matrices. The algorithm used in this eigenvalue assignment was initially developed in Ref. 4, but is modified here to accommodate partial eigenvector assignment as well.

4) At each step, after computing the gain matrix that assigns a pair of desired eigenvalues, the intermediate state matrix is transformed to a generalized Schur form with all earlier assigned eigenpairs in the top left block of the updated system matrix.

5) The overall gain matrix is constructed by accumulating the gains from each step.

6) This process can be continued until up to m eigenvalues have been assigned to the desired locations, where m denotes the number of inputs or outputs.

Eigenpair Assignment

This section describes the approach to select output feedback gains to assign one pair of complex-conjugate eigenvalues, while ensuring that the gain matrix is symmetric and the partial eigenvectors are as close as possible to their corresponding target vectors. Assume that the kth eigenpair is to be assigned. For notational simplicity, the system matrices will be denoted as E_{22}, A_{22}, B_2, and C_2, the output feedback gain matrix will be denoted as G, and the desired eigenvalue pair will be denoted $(\lambda, \bar{\lambda})$. The problem is to select a symmetric matrix G, such that $(\lambda, \bar{\lambda})$ is a generalized eigenpair of the closed-loop system matrix, $(E_{22}, A_{22} - B_2GC_2)$, and the eigenvectors are partially assigned to desired values, as given in Eq. (7).

Let ϕ be the eigenvector corresponding to the eigenvalue λ. The generalized eigenvalue problem becomes $(\lambda E_{22} - A_{22} + B_2GC_2)\phi = 0$. This expression can be rewritten as

$$[\lambda E_{22} - A_{22} \mid B_2] \begin{bmatrix} \phi \\ --- \\ GC_2\phi \end{bmatrix} \equiv \Gamma \begin{bmatrix} \phi \\ --- \\ GC_2\phi \end{bmatrix} = 0 \tag{11}$$

It is obvious from Eq. (11), that the vector on the right-hand side of the expression must lie in the right null space of Γ. Let N be a matrix whose columns form an orthogonal basis for the null space of Γ, that is, $\Gamma N = 0$. Note that unlike an actual control design problem where the number of inputs/outputs are usually fixed, we may choose the number inputs (parameters that can be changed in the model) large enough as to provide the freedom to assign the desired eigenvalues and specified elements of the corresponding eigenvectors. Although E_{22}, A_{22}, and B_2 are real matrices, Γ and N are complex matrices because the eigenvalue λ is a complex scalar. However, to ensure that the gain matrix is real, the eigenvector corresponding to the complex-conjugate eigenvalue is chosen to be the complex conjugate of ϕ, that is, $\bar{\phi}$ is chosen to be the eigenvector corresponding to $\bar{\lambda}$.

Because columns of N span the null space of Γ, it follows that

$$\begin{bmatrix} \phi \\ --- \\ GC_2\phi \end{bmatrix} = N\alpha = \begin{bmatrix} N_1 \\ --- \\ N_2 \end{bmatrix} \alpha \tag{12}$$

where α is an arbitrary vector of complex elements and the matrices N_1 and N_2 are formed by partitioning N compatibly with ϕ and $GC_2\phi$. From Eq. (12), $\phi = N_1\alpha$ and $GC_2\phi = N_2\alpha$, which leads to

$$GC_2N_1\alpha = N_2\alpha \tag{13}$$

The eigenassignment problem is now reduced to selecting α such that there exists a symmetric gain matrix G satisfying Eq. (13). With $\bar{\phi}$ being the eigenvector corresponding to $\bar{\lambda}$, real solutions for the gain matrix G can be obtained, and the equations can be written out to involve only real arithmetic operations as follows.

For the eigenvalue $\bar{\lambda}$, with eigenvector $\bar{\phi}$, the matrix $\bar{\Gamma} = [\bar{\lambda}E_{22} - A_{22} \mid B_2]$, and $\bar{N}$ is a matrix whose orthogonal columns span the null space of $\bar{\Gamma}$. If the arbitrary coefficient vector is chosen to be $\bar{\alpha}$, the complex conjugate of α, then it follows that

$$GC_2\bar{N}_1\bar{\alpha} = \bar{N}_2\bar{\alpha} \tag{14}$$

Equation (13) or Eq. (14) can be rewritten as

$$GC_2[\mathrm{Re}(N_1) \quad -\mathrm{Im}(N_1)] \begin{bmatrix} \mathrm{Re}(\alpha) \\ \mathrm{Im}(\alpha) \end{bmatrix} = [\mathrm{Re}(N_2) \quad -\mathrm{Im}(N_2)] \begin{bmatrix} \mathrm{Re}(\alpha) \\ \mathrm{Im}(\alpha) \end{bmatrix} \tag{15}$$

and

$$GC_2[\mathrm{Im}(N_1) \quad \mathrm{Re}(N_1)] \begin{bmatrix} \mathrm{Re}(\alpha) \\ \mathrm{Im}(\alpha) \end{bmatrix} = [\mathrm{Im}(N_2) \quad \mathrm{Re}(N_2)] \begin{bmatrix} \mathrm{Re}(\alpha) \\ \mathrm{Im}(\alpha) \end{bmatrix} \tag{16}$$

where Re( ) denotes real part of the argument and Im( ) denotes imaginary part of the argument. In compact form these equations are written as

$$GW_1\boldsymbol{p} = V_1\boldsymbol{p}, \qquad GW_2\boldsymbol{p} = V_2\boldsymbol{p} \tag{17}$$

where $\boldsymbol{p} = [\mathrm{Re}(\alpha^T) \;\; \mathrm{Im}(\alpha^T)]^T$, $W_1 = C_2[\mathrm{Re}(N_1) \;\; -\mathrm{Im}(N_1)]$, $V_1 = [\mathrm{Re}(N_2) \;\; -\mathrm{Im}(N_2)]$, $W_2 = C_2[\mathrm{Im}(N_1) \;\; \mathrm{Re}(N_1)]$, and $V_2 = [\mathrm{Im}(N_2) \;\; \mathrm{Re}(N_2)]$. Note that Eq. (17) is a system of quadratic equations in the unknown variables, namely, the elements of the gain matrix G and the coefficient vector $\boldsymbol{p}$. Furthermore, the elements of G should be constrained such that G is symmetric, and the solution of the system has to yield an eigenvector, for the whole system, Υ that satisfies the partial eigenvector conditions of Eq. (7), that is,

$$RU\Upsilon = \Psi_t^{2k-1} \tag{18}$$

where $R = [\bar{R} \;\; 0]$ is the corresponding coefficient in the descriptor form of the system equations, U is the right unitary matrix in the generalized Schur form at the kth step (that keeps the closed-loop matrices in real Schur form), and Ψ_t^{2k-1} is the target partial eigenvector for the kth step. Note that, if the condition of Eq. (18) is satisfied for one of the eigenvectors of the eigenpair, it would identically satisfy its complex conjugate. Here, we assume that the set of earlier assigned eigenvalues does not match the remaining eigenvalues of the system, either before or after the eigenpair assignment. This mild assumption ensures that the eigenvectors of the earlier assigned eigenvalues/eigenvectors remain unchanged as additional eigenvalues are assigned. Furthermore, this means that the eigenvector condition of the type in Eq. (18) can be imposed one mode at a time, and once imposed for an eigenvector, it need not be reimposed again. Now, considering the eigenvalue problem of the whole system for the eigenvalue being assigned, one can write

$$\begin{bmatrix} \lambda E_{11} - A_{11} & \lambda E_{12} - A_{12} + B_1GC_2 \\ 0 & \lambda E_{22} - A_{22} + B_2GC_2 \end{bmatrix} \begin{Bmatrix} \varphi \\ \phi \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{19}$$

