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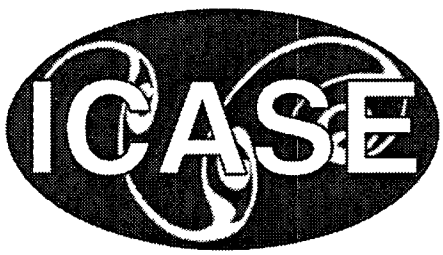

A Technique of Treating Negative Weights in WENO Schemes

Jing Shi, Changqing Hu, and Chi-Wang Shu
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December 2000

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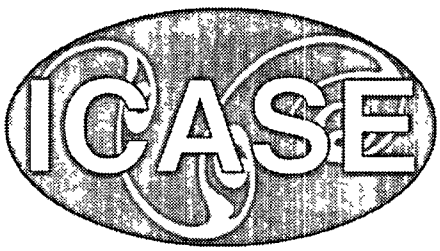

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A TECHNIQUE OF TREATING NEGATIVE WEIGHTS IN WENO SCHEMES*

JING SHI†, CHANGQING HU‡, AND CHI-WANG SHU§

Abstract. High order accurate weighted essentially non-oscillatory (WENO) schemes have recently been developed for finite difference and finite volume methods both in structured and in unstructured meshes A key idea in WENO scheme is a linear combination of lower order fluxes or reconstructions to obtain a higher order approximation The combination coefficients, also called linear weights, are determined by local geometry of the mesh and order of accuracy and may become negative WENO procedures cannot be applied directly to obtain a stable scheme if negative linear weights are present Previous strategy for handling this difficulty is by either regrouping of stencils or reducing the order of accuracy to get rid of the negative linear weights In this paper we present a simple and effective technique for handling negative linear weights without a need to get rid of them Test cases are shown to illustrate the stability and accuracy of this approach

Key words. weighted essentially non-oscillatory, negative weights, stability, high order accuracy, shock calculation

Subject classification. Applied and Numerical Mathematics

1. Introduction. High order accurate weighted essentially non-oscillatory (WENO) schemes have recently been developed to solve a hyperbolic conservation law

$$u_t + \nabla \quad f(u) = 0 \tag{1 1}$$

The first WENO scheme was constructed in [18] for a third order finite volume version in one space dimension In [10], third and fifth order finite difference WENO schemes in multi space dimensions are constructed, with a general framework for the design of the smoothness indicators and nonlinear weights Later, second, third and fourth order finite volume WENO schemes for 2D general triangulation have been developed in [4] and [8] Very high order finite difference WENO schemes (for orders between 7 and 13) have been developed in [1] Central WENO schemes have been developed in [12], [13] and [14]

WENO schemes are designed based on the successful ENO schemes in [7, 23, 24] Both ENO and WENO use the idea of adaptive stencils in the reconstruction procedure based on the local smoothness of the numerical solution to automatically achieve high order accuracy and non-oscillatory property near discontinuities ENO uses just one (optimal in some sense) out of many candidate stencils when doing the reconstruction, while WENO uses a convex combination of all the candidate stencils, each being assigned a nonlinear weight which depends on the local smoothness of the numerical solution based on that stencil WENO improves upon ENO in robustness, better smoothness of fluxes, better steady state convergence, better provable convergence properties, and more efficiency For a detailed review of ENO and WENO schemes, we refer to the lecture notes [21, 22]

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WENO schemes have already been widely used in applications Some of the examples include dynamical response of a stellar atmosphere to pressure perturbations [3], shock vortex interactions and other gas dynamics problems [5], [6], incompressible flow problems [26], Hamilton-Jacobi equations [9], magneto-hydrodynamics [11], underwater blast-wave focusing [15], the composite schemes and shallow water equations [16], [17], real gas computations [19], wave propagation using Fey's method of transport [20], etc

A key idea in WENO schemes is a linear combination of lower order fluxes or reconstructions to obtain a higher order approximation The combination coefficients, also called linear weights, are determined by local geometry of the mesh and order of accuracy and may become negative WENO procedures cannot be applied directly to obtain a stable scheme if negative linear weights are present Previous strategy for handling this difficulty is by either regrouping of stencils (e g in [8]) or reducing the order of accuracy (e g in [12]) to get rid of the negative linear weights In this paper we present a simple and effective technique for handling negative linear weights without a need to get rid of them Test cases will be shown to illustrate the stability and accuracy of this approach

We first summarize the general WENO reconstruction procedure, consisting of the following steps We assume we have a given cell Δ (which could be an interval in 1D, a rectangle in a 2D tensor product mesh, or a triangle in a 2D unstructured mesh) and a fixed point x^G within or on one edge of the cell

1 We identify several stencils $\mathcal{S}_j$, $j = 1, \quad , q$, such that Δ belongs to each stencil We denote by $\mathcal{T} = \bigcup_{j=1}^{q} \mathcal{S}_j$ the larger stencil which contains all the cells from the q stencils

2 We have a (relatively) lower order reconstruction or interpolation function (usually a polynomial), denoted by $p_j(x)$, associated with each of the stencils $\mathcal{S}_j$, for $j = 1, \quad , q$ We also have a (relatively) higher order reconstruction or interpolation function (again usually a polynomial), denoted by $Q(x)$, associated with the larger stencil $\mathcal{T}$

3 We find the combination coefficients, also called linear weights, denoted by $\gamma_1, \quad , \gamma_q$, such that

$$Q(x^G) = \sum_{j=1}^{q} \gamma_j p_j(x^G) \tag{1 2}$$

for all possible given data in the stencils These linear weights depend on the mesh geometry, the point x^G, and the specific reconstruction or interpolation requirements, but *not* on the given solution data in the stencils

4 We compute the smoothness indicator, denoted by β_j, for each stencil $\mathcal{S}_j$, which measures how smooth the function $p_j(x)$ is in the target cell Δ The smaller this smoothness indicator β_j, the smoother the function $p_j(x)$ is in the target cell In all of the current WENO schemes we are using the following smoothness indicator

$$\beta_j = \sum_{1 \le |\alpha| \le k} \int_{\Delta} |\Delta|^{|\alpha|-1} (D^\alpha p_j(x))^2 dx \tag{1 3}$$

for $j = 1, \quad , q$, where k is the degree of the polynomial $p_j(x)$ and $|\Delta|$ is the area of the cell Δ in 2D This factor is different for 1D or 3D the purpose of it is to bring the smoothness indicator invariant under spatial scaling

5 We compute the nonlinear weights based on the smoothness indicators

$$\omega_j = \frac{\tilde{\omega}_j}{\sum_j \tilde{\omega}_j}, \qquad \tilde{\omega}_j = \frac{\gamma_j}{(\varepsilon + \beta_j)^2} \tag{1 4}$$

where γ_j are the linear weights determined in step 3 above, and ε is a small number to avoid the denominator to become 0 We are using $\varepsilon = 10^{-6}$ in all the computations in this paper The final WENO approximation or reconstruction is then given by

$$R(x^G) = \sum_{j=1}^{q} \omega_j p_j(x^G) \tag{1 5}$$

We remark that all the coefficients in the above steps which depend on the mesh but not on the data of the numerical solution, should be computed and stored at the beginning of the code after the generation of the mesh but before the time evolution starts

