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AERODYNAMIC INDICIAL FUNCTIONS AND THEIR USE IN AEROELASTIC FORMULATION OF LIFTING SURFACES

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ABSTRACT

An investigation related to the use of linear indicial functions in the time and frequency domains, enabling one to derive the proper aerodynamic loads as to study the subcritical response and flutter of swept lifting surfaces, respectively, of the open/closed loop aeroelastic system is presented. The expressions of the lift and aerodynamic moment in the frequency domain are given in terms of the Theodorsen's function, while, in the time domain, these are obtained directly with the help of the Wagner's function. Closed form solutions of aerodynamic derivatives are obtained, graphical representations are supplied and conclusions and prospects for further developments are outlined.

NOMENCLATURE

a_n Dimensionless elastic axis position measured from the midchord, positive aft
c_n Chord length of wing, normal to the elastic axis, $2b_n$
$C_{L\alpha_n}$ Lift-curve slope
f_h, f_α Plunging and pitching deflection functions
h, h_0 Plunging displacement and its amplitude, respectively
I_α Mass moment of inertia per unit length of wing
l Wing semi-span measured along the midchord line
l_h Dimensionless aerodynamic lift, $L_h b/mU_\infty^2$
L_b, l_b Overpressure signature of the N-wave shock pulse and its dimensionless counterpart, $L_b b/mU_\infty^2$, respectively
m Airfoil mass per unit length
m_α Dimensionless aerodynamic moment, $M_\alpha b^2/I_\alpha U_\infty^2$
N Load Factor, $h^{\bullet}/g$
$P_m, \wp_m$ Peak reflected pressure in excess and its dimensionless value $P_m b/mU_\infty^2$, respectively
r Shock pulse length factor
r_α Dimensionless radius of gyration, $(I_\alpha/mb^2)^{1/2}$
$s, \mathscr{L}$ Laplace transform variable and operator
S_α, χ_α Static unbalance about the elastic axis and its dimensionless counterpart, S_α/mb
t, τ_0 Time variables
U_∞, U_n Freestream speed and its component normal to the elastic axis
V Dimensionless free-stream speed, $U_\infty/b\omega_\alpha$
x Coordinate parallel to freestream direction
$\bar{x}$ Chordwise coordinate normal to the elastic axis
y Coordinate perpendicular in the freestreem direction
$\bar{y}$ Spanwise coordinate along the elastic axis
w Downwash velocity
z Transverse normal coordinate to the midplane of the wing
Z Vertical displacement in z direction
α, α_0 Twist angle about the pitch axis and its amplitude, respectively
ζ_h, ζ_α Structural damping ratio in plunging, $c_h/2m\omega_h$ and in pitching, $c_\alpha/2I_\alpha\omega_\alpha$
η Dimensionless coordinate along the wing span, $\bar{y}/l$
λ Spanwise rate of change of twist
Λ Swept angle (positive for swept back)
μ Airfoil/air mass ratio, $m/\pi\rho b^2$
ξ Dimensionless plunge coordinate, h/b

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ρ	Air density
σ	Spanwise rate of change of bending
τ	Dimensionless time, $U_n t/b_n$
τ_p	Dimensionless positive phase duration of the pulse, measured from the time of the arrival
ω, k_n	Circular and reduced frequencies, $\omega b_n / U_n$
ω_h, ω_α	Uncoupled frequency in plunging, $(K_h/m)^{1/2}$ and pitching, $(K_\alpha/I_\alpha)^{1/2}$
$\bar{\omega}$	Plunging-pitching frequency ratio, ω_h/ω_α

Subscript

$(\cdot)_c$	Circulatory terms of lift and aerodynamic moment
$(\cdot)_{nc}$	Non-circulatory terms of lift and aerodynamic moment
$(\cdot)_n$	Quantity normal to the elastic axis
$_\Lambda(\cdot)$	Quantity associated with the swept wing

Superscript

$(\hat{\cdot})$	Variables in Laplace transformed space
$(\dot{\cdot}), (\cdot)'$	Derivatives with respect to the time t, and the dimensionless time τ, respectively

INTRODUCTION

In this paper, the linear indicial functions in the time and frequency domains are used to determine the corresponding unsteady aerodynamic derivatives for swept lifting surfaces. Such a treatment of the problem enables one to approach either the open/closed loop aeroelastic response in the subcritical flight speed regime to arbitrary time-dependent external excitations (such as e.g. gusts, airblasts due to explosions or sonic-booms), or the flutter instability of actively controlled/uncontrolled swept wings. Moreover, this study is intended to extend its scope as to include, by an analogous procedure, the case of the various flight speed regimes, i.e. in addition to the incompressible one, also the compressible, transonic (within the linearized concept of indicial functions), and supersonic.

The representations of the aeroelastic governing equations in the time-domain is useful toward the approach of both the subcritical aeroelastic response of the actively controlled/uncontrolled swept aircraft wings. In other worlds, in this framework, the open/closed loops dynamic response of aeroelastic systems can be analyzed.

The unsteady aerodynamic lift and moment in incompressible flight speed regime are expressed for the swept aircraft wing in the time and the frequency domains by using the Wagner and Theodorsen functions, respectively. For the response of dynamic systems it is only necessary to express the lift and moment via the indicial Wagner's function. For the approach of the flutter problem, the Theodorsen's function helps the conversion of the expressions of both the aerodynamic loads and the unsteady aerodynamic derivatives in the frequency domain. Herein the case of a 2-D lifting surface, including the plunging and pitching degrees of freedoms is considered.

PRELIMINARIES

In the next developments an extensive use of variables in both the time and frequency domains will occur. As shown in Edwards, Ashley & Breakwell[3], for zero initial conditions, the aeroelastic equations can be converted from the time to the frequency domain via a Laplace transform. This results in the possibility of using the correspondence $s \to ik_n$, where s and k_n are the Laplace variable and the reduced frequency, respectively. The Laplace transform operator $\mathscr{L}$ is defined as:

$$\mathscr{L}(\cdot) = \int_0^\infty (\cdot) e^{-s\tau} d\tau . \qquad (1)$$

In this sense, the Wagner's function $\phi(\tau)$ is connected with Theodorsen's, function $C(k_n)$ via Laplace transform as:

$$\frac{C(k_n)}{ik_n} = \frac{F(k_n) + iG(k_n)}{ik_n} = \int_0^\infty \phi(\tau) e^{-ik_n\tau} d\tau = \Phi(ik_n), \qquad (2)$$

and vice-versa:

$$\phi(\tau) = \mathscr{L}^{-1}\{C(k_n)/ik_n\}, \qquad \mathrm{Re}(ik_n) > 0 . \qquad (3)$$

and having in view the correspondence $s \leftrightarrow ik_n$ we can also write:

$$\Phi(ik_n)\underset{ik_n \to s}{=} \int_0^\infty \phi(\tau) e^{-s\tau} d\tau = \Phi(s), \qquad (4)$$

Using this relationship, it is possible to obtain the full expression of unsteady aerodynamic coefficients in terms of the Theodorsen's function $C(k_n)$ and its circulatory components $F(k_n)$ and $G(k_n)$.

It is interesting to note that the reduced frequency parameter k_n for swept and for straight wings coincide:

$$k_n = \frac{\omega b_n}{U_n} = \frac{\omega b \cos\Lambda}{U \cos\Lambda} = \frac{\omega b}{U} = k\,, \qquad (5)$$

$$i\omega t = ik_n\tau\,. \qquad (6)$$

This implies that the indicial Wagner's function $\phi(\tau)$ remains invariant to any change of the sweep angle.