Note that

$$\Upsilon = \begin{Bmatrix} \varphi \\ \phi \end{Bmatrix}$$

Solving for φ in terms of ϕ, one obtains

$$\varphi = -(\lambda E_{11} - A_{11})^{-1}(\lambda E_{12} - A_{12} + B_1 G C_2)\phi \tag{20}$$

and by substituting for $GC_2\phi$ and ϕ from Eq. (12), one has

$$\phi = N_1 \alpha$$

$$\varphi = -(\lambda E_{11} - A_{11})^{-1}([\lambda E_{12} - A_{12}]N_1 + B_1 N_2)\alpha \equiv Q\alpha \tag{21}$$

or

$$\Upsilon = \begin{bmatrix} Q \\ N_1 \end{bmatrix}\alpha \equiv S\alpha \tag{22}$$

Using Eq. (22) into Eq. (18) and expanding and separating the real and imaginary parts, one has

$$RU[\mathrm{Re}(S) \quad -\mathrm{Im}(S)]\begin{Bmatrix} \mathrm{Re}(\alpha) \\ \mathrm{Im}(\alpha) \end{Bmatrix} = \mathrm{Re}\left(\Psi_t^{2k-1}\right)$$

$$RU[\mathrm{Im}(S) \quad \mathrm{Re}(S)]\begin{Bmatrix} \mathrm{Re}(\alpha) \\ \mathrm{Im}(\alpha) \end{Bmatrix} = \mathrm{Im}\left(\Psi_t^{2k-1}\right) \tag{23}$$

Recalling the definition of vector $\boldsymbol{p}$ from Eq. (17) and combining these equations, one obtains

$$L\boldsymbol{p} = \boldsymbol{q} \tag{24}$$

where

$$L = \begin{bmatrix} RU\,\mathrm{Re}(S) & -RU\,\mathrm{Im}(S) \\ RU\,\mathrm{Im}(S) & RU\,\mathrm{Re}(S) \end{bmatrix}, \qquad \boldsymbol{q} = \begin{Bmatrix} \mathrm{Re}\left(\Psi_t^{2k-1}\right) \\ \mathrm{Im}\left(\Psi_t^{2k-1}\right) \end{Bmatrix} \tag{25}$$

Therefore, a coefficient vector $\boldsymbol{p}$ and a symmetric gain matrix G that satisfy Eqs. (17) and (24) have to be found. The condition for the existence of a symmetric gain matrix G that satisfies Eq. (17) has been established in Ref. 4 and is given as the existence of a vector $\boldsymbol{p}$ that satisfies

$$\boldsymbol{p}^T\left(V_1^T W_2 - V_2^T W_1\right)\boldsymbol{p} \equiv \boldsymbol{p}^T J \boldsymbol{p} = 0 \tag{26}$$

To summarize, the conditions for the placement of an eigenpair of the system to desired values, while partially assigning the corresponding eigenvectors to target values, reduces to computing a coefficient vector $\boldsymbol{p}$ that satisfies the quadratic equation given by Eq. (26) and the linear system of equations represented by Eq. (24).

One possible approach to obtaining a coefficient vector $\boldsymbol{p}$ that satisfies Eqs. (26) and (24) would be to first solve for $\boldsymbol{p}$ in Eq. (24), to obtain

$$\boldsymbol{p} = L^+\boldsymbol{q} + N_L\boldsymbol{\beta} \tag{27}$$

where $()^+$ is the pseudoinverse of $()$, N_L is a matrix collecting a set of basis vectors for the right null space of matrix L, and $\boldsymbol{\beta}$ is a coefficient vector associated with the basis vector, yet to be defined. Substituting for $\boldsymbol{p}$ from Eq. (27) into Eq. (26) yields

$$f(\boldsymbol{\beta}) \equiv \boldsymbol{\beta}^T N_L^T J N_L \boldsymbol{\beta} + \boldsymbol{\beta}^T N_L^T J L^+ \boldsymbol{q} + \boldsymbol{q}^T L^{+^T} J N_L \boldsymbol{\beta}$$
$$+ \boldsymbol{q}^T L^{+^T} J L^+ \boldsymbol{q} = 0 \tag{28}$$

Standard Newton methods for obtaining the solution of nonlinear equations may be used to obtain a solution vector $\boldsymbol{\beta}$. Analytic gradients of $f(\boldsymbol{\beta})$ are readily available because the gradient of any quadratic, $f(\boldsymbol{\beta}) = \boldsymbol{\beta}^T Q_1 \boldsymbol{\beta} + 2Q_2^T\boldsymbol{\beta} + c$, is given by $(\partial/\partial\boldsymbol{\beta}) f(\boldsymbol{\beta}) = (Q_1 + Q_1^T)\boldsymbol{\beta} + 2Q_2$. The nonlinear problem of Eq. (28) is very well behaved because the function is quadratic in $\boldsymbol{\beta}$ and analytic gradients are linear in $\boldsymbol{\beta}$.

Once a coefficient vector $\boldsymbol{p}$ (or $\boldsymbol{\beta}$) has been determined, the symmetric gain matrix G that assigns the desired complex-conjugate eigenvalues may be obtained as follows. Denote $\boldsymbol{y}_1 = V_1\boldsymbol{p}$, $\boldsymbol{y}_2 = V_2\boldsymbol{p}$, $\boldsymbol{x}_1 = W_1\boldsymbol{p}$ and $\boldsymbol{x}_2 = W_2\boldsymbol{p}$, and let $X = [\boldsymbol{x}_1 \;\; \boldsymbol{x}_2]$ and $Y = [\boldsymbol{y}_1 \;\; \boldsymbol{y}_2]$, then Eq. (17) is rewritten as $GX = Y$. Let Q be an orthogonal matrix, such that

$$Q^T Y = \begin{bmatrix} \tilde{Y}_1 \\ 0 \end{bmatrix} \tag{29}$$

where $\tilde{Y}_1$ is a nonsingular 2×2 matrix (otherwise, the problem is solved trivially). The matrix Q can be obtained by QR factorization of Y. Now, define $\tilde{X}_1$, $\tilde{X}_2$ as follows:

$$\begin{bmatrix} \tilde{X}_1 \\ \tilde{X}_2 \end{bmatrix} = Q^T X \tag{30}$$

where $\tilde{X}_1$ is a 2×2 matrix, and $\tilde{X}_2$ is a $(m-2) \times 2$ matrix. Now $\tilde{X}_1$ is nonsingular if $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ are linearly independent (otherwise, the problem is trivial). Defining $\tilde{G}_{11} = \tilde{Y}_1\tilde{X}_1^{-1}$, it can be seen that

$$\begin{bmatrix} \tilde{G}_{11} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \tilde{X}_1 \\ \tilde{X}_2 \end{bmatrix} = \begin{bmatrix} \tilde{Y}_1 \\ 0 \end{bmatrix} \tag{31}$$

Therefore, it follows that the matrix G defined as

$$G = Q\begin{bmatrix} \tilde{G}_{11} & 0 \\ 0 & 0 \end{bmatrix}Q^T \tag{32}$$

satisfies $GX = Y$

Note that if only eigenvalue assignment is required, that is, it is not required to perform partial eigenvector assignment, then the solution vector $\boldsymbol{p}$ does not have to satisfy Eq. (24), and only the quadratic equation given by Eq. (26) needs to be satisfied. The solution to this equation can be obtained through standard Newton methods as mentioned earlier. However, because Eq. (26) is a simple quadratic, the check for existence and computation of a solution can be achieved via examination of the matrix J. A solution vector exists if and only if the symmetric part of J is either indefinite or semidefinite. If the symmetric part of the matrix has zero eigenvalues, then any corresponding eigenvector is a solution for vector $\boldsymbol{p}$. If the symmetric part of J is indefinite, then any linear combination of eigenvectors of the symmetric part, whose corresponding eigenvalues are not all of the same sign, qualifies as a solution, if the coefficient of the linear combination are chosen such that the quadratic in Eq. (26) vanishes.