We now use a simple example to illustrate the steps outlined above We assume we are given a uniform mesh $I_i = (x_{i-1/2}, x_{i+1/2})$ and cell averages of a function $u(x)$ in these cells, denoted by $\bar{u}_i$ We would like to find a fifth order WENO reconstruction to the point value $u(x_{i+1/2})$, based on a stencil of five cells $\{I_{i-2}, I_{i-1}, I_i, I_{i+1}, I_{i+2}\}$, with the target cell containing the point $x_{i+1/2}$ chosen as $\triangle = I_i$

In step 1 above we could have the following three stencils

$$\mathcal{S}_1 = \{I_{i-2}, I_{i-1}, I_i\}, \qquad \mathcal{S}_2 = \{I_{i-1}, I_i, I_{i+1}\}, \qquad \mathcal{S}_3 = \{I_i, I_{i+1}, I_{i+2}\},$$

which make up a larger stencil

$$\mathcal{T} = \{I_{i-2}, I_{i-1}, I_i, I_{i+1}, I_{i+2}\}$$

In step 2 above we would have three polynomials $p_j(x)$ of degree at most two, with their cell averages agreeing with that of the function u in the three cells in each stencil $\mathcal{S}_j$ The higher order function $Q(x)$ is a polynomial of degree at most four, with its cell averages agreeing with that of the function u in the five cells in the larger stencil $\mathcal{T}$ The three lower order approximations to $u(x_{i+1/2})$, associated with $p_j(x)$, in terms of the given cell averages of u, are given by

$$\begin{aligned} p_1(x_{i+1/2}) &= \frac{1}{3}\bar{u}_{i-2} - \frac{7}{6}\bar{u}_{i-1} + \frac{11}{6}\bar{u}_i, \\ p_2(x_{i+1/2}) &= -\frac{1}{6}\bar{u}_{i-1} + \frac{5}{6}\bar{u}_i + \frac{1}{3}\bar{u}_{i+1}, \\ p_3(x_{i+1/2}) &= \frac{1}{3}\bar{u}_i + \frac{5}{6}\bar{u}_{i+1} - \frac{1}{6}\bar{u}_{i+2} \end{aligned} \tag{1 6}$$

Each of them is a third order approximation to $u(x_{i+1/2})$ The higher order approximation to $u(x_{i+1/2})$, associated with $Q(x)$, is given by

$$Q(x_{i+1/2}) = \frac{1}{30}\bar{u}_{i-2} - \frac{13}{60}\bar{u}_{i-1} + \frac{47}{60}\bar{u}_i + \frac{9}{20}\bar{u}_{i+1} - \frac{1}{20}\bar{u}_{i+2}, \tag{1 7}$$

which is a fifth order approximation to $u(x_{i+1/2})$

In step 3 above we would have

$$\gamma_1 = \frac{1}{10}, \qquad \gamma_2 = \frac{3}{5}, \qquad \gamma_3 = \frac{3}{10}$$

It can be readily verified, using (1 6) and (1 7), that

$$Q(x_{i+1/2}) = \gamma_1\ p_1(x_{i+1/2}) + \gamma_2\ p_2(x_{i+1/2}) + \gamma_3\ p_3(x_{i+1/2})$$

for all possible given data $\bar{u}_j$, $j = i-2, i-1, i, i+1, i+2$

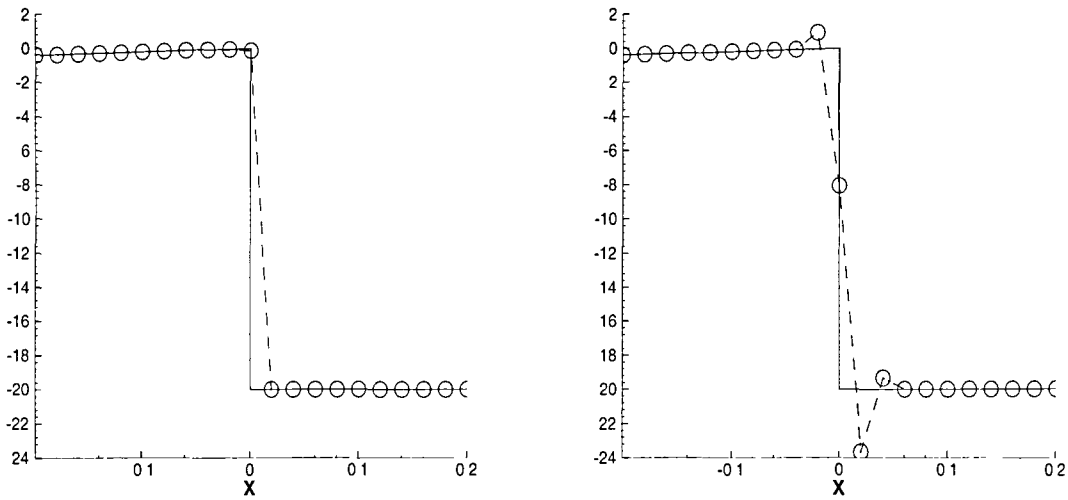

FIG 1 1 *Reconstructions to $u(x_{i+1/2})$ Solid lines exact function, symbols numerical approximations Left fifth order WENO Right fifth order traditional*

In step 4 above we could easily work out from (1 3) the three smoothness indicators given by

$$\beta_1 = \frac{13}{12}\left(\bar{u}_{i-2} - 2\bar{u}_{i-1} + \bar{u}_i\right)^2 + \frac{1}{4}\left(\bar{u}_{i-2} - 4\bar{u}_{i-1} + 3\bar{u}_i\right)^2,$$
$$\beta_2 = \frac{13}{12}\left(\bar{u}_{i-1} - 2\bar{u}_i + \bar{u}_{i+1}\right)^2 + \frac{1}{4}\left(\bar{u}_{i-1} - \bar{u}_{i+1}\right)^2,$$
$$\beta_3 = \frac{13}{12}\left(\bar{u}_i - 2\bar{u}_{i+1} + \bar{u}_{i+2}\right)^2 + \frac{1}{4}\left(3\bar{u}_i - 4\bar{u}_{i+1} + \bar{u}_{i+2}\right)^2$$

We notice in particular that the linear weights $\gamma_1, \gamma_2, \gamma_3$ in step 3 above are all positive In such cases, the WENO reconstruction procedure outlined above and the scheme based on it work very well In Fig 1 1 we plot the approximation to $u(x)$ for a discontinuous function $u(x) = 2x$ for $x \leq 0$ and $u(x) = -20$ otherwise, by the fifth order WENO reconstruction on the left and by the fifth order traditional reconstruction (1 7) on the right, with a mesh $x_i = (i - 0\ 4965)\Delta x$ with $\Delta x = 0\ 02$ We can clearly see that WENO avoids the over and undershoots near the discontinuity

We now look at another simple example where some of the linear weights in step 3 above would become negative We have exactly the same setting as above except now we seek the reconstruction not at the cell boundary but at the cell center x_i This is needed by the central schemes with staggered grids [12] Thus, step 1 would stay the same as above, step 2 would produce

$$\begin{aligned} p_1(x_i) &= -\frac{1}{24}\bar{u}_{i-2} + \frac{1}{12}\bar{u}_{i-1} + \frac{23}{24}\bar{u}_i, \\ p_2(x_i) &= -\frac{1}{24}\bar{u}_{i-1} + \frac{13}{12}\bar{u}_i - \frac{1}{24}\bar{u}_{i+1}, \\ p_3(x_i) &= \frac{23}{24}\bar{u}_i + \frac{1}{12}\bar{u}_{i+1} - \frac{1}{24}\bar{u}_{i+2} \end{aligned} \tag{1 8}$$