ANALYTICAL DEVELOPMENTS

For swept wings, the total lift per unit span, can be expressed in the form:

$$\begin{aligned} {}_\Lambda L_h(\bar{y},t) = {}_\Lambda L_c(\bar{y},t) + {}_\Lambda L_{nc1}(\bar{y},t) \\ + {}_\Lambda L_{nc2}(\bar{y},t) + {}_\Lambda L_{nc3}(\bar{y},t)\,. \end{aligned} \qquad (7)$$

where the indices c and nc identify the various contributions associated with the circulatory and noncirculatory terms respectively.

Using similar notations, the total moment per unit span about the elastic axis is:

$$\begin{aligned} {}_\Lambda M_\alpha(\bar{y},t) = {}_\Lambda M_c(\bar{y},t) + {}_\Lambda M_{nc1}(\bar{y},t) \\ + {}_\Lambda M_{nc2}(\bar{y},t) + {}_\Lambda M_{nc3}(\bar{y},t) + {}_\Lambda M_a(\bar{y},t), \end{aligned} \qquad (8)$$

${}_\Lambda M_a(\bar{y},t)$ being associated with the apparent moment of inertia. Herein the lift is positive in the negative z direction (considered positive downward), while the moment is positive nose up. For the sake of convenience, herein the plunging coordinate is positive when is downward (see Fig. 2).

Using the expression of the lift (Eq. (7)), the equation for the moment, Eq. (8), can be cast as:

$$\begin{aligned} {}_\Lambda M_\alpha(\bar{y},t) = -(1/2 + a_n)b_n\, {}_\Lambda L_c(\bar{y},t) - a_n b_n\, {}_\Lambda L_{nc1}(\bar{y},t) \\ + (1/2 - a_n)b_n\, {}_\Lambda L_{nc2}(\bar{y},t) + {}_\Lambda M_{nc3}(\bar{y},t) \\ + {}_\Lambda M_a(\bar{y},t). \end{aligned} \qquad (9)$$

Basic Considerations

Expressing the vertical displacement Z of a point on the center line of the wing as, (Fig.2):

$$Z(\bar{x},\bar{y},t) = h + \bar{x}\alpha\,, \qquad (10)$$

where $h \equiv h(\bar{y},t)$, $\alpha \equiv \alpha(\bar{y},t)$ are the displacements in plunging and pitching, respectively, and the origin of the $\bar{x}$ axis coincides with the elastic center, the downwash velocity w normal to the lifting surface becomes:

$$w(x,y,t) \equiv w(\bar{x},\bar{y},t) = \partial Z/\partial t + U_\infty\, \partial Z/\partial \bar{x}\,. \qquad (11)$$

In conjunction with the definition of Z, (Eq. 10), Eq. (11) becomes:

$$\begin{aligned} w(\bar{x},\bar{y},t) = \dot{h} + \bar{x}\dot{\alpha} + U_\infty \alpha \cos\Lambda \\ + U_\infty\left(\partial h/\partial\bar{y} + \underline{\bar{x}\,\partial\alpha/\partial\bar{y}}\right)\sin\Lambda\,. \end{aligned} \qquad (12)$$

where the superposed dots denote the derivatives with respect to time t. The quantity in Eq. (12) undescored by a solid line is usually discarded in the specialized literature (see Bisplinghff, Ashley & Halfman[2]), as being related with the wing camber effect (see also Flax[4]). However, herein this quantity will be included.

The in-plane coordinate $\bar{x}$ normal to the elastic axis (see Fig. 1) can be expressed as:

$$\bar{x} = b_n(1/2 - a_n)\,. \qquad (13)$$

Consequently, using the dimensionless time $\tau (\equiv U_n t/b_n)$ Eq. (12) becomes:

$$\begin{aligned} w(\bar{x},\bar{y},\tau) = U_n\left(\frac{h'}{b_n} + \alpha + \frac{\partial h}{\partial\bar{y}}\tan\Lambda \right.\\ \left. + (1/2 - a_n)\left(\alpha' + b_n\frac{\partial\alpha}{\partial\bar{y}}\tan\Lambda\right)\right), \end{aligned} \qquad (14)$$

where $(\cdot)' \equiv \partial(\cdot)/\partial\tau$.

In the following sections, the unsteady aerodynamic loads in incompressible flow can be obtained in time and, with the use of the Laplace transform space, in the frequency domain.

Unsteady Aerodynamic Loads in Incompressible Flow

Time Domain

The circulatory components of the lift and moment expressed in term of Wagner's indicial function $\phi(\tau)$ (called also *heredity* function) obtained in the time domain are:

$$_{\Lambda}L_c(\bar{y},\tau)=-C_{L\alpha_n}b_n\rho U_n\int_{-\infty}^{\tau}\phi(\tau-\tau_0)\frac{\partial w(\bar{x},\bar{y},\tau_0)}{\partial \tau_0}d\tau_0\,, \tag{15}$$

$$_{\Lambda}M_c(\bar{y},\tau)=\left(\frac{1}{2}+a_n\right)C_{L\alpha_n}b_n^2\rho U_n^2 \int_{-\infty}^{\tau}\phi(\tau-\tau_0)\frac{\partial w(\bar{x},\bar{y},\tau_0)}{\partial \tau_0}d\tau_0. \tag{16}$$

As concern the aerodynamic non-circulatory components, using the dimensionless time τ these are expressed as:

$$_{\Lambda}L_{nc1}(\bar{y},\tau)=-\frac{1}{2}C_{L\alpha_n}\rho U_n^2\left[h''-a_nb_n\alpha''\right], \tag{17 a}$$

$$_{\Lambda}L_{nc2}(\bar{y},\tau)=-\frac{1}{2}C_{L\alpha_n}\rho U_n^2b_n\alpha', \tag{17 b}$$

$$_{\Lambda}L_{nc3}(\bar{y},\tau)=-\frac{1}{2}C_{L\alpha_n}\rho U_n^2b_n^2\tan\Lambda\left[(\delta_r+1)\frac{\sigma'}{b_n}+\delta_r\lambda +\delta_r\frac{\partial\sigma}{\partial\bar{y}}\tan\Lambda\right]+\frac{1}{2}C_{L\alpha_n}\rho U_n^2b_n^3\tan\Lambda\left[(\delta_r+1)\frac{\lambda'}{b_n} +\delta_r\frac{\partial\lambda}{\partial\bar{y}}\tan\Lambda\right]. \tag{17 c}$$

The moments induced by the non-circulatory components of lift are expressed using the dimensionless time as:

$$_{\Lambda}M_{nc1}(\bar{y},\tau)=\frac{1}{2}C_{L\alpha_n}\rho U_n^2\left[h''-a_nb_n\alpha''\right]a_nb_n, \tag{18 a}$$

$$_{\Lambda}M_{nc2}(\bar{y},\tau)=-\frac{1}{2}C_{L\alpha_n}\rho U_n^2b_n^2\left(\frac{1}{2}-a_n\right)\alpha', \tag{18 b}$$

$$_{\Lambda}M_{nc3}(\bar{y},\tau)=-\frac{1}{2}C_{L\alpha_n}\rho U_n^2b_n^3\frac{1}{2}\lambda\tan\Lambda +\frac{1}{2}C_{L\alpha_n}\rho U_n^2b_n^3a_n\tan\Lambda\left[(\delta_r+1)\frac{\sigma'}{b_n}+\delta_r\lambda +\delta_r\frac{\partial\sigma}{\partial\bar{y}}\tan\Lambda\right]-\frac{1}{2}C_{L\alpha}\rho U_n^2b_n^4\left(\frac{1}{8}+a_n^2\right) \tan\Lambda\left[(\delta_r+1)\frac{\lambda'}{b_n}+\delta_r\frac{\partial\lambda}{\partial\bar{y}}\tan\Lambda\right], \tag{18 c}$$

