AEROELASTIC RESPONSE OF NONLINEAR WING SECTION BY FUNCTIONAL SERIES TECHNIQUE

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Abstract

This paper addresses the problem of the determination of the subcritical aeroelastic response and flutter instability of nonlinear two-dimensional lifting surfaces in an incompressible flow-field via indicial functions and Volterra series approach. The related aeroelastic governing equations are based upon the inclusion of structural and damping nonlinearities in plunging and pitching, of the linear unsteady aerodynamics and consideration of an arbitrary time-dependent external pressure pulse. Unsteady aeroelastic nonlinear kernels are determined, and based on these, frequency and time histories of the subcritical aeroelastic response are obtained, and in this context the influence of the considered nonlinearities is emphasized. Conclusions and results displaying the implications of the considered effects are supplied.

Nomenclature

a Dimensionless elastic axis position measured from the mid-chord, positive aft

c Chord length of 2-D lifting surface, $2b$

$c_{hi}, c_{\alpha i}, K_{hi}, K_{\alpha i}$ Damping and stiffness coefficients in plunging and pitching (i=1,2,3 - linear, quadratic, cubic), respectively

$C_{L\alpha}$ Lift-curve slope

$C(k), F(k), G(k)$ Theodorsen's function and its real and imaginary counterparts, respectively

h, ξ Plunging displacement and its dimensionless counterpart, (h/b), respectively

h_n, H_n n-*th* order Volterra kernel in time, and its Laplace transformed counterpart, respectively

I_α, r_α Mass moment of inertia per unit wingspan and the dimensionless radius of gyration, $\left(I_\alpha / mb^2\right)^{1/2}$, respectively

l_a, m_a Dimensionless aerodynamic lift and moment, $\left(L_a b / mU_\infty^2\right)$ and $\left(M_a b^2 / I_\alpha U_\infty^2\right)$, respectively

L_a, M_a Total lift and moment per unit span

L_b, l_b Overpressure signature of the N-wave shock pulse and its dimensionless counterpart, $\left(L_b b / mU_\infty^2\right)$, respectively

m, μ Airfoil mass per unit length and reduced mass ratio, $\left(m / \pi\rho b^2\right)$, respectively

N Load factor, $1 + h'' / g$

$P_m, \wp_m$ Peak reflected pressure amplitude and its dimensionless counterpart, $\left(P_m b / mU_\infty^2\right)$, respectively

r Shock pulse length factor

$s_j, \mathcal{L}$ Laplace transform variable and Laplace operator, respectively, $s_j = ik_j; i^2 = -1$

S_α, χ_α Static unbalance about the elastic axis and its dimensionless counterpart, S_α / mb, respectively

t, τ_0, τ Time variables and dimensionless counterpart, $\left(U_\infty t / b\right)$, respectively

t_p, τ_p Positive phase duration, measured from the time of the arrival of the pulse, and its dimensionless value, respectively

TF   Transfer function

U_∞, V Freestream speed and its dimensionless counterpart, $\left(U_\infty / b\omega_\alpha\right)$

$x(t)$ Time-dependent external pulse (traveling gusts and wake blast waves)

$y(t)$ Response of the considered degree of freedom (pitch α and/or plunge h)

α Twist angle about the pitch axis

ζ_h, ζ_α Structural damping ratios in plunging $\left(c_h / 2m\omega_h\right)$, and pitching $\left(c_\alpha / 2I_\alpha\omega_\alpha\right)$, respectively

ρ Air density

$\phi(\tau)$ Wagner's indicial function

ω, k Circular and reduced frequencies, $\left(\omega b / U_\infty\right)$, respectively

ω_h, ω_α Uncoupled frequencies in plunging and pitching, $\left(K_h / m\right)^{1/2}$ and $\left(K_\alpha / I_\alpha\right)^{1/2}$, respectively

$\overline{\omega}$ Plunging-pitching frequency ratio, $\left(\omega_h / \omega_\alpha\right)$

Superscript

$(\hat{\cdot})$ Variables in Laplace transformed space

$(\dot{\cdot})$, $(\cdot)'$ Derivatives with respect to the time t, and the dimensionless time τ, respectively

I. Introduction

It is a well-known fact that within the linearized approach of the aeroelasticity discipline, it is possible to obtain the divergence and the flutter instability boundaries, and also to get the linearized subcritical aeroelastic response to time-dependent external pulses, such as blast and sonic boom pressure signatures and gust loads. On the other hand, the nonlinear approach of the aeroelastic problem can provide important information: i) on the influence of the considered nonlinearities on the subcritical aeroelastic response, and ii) about the character of the instability flutter boundary, i.e. benign or catastrophic one. In other words, such an approach gives the possibility of determining in what conditions the flutter speed can be exceeded without the occurrence of a catastrophic failure of the wing, in which case the flutter is *benign*, as well as the conditions in which undamped oscillations may appear at velocities below the flutter velocity, in which case the flutter is *catastrophic*. In addition, the considered nonlinearities play a great role on the subcritical aeroelastic response of lifting surfaces. Due to the strong implications of various nonlinearities on the highly flexible lifting surfaces, their related aeroelastic phenomena cannot longer be analyzed only within the standard linearized aeroelasticity theory. Aircraft wing structures often contain nonlinearities, which affect their aeroelastic behavior and performance characteristics, and flutter boundaries. For these reasons, in order to investigate the aeroelastic behavior of lifting surfaces in the vicinity of the flutter boundary, the aeroelastic governing equations have to be necessarily considered in nonlinear form.

This investigation concerns the aeroelastic response of two-dimensional nonlinear lifting surfaces exposed to an incompressible flow field and subjected to an external pressure pulse[1-3].

Based on Volterra's functional series approach[4-8] pertinent information about the effects of nonlinearities on either the aeroelastic response in the subcritical flight speed regime, and their implication on the flutter boundary are supplied.

The advantage of the technique based on Volterra's series and indicial function (Lomax[9], Bisplinghoff[10], Marzocca et al.[11]) consists, among others, on the possibility to investigate, within a rigorous theoretical basis, the aeroelastic systems featuring a wide class of nonlinearities. As a limiting case, based upon the first order Volterra kernel the study of the linear aeroelastic stability of the systems can be carried out. This methodology can encompass the case of an arbitrary number of degrees of freedom and at the same time is conceptually clearer, computationally simpler and can provide more accurate and realistic results as compared to the

conventional techniques used in nonlinear aeroelastic systems based on perturbation and multiple scale methods. In addition, this method, as shown in this paper, features a faster convergence as compared to the other methodologies. Moreover, this method does not experience the limitations, usually characterizing the other techniques, such as the Hilbert transform[12,13], developed to identify nonlinear systems from the first order frequency response function (*FRF*), or the phase plane methods that can describe the motion as just a function of two variables. In contrast to these methods that are suitable mainly to one degrees of freedom systems, Volterra series approach overcomes the shortcomings facing the other methods, consisting of the impossibility to cope with exchange of energy between the different mode frequencies.

Toward the end of determining the nonlinear unsteady aeroelastic kernels, the harmonic probing algorithm, referred to as the method of growing exponentials advanced by Bedrosian and Rice[14,15], and the multidimensional Laplace transform[16] will be used.

In addition to the aeroelastic response and determination of the flutter instability boundary, Volterra Series considered in conjunction with this nonlinear aeroelastic model can be used to study the conditions rendering the flutter boundary a benign or a catastrophic one (Librescu[17,18]).

Moreover, when the closed-loop dynamic response of actively controlled lifting surface is analyzed, also the feedback control forces and moments should be included (Librescu[17,18], Gern and Librescu[19], Librescu and Na[20], Van Trees[21], Chua and Ng[22]) and Volterra's series approach can still be applied towards the control purpose.

Volterra's series approach provides a firm basis for the treatment of the nonlinear subcritical aeroelastic response, in the sense that it supplies an explicit relationship between the input (any type of time-dependent external pulses, i.e. blast load, sonic-boom, gust loads) and its response. With the so-called *Volterra Kernel* identification scheme, the modeling of an aeroelastic system using this approach becomes feasible. However, this methodology requires determination for each specific flight conditions of the corresponding nonlinear kernel of the Volterra's series. For this reason, in order to define the appropriate aerodynamic loads, the recent interest in the modeling of unsteady nonlinear aerodynamics by this approach has been focused on the identification of Volterra's kernels in the *time domain* (Silva[23-26]), and in the *frequency domain* (Tromp and Jenkins[27]).

A number of fundamental contributions related to Volterra's series, developed by outstanding mathematicians (Volterra[4], Wiener[5]) and used mainly in electrical engineering[6-8], are already available.

The original studies on functional series by Volterra[4], have been continued in the works by Volterra himself, and of those by Rugh[6], Schetzen[7], Boyd[8]. These concepts have been mainly used in the general nonlinear system theory.

Originally, the method of Volterra series and Volterra kernel identification were developed to identify the nonlinear behavior in electrical circuits. In the aerospace field, the fundamental contributions were brought by Silva[23], who has shown that the method is also applicable to aeroelastic systems (aerodynamic reactions and forced structural model). Silva's pioneering work[23-26] in this area has opened a very promising way of modeling and approaching nonlinear aeroelastic systems.