Each of them is a third order reconstruction to $u(x_i)$ The higher order reconstruction to $u(x_i)$, associated with $Q(x)$, is given by

$$Q(x_i) = \frac{3}{640}\bar{u}_{i-2} - \frac{29}{480}\bar{u}_{i-1} + \frac{1067}{960}\bar{u}_i - \frac{29}{480}\bar{u}_{i+1} + \frac{3}{640}\bar{u}_{i+2}, \tag{1 9}$$

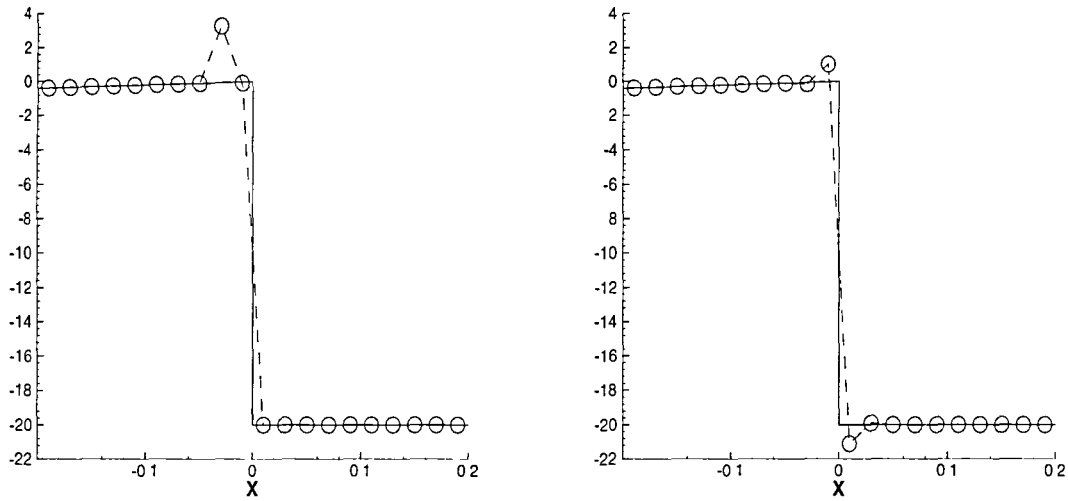

FIG 1 2 *Reconstructions to $u(x_i)$ Solid lines exact function, symbols numerical approximations Left fifth order WENO Right fifth order traditional*

which is a fifth order reconstruction to $u(x_i)$ Step 3 would produce the following weights

$$\gamma_1 = -\frac{9}{80}, \qquad \gamma_2 = \frac{49}{40}, \qquad \gamma_3 = -\frac{9}{80}$$

Notice that two of them are negative The smoothness indicators in step 4 will remain the same This time, the WENO approximation, shown at the left of Fig 1 2, is less satisfactory (in fact, even worse than a traditional fifth order reconstruction show on the right), because of the negative linear weights

We remark that negative linear weights do not appear in finite difference WENO schemes in any spatial dimensions for conservation laws for any order of accuracy [10], [1], and they do not appear in one dimensional as well as some multi-dimensional finite volume WENO schemes for conservation laws Unfortunately, they do appear in some other cases, such as the central WENO schemes using staggered meshes we have seen above, high order finite volume schemes for two dimensions described in [8] and in this paper, and finite difference WENO approximations for second derivatives

While on approximation alone the appearance of negative linear weights might be annoying but perhaps not fatal (Fig 1 2), in solving a PDE the result might be more serious As an example, in Fig 1 3 we show the results of using a fourth order finite volume WENO scheme [8] on a non-uniform triangular mesh shown at the left, which has negative linear weights, for solving the two dimensional Burgers equation

$$u_t + \left(\frac{u^2}{2}\right)_x + \left(\frac{u^2}{2}\right)_y = 0 \tag{1 10}$$

in the domain $[-2,2] \times [-2,2]$ with an initial condition $u_0(x,y) = 0\,3 + 0\,7 \sin\left(\frac{\pi}{2}(x+y)\right)$ and periodic boundary conditions We can see that serious oscillation appears in the numerical solution once the shock has developed The oscillation eventually leads to instability and blowing up of the numerical solution for this example

The main purpose of this paper is to develop a simple and effective technique for handling negative linear weights without a need to get rid of them Test cases will be shown to illustrate the stability and accuracy of this approach

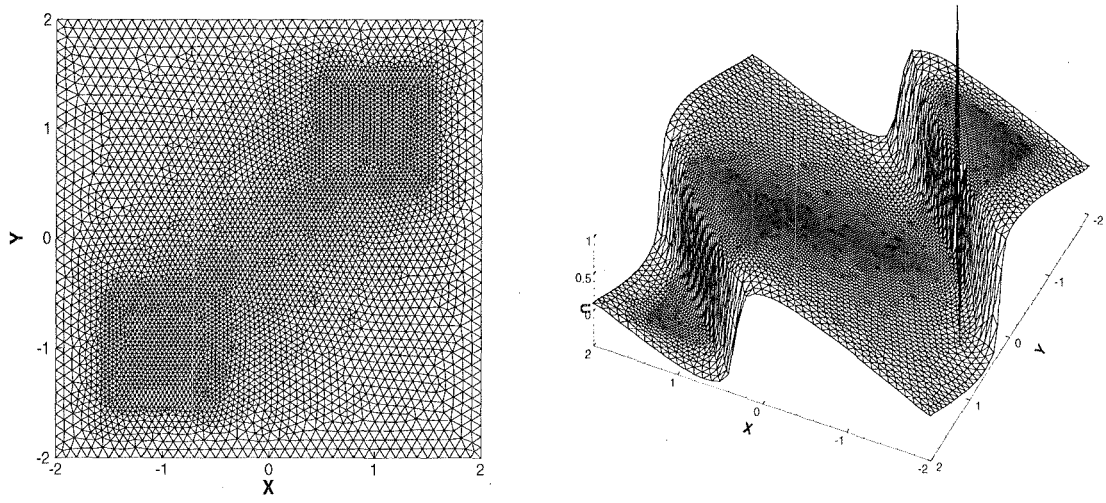

FIG. 1.3. *2D Burgers' equation. Left: non-uniform triangular mesh used in the computation. Right: fourth order WENO result at $t = 0.473$, cfl=0.2, without any special treatment for the negative linear weights.*

2. A splitting technique. We now introduce a splitting technique to treat the negative weights. It is very simple, involves little additional cost, yet is quite effective. The WENO procedure outlined in the previous section is only modified in step 5 in the following way:

5' If $min(\gamma_1, ..., \gamma_q) \geq 0$ proceed as before. Otherwise, we split the linear weights into two parts: positive and negative. Define

$$\tilde{\gamma}_i^+ = \frac{1}{2}\left(\gamma_i + \theta\,|\gamma_i|\right), \qquad \tilde{\gamma}_i^- = \tilde{\gamma}_i^+ - \gamma_i, \qquad i = 1, ..., q \tag{2.1}$$

where we take $\theta = 3$ all the numerical tests. We then scale them by

$$\sigma^{\pm} = \sum_{j=1}^{q} \tilde{\gamma}_j^{\pm}; \qquad \gamma_i^{\pm} = \tilde{\gamma}_i^{\pm}/\sigma^{\pm}, \qquad i = 1, ..., q. \tag{2.2}$$