$$_{\Lambda}M_a(\bar{y},\tau)=-\frac{1}{16}\rho C_{L\alpha_n}b_n^2U_n^2\alpha''. \tag{18 d}$$

where the spanwise rates of change of bending and twist, σ and λ, respectively, are expressed as $\sigma=\partial h/\partial\bar{y}$ and $\lambda=\partial\alpha/\partial\bar{y}$. In these equations as well as in the following ones, the terms affected by the tracer δ_r identify those generated by the last term in the expression of the downwash velocity (Eq. (12)), (term underscored by a solid line). In general these terms are discarded being considered negligibly small (see Bisplinghoff et al.[2], and Flax[4]), in what case $\delta_r=0$, otherwise $\delta_r=1$. Replacement of Eqs. (15) and (17) in Eq. (7) and of Eqs. (16) and (18) into Eq. (8), results in the unsteady lift and aerodynamic moment expressed in the time domain. Concerning the circulatory parts of the lift and aerodynamic moment, Eqs. (15) and (16), these can be explicitly determined by transforming these expressions in the Laplace domain using the relationship between the Laplace transform of the Wagner and Theodorsen functions, namely $C(-is)/s=\mathcal{L}(\phi(\tau))=\Phi(s)$, and afterwards inverting back the obtained expressions in the temporal space.
Alternatively, in order to ease the computations, the available approximate expressions for $\phi(\tau)$ (see Edwards et al.[3]) can be used in the Laplace transform process.
The expressions of lift and aerodynamic moment in the time domain, $_{\Lambda}L_h(\bar{y},\tau)$ and $_{\Lambda}M_\alpha(\bar{y},\tau)$ can be used to determine the subcritical aeroelastic response of swept wings. However, when the aeroelastic response of lifting surface to time-dependent external pulses, is needed, the unsteady aerodynamic loads in the time domain, $_{\Lambda}L_h$ and $_{\Lambda}M_\alpha$, have to be supplemented by the ones corresponding to above mentioned pulses.
This will be considered in the next developments and a simplified illustration of the capability of this method will be given in this work.

<u>Laplace Transformed Space</u>

Several preliminaries related to Laplace transform applied to aeroelastic quantities will be given next.

$$_{\Lambda}\{L_h(\bar{y},s),M_\alpha(\bar{y},s)\}= =\int_0^\infty\{_{\Lambda}L_h(\bar{y},\tau),_{\Lambda}M_\alpha(\bar{y},\tau)\}e^{-s\tau}d\tau. \tag{19}$$

We will express the Wagner's function and the plunging and pitching degrees of freedom in the Laplace transformed space as:

$$\phi(t)\overset{\mathcal{LT}}{\to}\Phi(s)\qquad h(t)\overset{\mathcal{LT}}{\to}\hat{h}(s)\qquad \alpha(t)\overset{\mathcal{LT}}{\to}\hat{\alpha}(s)$$

Considering zero initial condition, Laplace Transformed counterparts of Eqs. (15) and (16) are:

$$ {}_{\Lambda}L_c(\bar{y},s) = -C_{L\alpha_n} b_n \rho U_n^2 \left(s^2 \frac{\hat{h}}{b_n} + s\hat{\alpha} + s\sigma \tan\Lambda + \right. $$

$$ \left. + \left(\frac{1}{2} - a_n \right) \left(s^2\hat{\alpha} + b_n s\lambda \tan\Lambda \right) \right) \Phi(s) \quad , (20) $$

$$ {}_{\Lambda}M_c(\bar{y},s) = \left(\frac{1}{2} + a_n \right) C_{L\alpha_n} b_n \rho U_n^2 \left(s^2 \frac{\hat{h}}{b_n} + s\hat{\alpha} + s\sigma \tan\Lambda \right. $$

$$ \left. + \left(\frac{1}{2} - a_n \right) \left(s^2\hat{\alpha} + b_n s\lambda \tan\Lambda \right) \right) \Phi(s). \quad (21) $$

Herein, and in the next expressions, the plunging and pitching motions in the Laplace domain are expressed as $\hat{h} \equiv h(\bar{y},s)$ and $\hat{\alpha} \equiv \alpha(\bar{y},s)$.

The Laplace transformed counterpart of Eqs. (17) can be expressed as:

$$ {}_{\Lambda}L_{nc1}(\bar{y},s) = -\frac{1}{2} C_{L\alpha_n} \rho U_n^2 s^2 \left[\hat{h} - a_n b_n \hat{\alpha} \right], \quad (22\text{ a}) $$

$$ {}_{\Lambda}L_{nc2}(\bar{y},s) = -\frac{1}{2} C_{L\alpha_n} \rho U_n^2 b_n s\hat{\alpha}, \quad (22\text{ b}) $$

$$ {}_{\Lambda}L_{nc3}(\bar{y},s) = -\frac{1}{2} C_{L\alpha_n} \rho U_n^2 b_n^2 \tan\Lambda \left[(1+\delta_r) \frac{\sigma}{b_n} s + \delta_r \lambda \right. $$

$$ \left. + \delta_r \frac{\partial\sigma}{\partial\bar{y}} \tan\Lambda \right] + \frac{1}{2} C_{L\alpha_n} \rho U_n^2 b_n^3 \tan\Lambda \left[(1+\delta_r) \frac{\lambda}{b_n} s \right. $$

$$ \left. + \delta_r \frac{\partial\lambda}{\partial\bar{y}} \tan\Lambda \right]. \quad (22\text{ c}) $$

The Laplace transformed counterpart of Eqs. (18) can be written as:

$$ {}_{\Lambda}M_{nc1}(\bar{y},s) = \frac{1}{2} C_{L\alpha_n} \rho U_n^2 s^2 \left[\hat{h} - a_n b_n \hat{\alpha} \right] a_n b_n, \quad (23\text{ a}) $$

$$ {}_{\Lambda}M_{nc2}(\bar{y},s) = -\frac{1}{2} C_{L\alpha_n} \rho U_n^2 b_n^2 \left(\frac{1}{2} - a_n \right) s\hat{\alpha}, \quad (23\text{ b}) $$

$$ {}_{\Lambda}M_{nc3}(\bar{y},s) = -\frac{1}{2} C_{L\alpha_n} \rho U_n^2 b_n^3 \frac{1}{2} \lambda \tan\Lambda $$

$$ + \frac{1}{2} C_{L\alpha_n} \rho U_n^2 b_n^3 a_n \tan\Lambda \left[(\delta_r + 1) \frac{\sigma}{b_n} s + \delta_r \lambda + \right. $$

$$ \left. + \delta_r \frac{\partial\sigma}{\partial\bar{y}} \tan\Lambda \right] - \frac{1}{2} C_{L\alpha_n} \rho U_n^2 b_n^4 \left(\frac{1}{8} + a_n^2 \right) \tan\Lambda $$

$$ \left[(\delta_r + 1) \frac{\lambda}{b_n} s + \delta_r \frac{\partial\lambda}{\partial\bar{y}} \tan\Lambda \right], \quad (23\text{ c}) $$

$$ {}_{\Lambda}M_a(\bar{y},s) = -\frac{1}{16} \rho C_{L\alpha_n} b_n^2 U_n^2 s^2 \hat{\alpha}. \quad (23\text{ d}) $$

The equations of lift and aerodynamic moment help us to perform the conversion in the frequency domain. This will be done in the next section.