II. Basic Concepts

Having in view that for nonlinear systems the superposition principle is not applicable, and having in view the different types of responses induced by unsteady aerodynamic loads and external excitation, a combination of transfer functions is used. For the nonlinear aeroelastic systems these transfer functions and the time-histories response in time and frequency domains are determined by taking the multi-dimensional Laplace transform of Volterra kernels of the related aeroelastic system via a Mathematica® code developed by these authors[28]. Our approach intended to address the subcritical response of the nonlinear aeroelastic governing equations, is based on its exact representation as an infinite sum of multidimensional convolution integrals, the first one, (i.e. the linear kernel) being the analogous to the linear indicial aeroelastic function. The full nonlinear aeroelastic response will be composed of additional higher-order contributions. In the frequency domain, if the nonlinear function governing a system is "smooth", then for small inputs the system must be asymptotically linear[6]. One of the key issues is to determine, corresponding to the considered type of structural, damping and aerodynamic nonlinearities, the pertinent Volterra's kernels. When also the active control is implemented, the corresponding Volterra's kernel should also be determined.

III. The Theory

In an attempt to make the paper reasonably self-contained, several elements associated with the indicial functions and Volterra's series as applied to aeroelastic system, will be supplied here.

A. Indicial Theory and Aerodynamic Loads

Using the *aerodynamic indicial functions* corresponding to the transient aerodynamic reaction to a step pulse, the aerodynamic forces and moments induced in any maneuver and flight regime can be determined. Aerodynamic forces and moments acting on a rapidly maneuvering aircraft are, in general, nonlinear functions of the motion variables, their time rate of change, and the history of the maneuvering (Tobak and Chapman[29]). However, in this study, the linear aerodynamic theory is adopted. Once the response of the system to a step change in one of the disturbing variables (i.e. the indicial response) is known, the indicial method permits determination of the response to an arbitrary schedule of disturbances. There is a critical value of the flight speed, referred to as the flutter speed, above which the steady motion becomes unstable. In a nonlinear aeroelastic system the flutter phenomenon corresponds to the instability known as the Hopf bifurcation, resulting in the case of *supercritical Hopf bifurcation* in finite amplitude oscillations, and in the case of the *subcritical Hopf bifurcation* in oscillations with increasing amplitudes, even if the system operates before reaching the flutter speed[30-32].

We need to mention that within the *nonlinear indicial theory*[33], the response of a nonlinear system to an arbitrary input can be constructed by integrating a nonlinear functional that involves the knowledge of the time-dependent input and the kernel response. Whereas, within the linear indicial theory the linear kernel or linear impulse response can be convoluted with the input to predict the output of a linear system, the nonlinear indicial theory constitutes a generalization of this concept. It can also be shown that the traditional Volterra-Wiener theory of nonlinear systems constitutes a subset of the nonlinear indicial theory. It should also be mentioned that the nonlinear unsteady aerodynamics valid throughout the subsonic incompressible/compressible, transonic and supersonic flight speed regimes can be used and determined via the use of nonlinear indicial functions in conjunction with the Volterra's series approach.

B. Volterra Functional Series Theory

As it was shown (Rugh[6], Schetzen[7]), within Volterra's series approach the full response in the time domain, $y(t)$, of the nonlinear systems with memory can be cast as:

$$y(t) = \sum_{k=0}^{\infty} y_k(t),$$ (1)

where, $y_k(t)$ is expressed as:

$$y_k(t) = \underset{k \ times}{\iiint\int_{-\infty}^{\infty}} h_k(t-\tau_1, t-\tau_2, \cdots t-\tau_k) \prod_{i=1}^{k} x(\tau_i) d\tau_i.$$ (2)

By a change of variables, it is possible to express Eq. (2) in contracted form as:

$$y_k(t) = \underset{k \ times}{\iiint\int_0^{\infty}} h_k(\tau_1, \tau_2, \cdots \tau_k) \prod_{i=1}^{k} x(t-\tau_i) d\tau_i.$$ (3)

It is assumed that $x(t) = 0$ for $\tau < 0$, implying that the system is causal.

With this restriction, all the integrals in the subsequent discussions are different from zero over the time range $[0, \infty)$. Restricting the development of Eq. (3) to the first three terms one obtains:

$$y(t) = \int h_1(\tau_1) x(t-\tau_1) d\tau_1 + \iint h_2(\tau_1, \tau_2) x(t-\tau_1) x(t-\tau_2) d\tau_1 d\tau_2$$
$$+ \iiint h_3(\tau_1, \tau_2, \tau_3) x(t-\tau_1) x(t-\tau_2) x(t-\tau_3) d\tau_1 d\tau_2 d\tau_3 + \cdots \qquad .$$ (4)

On the other hand, the response of the system can be expressed also in the frequency domain.

Volterra' series is essentially a polynomial approximation of the system, extension of Taylor series to systems with memory, while Volterra's kernels $h_i(s_i)$ are a direct extension of the impulse response concept of the linear system theory to multiple dimensions (Volterra[5], Rugh[6], Schetzen[7], Boyd[8]). Consequently, a multidimensional analogue of the impulse response can be used to characterize a nonlinear system (Silva[23-26]).

Having in view that the memory of aeroelastic systems is not infinite and, at the same time, the time-dependent external excitations, such as impulse, gust, blast and sonic-boom pressure signatures are non persistent, (in the sense that their effect decay as time unfolds), it is possible to characterize a nonlinear aeroelastic system via Volterra series. This fact is reflected in the interpretation of the Volterra kernels as higher order impulse response functions, i.e. $h(\tau_1, \cdots, \tau_n) \to 0$ as $\tau_1, \cdots, \tau_n \to \infty$.

We will use the definition of the *nonlinear transfer function or higher-order impulse response functions* namely:

$$H_n(s_1, s_2, \cdots s_n) = \iint \cdots \int h_n(\tau_1, \tau_2, \cdots, \tau_n) e^{-s_1\tau_1} e^{-s_2\tau_2} \cdots e^{-s_n\tau_n} d\tau_1 d\tau_2 \cdots d\tau_n,$$ (5)

as well as of their inverted counterparts:

$$h_n(\tau_1, \tau_2, \cdots \tau_n) = \left(\frac{1}{2\pi i}\right)^n \int_{\sigma_n - i\infty}^{\sigma_n + i\infty} \cdots \int_{\sigma_2 - i\infty}^{\sigma_2 + i\infty} \int_{\sigma_1 - i\infty}^{\sigma_1 + i\infty} H_n(s_1, s_2, \cdots, s_n) e^{s_1 \tau_1} e^{s_2 \tau_2} \cdots e^{s_n \tau_n} ds_1 ds_2 \cdots ds_n. \quad (6)$$

Once Volterra's kernels are known, the response of the nonlinear aeroelastic system can fully be identified. As demonstrated in the Schetzen works[7], without loss of generality, the kernels will be taken as symmetric, in the sense of $H_n(s_1, s_2, \ldots) = H_n(s_2, s_1, \ldots)$, and a similar relationship being valid for h_n, as well.

If we focalize the attention on the linear system, the Laplace transform $\mathcal{L}$ of the first term of Eq. (4) yields the familiar Laplace domain expression $Y(s) = H(s)X(s)$, where $Y(s), H(s), X(s)$ are the Laplace transforms of $y(\tau), h(\tau), x(\tau)$, respectively. $H(s)$ is the transfer function TF of the system. It is well known fact that for the linear system, either the first transfer function or the first kernel in time $h(\tau)$ encode all the information about the aeroelastic system. Moreover, as is well known, if the system is linear, i.e. superposition principle holds valid and is time invariant, the external load is uniquely related to the response by a convolution integral. With the use of functional series, i.e. the Volterra series, this functional representation can be extended to nonlinear systems. The comparison between the prediction of the linear aeroelastic responses of 2-D lifting surface in incompressible flow field based on the Volterra's series approach (using Theodorsen's function $C(k)$) and on the exact solution, based on convolution integrals (using Wagner's function $\phi(\tau)$) is presented in Fig. 1.

The excellent agreement of these two predictions, assess both the accuracy of the aeroelastic model and also the power of the methodology that combines Volterra's series and the indicial function.

IV. Mathematical Formulation

A. General Theory for Wing Sections Including Structural and Damping Nonlinearities in Plunging and Pitching

The aeroelastic governing equation of motion for 1- and 2-DOF lifting surfaces featuring structural and damping nonlinearities, that include the stiffnesses and the damping in plunging and pitching, will be analyzed next. In this sense: a 1-DOF lifting surface featuring purely plunging motion and a 2-DOF lifting surface featuring inertial and aerodynamic coupling in plunging h and pitching α will be analyzed. As previously mentioned, the unsteady aerodynamic

is considered linear. A harmonic time dependent external concentrated load is also included in the analysis. This can be considered to correspond, for example, to an engine mounted on an aircraft wing. As a result, a harmonic type loading due to the engine oscillations has to involve the motion of the wing.