Once the gain matrix G is computed, the current refinements in the damping matrix, represented by $\tilde{G}_{D_k}$ is determined from Eq. (10), and the overall refinement is updated as

$$G_D \leftarrow G_D + \tilde{G}_{D_k} \tag{33}$$

The procedure described thus far determines a symmetric gain matrix that reassigns a desired subset of the eigenvalues of the model, along with partial mode shapes, from their baseline to target values. However, the symmetry of the gain matrix does not necessarily guarantee that the refined (combined) model remains stable. Because in most situations the flexible system is open-loop stable, any refinements to the analytical model should be such to maintain that stability. One approach to this could be to use the design freedom in the solution vector $\boldsymbol{p}$ and impose constraints on eigenvalues of the refined damping matrix. However, this could be cumbersome, particularly, when the size of the system is large (thousands or hundreds of thousands of degrees of freedom). Another approach could be to use the freedom beyond eigensystem assignment [quadratic equalities in Eqs. (26) or (28)] to reduce some measure of the gain matrix G that represents the refinement in the damping matrix. This can be accomplished, for example, by imposing inequality constraints in the form of

$$\left|\boldsymbol{p}^T V_i^T W_j \boldsymbol{p}\right| \geq \delta_{ij}\boldsymbol{p}^T V_i^T V_i \boldsymbol{p}, \qquad i = 1, 2, \quad j = 1, 2 \tag{34}$$

where δ_{ij} are positive scalars whose values can be used to adjust the norm of the gain matrix.

Alternatively, one can require that the current gain matrix $\tilde{G}_{D_k}$ be dissipative at every sequence. Although the dissipativity requirement can be constraining, it will guarantee that the refined system remains stable. In other words, at every sequence, a pair of eigenvalues are assigned via a symmetric and dissipative gain matrix. Reference 4 provides a set of constraints to impose dissipativity of the gain matrix in this setting. These constraints are in the form of quadratic inequality constraints in the solution vector $\boldsymbol{p}$ as follows:

$$f_1(\boldsymbol{p}) = \boldsymbol{p}^T\left\{V_1^T W_1 + \tfrac{1}{2}\left(V_1^T W_2 + V_2^T W_1\right)\right\}\boldsymbol{p} \geq 0$$

$$f_2(\boldsymbol{p}) = \boldsymbol{p}^T\left\{V_1^T W_1 - \tfrac{1}{2}\left(V_1^T W_2 + V_2^T W_1\right)\right\}\boldsymbol{p} \geq 0$$

$$f_3(\boldsymbol{p}) = \boldsymbol{p}^T\left\{V_2^T W_2 + \tfrac{1}{2}\left(V_1^T W_2 + V_2^T W_1\right)\right\}\boldsymbol{p} \geq 0$$

$$f_4(\boldsymbol{p}) = \boldsymbol{p}^T\left\{V_2^T W_2 - \tfrac{1}{2}\left(V_1^T W_2 + V_2^T W_1\right)\right\}\boldsymbol{p} \geq 0 \qquad (35)$$

These quadratic constraints go well with the quadratic symmetry condition given in Eq. (26), and hence, the appealing computational nature of the algorithm is retained. Another possible approach could be to require that at each sequence of the eigensystem assignment procedure the overall gain matrix G_D remains positive semidefinite, that is, the gain matrix is dissipative.

Sparsity Accommodation

In most applications involving flexible systems, the system matrices are not fully populated matrices. On the contrary, these matrices exhibit significant sparsity. However, the formulation presented thus far for model refinement does not consider or incorporate possible sparsity in the system. It is essential to have such a capability because it may not be practical to produce refined models that have different sparsity (bandwidth) than the nominal model. One exception can be in applications wherein identification of possible missing elements (connectivity) in the nominal model is of concern.

To accommodate sparsity in the damping refinement, assume that the refinement influence matrix L_D is defined such that each of its columns has only one nonzero element. In other words, no coupling in matrix L_D is considered between the degrees of freedom considered for refinement. This assumption would isolate any potential couplings between the degrees of freedom in the elements of the gain matrix G_D. Also, this assumption would not result in any loss of generality because any coupling in influence matrix L_D can be absorbed into the gain matrix. The procedure to incorporate sparsity in the damping matrix refinements follows.

As mentioned earlier, the placement procedure is sequential wherein at each sequence the model is refined such that a pair of complex-conjugate eigenvalues (and possibly partial eigenvectors) are reassigned to match system ID values. At step k of this procedure, the local gain matrix G must satisfy the conditions established in Eq. (17), such that

$$G[W_1\boldsymbol{p} \quad W_2\boldsymbol{p}] = [V_1\boldsymbol{p} \quad V_2\boldsymbol{p}] \qquad (36)$$

Note that, in this equation and the following development, the subscript k, denoting the step number, has been omitted to improve the clarity of presentation. Moreover, for this refinement not to effect the $(k-1)$ eigenpairs earlier matched by the algorithm, the overall gain matrix at step k, $\tilde{G}$, must be defined via Eq. (10). The required sparsity in the refinement matrix should be incorporated in the elements of matrix $\tilde{G}$. Now assume that sparsity requires that super diagonal elements $j_1, j_2, \ldots, j_q$ of the ith row of $\tilde{G}$ must be zero, where q is the number of zero-valued elements. It can easily be shown that Eq. (36) is satisfied identically if

$$\tilde{N}G\tilde{N}^T\tilde{N}[W_1\boldsymbol{p} \quad W_2\boldsymbol{p}] = \tilde{N}[V_1\boldsymbol{p} \quad V_2\boldsymbol{p}]$$

leading to

$$\tilde{G}[\tilde{W}_1\boldsymbol{p} \quad \tilde{W}_2\boldsymbol{p}] = [\tilde{V}_1\boldsymbol{p} \quad \tilde{V}_2\boldsymbol{p}] \rightarrow \tilde{G}\tilde{W} = \tilde{V} \qquad (37)$$

For a gain matrix $\tilde{G}$, with vanishing super diagonal elements j_1, $j_2, \ldots, j_q$ in its ith row, to exist while satisfying Eq. (37), the ith row of matrix $\tilde{V}$, $\tilde{V}_i$, must be in the range space of the rows of $\tilde{W}_r$, where $\tilde{W}_r$ includes all rows of $\tilde{W}$ except rows $j_1, j_2, \ldots, j_q$. From linear matrix theory, $\tilde{V}_i$ is in the range space of $\tilde{W}_r$ iff all rows of $\tilde{W}_r$ are orthogonal to the right null space of $\tilde{V}_i$, that is,

$$\tilde{W}_r\tilde{V}_i^{\perp} = 0 \qquad (38)$$

The ith row of matrix $\tilde{V}$ is a 1×2 vector defined as $[\tilde{V}_{1i}\boldsymbol{p} \quad \tilde{V}_{2i}\boldsymbol{p}]$, where the subscript i refers to the ith row of the matrix. The right null vector of $\tilde{V}_i$ may be written as

$$\tilde{V}_i^{\perp} = \begin{Bmatrix} \tilde{V}_{2i}\boldsymbol{p} \\ -\tilde{V}_{1i}\boldsymbol{p} \end{Bmatrix} \qquad (39)$$