Frequency Domain

Upon replacing $s \to ik_n$ in Eqs. (22) and (23); using the relationship between Laplace transform of Wagner and Theodorsen's functions (Eq. (2)); representing the time dependence of displacement quantities as:

$$ \alpha(\bar{y},\tau) = f_\alpha(\bar{y}) \tilde{\alpha}(\tau,k_n) = f_\alpha(\bar{y}) \alpha_0 e^{ik_n\tau}, \quad (24\text{ a}) $$

$$ h(\bar{y},\tau) = f_h(\bar{y}) \tilde{h}(\tau,k_n) = f_h(\bar{y}) h_0 e^{ik_n\tau}, \quad (24\text{ b}) $$

and expressing:

$$ {}_{\Lambda}L_h(\bar{y},k_n,\tau) = {}_{\Lambda}\overline{L}_h(\bar{y},k_n) e^{ik_n\tau}, \quad (25\text{ a}) $$

$$ {}_{\Lambda}M_\alpha(\bar{y},k_n,\tau) = {}_{\Lambda}\overline{M}_\alpha(\bar{y},k_n) e^{ik_n\tau}, \quad (25\text{ b}) $$

the equations for the unsteady lift and moment amplitudes can be expressed in the frequency domain. These expressions that coincide with the ones obtained differently in Barmby et al.[1], can be used in the flutter analysis of swept aircraft wings.
In this analysis $f_\alpha(\bar{y})$ and $f_h(\bar{y})$ are chosen to be the decoupled eigenmodes in plunging and twisting of the counterpart structure, and are determined as to fulfill identically the boundary conditions. Using the spanwise dimensionless coordinate $\eta = \bar{y}/l$, these are expressed as:

$$ f_h(\bar{y}) = F_h(\eta) = C_1 \left((\cos\beta_1\eta - \cosh\beta_1\eta) \frac{\sinh\beta_1 + \sin\beta_1}{\cosh\beta_1 + \cos\beta_1} \right. $$

$$ \left. + \sinh\beta_1\eta - \sin\beta_1\eta \right) \quad (26) $$

$$ f_\alpha(\bar{y}) = F_\alpha(\eta) = C_2 \sin\beta_2\eta, \quad (27) $$

where for the first bending and torsion we have $\beta_1 = 0.5969\pi$ and $\beta_2 = \pi/2$.
The constants C_1 and C_2 are chosen as to normalize $f_h(\bar{y})$ and $f_\alpha(\bar{y})$, and so to get the unitary maximum deflection at the wing tip. The uncoupled first bending and torsion mode shapes needed for the evaluations of the aerodynamic lift and moment are shown in Fig. 3.
The following expressions will be useful in the next developments:

$$ \lambda' = \frac{\partial^2\alpha}{\partial\bar{y}\partial\tau} = \frac{\partial f_\alpha(\bar{y})}{\partial\bar{y}} \alpha_0 ik_n e^{ik_n\tau}, \quad (28) $$

$$\sigma' = \frac{\partial^2 h}{\partial \bar{y} \partial \tau} = \frac{\partial f_h(\bar{y})}{\partial \bar{y}} h_0 i k_n e^{i k_n \tau}, \tag{29}$$

$$\frac{\partial \lambda}{\partial \bar{y}} = \frac{\partial^2 f_\alpha(\bar{y})}{\partial \bar{y}^2} \alpha_0 e^{i k_n \tau}, \tag{30}$$

$$\frac{\partial \sigma}{\partial \bar{y}} = \frac{\partial^2 f_h(\bar{y})}{\partial \bar{y}^2} h_0 e^{i k_n \tau}. \tag{31}$$

Unsteady Aerodynamic Derivatives in the Frequency Domain

At this point, a careful inspection of Eqs. (7) - (8), suggests the following representations for the lift and aerodynamic moment:

$$_{\Lambda}L_h(\bar{y}, k_n, \tau) = \frac{1}{2} \rho U_n^2 2 b_n \left(k_n H_1 \frac{h'}{b_n} + k_n H_2 \alpha' + k_n^2 H_3 \alpha + k_n^2 H_4 \frac{h}{b_n} + H_5 \alpha'' + H_6 \frac{h''}{b_n} \right), \tag{32}$$

$$_{\Lambda}M_\alpha(\bar{y}, k_n, \tau) = \frac{1}{2} \rho U_n^2 2 b_n^2 \left(k_n A_1 \frac{h'}{b_n} + k_n A_2 \alpha' + k_n^2 A_3 \alpha + k_n^2 A_4 \frac{h}{b_n} + A_5 \alpha'' + A_6 \frac{h''}{b_n} \right). \tag{33}$$

Herein the dimensionless unsteady aerodynamic coefficients H_i, A_i have been introduced, and k_n has been included as to render the quantities in brackets nondimensional. Herein, b_n is the half-chord of the airfoil and U_n is the component of the flow speed, both normal to the elastic axis.

In a simplified context, such a mixed form of the lift and moment was used in Simiu & Scanlan[9], and Scanlan[8]. The unsteady aerodynamic derivatives for swept wings are obtainable from the previous equations of lift and aerodynamic moment, by assuming harmonic time dependence of displacements quantities. In such a way, the frequency domain counterpart of Eqs. (32)-(33), expressed in compact form, becomes:

$$_{\Lambda}\overline{L}_h(\bar{y}, k_n, \tau) = \rho U_n^2 k_n^2 b_n \left(\frac{h_0}{b_n} L_1 + \alpha_0 L_2 \right), \tag{34}$$

$$_{\Lambda}\overline{M}_\alpha(\bar{y}, k_n, \tau) = \rho U_n^2 k_n^2 b_n^2 \left(\frac{h_0}{b_n} M_1 + \alpha_0 M_2 \right). \tag{35}$$

Herein, the unsteady aerodynamic complex coefficients L_i and M_i can be expressed as:

$$L_1 = i\hat{H}_1 + \hat{H}_4; \qquad L_2 = i\hat{H}_2 + \hat{H}_3 \tag{36}$$

$$M_1 = i\hat{A}_1 + \hat{A}_4; \qquad M_2 = i\hat{A}_2 + \hat{A}_3. \tag{37}$$

where, for the sake of convenience, the unsteady aerodynamic derivative are written as:

$$\hat{H}_1 = H_1; \hat{H}_2 = H_2; \hat{H}_3 = (H_3 - H_5); \hat{H}_4 = (H_4 - H_6),$$
$$\hat{A}_1 = A_1; \hat{A}_2 = A_2; \hat{A}_3 = (A_3 - A_5); \hat{A}_4 = (A_4 - A_6). \tag{38}$$

The unsteady aerodynamic derivatives in the frequency domain for swept wing will be obtained from the following equations, expressed in terms of Wagner's function $\Phi(ik_n)$. Comparing Eqs. (36) with that expressing the lift in the frequency domain, yields:

$$L_1 = \frac{1}{2} C_{L\alpha_n} f_h - \frac{1}{2} \frac{C_{L\alpha_n}}{k_n^2} b_n \tan\Lambda \left[(\delta_r + 1) \frac{\partial f_h}{\partial \bar{y}} ik + \delta_r b_n \frac{\partial^2 f_h}{\partial \bar{y}^2} \tan\Lambda \right] + \frac{C_{L\alpha_n}}{k_n} \Phi(ik_n) \left[k_n f_h - i b_n \frac{\partial f_h}{\partial \bar{y}} \tan\Lambda \right] \tag{39 a}$$

$$L_2 = -\frac{C_{L\alpha_n}}{k_n} \Phi(ik_n) \left[i f_\alpha - \left(\frac{1}{2} - a_n \right) \left(k_n f_\alpha - i b_n \frac{\partial f_\alpha}{\partial \bar{y}} \tan\Lambda \right) \right] + \frac{1}{2} C_{L\alpha_n} a_n \frac{b_n}{k_n^2} \tan\Lambda \left[(\delta_r + 1) \frac{\partial f_\alpha}{\partial \bar{y}} i k_n + \delta_r b_n \frac{\partial^2 f_\alpha}{\partial \bar{y}^2} \tan\Lambda \right] - \delta_r \frac{1}{2} C_{L\alpha_n} \frac{b_n}{k_n^2} \frac{\partial f_\alpha}{\partial \bar{y}} \tan\Lambda - \frac{1}{2} C_{L\alpha_n} a_n f_\alpha - \frac{1}{2} \frac{C_{L\alpha_n}}{k_n} f_\alpha. \tag{39 b}$$