As a characteristic of this approach, the *transfer functions* of the system would exist and would be the same for any excitations[34-36], such as impulse, gusts, airblast or sonic-booms (random or deterministic ones). This is due to the fact that transfer functions are independent of the input to the system, being a characteristic of the system itself. As a reminder, the validity of this method is based on the use of continuous polynomial type nonlinearities. For nonlinear ordinary differential systems, there are in general, an infinite number of Volterra kernels. In practice, one can handle only a finite number of terms in the series, which leads to the problem of truncation accuracy. However, Wiener[5] suggest that the first terms of the series may be sufficient to represent the output of a nonlinear system if the nonlinearities are not too strong.

The use of the multidimensional Laplace transform as a function of several variables is a tool useful in stationary nonlinear system theory. The multivariable convolutions can be represented in terms of products of Laplace transforms.

It is well known that the nonlinear aeroelastic systems cannot be described by a simple transfer function for two main reasons: a) the response encode both the unsteady aerodynamic loads and the external excitation effects, and b), in the nonlinear case the superposition principle is not applicable. It is also well known that any time-dependent external excitation, i.e. periodic or otherwise, can be represented, to an arbitrary degree of accuracy, by a sum of sinusoidal waves[34]. In this context, if the external load is expressed in terms of multiple sinusoidal forms (for example, a traveling gust loads) this is easily convertible in the exponential form, i.e.:

$$u(t) = A\cos(\omega_A t) + B\cos(\omega_B t) \Leftrightarrow u(t) = \frac{A}{2}\left(e^{s_A t} + e^{-s_A t}\right) + \frac{B}{2}\left(e^{s_B t} + e^{-s_B t}\right), \qquad (7)$$

where $s_A = i\omega_A$ and $s_B = i\omega_B$. For clarity of exposition, it is convenient to adopt this approach for a system with one degree of freedom *(1-DOF)*. These results have more general bearing and can be extended for systems with multi-degrees of freedom *(M-DOF)*. In fact, by using the classical approach of the one dimensional frequency response function, it is possible to derive an analytical form of the multi-dimensional frequency response characteristics of nonlinear systems.

The systems based on 1-DOF (plunging h) and 2-DOF (pitching α and plunging h), will be considered in the next sections.

B. Plunging Airfoil Motion in an Incompressible Flow Field

The nonlinear equation of an airfoil featuring plunging motion can be expressed as:

$$m\ddot{h}(t) + \sum_{i=1}^{n} \left(c_{hi} \left[\dot{h}(t) \right]^i + k_{hi} \left[h(t) \right]^i \right) - L_a(t) = L_b(t), \tag{8}$$

where i defines the degree of the considered nonlinearity. In the numerical simulations, i will assume the values 1,2,3, implying linear, quadratic and cubic nonlinearities. In addition, m is mass parameter and c_{hi}, k_{hi}, are the damping and stiffness parameters, respectively, that are associated to the damping and deflection in plunging corresponding to the i-th power. In the right hand side member of these equations, $L_b(\tau)$ denotes the external time-dependent load acting on the rigid wing counterpart. In Eq. (8) the related unsteady aerodynamic lift is represented as a function of the plunging degree of freedom h, only:

$$L_a(\tau) = -C_{L\alpha}\rho U_\infty^2 \int_{-\infty}^{\tau} \phi(\tau - \tau_0) h'' d\tau_0 - \tfrac{1}{2}\rho C_{L\alpha} U_\infty^2 h''. \tag{9}$$

The non-circulatory component present in Eq. (9) has been represented in term of convolution integral of the indicial Wagner's function.

In order to explain how this methodology works, let us to determine, in terms of Volterra series, how a system responds to a harmonic or periodic time-dependent load. Let consider a periodic external excitation in the form:

$$L_b(t) = \sum_{j=1}^{n} X_j e^{s_j t}. \tag{10}$$

As is well known, the information acquired by the case of the response to a harmonically time-dependent load can be used to obtain the response to any time-dependent external excitation. In fact, considering the case of a concentrated load arbitrarily located in the x, y plane of the wing, we have:

$$u(x, y, t) = A\delta(x - x_0, y - y_0) e^{i\omega t}, \tag{11}$$

where $\delta(\cdot)$, x_0 , y_0 , A, ω denote Dirac's distribution, location of the load, its amplitude and excitation frequency, respectively. Once determined the transfer function (labeled as *TF*)

corresponding to a given excitation frequency, its counterpart in the time domain can be obtained as the inverse Laplace transform $\mathcal{L}^{-1}$:

$$TF(x,y,t) = \mathcal{L}^{-1}\{TF(x,y,s)\} = \frac{1}{2\pi i}\int_{\sigma_i-i\infty}^{\sigma_i+\infty} TF(x,y,s)e^{st}ds. \tag{12}$$

In addition to the direct role in the determining of the response, the transfer function TF has then the role in determining the response to arbitrary time-dependent external excitations.

The general procedure to identify the aeroelastic kernels of various order $\overline{(1,n)}$, is to consider a general input in the form of Eq. (10) and to equate, for the generic term of n order, the coefficients of $X_1 X_2 \cdots X_n e^{(s_1+s_2+\cdots+s_n)t}$. As an example, the first aeroelastic Volterra's kernels that describes the linear system, obtained by neglecting the nonlinear terms in the aeroelastic governing equations, is obtained by considering the input load as $L_b(t) = X_1 e^{s_1t}$ (which in dimensionless form is expressed as $l_b(t) = (b/mU_\infty^2)X_1 e^{s_1t}$); the response of the system is postulated in the form $h(t) = H_1(s_1)X_1 e^{s_1t} + h.o.t$. Substituting $h(t)$ and its derivatives in the governing equation of motion, one determines the coefficient of $X_1 e^{s_1t}$.

In a linear aeroelastic formulation, the system is completely characterized by a transfer function $H_1(s_1)$ that contains the aerodynamic term as follow:

$$H_1(s_1) = \left(k_{h1} + ms_1^2 + c_{h1}s_1 + s_1\rho C_{L\alpha}bU_\infty C[-is_1b/U_\infty] + \tfrac{1}{2}\rho C_{L\alpha}s_1^2 b^2\right)^{-1}. \tag{13}$$

Herein the Theodorsen's function C, connected with the Wagner's indicial function $\phi(\tau)$ via the Laplace's transform as $C(-is) = s\int_0^\infty \phi(\tau)e^{-s\tau}d\tau$, has been included in the formulation. The terms underscored by the solid line correspond to the unsteady aerodynamic loads component (circulatory term), while the dotted line corresponds to the effect of the added mass. When the aerodynamic loads are neglected and for $s = i\omega$, this result coincides with that of the linear FRF, derived via the conventional Modal Analysis. Notice that, the condition $s = i\omega$ correspond to zero initial conditions; the effect of the initial condition on the nonlinear aeroelastic response can be included remembering the complete form of the complex variable s, in which also the real part of the variable is included, namely, $s = \sigma + i\omega$.

For purely mechanical systems, in the frequency domain, the response via Volterra series has been carried out by several authors. In the present study an alternative procedure, based on the

multivariable kernel transforms techniques, referred to as Higher-order Transfer Functions (HTFs) is pursued. The two above mentioned approaches can be correlated each other, and this is shown also in this work. Assuming zero initial condition, the frequency response functions (FRFs) are obtained from the transfer functions TFs, by replacing the Laplace transform variable s with $j\omega$ (Worden et al.[36]).

In the present nonlinear aeroelastic system, toward the estimation of higher order frequency response functions (HO FRFs) that are defined as the multi-dimensional Fourier Transform (MDFT) of the Volterra's kernels, a sequence of transfer functions are employed.

By the use of the linear frequency-response-function $H_1(s_1)$, the behavior of the linear system is easily determined. It will be necessary to find a complete set of Volterra kernel transforms $H_n(s_1, s_2, \cdots s_n)$ for nonlinear systems and for this, in practice, we will use a convergent truncated series.

Probing the system with a single harmonic yields only the information about the value of the transfer functions terms on the diagonal line of the plane s_1, s_2, in the Laplace transformed space, where $s_1 = s_2$. However, in order to obtain information elsewhere in this space, one should use multi-frequency excitations.

