The Axisymmetric Pulsar Magnetosphere

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Abstract.
  We present the structure of the axisymmetric force–free magnetosphere of an aligned rotating magnetic dipole, in the case in which there exists a sufficiently large charge density (whose origin we do not question) to satisfy the ideal MHD condition, $\mathbf{E} \cdot \mathbf{B} = 0$, everywhere. The unique distribution of electric current along the open magnetic field lines which is required for the solution to be continuous and smooth is obtained numerically. We expect that our solution will be useful as the starting point for detailed studies of pulsar magnetospheres under more general conditions, namely when either the force-free and/or the ideal MHD condition $\mathbf{E} \cdot \mathbf{B} = 0$ are not valid in the entire magnetosphere. Based on our solution, we consider that the most likely positions of such an occurrence are the polar cap, the crossings of the zero space charge surface by open field lines, and the return current boundary, but *not* the light cylinder.

I INTRODUCTION

We present a summary of the solution of the exact structre of an axisymmetric pulsar magnetoshpere. The original work where this solution and the associated details are presented is [1]. The interested reader is encouraged to look at the details given in this work. The basic physics of this problem were given by [2], roughly thirty years ago. The discovery of radiation emission from radio to gamma-rays with the pulsar period has motivated the modification of the original GJ model to include features, such as charge gaps, which would lead to the acceleration of particles necessary to produce the observed radiation.

  The ubiquitous presence of high energy radiation from pulsars, in agreement with simple scaling laws [3] which make no particular demands on the magnetic field structure, has prompted a number of authors to suggest that the γ−ray emission results from an unavoidable violation of the assumption of strictly dissipation–free flow, which 'would lead to singularities in both the flow and the magnetic field, occuring a short distance beyond the light–cylinder' [4]. Based on the conviction that such singularities might simply reflect the shortcomings of our numerical methods

and not the physical path nature chooses, we have decided to investigate the issue ourselves.

II THE PULSAR EQUATION

We work in ideal MHD conditions, i.e. $\mathbf{E} \cdot \mathbf{B} = 0$ over length scales much larger than the size of the various 'gaps', and that the main stresses in the magnetosphere are magnetic and electric and a valid magnetospheric model will be one where

$$\frac{1}{c}\mathbf{J} \times \mathbf{B} + \rho_e \mathbf{E} = 0 \ . \tag{1}$$

Here, $\mathbf{J}$, $\mathbf{B}$, and $\mathbf{E}$ are the electric current density, magnetic and electric fields respectively and $\rho_e = \nabla \cdot \mathbf{E}/(4\pi)$.

A convenient approach in steady–state axisymmetric MHD is to work with the flux function Ψ defined through

$$\mathbf{B_p} = \frac{\nabla\Psi \times \hat{\phi}}{R} \ , \tag{2}$$

where, $\mathbf{B_p}$ is the poloidal (R, Z) component of the magnetic field in a cylindrical coordinate system (R, ϕ, Z). Magnetic field lines lie along magnetic flux surfaces of constant Ψ. At each point, Ψ is proportional to the total magnetic flux contained inside that point; it is also related to the ϕ component of the vector potential. Ideal force–free MHD requires that

$$B_\phi = \frac{A(\Psi)}{R} \ , \tag{3}$$

where $A(\Psi)$ is a yet to be determined function. The poloidal electric current $I \equiv cA/2$ is also a function of Ψ, which means that poloidal electric currents are required to flow along (and not across) flux surfaces. Finally, the electric field is given by

$$\mathbf{E} = \frac{R\Omega}{c}\mathbf{B_p} \times \hat{\phi} \ , \tag{4}$$

and is clearly perpendicular to $\mathbf{B}$. Ω is the angular velocity of rotation of the neutron star on to which the magnetosphere is anchored, and can directly be thought of as the angular velocity of *rigid* rotation of the magnetic field lines (*not* of the magnetospheric plasma!). Eq. (1) can now be written in the equivalent form

$$(1 - x^2)\left(\frac{\partial^2\Psi}{\partial x^2} - \frac{1}{x}\frac{\partial\Psi}{\partial x} + \frac{\partial^2\Psi}{\partial z^2}\right) - 2x\frac{\partial\Psi}{\partial x} = -R_{LC}^2 AA' \ , \tag{5}$$

where we have introduced the convenient notation $x \equiv R/R_{LC}$ and $z \equiv Z/R_{LC}$, with $R_{LC} \equiv c/\Omega$ the distance from the axis where a particle corotating with the star

would rotate at the speed of light (the so called 'light cylinder'); and $(\ldots)' \equiv \mathrm{d}/\mathrm{d}\Psi$. Eq. (5) is the well known pulsar equation [5]. Solutions to this equation have been found for specific functional forms of the current distribution $A(\Psi)$ in particular for $A = \text{const.}$ and $A = -2\Psi$ for which this equation becomes linear and the usual techniques can be applied to derive the form of the field geometry for $x \leq 1$, as well as for a quadratic form of $A(\Psi)$ corresponding to a magnetic monopole solution [5,6].