From a similar analysis, comparing the Eqs. (39) with that expressing the aerodynamic moment in frequency domain, yields:

$$M_1 = -\frac{C_{L\alpha_n}}{k_n} \left(\frac{1}{2} + a_n \right) \Phi(ik_n) \left[k_n f_h - i b_n \frac{\partial f_h}{\partial \bar{y}} \tan\Lambda \right] - \frac{1}{2} C_{L\alpha_n} a_n f_h + \frac{1}{2} \frac{C_{L\alpha_n}}{k_n^2} a_n b_n \tan\Lambda \left[(\delta_r + 1) \frac{\partial f_h}{\partial \bar{y}} i k_n + \delta_r b_n \frac{\partial^2 f_h}{\partial \bar{y}^2} \tan\Lambda \right], \tag{40 a}$$

$$M_2 = \frac{C_{L\alpha_n}}{k_n} \left(\frac{1}{2} + a_n \right) \Phi(ik_n) \left[i f_\alpha - \left(\frac{1}{2} - a_n \right) \left(k_n f_\alpha - i b_n \frac{\partial f_\alpha}{\partial \bar{y}} \tan\Lambda \right) \right] + \frac{1}{16} C_{L\alpha_n} f_\alpha - \left(\frac{1}{2} - a_n \right) \frac{1}{2} \frac{C_{L\alpha_n}}{k_n} i f_\alpha + \frac{1}{2} C_{L\alpha_n} a_n^2 f_\alpha - \frac{1}{2} \frac{C_{L\alpha_n}}{k_n^2} b_n \frac{\partial f_\alpha}{\partial \bar{y}} \tan\Lambda \left(\frac{1}{2} - \delta_r a_n \right) - \frac{1}{2} \frac{C_{L\alpha_n}}{k_n^2} b_n \left(\frac{1}{8} + a_n^2 \right) f_\alpha \tan\Lambda \left[(\delta_r + 1) \frac{\partial f_\alpha}{\partial \bar{y}} i k_n + \delta_r b_n \frac{\partial^2 f_\alpha}{\partial \bar{y}^2} \tan\Lambda \right]. \tag{40 b}$$

Using in Eqs (39)-(40) equations (2), separating the real and the imaginary parts of the above expressions, the unsteady aerodynamic derivatives result as:

$$\hat{H}_1 = -C_{L\alpha_n}\left(\frac{F(k_n)}{k_n} f_h + \frac{b_n}{k_n}\tan\Lambda \frac{\partial f_h}{\partial y}\left(\frac{G(k_n)}{k_n} + \frac{1}{2}(1+\delta_r)\right)\right),$$

$$\hat{H}_2 = -\frac{C_{L\alpha_n}}{k_n}\left(\left(\frac{1}{2}-a_n\right)F(k_n) f_\alpha + \frac{G(k_n)}{k_n} f_\alpha + \frac{1}{2} f_\alpha + b_n \frac{\partial f_\alpha}{\partial y}\tan\Lambda\left(\frac{2G(k_n)}{k_n}\left(\frac{1}{2}-a_n\right) - \frac{1}{2}a_n(1+\delta_r)\right)\right),$$

$$\hat{H}_3 = -C_{L\alpha_n}\left(\frac{F(k_n)}{k_n^2} f_\alpha - \frac{G(k_n)}{k_n}\left(\frac{1}{2}-a_n\right) f_\alpha + \frac{1}{2}a_n f_\alpha + \frac{b_n}{k_n^2}\frac{\partial f_\alpha}{\partial y}\tan\Lambda\left(F(k_n)\left(\frac{1}{2}-a_n\right) + \frac{1}{2}\delta_r - \delta_r a_n \frac{1}{2}\frac{b_n^2}{k_n^2}\frac{\partial^2 f_\alpha}{\partial y^2}\tan^2\Lambda\right)\right),$$

$$\hat{H}_4 = C_{L\alpha_n}\left(\frac{1}{2} f_h + \frac{G(k_n)}{k_n} f_h - \frac{b_n \tan\Lambda}{k_n^2}\left(F(k_n)\frac{\partial f_h}{\partial y} + \delta_r \frac{1}{2} b_n \frac{\partial^2 f_h}{\partial y^2}\tan\Lambda\right)\right) \quad (41)$$

$$\hat{A}_1 = \frac{C_{L\alpha_n}}{k_n}\left(\left(\frac{1}{2}+a_n\right)F(k_n) f_h + b_n \frac{\partial f_h}{\partial y}\tan\Lambda\left(\left(\frac{1}{2}+a_n\right)\frac{G(k_n)}{k_n} + \frac{1}{2}a_n(\delta_r+1)\right)\right),$$

$$\hat{A}_2 = \frac{C_{L\alpha_n}}{2k_n}\left(\left(a_n-\frac{1}{2}\right) f_\alpha + \left(\frac{1}{2}+a_n\right)\frac{2G(k_n)}{k_n} f_\alpha - \left(a_n^2-\frac{1}{4}\right)2F(k_n) f_\alpha + b_n \frac{\partial f_\alpha}{\partial y}\tan\Lambda\left(\left(\frac{1}{4}-a_n^2\right)\frac{2G(k_n)}{k_n} - (\delta_r+1)\left(\frac{1}{8}+a_n^2\right)\right)\right),$$

$$\hat{A}_3 = \frac{C_{L\alpha_n}}{4k_n^2}\Big((1+2a_n)2F(k_n) f_\alpha + (4a^2-1)G(k_n)k_n f_\alpha + \left(2a_n^2+\frac{1}{4}\right)k_n^2 f_\alpha + b_n \frac{\partial f_\alpha}{\partial y}\tan\Lambda\Big((1-4a_n^2)F(k_n) - (1-\delta_r 2a_n)\Big) - \delta_r b_n^2\left(\frac{1}{4}+2a_n^2\right)\frac{\partial^2 f_\alpha}{\partial y^2}\tan^2\Lambda\Big),$$

$$\hat{A}_4 = -C_{L\alpha_n}\left(\frac{1}{2}a_n f_h + \left(\frac{1}{2}+a_n\right)\frac{G(k_n)}{k_n} f_h - \frac{b_n}{k_n^2}\tan\Lambda\left(\left(\frac{1}{2}+a_n\right)F(k_n)\frac{\partial f_h}{\partial y} + \delta_r \frac{1}{2}a_n b_n \frac{\partial^2 f_h}{\partial y^2}\tan\Lambda\right)\right). \quad (42)$$

The coupling terms, due to the sweep effects, are separated in the overall expressions of aerodynamic derivatives. For $\Lambda = 0$ the expressions of aerodynamic coefficients corresponding to straight wings are obtained. The unsteady aerodynamic derivatives recorded above coincide with the ones obtained differently in Barmby et al.[1], where the spanwise rates of change of bending and twist, σ and λ, that are associated with the sweep effect, also intervene. When specialized for $\delta_r = 0$, there coincide with the ones by Bisplinghoff et al.[2]

Aeroelastic Response of an Airfoil Featuring Plunging and Pitching Coupled Motions to Sonic-Boom Pressure Pulses.