In the same way, the second order Volterra Kernel can be determined applying a load depending on two different frequencies expressed as: $L_b(t) = X_1 e^{s_1 t} + X_2 e^{s_2 t}$. In this case we can express the plunging response in the form:

$$h(t) = H_1(s_1)X_1 e^{s_1 t} + H_1(s_2)X_2 e^{s_2 t} + H_2(s_1, s_1)X_1^2 e^{2s_1 t} + H_2(s_2, s_2)X_2^2 e^{2s_2 t}$$
$$+ H_2(s_1, s_2)X_1 X_2 e^{(s_1+s_2)t} + H_2(s_2, s_1)X_2 X_1 e^{(s_2+s_1)t} + h.o.t. \qquad (14)$$

Substituting Eq. (14) in Eq. (8) and equating the terms containing $X_1 X_2 e^{(s_1+s_2)t}$ the second order aeroelastic Volterra Kernel in the Laplace transformed space is obtained:

$$H_2(s_1, s_2) = -(s_1 s_2 c_{h2} + k_{h2})H_1(s_1)H_1(s_2)H_1(s_1 + s_2), \qquad (15)$$

where:

$$H_1(s_1 + s_2) = \left(k_{h1} + (s_1 + s_2)^2 m + c_{h1}(s_1 + s_2)\right.$$
$$\left. + (s_1 + s_2)\rho C_{L\alpha}bU_\infty C[-i(s_1 + s_2)b/U_\infty] + \tfrac{1}{2}\rho C_{L\alpha}(s_1 + s_2)^2 b^2\right)^{-1}, \qquad (16)$$

is the first order Volterra Kernel in the Laplace transformed space at the frequency $\omega_1 + \omega_2$ (that is obtained from Eq. (13) in which s_1 is replaced by $s_1 + s_2$). Notice that the terms $H_2(s_1, s_1)$ and

$H_2(s_2,s_2)$ can be determined from Eq. (15) replacing s_2 with s_1 and vice versa, respectively. Following the same steps, applying the load $L_b(t) = X_1 e^{s_1 t} + X_2 e^{s_2 t} + X_3 e^{s_3 t}$, equating the terms in the form $X_1 X_2 X_3 e^{(s_1+s_2+s_3)t}$ and remembering that:

$$H_1(s_1 + s_2 + s_3) = \left(k_1 + (s_1 + s_2 + s_3)^2 m + c_1(s_1 + s_2 + s_3)\right.$$
$$\left. + (s_1 + s_2 + s_3)\rho C_{L\alpha} b U_\infty C[-i(s_1 + s_2 + s_3)b/U_\infty] + \tfrac{1}{2}\rho b^2 C_{L\alpha}(s_1 + s_2 + s_3)^2\right)^{-1}, \qquad (17)$$

the expressions for the third order Volterra Kernel in the Laplace transformed space results:

$$H_3(s_1,s_2,s_3) = -\frac{2}{3}(H_1(s_3)(3H_1(s_1)H_1(s_2)(k_{h3} + c_{h3}s_1 s_2 s_3)$$
$$+ 2H_2(s_1,s_2)(k_{h2} + c_{h2}(s_1 + s_2)s_3)) + 2(H_1(s_2)H_2(s_3,s_1)(k_{h2} + c_{h2}s_2(s_1 + s_3))$$
$$- H_1(s_2)H_2(s_2,s_3)(k_{h2} + c_{h2}s_1(s_2 + s_3))))/(H_1(s_1 + s_2 + s_3)) \qquad (18)$$

Notice that the constants k_{h2} and c_{h2} multiply the whole expression of H_2, and this term vanishes if the quadratic nonlinear term is absent in the aeroelastic governing equations. Herein one of the general properties of Volterra's series is recalled, namely that if all nonlinear terms in the equation of motion for the system consist of odd powers of x and y, then the associated Volterra series have no even-order kernels, and as a result it will possess no even-order *TFs*. It is also a general property of systems that all higher-order *TFs* can be expressed in terms of H_1. The expressions are function of the system and can be obtained by using the harmonic probing algorithm. It clearly appears that the higher order of FRFs, defined from the Volterra series, are independent of the input to the system.

C. Plunging-Pitching Airfoil Motion in an Incompressible Flow Field

The governing aeroelastic system of an airfoil featuring plunging - twisting coupled motion, exposed to a harmonic time dependent external excitation is:

$$m\ddot{h} + S_\alpha \ddot{\alpha} + \sum_{i=1}^{n}\left(c_{hj}\dot{h}^i + k_{hj}h^i\right) - L_a = L_b, \qquad (19)$$

$$S_\alpha \ddot{h} + I_\alpha \ddot{\alpha} + \sum_{i=1}^{n}\left(c_{\alpha j}\dot{\alpha}^i + k_{\alpha j}\alpha^i\right) - M_a = 0. \qquad (20)$$

Considering, as usual, the blast load $L_b(\tau)$ as uniformly distributed in the chordwise direction, no moment contribution $M_b(\tau)$ is introduced in Eq. (20).

Following the steps adopted for 1-DOF, applying a load depending on one frequency $L_b = X_1 e^{s_1 t}$, and expressing the plunging and pitching displacements in terms of transfer functions as:

$$h(t) = X_1 H_1^h(s_1) e^{s_1 t} + X_1^2 H_2^h(s_1, s_1) e^{2s_1 t} + X_1^3 H_3^h(s_1, s_1, s_1) e^{3s_1 t}, \tag{21}$$

$$\alpha(t) = X_1 H_1^\alpha(s_1) e^{s_1 t} + X_1^2 H_2^\alpha(s_1, s_1) e^{2s_1 t} + X_1^3 H_3^\alpha(s_1, s_1, s_1) e^{3s_1 t}, \tag{22}$$

the relative kernels and the aeroelastic responses can be determined.

The aeroelastic governing system including the blast pressure signatures can be expressed in the Laplace transformed space as:

$$s^2 \hat{\xi} + \chi_\alpha s^2 \hat{\alpha} + 2\zeta_h \frac{\overline{\omega}}{V} s \hat{\xi} + \left(\frac{\overline{\omega}}{V} \right)^2 \hat{\xi} + \frac{2}{\mu} \left(s\hat{\alpha} + s^2 \hat{\xi} + s^2 \left(\frac{1}{2} - a \right) \hat{\alpha} \right) \Phi(s) + \frac{1}{\mu} s^2 (\hat{\xi} - a\hat{\alpha}) + \frac{1}{\mu} s \hat{\alpha} = l_b(s), \tag{23}$$

$$(\chi_\alpha / r_\alpha^2) s^2 \hat{\xi} + s^2 \hat{\alpha} + (2\zeta_\alpha / V) s\hat{\alpha} + \hat{\alpha} / V^2 - \left(\frac{1}{2} + a \right) \frac{2}{\mu} \frac{1}{r_\alpha^2} \left(s\hat{\alpha} + s^2 \hat{\xi} + s^2 \left(\frac{1}{2} - a \right) \hat{\alpha} \right) \Phi(s)$$

$$- \frac{1}{r_\alpha^2} \frac{1}{\mu} a s^2 (\hat{\xi} - a\hat{\alpha}) + \left(\frac{1}{2} - a \right) \frac{1}{r_\alpha^2} \frac{1}{\mu} s\hat{\alpha} + \frac{1}{8} \frac{1}{r_\alpha^2} \frac{1}{\mu} s^2 \hat{\alpha} = 0. \tag{24}$$

Herein $\hat{(\cdot)} = \mathcal{L}(\cdot)$, and consequently $\hat{\xi} = \mathcal{L}(\xi(t))$ and $\hat{\alpha} = \mathcal{L}(\alpha(t))$.

Following the same steps, applying the loads $L_b(t) = X_1 e^{s_1 t} + X_2 e^{s_2 t}$ and $L_b(t) = X_1 e^{s_1 t} + X_2 e^{s_2 t} + X_3 e^{s_3 t}$, equating the terms in the forms $X_1 X_2 e^{(s_1 + s_2)t}$ and $X_1 X_2 X_3 e^{(s_1 + s_2 + s_3)t}$, the expressions for the second and third order Volterra Kernel in the Laplace transformed space can be obtained.

D. Generalization to Multi Degrees of Freedoms Systems (M-DOFs)

The method shown for 1-DOF and 2-DOF lifting surfaces can be extended to systems featuring multi degrees of freedoms in general, and to a 3-D aircraft wing in particular. The method of deriving the n-*th* order nonlinear aeroelastic transfer functions is based upon the fact that when the aeroelastic system described by the response $y(t)$ (expressed via Volterra series), is excited by a set of k unit amplitude exponentials at the arbitrary frequencies s_1, s_2, ... s_k, the output will contain exponential components of the form:

$$y(t) = \sum_{n=1}^{\infty} \sum_{m=1}^{k} \frac{n!}{m_1! m_2! \dots m_n!} H_n(s_1, s_2, \cdots s_n) e^{(m_1 s_1 + m_2 s_2 + \cdots + m_n s_n)t}. \tag{25}$$

The presence of nonlinearities causes harmonic excitations and sums of harmonics to appear in the response of the aeroelastic system. Due to the nonlinear formulation, different frequencies can be expected as well.

From the energetic point of view, we can observe that $H_1(s_1)$ produces a single frequency output in response to the simple input $e^{s_1 t}$. However, because the system is nonlinear, $H_2(s_1, s_2)$ takes into account the terms that produce an output energy corresponding to the sum of frequencies $\omega_1 + \omega_2$, or in other words to the input $e^{(s_1 + s_2)t}$. Similarly, the third order nonlinear aeroelastic kernel, will inject a mix of three input frequencies into the total system output (see Figs. 6-8). This is the great advantage of this methodology over the other approaches based on the first order frequency response function. In contrast to these methodologies Volterra's series approach is able to capture the transfer of energy between frequencies, that is typical for nonlinear systems.