We now have two split polynomials

$$Q^{\pm}(x^G) = \sum_{j=1}^{q} \gamma_j^{\pm} p_j(x^G) \tag{2.3}$$

which satisfy

$$Q(x^G) = \sigma^+ Q^+(x^G) - \sigma^- Q^-(x^G). \tag{2.4}$$

We can then define the nonlinear weights (1.4) for the positive and negative groups $\gamma_j^{\pm}$ separately, denoted by $\omega_j^{\pm}$, based on the same smoothness indicator β_j. We will then define the WENO approximation $R^{\pm}(x^G)$ separately by (1.5), using $\omega_j^{\pm}$, and form the final WENO approximation by

$$R(x^G) = \sigma^+ R^+(x^G) - \sigma^- R^-(x^G).$$

We remark that the key idea of this decomposition is to make sure that every stencil has a significant representation in both the positive and the negative weight groups. Within each group, the WENO idea of

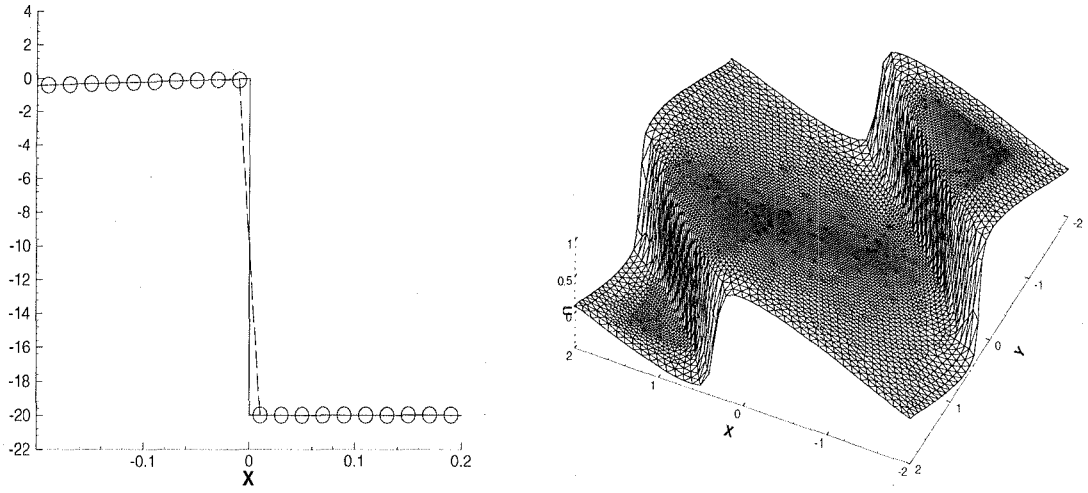

FIG. 2.1. *WENO approximations with the splitting treatment for negative linear weights. Left: approximation to* $u(x_i)$. *Right: Burgers equation, solution at* $t = 5/\pi^2$, *cfl=0.2.*

redistributing the weights subject to a fixed sum according to the smoothness of the approximation is still followed as before.

For the simple example of fifth order WENO reconstruction to $u(x_i)$, the split linear weights corresponding to (2.1) are, before the scaling,

$$\tilde{\gamma}_1^+ = \frac{9}{80},\ \tilde{\gamma}_1^- = \frac{9}{40}, \qquad \tilde{\gamma}_2^+ = \frac{49}{20},\ \tilde{\gamma}_2^- = \frac{49}{40}, \qquad \tilde{\gamma}_3^+ = \frac{9}{80},\ \tilde{\gamma}_3^- = \frac{9}{40}.$$

We notice that, as the most expensive part of the WENO procedure, namely the computation of the smoothness indicators (1.3), has not changed, the extra cost of this positive/negative weight splitting is very small.

However this simple and inexpensive change makes a big difference to the computations. In Fig. 2.1 we show the result of the two previous unsatisfactory cases, the fifth order WENO reconstruction to $u(x_i)$ in Fig. 1.2 left, and the approximation to the Burgers equation in Fig. 1.3 right, now using WENO schemes with this splitting treatment. We can see clearly that the results are now as good as one would get from WENO schemes having only positive linear weights.

It is easy to prove that the splitting maintains the accuracy of the approximation in smooth regions. We will demonstrate this fact in the following sections. We will also demonstrate the effectiveness of this simple splitting technique through a few selected numerical examples in the next sections. The main WENO schemes we will consider are fifth order finite volume WENO schemes on Cartesian meshes, and the third and fourth order finite volume WENO schemes on triangular meshes. In both cases negative linear weights appear regularly.

The calculations are performed on SUN Ultra workstations and also on the IBM SP parallel computer at TCASCV of Brown University. The parallel efficiency of the method is excellent (more than 90%).

3. 2D finite volume WENO schemes on Cartesian meshes.

3.1. The schemes. We describe two different ways to construct fifth order finite volume WENO schemes on Cartesian meshes. Comparing with finite difference WENO methods [10], finite volume meth-

ods have the advantage of an applicability of using arbitrary non-uniform meshes, at the price of increased computational cost [2]

We define the cell

$$I_{i,j} \equiv [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}] \times [y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}] \tag{3 1}$$

for $i = 1, \quad , m, j = 1, \quad , n$, Where $I_{i,j}$ needs not be uniform or smooth varying

The three-point Gaussian quadrature rule is used at each cell edge when evaluating the numerical flux in order to maintain fifth order accuracy Let (x^G, y^G) denote one of the Gaussian quadrature points at the cell boundary of $I_{i,j}$ given by $\Gamma \equiv \{x = x_{i-\frac{1}{2}}, y_{j-\frac{1}{2}} \leq y \leq y_{j+\frac{1}{2}}\}$ There are two ways to perform a WENO reconstruction at the point (x^G, y^G)

Genuine 2D The first WENO reconstruction is genuine 2D finite volume We can see that there are totally nine stencils $\mathcal{S}_{s,t}$ $(s,t = -1,0,1)$ Each stencil $\mathcal{S}_{s,t}$ contains 3×3 cells centered around $I_{i+s,j+t}$ On each stencil we can construct a Q^2 polynomial (tensor product of second order polynomials in x and y) satisfying the cell average condition (i e its cell average in each cell inside the stencil equals to the given value) Let $\mathcal{T} = \bigcup_{s,t=-1}^{1} \mathcal{S}_{s,t}$, which contains 5×5 cells centered around $I_{i,j}$ On $\mathcal{T}$ we can construct a Q^4 polynomial satisfying the cell average condition The WENO reconstruction is then performed according to the steps outlined in sections 1 and 2

We would like to make the following remarks

1 By using a Lagrange interpolation basis, we can easily find the unique linear weights
2 Even for a uniform mesh, a negative linear weight appears for the middle Gaussian point $(x^G, y^G) = (x_{i-\frac{1}{2}}, y_j)$ Such appearance of negative linear weights has also been observed in the central WENO schemes [12], see the example in sections 1 and 2 before
3 By Taylor expansions, we can prove that the smoothness indicators yield a uniform fifth order accuracy in smooth regions See [10] for the method of proof