An applications on the aeroelastic response of an airfoil in an incompressible flow featuring plunging and pitching coupled motions to sonic-boom and blast pressure pulses will be given in the next developments. The aeroelastic governing system of equations of an airfoil featuring plunging and twisting degrees of freedom to sonic-boom, blast pressure pulses, expressed in dimensionless form, can be cast as:

$$\xi''(\tau) + \chi_\alpha \alpha''(\tau) + 2\zeta_h(\bar{\omega}/V)\xi'(\tau) + (\bar{\omega}/V)^2\xi(\tau) - l_h(\tau) = l_b(\tau) \quad (43)$$

$$(\chi_\alpha/r_\alpha^2)\xi''(\tau) + \alpha''(\tau) + (2\zeta_\alpha/V)\alpha'(\tau) + \alpha/V^2 - m_\alpha(\tau) = 0 \quad (44)$$

In the above expressions this following dimensionless parameters have been used:

$l_h = L_h b/mU_\infty^2$; $\wp_m = P_m b/mU_\infty^2$; $V = U_\infty/b\omega_\alpha$;
$\mu = m/\pi\rho b^2$; $\zeta_h = c_h/2m\omega_h$; $\bar{\omega} = \omega_h/\omega_\alpha$;
$\omega_h = (K_h/m)^{1/2}$; $\omega_\alpha = (K_\alpha/I_\alpha)^{1/2}$; $\zeta_\alpha = c_\alpha/2I_\alpha\omega_\alpha$;
$r_\alpha = (I_\alpha/mb^2)^{1/2}$; $\chi_\alpha = S_\alpha/mb$; $m_\alpha = M_\alpha b^2/I_\alpha U_\infty^2$.

The dimensionless *sonic-boom* overpressure signature of the N-wave shock pulse, can be described as follow:

$$l_b(\tau) = H(\tau)\wp_m\left(1-\frac{\tau}{\tau_p}\right) - \delta_b H(\tau - r\tau_p)\wp_m\left(1-\frac{\tau}{\tau_p}\right). \quad (45)$$

Herein, the Heaviside step function $H(\tau)$ has been introduced in order to describe the typical pressure time-history for blast or sonic-boom loads; δ_b is a tracer that should be taken as one when the sonic-boom is considered, and zero when the blast load is included; $\wp_m$ denotes the dimensionless peak reflected pressure in excess of the ambient one; τ_p denotes the positive phase duration of the pulse measured from the time of impact of the structure; r denotes the shock pulse length factor. For $r=1$ the N-shaped pulse degenerates into a triangular pulse which corresponds to an explosive pulse (Fig. 4.a), and for $r=2$ a symmetric N-shaped pulse is obtained. A depiction of $l_b/\wp_m$ vs. time is displayed in Fig. 4.b.

The Eqs. (43) and (44) can be converted in the Laplace transformed space and solved for their unknowns, $\hat{\xi}\,(\equiv\mathcal{L}\,(\xi))$ and $\hat{\alpha}\,(\equiv\mathcal{L}\,(\alpha))$; inverted back in time domain one obtain the plunging and pitching time-histories and the load factor time-history due to the sonic-boom pressure pulse, $\xi(\tau)\equiv\mathcal{L}^{-1}\{\hat{\xi}(s)\}$ and $\alpha(\tau)\equiv\mathcal{L}^{-1}\{\hat{\alpha}(s)\}$, respectively.

When the dynamic response of the actively controlled lifting surface is analyzed, also the feedback control forces and moments, that are time dependent, have to be included in the Eqs. (43) and (44). This will be considered in the future work.

RESULTS AND DISCUSSION

Herein, an unified way enabling one to obtain the unsteady lift and aerodynamic moment in the time and frequency domains for swept aircraft wing was developed. This was done via the use of the indicial function approach. The time domain representation is essential towards determination of the dynamic aeroelastic response to time dependent external loads, and in the case of the application of a feedback control methodology, of the dynamic aeroelastic response to both external time dependent loads and control forces. The frequency domain representation is essential towards determination of the flutter instability.

The unsteady aerodynamic derivatives for different values of a_n , Λ and $C_{L\alpha_n}$ have been plotted in Figs. 5 and 6, as a function of k_n. For swept wings, the local lift-curve slope $C_{L\alpha_n}$ for sections normal to the elastic axis are obtained from the aerodynamics of swept wings (see Yates[10]).

The maximum influence of the corrective term (identified by the tracer δ_r) is present the first plunging coefficients H_1 where the aerodynamic coefficient changes also its sign. Usually, for all coefficients, the effect of these terms becomes higher for high sweep angles.

For swept wings, the local lift-curve slope $C_{L\alpha_n}$ for sections normal to the elastic axis are expressed as:

$$C_{L\alpha_n} = C_{L\alpha}/\cos\Lambda\,. \tag{49}$$

The variation of the unsteady aerodynamic derivatives $\hat{H}_i$ and $\hat{A}_i$ as a function of k_n are depicted. In these developments, all the terms, including the aerodynamic ones associated with $\ddot{h}$ and $\ddot{\alpha}$, usually neglected, have been retained. As a result, the coefficients H_5, H_6 and A_5, A_6 are also included. Whereas the aerodynamic coefficients $\hat{H}_1$ and $\hat{A}_2$ are the principal uncoupled aerodynamic damping coefficients in plunging and torsion, respectively, $\hat{H}_2$ and $\hat{A}_1$ are the coupled damping coefficients. For straight wings, these terms remains negative for all values of $2\pi/k_n$ and for different values of a_n, but for swept wings only $\hat{A}_2$ continues to remain negative. The elastic axis position is not involved in the expression of the unsteady aerodynamic coefficients $\hat{H}_1$ and $\hat{H}_4$, a fact, which clearly appears from the equations. The variation of the aerodynamic derivatives as a function of k_n is, in general, a smooth one. Among these coefficients, only $\hat{H}_2$ features, with the variation of the reduced frequency k_n, a change of sign. As concerns, the depiction of $\hat{H}_i$ and $\hat{A}_i$ versus $2\pi/k_n\ (=U_n/nb_n)$, this representation enables one to get an idea of the variation of the respective quantity with that of the normal freestream speed U_n. The differences in the unsteady aerodynamic coefficients induced by the discard or inclusion of the terms generated from the downwash velocity (i.e. that underscored by a solid line in Eq. (12)) are properly indicated in Figs. 7. As shown, the corrective term does not modify the trend of coefficients $\hat{A}_1$ and $\hat{A}_4$. It should be mentioned that the expressions of the lift and aerodynamic moment in the frequency domain obtained by Barmby, Cunningham and Garrick[1] coincide with the ones obtained here via indicial function approach.

The graphs depicting the aeroelastic response time-history to blast pulses (i.e. explosive and sonic-boom blasts) are displayed (Figs. 8 - 10). The graphs supply the dimensionless plunging displacement $(\xi\equiv h/b)$, and the load factor ($N\equiv h''/g$, where g is the acceleration of gravity). The predictions of ξ and N based on pure plunging and coupled plunging-twist models are depicted on the same plots, and the closeness of the two predictions becomes apparent from the graphs. Herein, the dotted and solid curves correspond to pure plunging and coupled plunging-pitch models, respectively. As a result, the coupling helps to reduce the amplitude of the aeroelastic response. The pitching displacement has its maximum for $\tau = 0''$. The same trend is valid also for the load factor N. It should be indicated that the response to sonic-boom pressure pulse involves two different regimes; one for which $0 < \tau < 30''$ that corresponds to the forced motion, and the other one to $\tau > 30''$ belonging to the free motion. The jump in the time-history of N is due to the discontinuity in the load occurring at $\tau = 30''$. This jump doesn't appear for explosive pressure pulses, where r = 1. The increase of the mass ratio results in the increase of the plunging displacement amplitude and the decrease of the pitching amplitude. At the same time, for higher mass ratios, the differences in the plunging predictions based upon 1DOF and 2 DOF disappear (Figs. 8 and 9). Moreover, for higher mass ratios, the motion damps out at larger times. Figs. 10 highlights the effect of the structural damping coefficient in plunging and pitching.