V. Results and Discussion

To assess the versatility and provide a validation of this methodology, a comparison of the predictions of the aeroelastic response of nonlinear 2-D lifting surface using three approximations are shown in Figs. 2 and 3. The excellent agreement of the predictions, assess both the accuracy of the aeroelastic model and also the power of the methodology based on the Volterra series and indicial function approach. The first, the second and the third approximations of the aeroelastic response to the two loads *1-COSINE* gust load and triangular blast load are plotted for different parameters, together with the "exact" response of the aeroelastic system as obtained through digital-computer solution of the nonlinear aeroelastic governing equations. Both figures reveal the rapid convergence of the approximation. The same approach has been applied to a nonlinear time-varying system represented in Ref. 37 in which a transient response analysis of a continuous system has been addressed via functional techniques and multidimensional Laplace transformation. The impulse response of the system, represented by the differential equation:

$$\frac{dc(t)}{dt} + A c(t) + k m(t) c(t) + \varepsilon\, c^2(t) = R\,\delta(t), \tag{26}$$

evaluated with the present analysis coincides with that shown in the Ref. 37 in which the parameters in use are: $A = 1, R = 1, k = 1, 10, \varepsilon = 1, m(t) = \sin(\omega_0 t), \omega_0 = 2\pi, 20\pi$. Figs. 4 and 5

show the excellent agreement of these two approaches. In Figs. 5 the phase-space of the two responses have been supplied.

The results provided in these figures constitute a strong test of the speed of convergence of the present method[37,38]. The coefficients of the nonlinear elements are not small, nor is the period of the time-varying parameter long compared with the natural time constant of the system. For the 2-D lifting surface encompassing pure plunging, the first three aeroelastic kernels in magnitude and phase are depicted in Fig. 6 as a function of the frequency, considering the representation along the diagonal of the plane ω_1, ω_2, i.e. for $\omega = \omega_1 = \omega_2 = \omega_3$. As is clearly seen, a reduced influence on the response of the third kernel is experienced.

In Figs. 7 the Volterra's kernels for the lifting surface featuring plunging - pitching coupled motions are depicted. Also in this case the plots include the magnitude and phase for the kernels in plunging H_i^h and pitching H_i^α, in which i identifies the order of the kernel. In Figure 8, 3-D plots of the magnitude and phase of the second order kernel vs. the two frequencies ω_1 and ω_2 are provided. The contour plots reveal the symmetry of this kernel respect to the diagonal represented by $\omega_1 = \omega_2$. The third order Volterra kernel for the case in which $\omega_3 = \omega_1$ are depicted in a 3-D plot in Figs. 9.

A. Response in Time and Frequency Domains

Determination of the subcritical aeroelastic response to any time-dependent externally applied load is useful in the design of wing structures and of the associated feedback control systems. In some circumstances of the nonlinear analysis we are only interested in the special case implying $\tau_1 = \tau_2 = \cdots = \tau_n = \tau$ [16,39]. This case can be represented as:

$$g(\tau) \equiv h_n(\tau_1, \tau_2, \cdots \tau_n)\big|_{\tau_1 = \tau_2 = \cdots = \tau_n = \tau}.$$ (27)

This function $g(\tau)$ has a corresponding Laplace transform $G(s)$ (also called *associated transform*) in the single-dimensional Laplace transform space: $G(s) = \mathcal{L}[g(\tau)]$. The response in time can be obtained from $H(s_1, s_2, \cdots, s_n)$ by determining $G(s)$ first and evaluating the single dimensional inverse Laplace transform $g(\tau)$. This approach is called *association of variable*[16,39]. The nonlinear aeroelastic response in the time domain is depicted in Figs. 10 for a 2-D lifting surface featuring the plunging degree of freedom. In this figure the first plot represents the linear impulse response that corresponds to the convolution integral for the linear analysis. The other

three plots represent the components of the response due to the second and the third order kernels and the total response as a combination of the three partial responses. The influence of the stiffness and of the damping coefficients on the response are displayed in Figs. 11 and 12. An increase of the nonlinear damping or of the stiffness coefficients contributes to the decrease of the magnitude of the kernels and consequently, of the response amplitude. This shows that the nonlinearities in the stiffness and damping play a beneficial role on the subcritical aeroelastic response[18]. Figure 13 highlights the effect of the speed parameter V $\left(\equiv U/b\,\omega_\alpha\right)$ on the lifting surfaces subjected to sonic-boom pressure signature as shown in the inset. Herein τ_p denotes the positive phase duration of the pulse measured from the time of impact of the structure; r denotes the shock pulse length factor. For $r = 2$ the N-shaped pulse degenerates into a symmetric sonic-boom pulse, in the sense that its positive phase has the same characteristics as its negative one, and for $r = 1$ a triangular pulse that corresponds to an explosive pulse is obtained. It becomes apparent that the amplitude of the response time-history (that have been evaluated for practical use with three kernels) increases with the increase of V. Moreover, in a certain range of speeds, as time unfolds, a decay of the amplitude is experienced, which reflects the fact that in this case the subcritical response is involved. However, for the dimensionless speed parameter V greater then the flutter speed (this one was determined using the linearized aeroelastic system), the response becomes unbounded implying that the occurrence of the flutter instability is impending. Also in this case the nonlinear stiffness and damping coefficients play a beneficial role on the subcritical aeroelastic response.

VI. Conclusions

In this work several issues related to the approach of the nonlinear aeroelastic response via Volterra's series approach have been presented. Following the same approach presented here, the character of the instability boundary, i.e. benign or catastrophic can also be addressed. It was also shown that the method based on Volterra series opens large opportunities toward approaching in an unified and efficient way problems of nonlinear aeroelastic response and flutter instability. Moreover, this approach can be extended as to include also active control capabilities. Comparisons of results carried out via Volterra series in conjunction with indicial functions approach and classical approach have been provided in Fig. 1 for the linearized model. For the full nonlinear model, such validations have been displayed in Figs. 2-5, in which a

comparison with the exact digital evaluation of the nonlinear aeroelastic response has been supplied. In addition, it is worth noting that the aerodynamic indicial functions (for incompressible/compressible flow fields) considered in conjunction with Volterra's series approach can be used as a powerful analytical tool for developing unsteady aerodynamic models and a unified nonlinear aeroelastic model. To the best of the authors' knowledge, with the exception of this paper, the problem of the aeroelastic response of lifting surfaces to external pulses via Volterra's series and indicial function approach was not yet addressed in the literature.

Acknowledgment

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Figure Captions

Fig. 1 Aeroelastic response to Dirac delta impulse, as represented in inset. Comparison of response prediction based on the first Volterra kernel and the exact solution $(m = 1; c_{h1} = 0.1; k_{h1} = 10^2; b = 1; \rho = 0.125; U_\infty = 0.4U_F; C_{L\alpha} = 2\pi)$.

Fig. 2 Convergence study involving the first three kernels and comparison with the "exact" nonlinear aeroelastic response to a *1-COSINE* gust pulse, as shown in the inset; parameters as in Fig. 1.

Fig. 3 Convergence study involving the first three kernels and comparison with the "exact" nonlinear aeroelastic response to a *triangular* blast load, as shown in the inset; parameters as in Fig. 1.

Fig. 4 Convergence study involving the first three kernels and comparison with the "exact" nonlinear aeroelastic response of the impulse response described in Eq. (26) $(A = 1; R = 1; k = 1;$

$\varepsilon = 1; \omega_0 = 2\pi)$.

Fig. 5 Convergence study involving the first three kernels and comparison with the "exact" nonlinear aeroelastic response of the impulse response described in Eq. (26) (phase-space representation) $(A = 1; R = 1; k = 10; \varepsilon = 1; \omega_0 = 20\pi)$.

Fig. 6 Comparison of the first three aeroelastic kernels for pure plunging motion $(m = 1; c_{h1} = 10; k_{h1} = 10^4; c_{h2} = 10; k_{h2} = 10^7; c_{h3} = 10; k_{h3} = 10^{10}; b = 1; \rho = 0.125; U_\infty = 0.4U_F; C_{L\alpha} = 2\pi)$. Representation for $s_1 = s_2 = s_3$, i.e. $\omega_1 = \omega_2 = \omega_3$.

Fig. 7 First two aeroelastic kernels for plunging - pitching coupled motion $(m = 1; a = -0.2; c_{h1} = 10;$ $(k_{h1} = 10^4; c_{\alpha1} = 10; k_{\alpha1} = 10^4; c_{h2} = 10; k_{h2} = 10^7; c_{\alpha2} = 10; k_{\alpha2} = 10^7; b = 1; \rho = 0.125; U_\infty = 0.4U_F; C_{L\alpha} = 2\pi)$.

Fig. 8 3-D and contour plots of second order aeroelastic kernel; parameters as in Fig. 6.

Fig. 9 3-D and contour plots of third order aeroelastic kernel; parameters as in Fig. 6.