Eq. (5) is elliptic, and according to the theory of elliptic equations (albeit the linear ones), the solution at all x and z is uniquely determined when one specifies the values of either Ψ (Dirichlet boundary conditions) or the normal derivative of Ψ (Neumann boundary conditions) along the boundaries, i.e. along the axis $x = 0$, the equatorial plane $z = 0$, and infinity (as one expects the boundary conditions at infinity will not affect the solution near the origin and the light cylinder). Unfortunately, this procedure does not work since eq. (5) is singular at the position of the light cylinder $x = 1$. The singularity at $x = 1$ imposes the important constraint that

$$\frac{\partial \Psi}{\partial x} = \frac{1}{2}AA' , \qquad\qquad (6)$$

at all points along the light cylinder, and as a result, not all distributions of electric current along the open field lines $A = A(\Psi)$ will lead to solutions which cross the light cylinder without kinks or discontinuities. In fact, *there exists a unique (?) distribution $A = A(\Psi)$ which allows for the continuous and smooth crossing of the light cylinder.*

One sees directly that eq. (6) has precisely the form of a boundary condition along the light cylinder which allows for the solution of eq. (5) *inside* and *outside* the light cylinder. In other words, eq. (6) determines the normal derivative of Ψ along the light cylinder, when $A = A(\Psi)$ is known, which can be used to solve the original elliptic equation both inside and outside $x = 1$. The inside solution will yield the distribution of $\Psi(x = 1^-, z)$, the outside solution the distribution of $\Psi(x = 1^+, z)$, and in general, $\Psi(x = 1^-, z)$ will not be equal to $\Psi(x = 1^+, z)$, unless of course one is 'lucky enough' to try *the* correct distribution of $A(\Psi)$. Several unsuccessful attempts to solve eq. (5) in all space, have concluded in favor of the 'inevitability of the break–down of continuity and smoothness' of these solutions.

Motivated by the fact that the singularity at the light cylinder is none other than the relativistic unique generalization of the familiar Alfvén point of the non-relativistic theory, we were more optimistic in that such a continuous and smooth solution actually exists.

We chose some (any) initial trial electric current distribution and solve the problem both inside and outside the light cylinder, and thus obtain the two distributions $\Psi(x = 1^-, z)$ and $\Psi(x = 1^+, z)$ along the light cylinder, which in general differ. We correct for the distribution of $A(\Psi)$ along field lines which cross the light cylinder by an average of the 'in' and 'out' values and we repeat the procedure until these 'in' and 'out' values of $AA'(\Psi)$ become identical.
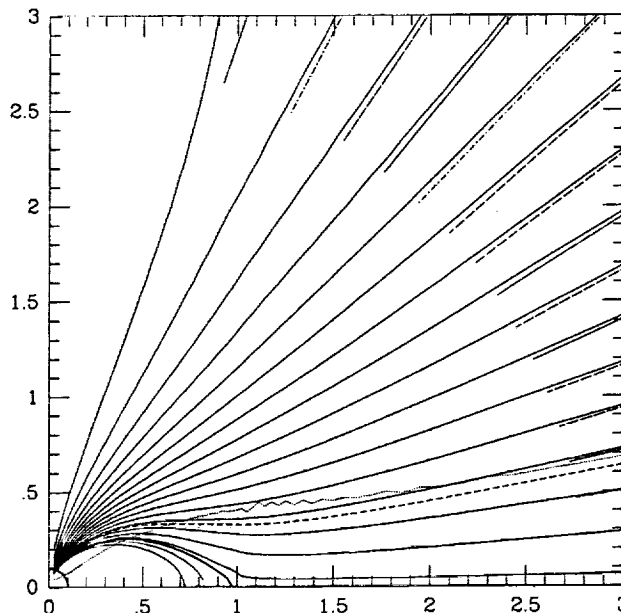
FIGURE 1. The final numerical solution for the structure of the axisymmetric force–free magnetosphere of an aligned rotating magnetic dipole. Thin lines represent flux surfaces in intervals of $0.1\Psi_{pc}$, with $\Psi = 0$ along the axis. A small amount to return current flows between the dashed field line $\Psi = 1.08\Psi_{pc}$ and the thick line at $\Psi_{open} = 1.36\Psi_{pc}$. The null line, along which $\rho_e = 0$, is shown dotted.

Figure 1 presents the results of the numerical solution outlined above. Nothing special happens at the light cylinder. The field lines cross this surface smoothly and so does the fluid. The dotted line is the zero charge surface of our solution. The sign of charge of the plasma necessary to short out the component of the electric field parallel to the magnetic field changes as one crosses this line. The point to note is that this (the zero charge) line crosses open field lines. Because plasma outflows only *along* the open field lines, the change in the charge sign implies the presence of charge sources along this dotted line. This source is likely to be due to pair production of high energy photons, or the acceleration of electrons which will result in the production of pairs by the same process. We therefore think it likely that this region of the magnetosphere will yield high energy pairs and possibly the observed γ–rays that seem to accompany the pulsar emission.

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