Using the idea developed by Yates[10], a modified strip theory can be accommodated as to address the problem of the aeroelastic response (for open/closed loop aeroelastic systems), by capturing also the 3-D effects.

Alternatively, an exact solution methodology enabling one to determine both the flutter instability and the aeroelastic dynamic response based on a double Laplace transform, in time and space, can be used. Such a method was devoted to the solution of aeroelastic eigenvalue problems in Karpouzian & Librescu[5,6], Librescu & Thangjitham[7]. On this basis, the obtained results can be extended as to approach the subcritical aeroelastic response and flutter, respectively, of 3-D advanced lifting surfaces, in various flight speed regimes.

CONCLUSIONS

A unified treatment of the aeroelasticity of 2-D lifting surfaces in time and frequency domains has been presented and the usefulness in this context of the aerodynamic indicial functions concept was emphasized.

Applications assessing the versatility of this approach enabling one to treat both subcritical aeroelastic responses and flutter instability were presented, and prospects for extending this treatment to 3-D aeroelastic problems are contemplated in forthcoming developments.

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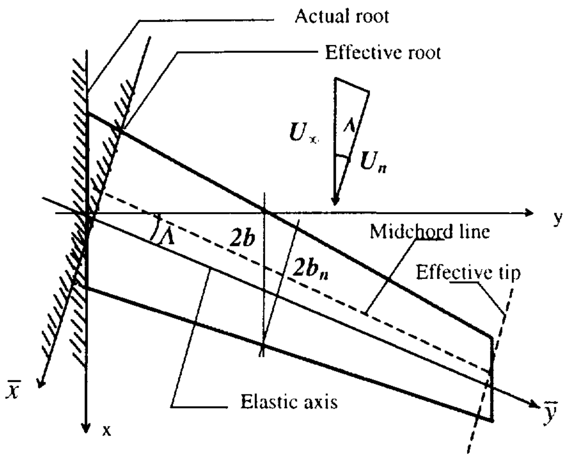

Fig. 1 Nonuniform Swept Wing

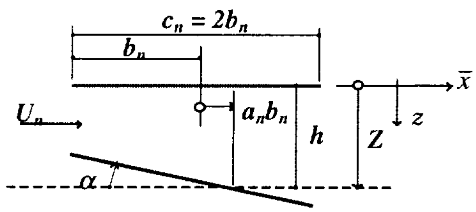

Fig. 2 Airfoil Section

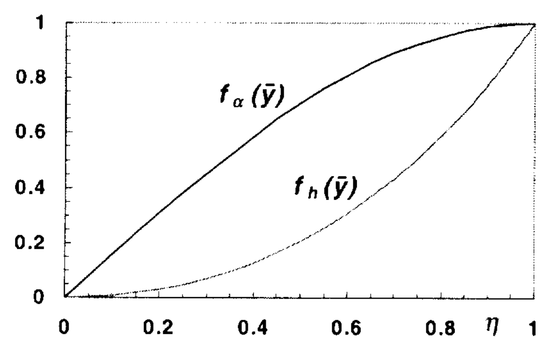

Fig. 3 Bending-Mode and Torsion-Mode

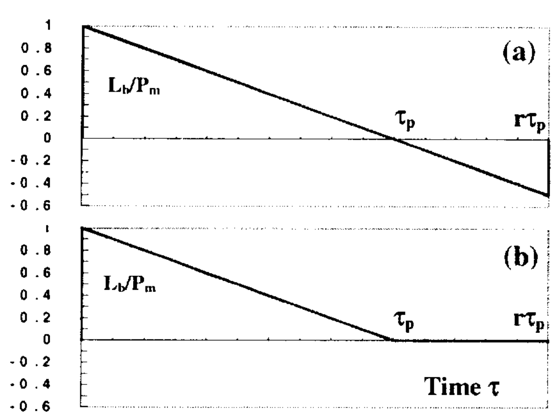

Fig. 4 (a) Sonic-Boom and
(b) Triangular Blast Pressure Pulses

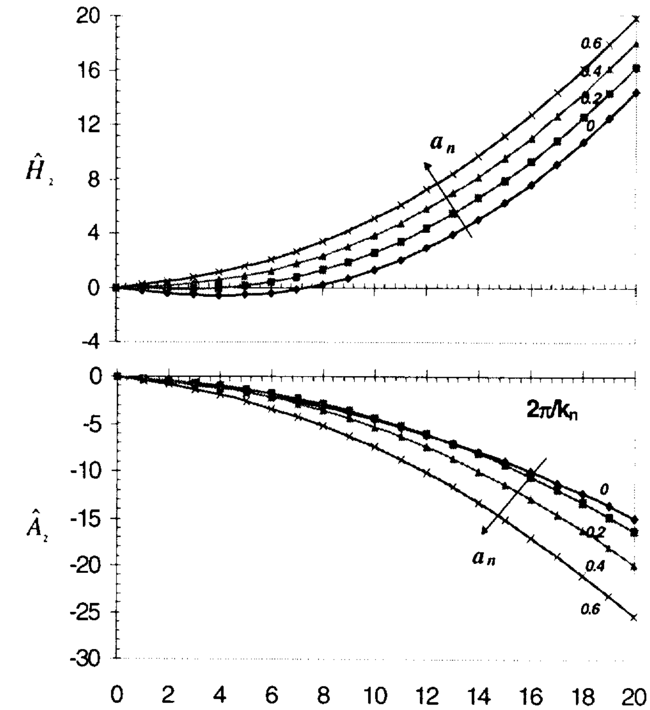

Fig. 5 Unsteady aerodynamic derivative for swept lifting surfaces ($\Lambda = 45^o$) and different elastic axis position (a_n), vs. $2\pi/k_n$

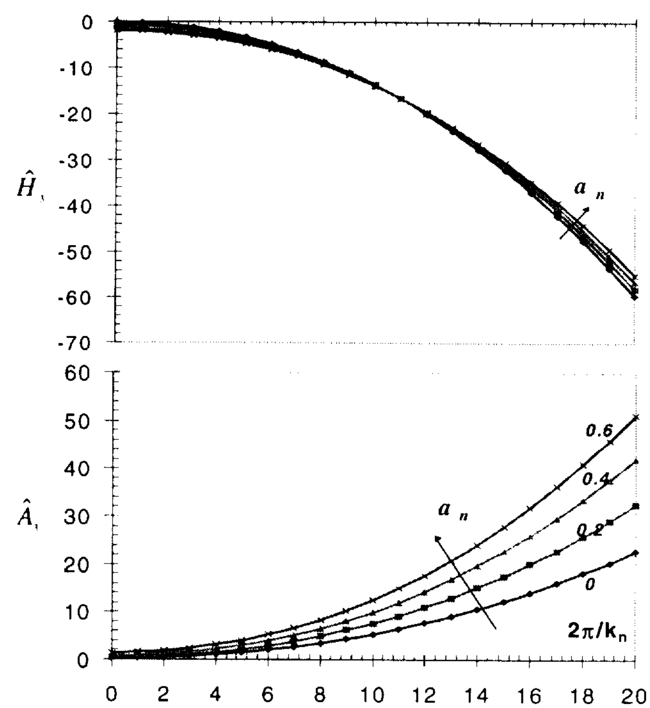

Fig. 6 Unsteady aerodynamic derivative for swept lifting surfaces ($\Lambda = 15^o$) and different elastic axis position (a_n), vs. $2\pi/k_n$