Dimension by Dimension The second WENO reconstruction exploits the tensor product nature of the interpolation we use This WENO procedure is performed on a dimension by dimension fashion The WENO schemes applied in [5], [6] belong to this class Consider the point (x^G, y^G) as above First we perform a one dimensional WENO reconstruction in the y direction, in order to get the one dimensional cell averages (in the x direction) $w(\bullet, y^G)$ Then we perform another one dimensional WENO reconstruction to w in the x direction, to obtain the final reconstructed point value at (x^G, y^G)

We would like to make the following remarks

1 For a scalar equation, the underlying linear reconstructions of the above two versions are equivalent For nonlinear WENO reconstructions they are not equivalent Both of them are fifth order accurate but the actual errors on the same mesh may be different, see Table 3 1 below
2 For systems of conservation laws such as the Euler equations of gas dynamics, both versions of the WENO reconstruction should be performed in local characteristic fields
3 The dimension by dimension version of the WENO reconstruction is less expensive and requires smaller memory than the genuine two dimensional version The CPU time saving is about a factor of 4 for the Euler equations in our implementation The computed results are mostly similar from both versions

In the following, we will give numerical examples computed by the above WENO schemes Splitting technique has been used in all the computations when negative linear weights appear We will show the

TABLE 3 1
2D vortex evolution

		genuine FV		dim-by-dim	
N	Δx	L^∞ error	order	L^∞ error	order
20	6 71E-1	4 38E-2		5 26E-2	
40	3 77E-1	3 10E-3	4 59	5 66E-3	3 86
80	2 01E-1	1 20E-4	5 15	3 96E-4	4 22
160	1 00E-1	4 39E-6	4 76	7 96E-6	5 62
320	5 00E-2	1 88E-7	4 53	2 90E-7	4 77

results for both smooth and discontinuous problems

3.2. 2D vortex evolution. First, we check the accuracy of the WENO schemes constructed above The two dimensional vortex evolution problem [21], [8] is used as a test problem

We solve the Euler equations for compressible flow in 2D

$$U_t + f(U)_x + g(U)_y = 0, \tag{3 2}$$

where

$$U = (\rho, \rho u, \rho v, E)^T,$$

$$f(U) = (\rho u, \rho u^2 + p, \rho uv, u(E + p))^T,$$

$$g(U) = (\rho v, \rho uv, \rho v^2 + p, v(E + p))^T$$

Here ρ is the density, (u, v) is the velocity, E is the total energy, p is the pressure, related to the total energy by $E = \frac{p}{\gamma-1} + \frac{1}{2}\rho(u^2 + v^2)$ with $\gamma = 1\ 4$

The setup of the problem is the mean flow is $\rho = 1$, $p = 1$, $(u, v) = (1, 1)$ and the computational domain is $[0, 10] \times [0, 10]$ We add, to the mean flow, an isentropic vortex (perturbations in (u, v) and the temperature $T = \frac{p}{\rho}$, no perturbation in the entropy $S = \frac{p}{\rho^\gamma}$)

$$(\delta u, \delta v) = \frac{\epsilon}{2\pi} e^{0\ 5(1-r^2)}(-\overline{y}, \overline{x}), \qquad \delta T = -\frac{(\gamma - 1)\epsilon^2}{8\gamma\pi^2} e^{1-r^2}, \qquad \delta S = 0$$

where $(\overline{x}, \overline{y}) = (x - 5, y - 5)$, $r^2 = \overline{x}^2 + \overline{y}^2$, and the vortex strength $\epsilon = 5$

We use non-uniform meshes which are obtained by an independent random shifting of each point from a uniform mesh in each direction within 30% of the mesh sizes The solution is computed up to $t = 2$ Table 3 1 shows the L^∞ errors of ρ We can see that both the genuine two dimensional finite volume WENO scheme and the dimension by dimension finite volume WENO scheme can achieve the desired order of accuracy while the genuine two dimensional scheme gives smaller errors for the same mesh

3.3. Oblique shock tubes. The purpose for this test is to see the capability of the rectangular WENO schemes in resolving waves that are oblique to the computational meshes For details of the problem, we refer to [10] The 2D Sod's shock tube problem is solved where the initial jump makes an angle θ against the x axis We take our computational domain to be $[0, 6] \times [0, 1]$ and the initial jump starting at $(x, y) = (2\ 25, 0)$

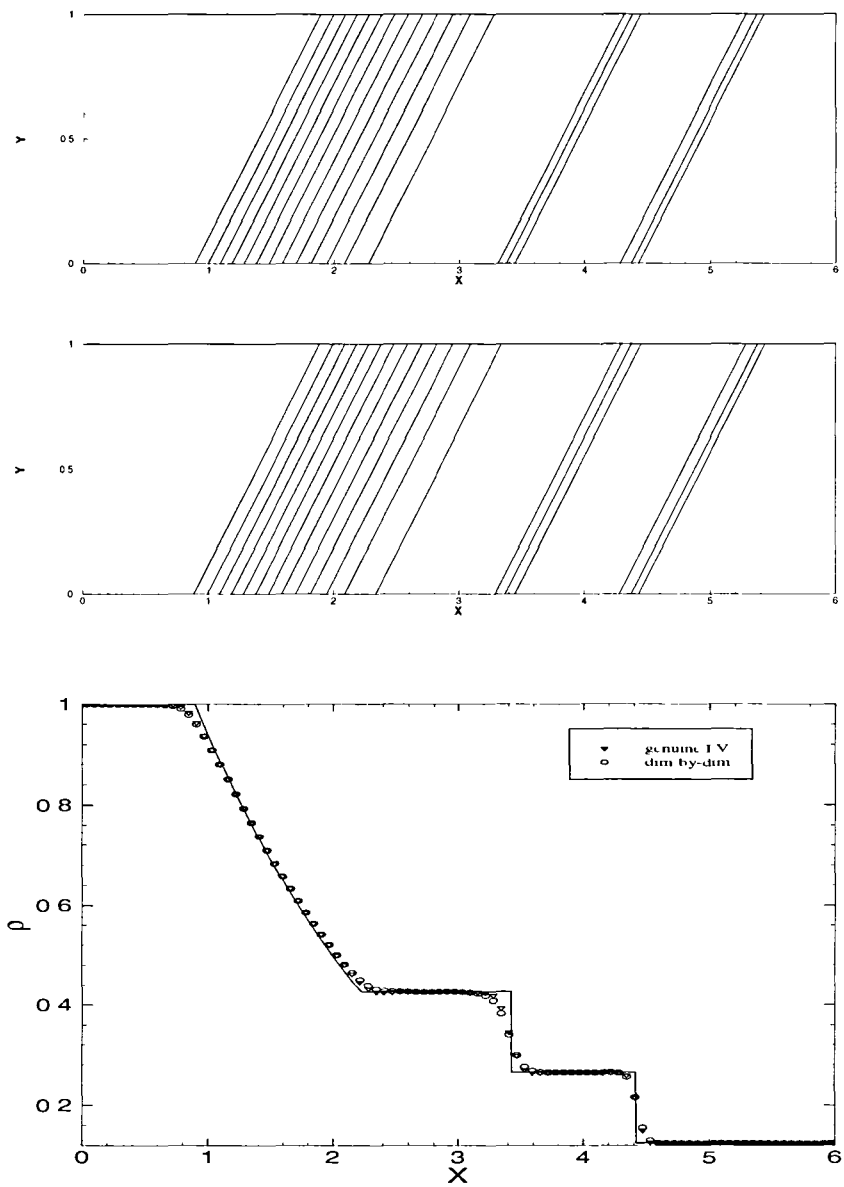

FIG 3 1 *Oblique Sod's problem Density ρ Top contour, genuine two dimensional WENO, middle contour, dimension by dimension WENO, bottom cut at the bottom of the computational domain, the solid line is the exact solution, the triangles are the genuine two dimensional WENO results, and the circles are the dimension by dimension WENO results*

and making a $\theta = \frac{\pi}{4}$ angle with the x axis The solution is computed up to $t = 1\,2$ on a 96×16 uniform mesh In Fig 3 1 we plot the density contours computed by the above two WENO schemes and the density cut at the bottom of the computational domain We can see that both schemes perform equally well in resolving the waves The genuine two dimensional scheme gives a slightly better resolution in the contact discontinuity and the rarefaction wave