Fig. 10 Time-history of the nonlinear aeroelastic response $(U_\infty = 0.4U_F; C_{L\alpha} = 2\pi; m = 1; c_{h1} = 10;$ $k_{h1} = 10^4; c_{h2} = 10; k_{h2} = 10^7; c_{h3} = 10; k_{h3} = 10^{10}; b = 1; \rho = 0.125)$.

Fig. 11 Influence of the nonlinear stiffness coefficient k_{hi} on the nonlinear aeroelastic response; parameters as in Fig. 7.

Fig. 12 Influence of the nonlinear damping coefficient c_{hi} on the nonlinear aeroelastic response; parameters as in Fig. 7.

Fig. 13 Influence of the flight speed on the nonlinear aeroelastic response to a *sonic-boom* $(\tau_p = 15\sec; r = 2)$, as shown in the inset, evaluated with three kernels; parameters as in Fig. 7.

LINEAR AEROELASTIC RESPONSE TIME-HISTORY: COMPARISON
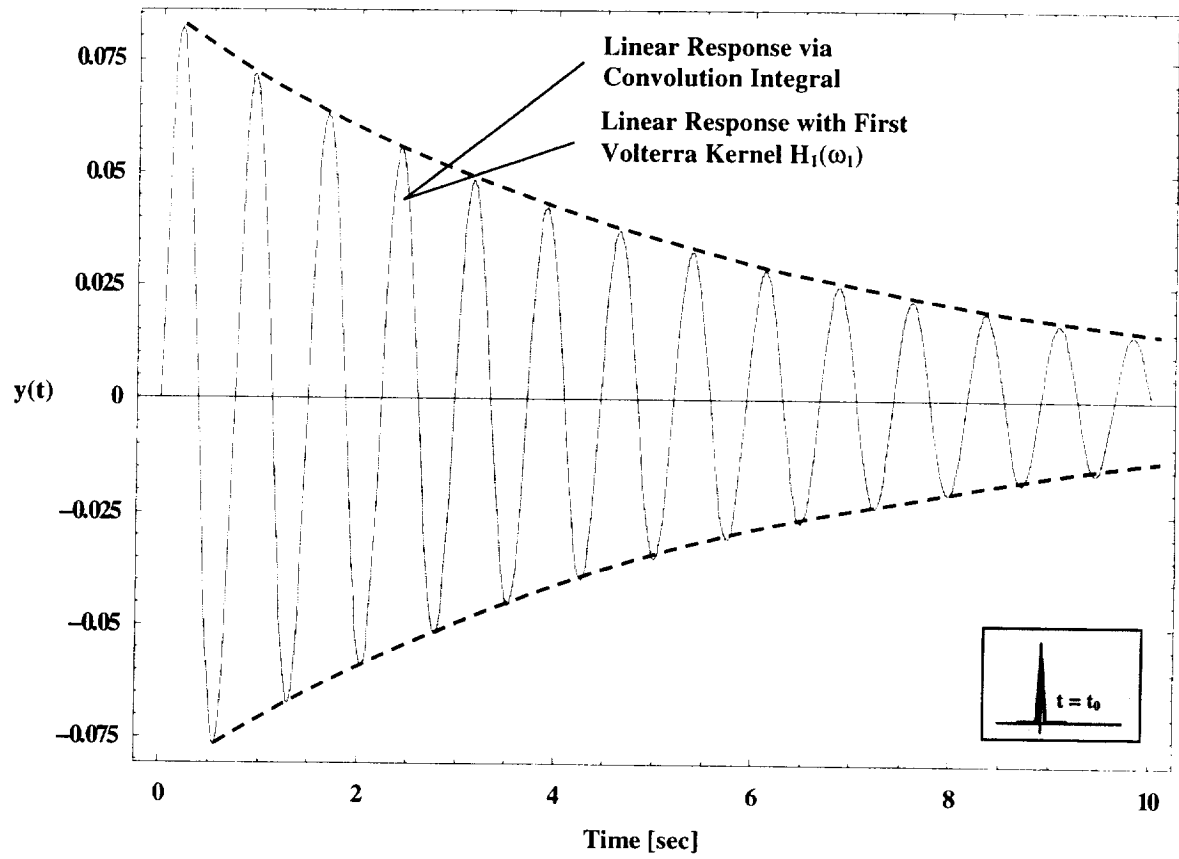
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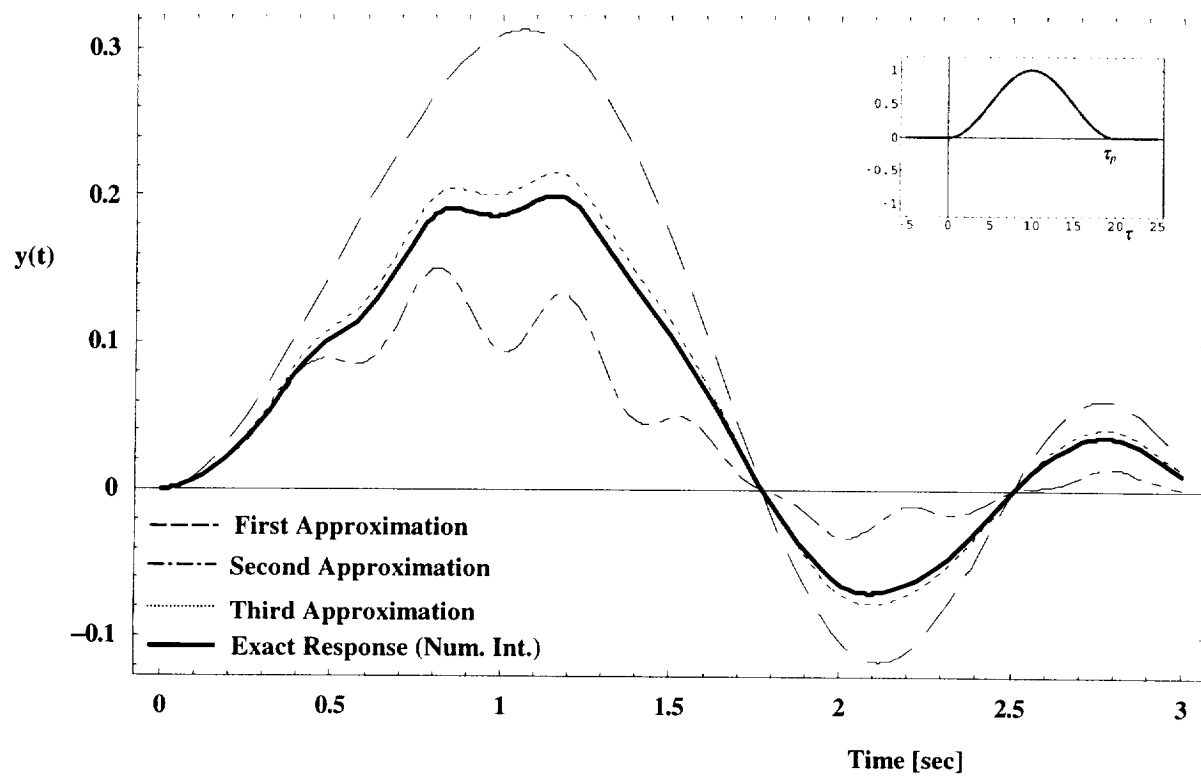
Fig. 2
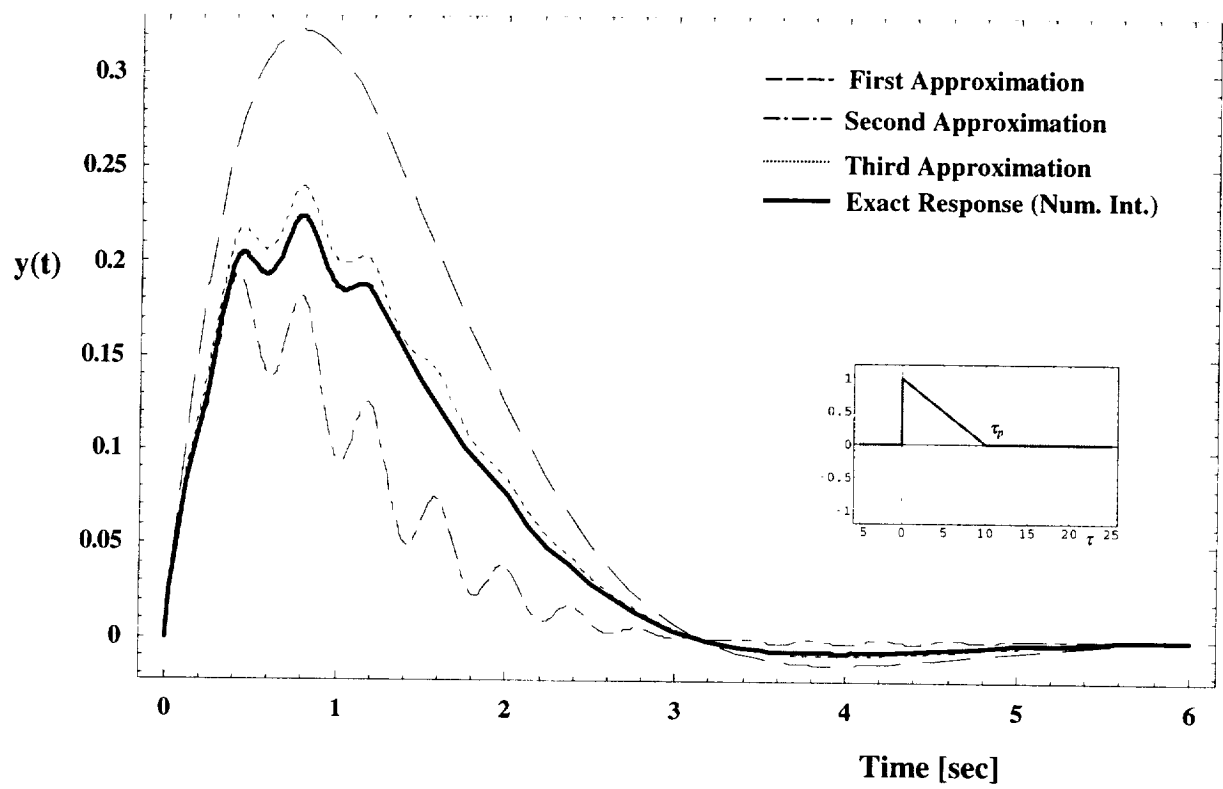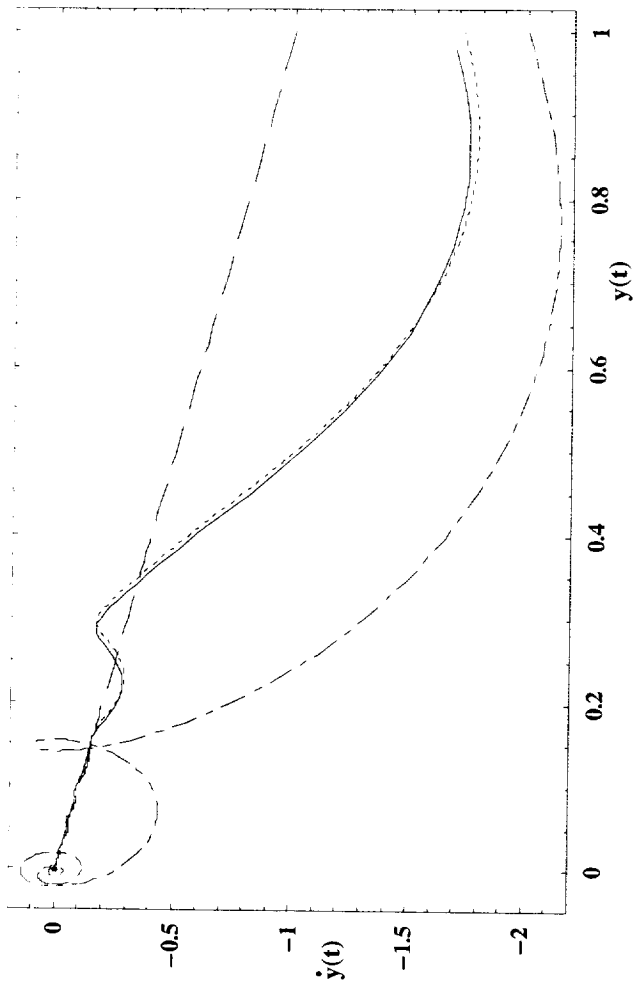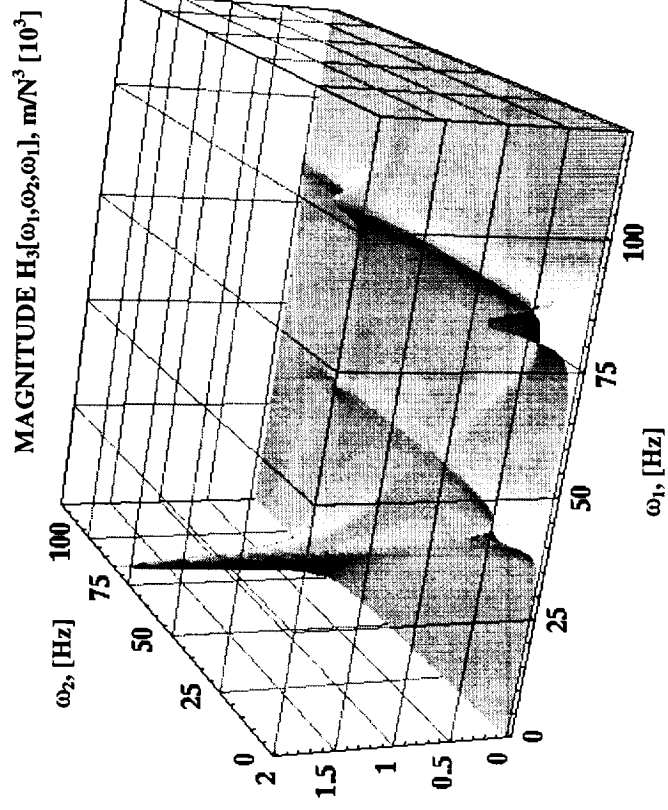
Fig. 3