Using Eq. (39) in Eq. (38) and expanding $\tilde{W}_r$ as $[\tilde{W}_{1r}\boldsymbol{p} \quad \tilde{W}_{2r}\boldsymbol{p}]$, one obtains

$$\tilde{W}_{1r}\boldsymbol{p}\tilde{V}_{2i}\boldsymbol{p} - \tilde{W}_{2r}\boldsymbol{p}\tilde{V}_{i1}\boldsymbol{p} = 0 \qquad (40)$$

Therefore, the condition for existence of a gain matrix $\tilde{G}$, with vanishing super diagonal elements $j_1, j_2, \ldots, j_q$ in its ith row, is a system of quadratic equalities in the coefficient vector $\boldsymbol{p}$. Similar conditions can be established for vanishing elements in other rows of matrix $\tilde{G}$. These quadratic conditions, together with the quadratic equation given by Eq. (26) for symmetry, the quadratic inequalities given by Eq. (35) for dissipativity, and the linear system of equations provided by Eq. (24) for partial eigenvector assignment constitute the complete conditions for sparse damping matrix refinement.

The computation of the gain matrix in the presence of sparsity conditions requires a different treatment than described earlier [see Eq. (32)]. Here, the gain is directly computed from the solution of simultaneous equations that impose eigenvalue assignment, symmetry, and sparsity conditions. Once, a feasible coefficient vector $\boldsymbol{p}$, satisfying the quadratic and linear requirements described earlier for sparse refinement, has been obtained. Then, the system of simultaneous equations that yields the elements of the local gain matrix G are derived as follows. Denote $X = [W_1\boldsymbol{p} \quad W_2\boldsymbol{p}]$ and $Y = [V_1\boldsymbol{p} \quad V_2\boldsymbol{p}]$; then transposing Eq. (36) gives

$$X^T G = Y^T \qquad (41)$$

Packing the columns of matrix G to obtain vector $\boldsymbol{g}$, Eq. (41) may be rewritten as

$$\begin{bmatrix} X^T & 0 & \cdots & 0 \\ 0 & X^T & \vdots & \vdots \\ \vdots & \cdots & \ddots & 0 \\ 0 & \cdots & 0 & X^T \end{bmatrix} \boldsymbol{g} = \boldsymbol{Y}_p \qquad (42)$$

where vector $\boldsymbol{Y}_p$ denotes the columns of matrix Y^T packed in a columnwise manner. The condition for symmetry of matrix G may be written as

$$S\boldsymbol{g} = 0 \qquad (43)$$

where S is a coefficient matrix with -1, 0, or 1 for its elements. Finally, the condition for proper sparsity of matrix $\tilde{G}$ or $\tilde{N}G\tilde{N}^T$ may be written as

$$P\boldsymbol{g} = 0 \qquad (44)$$

where P is a coefficient matrix whose elements depend on the elements of the matrix $\tilde{N}$. Now the gain matrix G (or gain vector $\boldsymbol{g}$) can be computed by solving the system of simultaneous equations given by Eqs. (42–44). Note that typically there would a great deal of sparsity in the coefficient matrices of Eqs. (42–44) so that sparse solvers would be ideal for obtaining the solution. Alternatively, one can solve first for $\boldsymbol{g}$ from Eqs. (43) and (44), that is,

$$\boldsymbol{g} = N_{SP}\boldsymbol{\theta} \qquad (45)$$

where the matrix N_{SP} is a set of basis vectors for the right null space of matrix

$$\begin{bmatrix} S \\ P \end{bmatrix}$$

and θ is the coefficient vector yet to be determined. Substitute for g from Eq. (45) into Eq. (42) and solve for θ to obtain the gain solution. This alternative may be the best approach because there may be times where a completely feasible solution for p cannot be obtained. In such a case, its best to guarantee that the refinement is feasible by being symmetric and having proper sparsity [Eqs. (43) and (44)], and then attempt to match the system ID data as close as possible [Eq. (42)].

Before ending this section on incorporation of sparsity, it should be mentioned that if degrees of freedom for model refinement can be chosen such that the refinement matrices take a block-diagonal form, then a much easier procedure, one that is very similar to the case with no sparsity, may be followed. For example, assume that the gain matrix at step k is supposed to have the form

$$\tilde{N}G\tilde{N}^T = \begin{bmatrix} \tilde{G}_1 & 0 \\ 0 & \tilde{G}_2 \end{bmatrix} \tag{46}$$

As stated earlier, it can easily be shown that the condition for eigenpair assignment [Eq. (36)] is satisfied identically if

$$\tilde{N}G\tilde{N}^T\tilde{N}[W_1 p \quad W_2 p] = \tilde{N}[V_1 p \quad V_2 p]$$

which, along with Eq. (46), becomes

$$\begin{bmatrix} \tilde{G}_1 & 0 \\ 0 & \tilde{G}_2 \end{bmatrix}\tilde{N}[W_1 p \quad W_2 p] = \tilde{N}[V_1 p \quad V_2 p]$$

leading to

$$\tilde{G}_1\tilde{N}_1[W_1 p \quad W_2 p] = \tilde{N}_1[V_1 p \quad V_2 p]$$

$$\tilde{G}_2\tilde{N}_2[W_1 p \quad W_2 p] = \tilde{N}_2[V_1 p \quad V_2 p] \tag{47}$$

where matrices $\tilde{N}_1$ and $\tilde{N}_2$ are partitions of $\tilde{N}$ according to rows of matrices $\tilde{G}_1$ and $\tilde{G}_2$, respectively. These equation are almost identical to those obtained for the case with no sparsity [see Eq. (17)]. The main difference here is that there are two gain matrices for which the conditions of symmetry and/or dissipativity have to satisfied.

General Model Refinements

The approach for a total model refinement, which includes refinements in the mass, damping, and stiffness matrices, parallels the one described in the preceding sections for the refinement in the damping matrix alone. In each step of the sequential procedure, one conjugate pair of eigenvalues is assigned to desired values while making sure that the earlier assigned eigenvalues are not disturbed. The procedure uses a first-order descriptor representation of the system, obtained from Eq. (6)

$$\begin{bmatrix} I & 0 \\ 0 & M \end{bmatrix}\begin{Bmatrix} \dot{x} \\ \ddot{x} \end{Bmatrix} = \begin{bmatrix} 0 & I \\ -K & -D \end{bmatrix}\begin{Bmatrix} x \\ \dot{x} \end{Bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ L_M & L_D & L_K \end{bmatrix}\begin{Bmatrix} u_M \\ u_D \\ u_K \end{Bmatrix} + \begin{bmatrix} 0 \\ H \end{bmatrix} f$$

$$u_M = -G_M L_M^T \ddot{x} = -G_M \begin{bmatrix} 0 & L_M^T \end{bmatrix}\begin{Bmatrix} \dot{x} \\ \ddot{x} \end{Bmatrix} \equiv -G_M C_M \begin{Bmatrix} \dot{x} \\ \ddot{x} \end{Bmatrix}$$

$$u_D = -G_D L_D^T \dot{x} = -G_D \begin{bmatrix} 0 & L_D^T \end{bmatrix}\begin{Bmatrix} x \\ \dot{x} \end{Bmatrix} \equiv -G_D C_D \begin{Bmatrix} x \\ \dot{x} \end{Bmatrix}$$

$$u_K = -G_K L_K^T x = -G_D \begin{bmatrix} L_K^T & 0 \end{bmatrix}\begin{Bmatrix} x \\ \dot{x} \end{Bmatrix} \equiv -G_D C_K \begin{Bmatrix} x \\ \dot{x} \end{Bmatrix} \tag{48}$$

that is, the descriptor form

$$E\dot{z} = Az + Bu + Pf, \qquad u = \begin{Bmatrix} u_M \\ u_D \\ u_K \end{Bmatrix} \tag{49}$$