3.4. A Mach 3 wind tunnel with a step. This model problem is originally from [25] The setup of the problem is The wind tunnel is 1 length unit wide and 3 length units long The step is 0 2 length

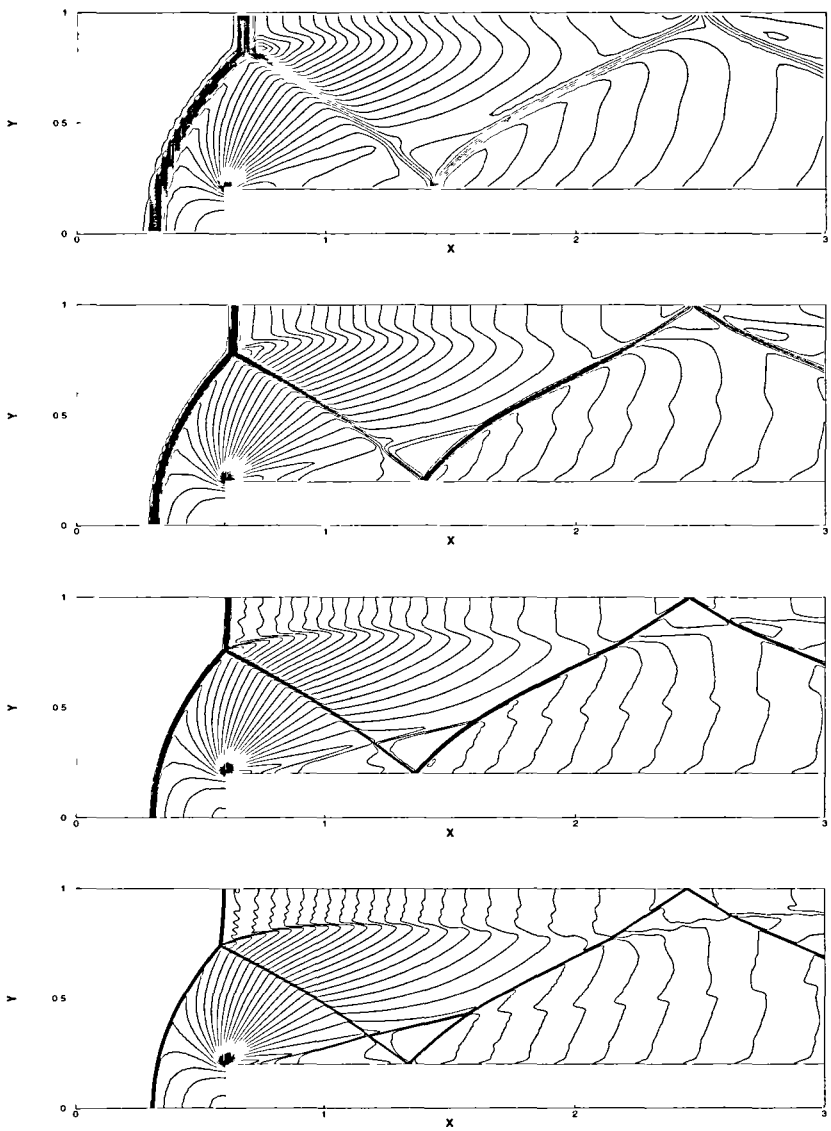

FIG 3 2 *Forward step problem,* $\Delta x = \Delta y = \frac{1}{40}, \frac{1}{80}, \frac{1}{160}, \frac{1}{320}$ *from top to bottom 30 contours from 0 12 to 6 41, dimension by dimension WENO*

units high and is located 0 6 length units from the left-hand end of the tunnel The problem is initialized by a right-going Mach 3 flow Reflective boundary conditions are applied along the wall of the tunnel and inflow/outflow boundary conditions are applied at the entrance/exit The corner of the step is a singular point and we treat it the same way as in [25], which is based on the assumption of a nearly steady flow in the region near the corner We show the density contours at time $t = 4$ in Fig 3 2 Only the results from the dimension by dimension WENO scheme are shown Uniform meshes of $\Delta x = \Delta y = \frac{1}{40}, \frac{1}{80}, \frac{1}{160}, \frac{1}{320}$ are used

3.5. Double Mach reflection. This problem is also originally from [25] The computational domain for this problem is chosen to be $[0,4] \times [0,1]$ The reflecting wall lies at the bottom, starting from $x = \frac{1}{6}$ Initially a right-moving Mach 10 shock is positioned at $x = \frac{1}{6}, y = 0$ and makes a 60° angle with the x axis