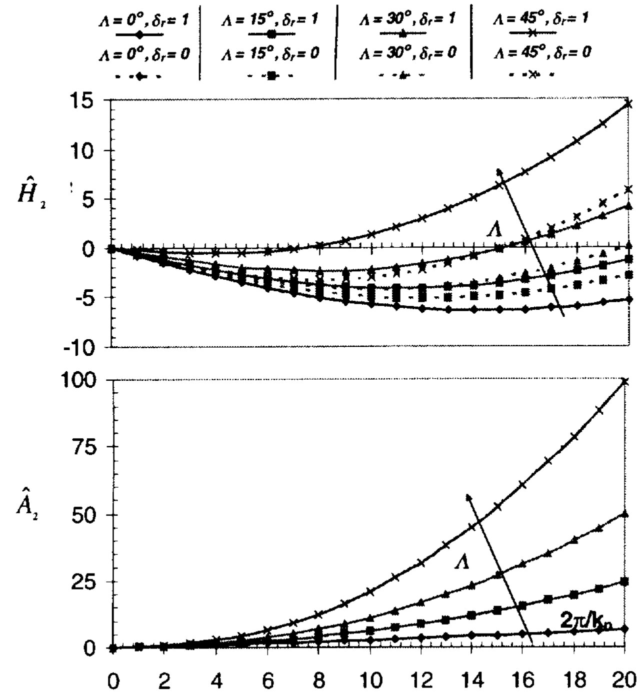

Fig. 7 Results based on the discard of the corrective effect ($\delta_r = 0$)——, and including that effect ($\delta_r = 1$) -·-·- , vs. $2\pi/k_n$.

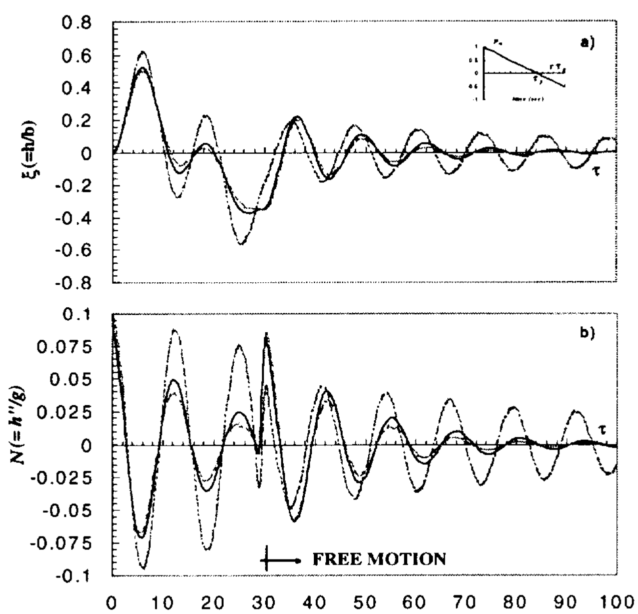

Fig. 8 Influence of mass ratio on the time-history response of (a) the nondimensional plunging displacement and (b) of the load factor, *to sonic-boom pressure pulse*

—— $\mu = 10$, 1DOF; - - - $\mu = 10$, 2DOF; —·· $\mu = 50$, 1DOF; ····· $\mu = 50$, 2DOF, ($\wp_m = 1$; $\tau_p = 15''$; $\zeta_h = \zeta_\alpha = 0$; $V = 1$; $r = 2$; $\overline{\omega} = 0.5$; $\chi_\alpha = 0$; $r_\alpha = 0.5$; $a = -0.2$).

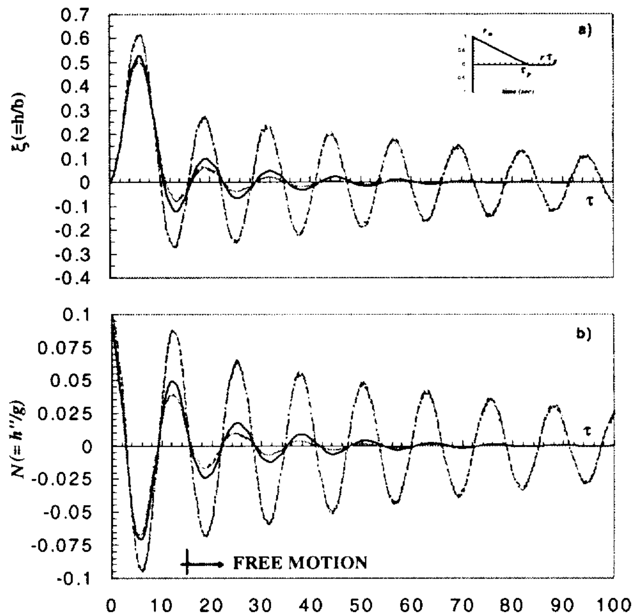

Fig. 9 Influence of mass ratio on the time-history response of (a) the nondimensional plunging displacement and (b) of the load factor, *to explosive pressure pulse*

—— $\mu = 10$, 1DOF; - - - $\mu = 10$, 2DOF; —·· $\mu = 50$, 1DOF; ····· $\mu = 50$, 2DOF, ($\wp_m = 1$; $\tau_p = 15''$; $\zeta_h = \zeta_\alpha = 0$; $V = 1$; $r = 1$; $\overline{\omega} = 0.5$; $\chi_\alpha = 0$; $r_\alpha = 0.5$; $a = -0.2$).

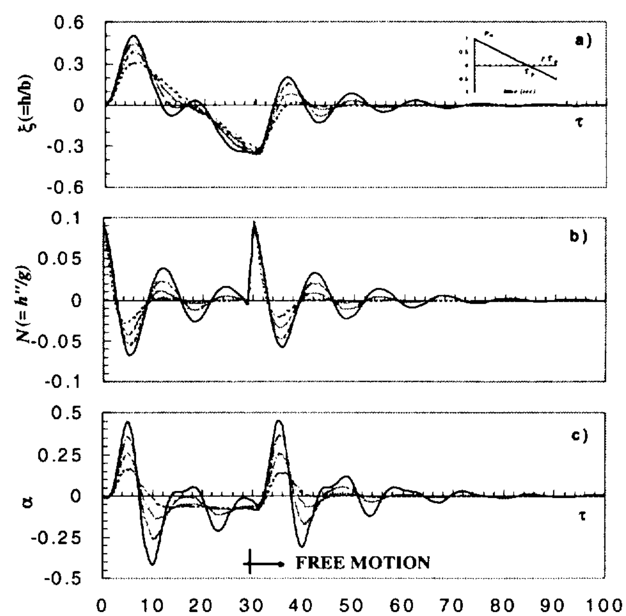

Fig. 10 Influence of structural damping coefficient on the time-history response of (a) the nondimensional plunging displacement, (b) of the load factor and (c) of the nondimensional pitching displacement *to sonic-boom pressure pulse*

—— $\zeta_h = \zeta_\alpha = 0$; - - - $\zeta_h = \zeta_\alpha = 0.1$; ····· $\zeta_h = \zeta_\alpha = 0.25$; —·· $\zeta_h = \zeta_\alpha = 0.5$; ($\wp_m = 1$; $\tau_p = 15''$; $\zeta_h = \zeta_\alpha = 0$; $V = 1$; $r = 2$; $\overline{\omega} = 0.5$; $\chi_\alpha = 0$; $r_\alpha = 0.5$; $a = -0.2$).