The same sequential procedure is now used to assign the eigenpairs, one at a time. As in the damping case, the structure of the three gain matrices, G_M, G_D and G_K, is prescribed such that it only affects the eigenvalues in the lower bottom partition of the system matrices. For example, assume that $(k-1)$ conjugate pairs of the eigenvalues have been placed in the earlier steps and that they are in the top left block of $(\bar{E}, \bar{A})$. Let $\tilde{N}_{M_k}$, $\tilde{N}_{D_k}$, and $\tilde{N}_{K_k}$ be matrices whose columns form an orthogonal basis for the left null space of $\bar{C}_{M_{k1}}$, $\bar{C}_{D_{k1}}$, and $\bar{C}_{K_{k1}}$ respectively. Here, $\bar{C}_{M_{k1}}$, $\bar{C}_{D_{k1}}$, and $\bar{C}_{K_{k1}}$ are the first $2(k-1)$ column partitions of the output influence matrices in transformed coordinates. If the gain matrices (in the transformed coordinates) are constructed as

$$\tilde{G}_{M_k} = \tilde{N}_{M_k}\hat{G}_{M_k}\tilde{N}_{M_k}^T, \qquad \tilde{G}_{D_k} = \tilde{N}_{D_k}\hat{G}_{D_k}\tilde{N}_{D_k}^T$$

$$\tilde{G}_{K_k} = \tilde{N}_{K_k}\hat{G}_{K_k}\tilde{N}_{K_k}^T \tag{50}$$

with $\hat{G}_{M_k}$, $\hat{G}_{D_k}$, and $\hat{G}_{K_k}$ arbitrary matrices, then output feedback with the gain matrices will not affect the $(k-1)$ eigenvalue pairs assigned in the earlier steps.

The approach to select output feedback gains is described next, to assign one pair of complex-conjugate eigenvalues, while ensuring that the gain matrices are symmetric and the partial eigenvectors are as close as possible to their corresponding target vectors. Assume that the kth eigenpair is to be assigned. For notational simplicity, the system matrices will be denoted as E_{22}, A_{22}, B_{M_2}, C_{M_2}, B_{D_2}, C_{D_2}, B_{K_2}, and C_{K_2}, the output feedback gain matrices $\hat{G}_{M_k}$, $\hat{G}_{D_k}$, and $\hat{G}_{K_k}$ will be denoted, respectively, as G_a, G_v, and G_d, and the desired eigenvalue pair will be denoted $(\lambda, \bar{\lambda})$. The problem is to select symmetric matrices G_a, G_v, and G_d, such that $(\lambda, \bar{\lambda})$ is a generalized eigenpair of the closed-loop system matrix, $(E_{22} + B_{M_2}G_aC_{M_2}, A_{22} - B_{D_2}G_vC_{D_2} - B_{K_2}G_dC_{K_2})$, and the eigenvectors are partially assigned to desired values, as given in Eq. (7).

Let ϕ be the eigenvector corresponding to the eigenvalue λ. The generalized eigenvalue problem becomes

$$\left[\lambda\left(E_{22} + B_{M_2}G_aC_{M_2}\right) - A_{22} + B_{D_2}G_vC_{D_2} + B_{K_2}G_dC_{K_2}\right]\phi = 0 \tag{51}$$

This expression can be rewritten as

$$\begin{bmatrix} \lambda E_{22} - A_{22} & \lambda B_{M_2} & B_{D_2} & B_{K_2} \end{bmatrix}\begin{bmatrix} \phi \\ G_aC_{M_2}\phi \\ G_vC_{D_2}\phi \\ G_dC_{K_2}\phi \end{bmatrix} \equiv \Gamma\begin{bmatrix} \phi \\ G_aC_{M_2}\phi \\ G_vC_{D_2}\phi \\ G_dC_{K_2}\phi \end{bmatrix} = 0 \tag{52}$$

It is obvious from Eq. (52), that the vector on the right-hand side of the preceding expression must lie in the right null space of Γ. Let N be a matrix whose columns form an orthogonal basis for the null space of Γ, that is, $\Gamma N = 0$. Because columns of N span the null space of Γ, it follows that

$$\begin{bmatrix} \phi \\ G_aC_{M_2}\phi \\ G_vC_{D_2}\phi \\ G_dC_{K_2}\phi \end{bmatrix} = N\alpha = \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix}\alpha \tag{53}$$

where α is an arbitrary vector of complex elements and the matrices N_1, N_2, N_3, N_4 are formed by partitioning N compatibly with ϕ, $G_aC_{M_2}\phi$, $G_vC_D\phi$, and $G_dC_{K_2}\phi$. From Eq. (53), one has

$$\phi = N_1\alpha, \qquad G_aC_{M_2}\phi = N_2\alpha$$

$$G_vC_{D_2}\phi = N_3\alpha, \qquad G_dC_{K_2}\phi = N_4\alpha \tag{54}$$

or

$$G_aC_{M_2}N_1\alpha = N_2\alpha, \qquad G_vC_{D_2}N_1\alpha = N_3\alpha$$

$$G_dC_{K_2}N_1\alpha = N_4\alpha \tag{55}$$

Following algebraic manipulations, similar to those outlined in Eqs. (13–17) for the damping refinement case, the eigenpair assignment problem reduces to the solution of

$$G_a W_{M_1} p = V_{M_1} p, \qquad G_a W_{M_2} p = V_{M_2} p, \qquad G_v W_{D_1} p = V_{D_1} p$$
$$G_v W_{D_2} p = V_{D_2} p, \qquad G_d W_{K_1} p = V_{K_1} p, \qquad G_d W_{K_2} p = V_{K_2} p \tag{56}$$

where $p = [\mathrm{Re}^T(\alpha)\ \mathrm{Im}^T(\alpha)]^T$. The matrices W_{M_1}, W_{M_2}, W_{D_1}, W_{D_2}, W_{K_1}, W_{K_2}, V_{M_1}, V_{M_2}, V_{D_1}, V_{D_2}, V_{K_1}, and V_{K_2} are formed from the imaginary and real parts of the matrices C_{M_2}, C_{D_2}, C_{K_2}, N_1, N_2, N_3, and N_4, similar to what was done for the damping refinement case in Eqs. (15) and (16). Note that Eq. (56) is a system of quadratic equations in the unknown variables, namely, the elements of the gain matrices G_a, G_v, and G_d and the coefficient vector p. The elements of the gain matrices should be constrained such that they are symmetric, and the solution of the system has to yield an eigenvector Υ for the whole system, which satisfies the partial eigenvector conditions of Eq. (18) for the kth eigensystem assignment. The assumption that the set of earlier assigned eigenvalues does not match the remaining eigenvalues of the system, either before or after the eigenpair assignment still holds here. Now, considering the eigenvalue problem of the whole system for the eigenvalue being assigned, one can write

$$\begin{bmatrix} X_{11} & X_{12} \\ 0 & X_{22} \end{bmatrix} \begin{Bmatrix} \varphi \\ \phi \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}, \qquad X_{11} = \lambda E_{11} - A_{11}$$
$$X_{12} = \lambda\left(E_{12} + B_{M_{21}} G_a C_{M_2}\right) - A_{12} + B_{D_{21}} G_v C_{D_2} + B_{K_{21}} G_a C_{K_2}$$
$$X_{22} = \lambda\left(E_{22} + B_{M_{22}} G_a C_{M_2}\right) - A_{22} + B_{D_{22}} G_v C_{D_2} + B_{K_{22}} G_a C_{K_2} \tag{57}$$