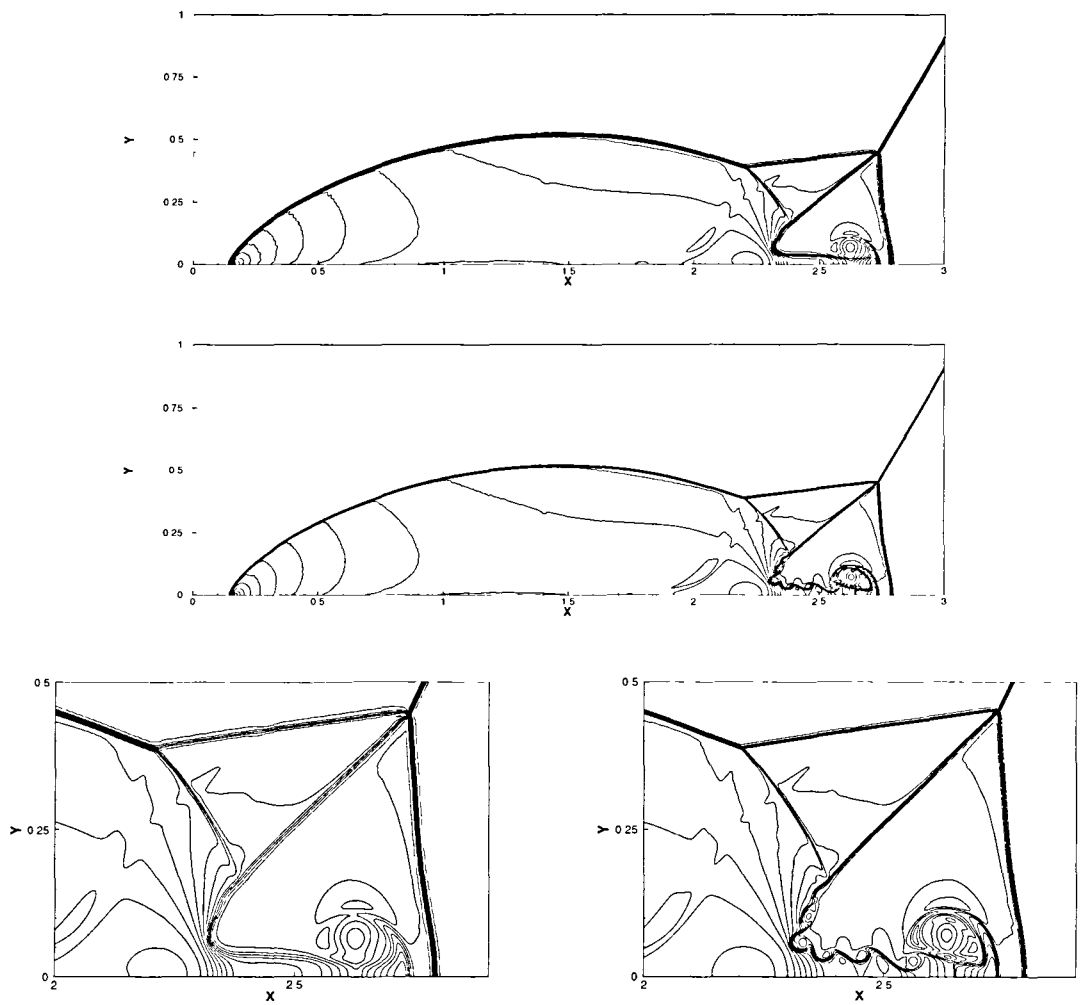

FIG 3 3 *Double Mach reflection,* $\Delta x = \Delta y = \frac{1}{240}$ *(top and lower left) and* $\frac{1}{480}$ *(middle and lower right) Genuine two dimensional WENO Blow-up regions at the bottom for details*

For the bottom boundary, the exact post-shock condition is imposed for the part from $x = 0$ to $x = \frac{1}{6}$ and a reflective boundary condition is used for the rest At the top boundary, the flow values are set to describe the exact motion of a Mach 10 shock We compute the solution up to $t = 0\,2$ Fig 3 3 and Fig 3 4 show the equally spaced 30 density contours from 1 5 to 22 7 computed by the genuine two dimensional and the dimension by dimension WENO schemes We use uniform meshes with $\Delta x = \Delta y = \frac{1}{240}, \frac{1}{480}$ We can see that the results from both schemes are comparable

4. 2D finite volume WENO schemes on triangular meshes. Both third and fourth order finite volume WENO schemes on triangular meshes have been constructed in [8] The optional linear weights in such schemes are not unique These are then chosen to avoid negative weights whenever possible, and if that fails, a grouping (of stencils) technique is used in [8], which works fairly well in the third order case with quite general triangulation but can yield positive weights for the fourth order case only with fairly uniform triangulation In this section, we do not seek positive linear weights as in [8], but rather use the splitting technique to treat the negative linear weights when they appear For scalar equation, the scheme is stable

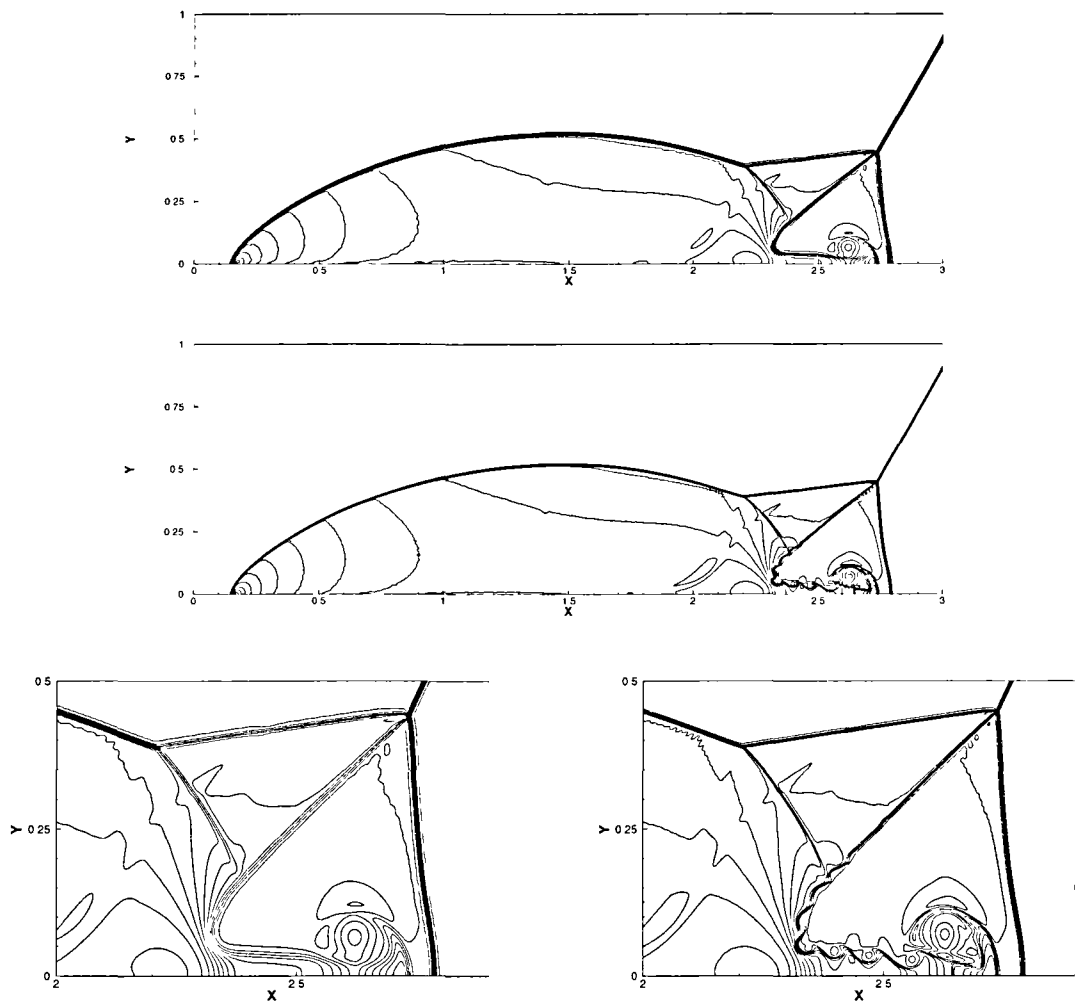

FIG 3 4 *Double Mach reflection, $\Delta x = \Delta y = \frac{1}{240}$ (top and lower left) and $\frac{1}{480}$ (middle and lower right) Dimension by dimension WENO Blow-up regions at the bottom for details*

in all runs For systems of conservation laws, there are still occasional cases of overshoot and instability, the reason seems to be related to characteristic decompositions and is still being investigated

4.1. Accuracy check for a smooth problem. We solve the 2D Burgers equation (1 10) with the same initial and boundary conditions as before using the fourth order finite volume WENO scheme [8] The solution is computed up to $t = \frac{0\,5}{\pi^2}$ when no shock has appeared The meshes used are 1) uniform meshes with equilateral triangulation and 2) random triangulation For the uniform meshes we do not seek positive weights as was done in [8], rather we use the splitting technique to treat the negative linear weights when they appear Table 4 1 indicates that close to fourth order accuracy can be achieved