Note that

$$\Upsilon = \begin{Bmatrix} \varphi \\ \phi \end{Bmatrix}$$

Solving for φ in terms of ϕ, one obtains

$$\varphi = -X_{11}^{-1} X_{12} \phi \tag{58}$$

and by using Eq. (54), one has

$$\phi = N_1 \alpha$$
$$\varphi = -X_{11}^{-1}\left([\lambda E_{12} - A_{12}] N_1 + \lambda B_{M_{21}} N_2 + B_{D_{21}} N_3 + B_{K_{21}} N_4\right)\alpha \equiv Q\alpha \tag{59}$$

or

$$\Upsilon = \begin{bmatrix} Q \\ N_1 \end{bmatrix} \alpha \equiv S\alpha \tag{60}$$

Using Eq. (60) into Eq. (18) and expanding and separating the real and imaginary parts, one obtains an expression similar to the one for the damping case [see Eq. (24)]

$$Lp = q \tag{61}$$

where the matrix L and vector q have been defined in Eq. (25). The condition for the existence of symmetric gain matrices G_a, G_v, and G_d that satisfies Eq. (56) reduces to the existence of a vector p that satisfies

$$p^T\left(V_{M_1}^T W_{M_2} - V_{M_2}^T W_{M_1}\right) p \equiv p^T J_1 p = 0$$
$$p^T\left(V_{D_1}^T W_{D_2} - V_{D_2}^T W_{D_1}\right) p \equiv p^T J_2 p = 0$$
$$p^T\left(V_{K_1}^T W_{K_2} - V_{K_2}^T W_{K_1}\right) p \equiv p^T J_3 p = 0 \tag{62}$$

To summarize, the conditions for the placement of an eigenpair of the system to desired values, while partially assigning the corresponding eigenvectors to target values, reduces to computing a coefficient vector p that satisfies the three quadratic equations given by Eq. (62) and the linear system of equations represented by Eq. (61). This is very similar to the problem obtained for the damping refinement case, with the exception that instead of one quadratic equation we have three quadratic equations. Hence, the approach proposed for the damping case, which involved a combination of the solution of the linear system of equations along with standard Newton methods, may be used to solve for a feasible coefficient vector p. Once a coefficient vector p is obtained, the procedure to compute the gain matrices G_a, G_v, and G_d is straightforward and follows the treatment described for computing the gain matrix in the damping refinement case [see Eqs. (27–32)]. Once the gain matrices G_a, G_v, and G_d (or matrices $\hat{G}_{M_k}$, $\hat{G}_{D_k}$, and $\hat{G}_{K_k}$) are computed, the current refinements in the mass, damping, and stiffness matrices, represented by $\tilde{G}_{M_k}$, $\tilde{G}_{D_k}$, and $\tilde{G}_{K_k}$, are determined from Eq. (50), and the overall refinements are updated as

$$G_M \leftarrow G_M + \tilde{G}_{M_k}, \qquad G_D \leftarrow G_D + \tilde{G}_{D_k}, \qquad G_K \leftarrow G_K + \tilde{G}_{K_k} \tag{63}$$

The procedure outlined determines symmetric gain matrices G_M, G_D, and G_K that reassigns a desired subset of the eigenvalues of the model, along with partial mode shapes, from their baseline to target values. As described for the case of damping matrix refinements, the symmetry of the gain matrices does not necessarily guarantee that the refined (combined) model remains stable. Because, in most situations, the flexible system is open-loop stable, any refinements to the analytical model should be such to maintain that stability. One approach to this could be to use the design freedom in the solution vector p and impose constraints on eigenvalues of the refined mass, damping, and stiffness matrices. However, this could be cumbersome, particularly when the size of the system is large (thousands or hundreds of thousands of degrees of freedom). Another approach could be to use the freedom beyond eigensystem assignment to reduce some measures of the gain matrices G_M, G_D, and G_K that represent the refinement in the system matrices. This can be accomplished, for example, by imposing inequality constraints similar to those presented in Eq. (34) for each of the matrices.

Alternatively, one can require that the current gain matrices $\tilde{G}_{M_k}$, $\tilde{G}_{D_k}$, and $\tilde{G}_{K_k}$ be dissipative at every sequence. Although dissipativity requirement can be constraining, it will guarantee that the refined system remains stable. In other words, at every sequence, a pair of eigenvalues are assigned via a symmetric and dissipative gain matrices. Similar to the damping case, dissipativity of the gain matrices can be achieved through a set of 12 (4 per gain matrix) quadratic inequality constraints in the solution vector p. The form of the inequality constraints for each gain matrix is exactly the same as the ones given in Eq. (35), except that the appropriate coefficient matrices are used instead of matrices W_1, W_2, V_1, and V_2. Yet another approach could be to require that at each sequence of the eigensystem assignment procedure the overall gain matrices G_M, G_D, and G_K remain positive semidefinite, that is, the gain matrices are dissipative.

Finally, the conditions for accommodating sparsity requirements of each of the system matrices directly follows the development for the damping refinement. These conditions are expressed as quadratic equalities in the coefficient vector p, similar to those given in Eqs. (40) or (47).

Numerical Example

The approach for model refinement using eigensystem assignment has been applied to a finite element model of the phase II CEM, a testbed for control of flexible space structures at NASA Langley Research Center. Here, the proposed approach is used to refine the damping and stiffness matrices of the structure using simulated identified modal frequencies and damping ratios.

The phase II CEM structure consists of a 62-bay central truss (each bay is 10-in. long), along with two horizontal booms for suspension, a vertical laser, and a vertical reflector tower, as shown in Fig. 1. This structure has 10 modes with frequencies up to about 5 Hz and 95 modes with frequencies under 60 Hz. The first six modes are rigid-body modes, due to suspension of the structure from the laboratory ceiling, that have frequencies up to about 0.3 Hz. Eight control stations housing collocated and compatible sensors and actuators are located at the bays shown in Fig. 1. Air thrusters, providing linear forces, are available at these locations along the directions shown in Fig. 1. Linear velocities are assumed to be available at these locations along the same directions.

An eight-degree-of-freedom structural model, which includes the first eight modes of the structure, is obtained following dynamic condensation techniques and is used in this numerical example. A low inherent damping ratio of 0.1% has been assumed for the each of the eight modes. The nominal eigenvalues along with damping and frequencies are shown in Table 1. Assume that only modes 2 and 8 are to be considered for refinement and that the frequency of mode 2 is low by 10% and its damping ratio is low by almost 25%, and the frequency of mode 8 is high by 12% and its damping ratio is low by almost 10%. Moreover, assume that the mass matrix is perfectly known, such that no refinements in the matrix is required. However, it is desired to refine the damping and stiffness matrices, using the proposed eigensystem assignment technique, such that the frequencies and damping ratios of modes 2 and 8 of the refined system matches the identified values. No identified eigenvectors are included, that is, there is no need for partial eigenvector assignment.

Assume that there is uncertainty in the elements of the damping and stiffness matrices corresponding to degrees of freedom 1, 2, 7, and 8. The input/output influence matrices [see Eq. (3)] were then chosen as

$$L_K = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad L_D = 0.1 \times L_K \tag{64}$$

to provide freedom for the appropriate elements of damping and stiffness matrices to change.