4.2. Discontinuous problem 1: Scalar equation in 2D. Having shown the stable results with the splitting treatment of negative linear weights for a fourth order finite volume WENO scheme for the Burgers equation in section 2, we now test the fourth order WENO scheme on the Buckley-Leverett problem whose

TABLE 4 1
2D Burgers equation accuracy check

uniform mesh			nonuniform mesh		
Δx	L^∞ error	order	Δx	L^∞ error	order
2 57E-1	6 22E-4		2 67E-1	2 11E-3	
1 29E-1	4 61E-5	3 75	1 26E-1	2 35E-4	2 92
6 43E-2	2 18E-6	4 40	6 32E-2	2 90E-5	3 03
3 21E-2	1 38E-7	3 98	3 34E-2	2 61E-6	3 78
1 61E-2	6 93E-9	4 32	1 66E-2	2 71E-7	3 24
8 08E-3	6 70E-10	3 40	7 44E-3	1 57E-8	3 55

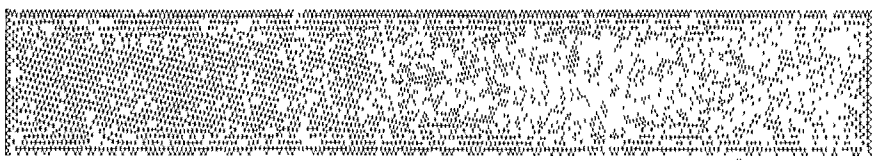

FIG 4 1 *2D Buckley-Leverett equation the mesh*

flux is non-convex

$$f(u) = \frac{u^2}{u^2 + 0\ 25(1-u)^2}, \qquad g(u) = 0$$

with the initial data $u = 1$ when $-\frac{1}{2} \le x \le 0$ and $u = 0$ elsewhere The solution is computed up to $t = 0\ 4$ The exact solution is a shock-rarefaction-contact discontinuity mixture The mesh we use here is a non-uniform triangulation, shown in Fig 4 1 Fig 4 2 shows that the waves have been resolved very well

4.3. Discontinuous problem 2: System of equations in 2D. We consider the 2D Euler equations in the domain $[-1, 1] \times [0, 0\ 2]$ The Sod and Lax shock tube initial data is set in the x direction and periodic boundary condition is applied in the y direction We use the fourth order finite volume scheme on triangular meshes to solve the above problem The mesh we use here is uniform But we do not seek positive weights as was done in [8], rather we use the splitting technique in section 2 to treat the negative linear weights when they appear In fact we set deliberately certain linear weights to be negative to test the splitting technique Fig 4 3 shows the numerical results of the Sod and Lax problems

It seems that there are still oscillations and instability for some non-uniform triangular meshes for the fourth order WENO schemes applied to Euler equations As the method works well for the same meshes with a scalar equation, the problem might be from the characteristic decompositions This is still under investigation

5. Concluding remarks. We have devised and tested a simple splitting technique to treat the negative linear weights in WENO schemes This technique involves very little additional CPU time and gives good results in most numerical tests The only case where it still yields oscillations and instability is when a fourth order finite volume WENO method is used on some non-uniform triangular meshes for Euler equations, the reason of which, presumably related to characteristic decompositions, is still under investigation

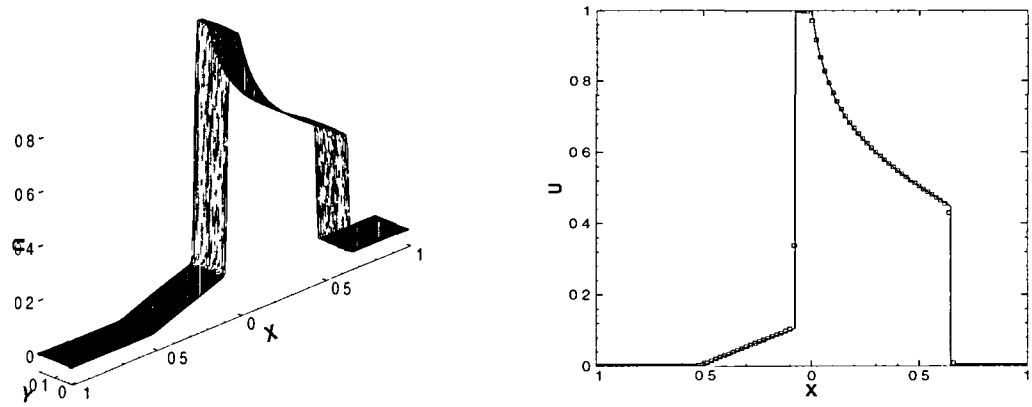

FIG 4 2 *2D Buckley-Leverett equation at t=0 4, with splitting Left the solution surface, Right the cut at y = 0 1 (solid line exact solution, symbols numerical solution)*

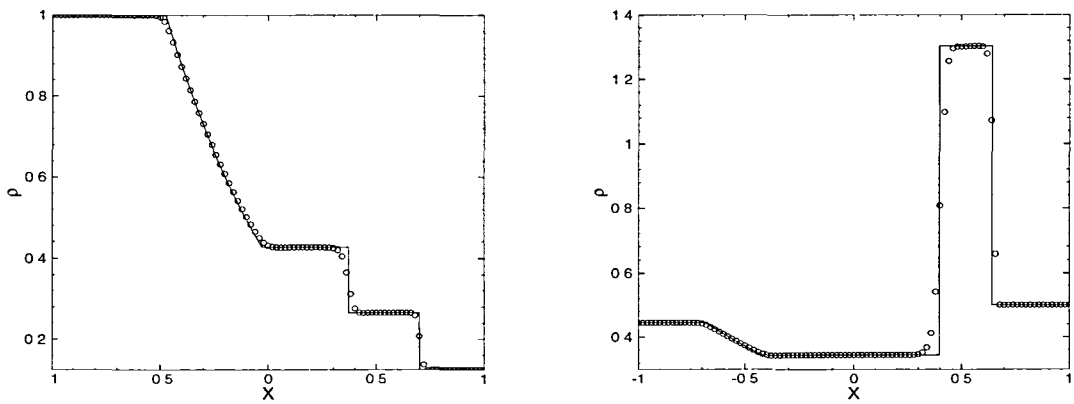

FIG 4 3 *Density plot, Left Sod problem, Right Lax problem, with splitting Roughly 100 points in the x direction*

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13 ABSTRACT *(Maximum 200 words)*

High order accurate weighted essentially non-oscillatory (WENO) schemes have recently been developed for finite difference and finite volume methods both in structured and in unstructured meshes A key idea in WENO scheme is a linear combination of lower order fluxes or reconstructions to obtain a higher order approximation The combination coefficients, also called linear weights, are determined by local geometry of the mesh and order of accuracy and may become negative WENO procedures cannot be applied directly to obtain a stable scheme if negative linear weights are present Previous strategy for handling this difficulty is by either regrouping of stencils or reducing the order of accuracy to get rid of the negative linear weights In this paper we present a simple and effective technique for handling negative linear weights without a need to get rid of them Test cases are shown to illustrate the stability and accuracy of this approach

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