Symmetric Refinements

The first objective was to decrease the natural frequency of the second mode by 10% and increase its damping ratio to 25% so that the first target pair of eigenvalues was $\lambda_{1,2} = -0.228 \pm 0.8841j$. First, symmetric position and rate gain matrices were sought to reassign this pair of eigenvalues. Following the procedure described in the preceding section, the gain matrices were computed from the solution of system of quadratic equation given in Eq. (62), except for no equations corresponding to mass matrix refinements. The quadratic equations were solved using the MATLAB® (Ref. 5) nonlinear equation solver routine FSOLVE, which uses a Levenberg–Marquardt method. The eigenvalues of the system, with the intermediate position and rate gain matrices in place, are provided in Table 2. Table 2 indicates that the complex-conjugate pair were successfully reassigned to desired values. However, the resulting refined system has an unstable pole on the real axis. This is to be expected because, as mentioned earlier, the symmetry of gain matrices does not typically guarantee the stability of the system.

For the second step, the damping ratio in the mode 8 of the system was to be increased to 10%, while its frequency was to be decreased by 12%, resulting in the second pair of desired eigenvalues to be $\lambda_{3,4} = -1.3149 \pm 13.0835j$. Following the sequential approach outlined, first the pair of complex-conjugate eigenvalues were placed on the top left partition of the real Schur form of the system using orthogonal transformations.[6] Then, the gain matrices were defined

Table 1 Nominal eigenvalues

Open-loop eigenvalues	Damping ratio	Frequency, rad/s
$-0.0008 \pm 0.8180j$	0.0010	0.8180
$-0.0008 \pm 0.8301j$	0.0010	0.8301
$-0.0009 \pm 0.8565j$	0.0010	0.8565
$-0.0011 \pm 1.1308j$	0.0010	1.1308
$-0.0011 \pm 1.1401j$	0.0010	1.1401
$-0.0019 \pm 1.9100j$	0.0010	1.9100
$-0.0107 \pm 10.7278j$	0.0010	10.7278
$-0.0149 \pm 14.9425j$	0.0010	14.9425

Table 2 Eigenvalues of refined system with intermediate gains

Closed-loop eigenvalues	Damping ratio	Frequency, rad/s
$-0.5018 \pm 0.6423j$	0.6156	0.8151
$-0.0068 \pm 0.8404j$	0.0081	0.8404
$-0.1148 \pm 0.8554j$	0.1330	0.8630
$-0.0068 \pm 0.8905j$	0.0076	0.8906
$-0.2283 \pm 0.8841j$	0.2500	0.9131
$-0.0222 \pm 10.6262j$	0.0021	10.6262
$-0.3956 \pm 14.4418j$	0.0274	14.4472
6.2235	−1.0000	6.2235
−19.3636	1.0000	19.3636

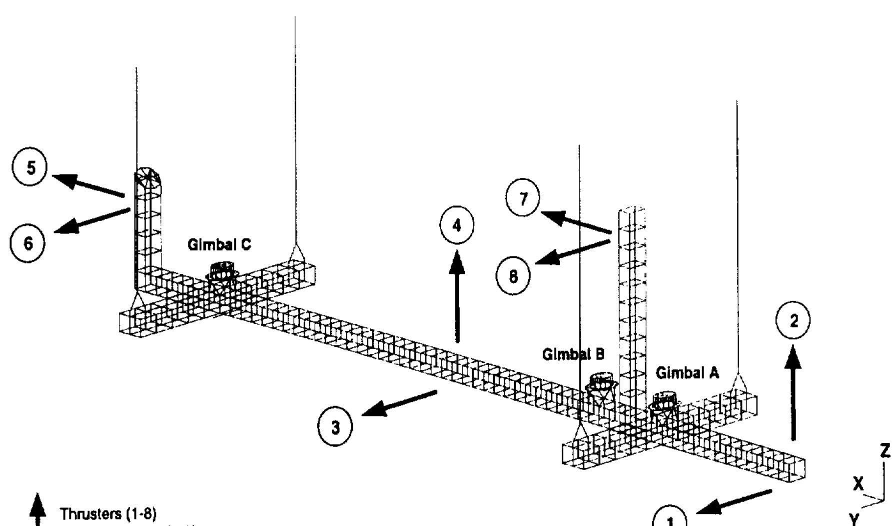

Fig. 1 Schematic of phase II CEM structure, with location of eight control stations.

Table 3 Eigenvalues of refined system

Closed-loop eigenvalues	Damping ratio	Frequency, rad/s
$-0.4955 \pm 0.5715j$	0.6551	0.7564
$-0.1219 \pm 0.8600j$	0.1403	0.8685
$-0.0054 \pm 0.8813j$	0.0062	0.8813
$-0.2283 \pm 0.8841j$	0.2500	0.9131
$-0.1458 \pm 4.9129j$	0.0297	4.9151
$-0.1640 \pm 12.8917j$	0.0127	12.8927
$-1.3149 \pm 13.0835j$	0.1000	13.1494
$-5.3957 \pm 15.7620j$	0.3239	16.6600

such that the eigenpair remain unchanged [see Eq. (50)] while the new pair of eigenvalues were assigned. The cumulative position and rate gain matrices G_D and G_K, which assign the two pairs of complex conjugate eigenvalues, were computed to be

$$G_D = \begin{bmatrix} 30.367 & 78.615 & 68.806 & 24.593 \\ 78.615 & 369.249 & 220.096 & 63.317 \\ 68.806 & 220.096 & 233.235 & 94.726 \\ 24.593 & 63.317 & 94.726 & 40.439 \end{bmatrix}$$

$$G_K = \begin{bmatrix} 134.819 & -26.937 & -27.664 & -81.395 \\ -26.937 & 15.249 & 23.763 & 9.922 \\ -27.664 & 23.763 & 38.679 & 6.733 \\ -81.395 & 9.922 & 6.733 & 36.126 \end{bmatrix} \tag{65}$$

The eigenvalues of the system, with the position and rate gain matrices in place, are provided in Table 3, where it is observed that the two pairs of eigenvalues had been successfully assigned to the identified values. Furthermore, the resulting refined system is stable, although there were no measures imposed to guarantee such stability. Also, note that the remaining eigenvalues (those that were not reassigned) have changed, some significantly. One could make some of those eigenvalues invariant during refinement by allowing more elements of the damping and stiffness matrices to change. The refinements in the damping and stiffness matrices are computed from Eq. (3). From these equations, the refinements in the damping and stiffness matrices, namely, ΔD and ΔK, are of the same order as the matrices themselves. However, because of the structures of the assumed L_D and L_K, only the elements corresponding to degrees of freedom 1, 2, 7, and 8 are nonzero, and are given as

$$\Delta D_1 = 0.01 \times G_D, \qquad \Delta K_1 = G_K \tag{66}$$

where G_D and G_K are given in Eq. (65).

Dissipative Refinements

As mentioned earlier, there are a number of ways of guaranteeing that the refined system remains stable. One of the proposed approaches was to take advantage of the freedom beyond eigensystem assignment and to determine the solution vector $\boldsymbol{p}$ such that the gain matrices, representing the refinements in the model, are dissipative. In the second example, the same model refinement problem as in the first case was considered with the exception that position and rate gain matrices were constrained to be dissipative. The position and rate gain matrices were computed from the solution of the system of quadratic equalities, given by Eq. (62), and quadratic inequalities, given by Eq. (35), except for no equations corresponding to mass matrix refinements. The gain matrices were determined to reassign the eigenvalues of the second mode to its target values at $\lambda_{1,2} = -0.228 \pm 0.8841j$. The system of quadratic equalities and inequalities were posed in the form of a minimax problem and was solved using MATLAB's minimax solver routine MINIMAX. The eigenvalues of the system, with the intermediate position and rate gain matrices in place, are provided in Table 4. Table 4 indicates that the complex-conjugate pair were successfully reassigned to desired values. The remaining eigenvalues were all stable, that is, the resulting refined system was stable. This is to be expected because the dissipative nature of the gain matrices guarantees the stability of the system. Next, the second pair of eigenvalues was reassigned to $\lambda_{3,4} = -1.3149 \pm 13.0835j$. Following the sequential approach outlined, first the pair of complex-conjugate eigenvalues were placed on the top left partition of the real Schur form of the system using orthogonal transformations.[6] Then, the gain matrices were defined such that the first eigenpair remain unchanged [see Eq. (50)] while the new pair of eigenvalues were assigned. The cumulative position and rate gain matrices G_D and G_K, which assign the two pairs of complex-conjugate eigenvalues, were computed to be

Table 4 Eigenvalues of refined system with intermediate gains

Closed-loop eigenvalues	Damping ratio	Frequency, rad/s
$-0.0025 \pm 0.8346j$	0.0029	0.8346
$-0.0137 \pm 0.8636j$	0.0158	0.8637
$-0.2283 \pm 0.8841j$	0.2500	0.9131
$-0.2313 \pm 1.1556j$	0.2002	1.1322
$-0.0127 \pm 1.9112j$	0.0066	1.9113
$-2.3693 \pm 4.8741j$	0.4372	5.4195
$-0.0633 \pm 10.8268j$	0.0058	10.8270
$-0.3471 \pm 14.8576j$	0.0234	14.8616

Table 5 Eigenvalues of refined system

Closed-loop eigenvalues	Damping ratio	Frequency, rad/s
$-0.0023 \pm 0.8454j$	0.0027	0.8454
$-0.0094 \pm 0.8707j$	0.0108	0.8707
$-0.2283 \pm 0.8841j$	0.2500	0.9131
$-0.3734 \pm 1.1556j$	0.2943	1.2686
$-0.1416 \pm 6.3839j$	0.0222	6.3855
$-1.3149 \pm 13.0835j$	0.1000	13.1494
$-0.9821 \pm 13.2494j$	0.0739	13.2857
$-0.7522 \pm 14.4382j$	0.0520	14.4577

$$G_D = \begin{bmatrix} 77.920 & 27.566 & 24.411 & -60.682 \\ 27.566 & 46.150 & 22.419 & 12.532 \\ 24.411 & 22.419 & 63.469 & 6.453 \\ -60.682 & 12.532 & 6.453 & 257.219 \end{bmatrix}$$

$$G_K = \begin{bmatrix} 60.661 & -32.170 & 3.756 & -121.149 \\ -32.170 & 42.725 & -3.516 & -7.936 \\ 3.756 & -3.516 & 0.696 & -1.421 \\ -121.149 & -7.936 & -1.421 & 455.059 \end{bmatrix} \tag{67}$$

The eigenvalues of the system, with the position and rate gain matrices in place, are provided in Table 5, where it is observed that the two pairs of eigenvalues had been successfully assigned to the identified values. Furthermore, the resulting refined system is stable, as expected. Also, note that the remaining eigenvalues (those that were not reassigned) have changed, some significantly. Again, one could make some of those eigenvalues invariant during refinement by allowing more elements of the damping and stiffness matrices to change. The refinements in the damping and stiffness matrices are computed from Eq. (3) and are given in Eq. (66), with the gain matrices from Eq. (67).

Comparisons of the refinements in each example indicate that no conclusions can be made in regards to the direction or magnitude of the computed refinements. In these examples, the computed refinements in damping matrix for the second example are typically lower than those obtained for the first example. However, the situation is reversed for the refinements in the stiffness matrix. This may be attributed to the variability in the solutions of the minimax optimization algorithms as well as the nonlinear equation solvers, in the sense that they may converge to different solutions depending on the starting points. In these examples, the starting estimate for the solution vector $\boldsymbol{p}$ was randomly chosen, in each example. Conceivably, one could attempt to exploit the freedom beyond eigensystem assignment to minimize, in some sense, the norm of the gain matrices to minimize the effective refinement needed for partial model matching.

Conclusions

This paper presented a novel approach for the refinement of the dynamic model of flexible structures using an eigensystem assignment technique. The approach presumes that modal parameters, such as frequencies, damping ratios, and mode shapes (at sensor locations), have been identified for modes in the range of interest. The proposed approach models the possible refinements in the mass, damping, and stiffness matrices of the finite element model in the form of a constant gain feedback with acceleration, velocity, and displacement measurements, respectively. The freedom to change model parameters, as well as the relative degree of change desired in one parameter with respect to the rest, is embedded in the elements of the input and output influence matrices for the various measurements. Once the elements of the input and output influence matrices have been defined and fixed, the problem of model refinement reduces to obtaining position, velocity, and acceleration gain matrices, with appropriate symmetry (and/or dissipativity) and sparsity that reassign a desired subset of the eigenvalues of the model, along with partial mode shapes, from their baseline values to those obtained from system identification test data. Hence, the problem of mode refinement becomes a problem of eigensystem assignment with output feedback. The proposed procedure assigns one conjugate pair of eigenvalues at each step using symmetric (or symmetric and dissipative) output feedback gain matrices, while ensuring that the eigenvalues assigned in the earlier steps are not disturbed. Moreover, the procedure provides that original sparsity of the system matrices in nominal model is preserved. The advantages of the proposed approach are that 1) it provides a systematic and computationally tractable means for exact model refinement, that is, the refined model would match the identified values exactly, without dependence on a nonlinear optimizer, and 2) it characterizes the freedom beyond model refinement for possible exploitation, which is inherent in the elements of the input and output matrices, as well as the elements of the position, velocity, and acceleration gain matrices. This freedom may be exploited to minimize the sensitivity of the refined model, to minimize global or local changes to the system matrices, etc. A numerical example, involving finite element model refinement for a structural testbed at NASA Langley Research Center (CEM) was presented to demonstrate the feasibility of the proposed approach.

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References

[1]Joshi, S. M., *Control of Large Flexible Space Structures*, edited by M. Thoma and A. Wyner, Vol. 131, Lecture Notes in Control and Information Sciences, Springer–Verlag, Berlin, 1989.

[2]Chen, T. Y., and Wang, B. P., "Finite Element Model Refinement Using Modal Analysis Data," AIAA Paper 88-2363, 1988.

[3]Kosmatka, J. B., and Stiles, P. A., "Comparison of Model Refinement Techniques," AIAA Paper 89-1279, 1989.

[4]Maghami, P. G., and Gupta, S., "Design of Constant Gain Dissipative Controllers for Eigensystem Assignment in Passive Systems," *Journal of Guidance, Control, and Dynamics*, Vol. 10, No. 4, 1987, pp. 648–657.

[5]Demmel, J., and Kagstrom, B., "The Generalized Schur Decomposition of an Arbitrary Pencil $A-\lambda B$: Robust Software with Error Bounds and Applications," Univ. of Umea, Rept. UMINF—170.90, Inst. of Information Processing, Sweden, Jan. 1990.

[6]MATLAB Reference Guide—High-Performance Numeric Computation and Visualization Software, MathWorks, Natick, MA, 1993.