PHASE-SPACE REPRESENTATION

RESPONSE TIME-HISTORIES

Fig. 5

Fig. 4

UNSTEADY AEROELASTIC KERNELS

MAGNITUDE [10^3]

First Order Kernel $H_1(\omega_1)$

Second Order Kernel $H_2(\omega_1, \omega_2)$

Third Order Kernel $H_3(\omega_1, \omega_2, \omega_3)$

Frequency, $\omega_1=\omega_2=\omega_3=\omega$, [Hz]

PHASE [deg]

Second Order Kernel
$H_2(\omega_1, \omega_2)$

First Order Kernel
$H_1(\omega_1)$

Third Order Kernel
$H_3(\omega_1, \omega_2, \omega_3)$

Frequency, $\omega_1=\omega_2=\omega_3=\omega$, [Hz]
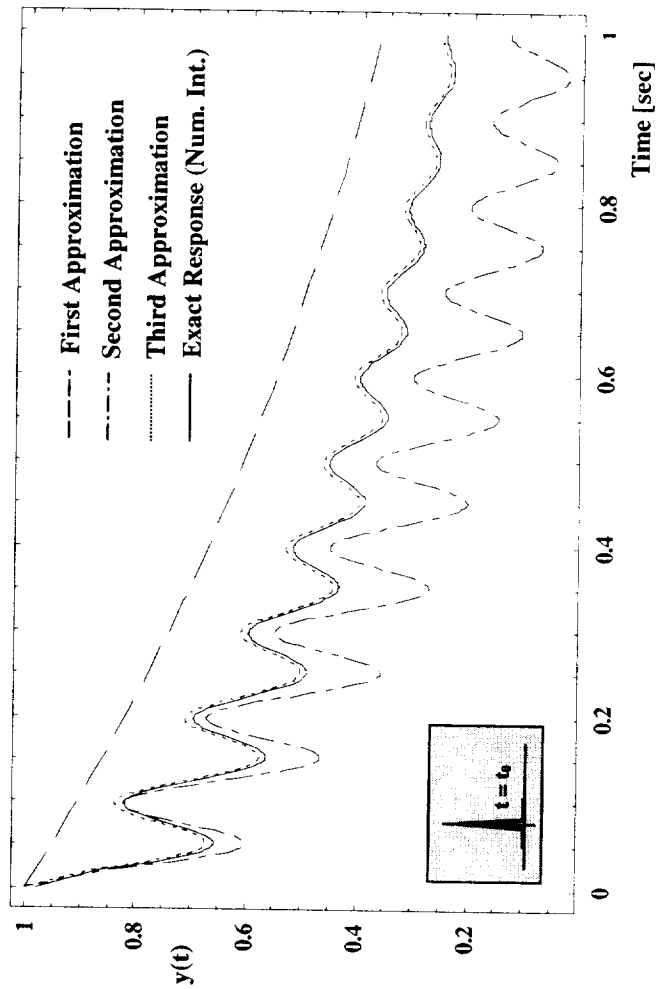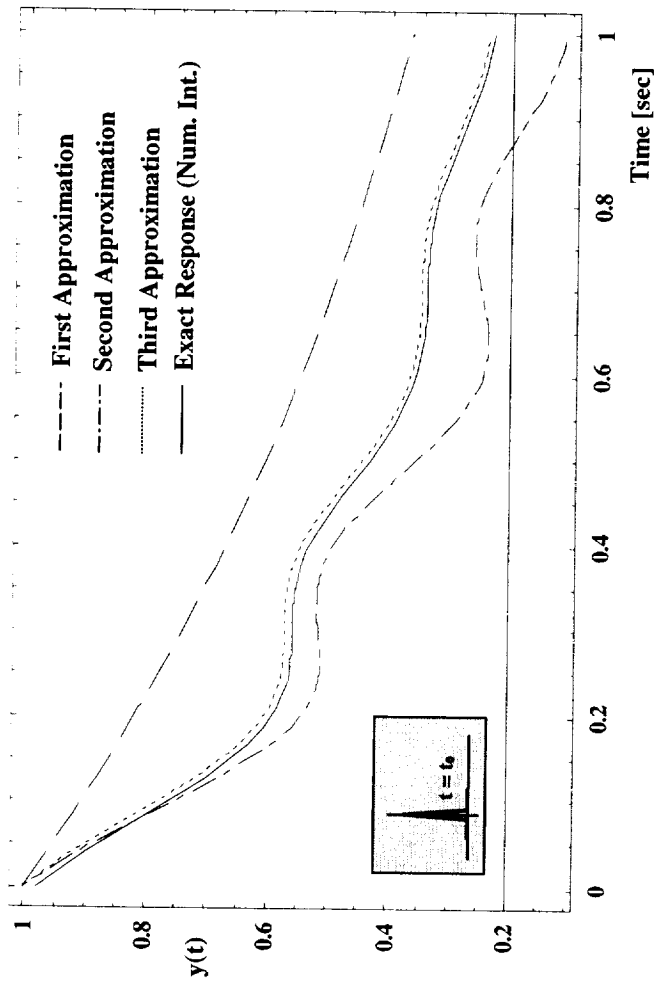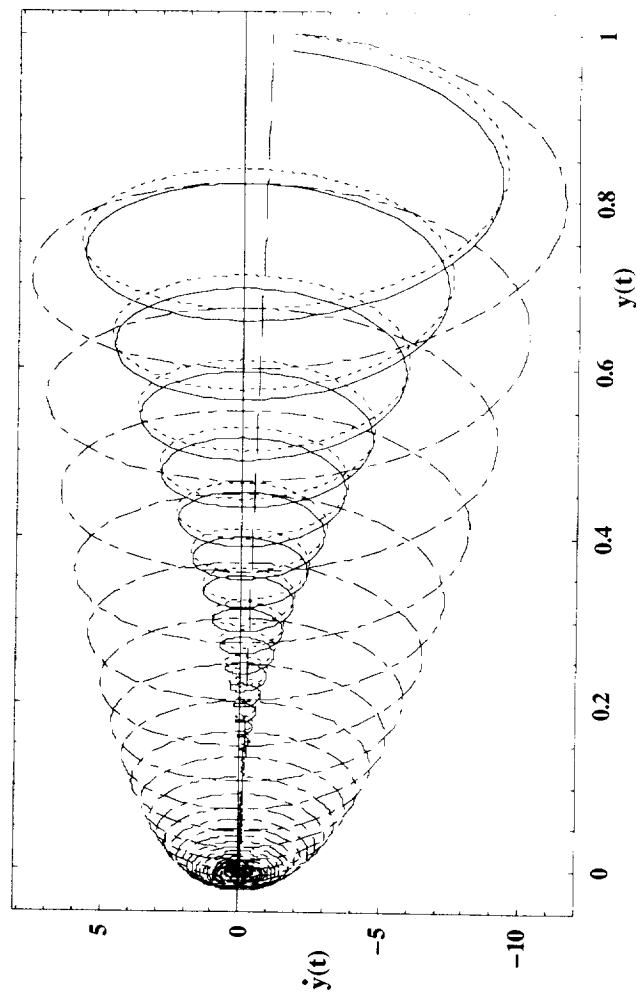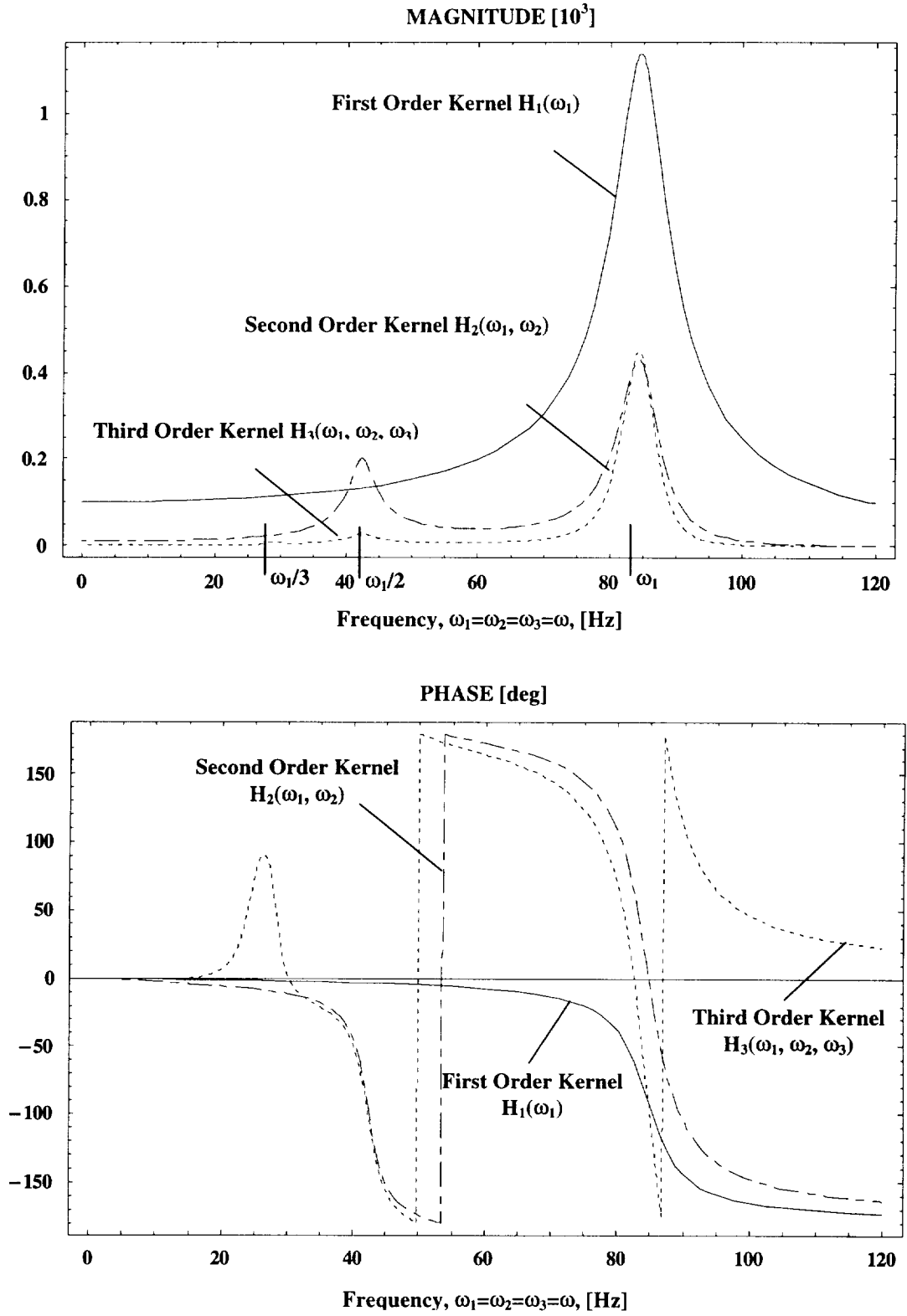
Fig. 6

AeroelasticSystem – Volterra Kernels

Fig. 7

MAGNITUDE H₂[ω₁,ω₂]

$\omega_1 = \omega_2$

$\omega_1 = -\omega_2$

ω_2, [Hz]

ω_1, [Hz]

PHASE H₂[ω₁,ω₂]

$\omega_1 = \omega_2$

$\omega_1 = -\omega_2$

ω_2, [Hz]

ω_1, [Hz]

MAGNITUDE H₂[ω₁,ω₂], m/N² [10³]

ω_2, [Hz]

ω_1, [Hz]

PHASE H₂[ω₁,ω₂], [deg]

ω_2, [Hz]

ω_1, [Hz]

Fig. 8

MAGNITUDE H₃[ω₁,ω₂,ω₁]

MAGNITUDE H₃[ω₁,ω₂,ω₁], m/N³ [10³]

PHASE H₃[ω₁,ω₂,ω₁]

PHASE H₂[ω₁,ω₂,ω₁], [deg]

ω₂, [Hz]

ω₁, [Hz]

ω₂, [Hz]

ω₁, [Hz]

Fig. 9

NONLINEAR AEROELASTIC RESPONSE TIME-HISTORIES

IMPULSE RESPONSE - FIRST ORDER KERNEL: LINEAR RESPONSE

Envelope of the linear response

$y_1(t)$

$t = t_0$

Time, [s]

IMPULSE RESPONSE - SECOND ORDER KERNEL

$y_2(t)$

$t = t_0$

Time, [s]

IMPULSE RESPONSE - THIRD ORDER KERNEL

$y_3(t)$

$t = t_0$

Time, [s]

NONLINEAR IMPULSE RESPONSE

$$y(t) = y_1(t) + y_2(t) + y_3(t)$$

$y(t)$

$t = t_0$

Time, [s]

NONLINEAR AEROELASTIC RESPONSE TIME-HISTORIES

IMPULSE RESPONSE - FIRST ORDER KERNEL:
LINEAR RESPONSE

IMPULSE RESPONSE - SECOND ORDER KERNEL
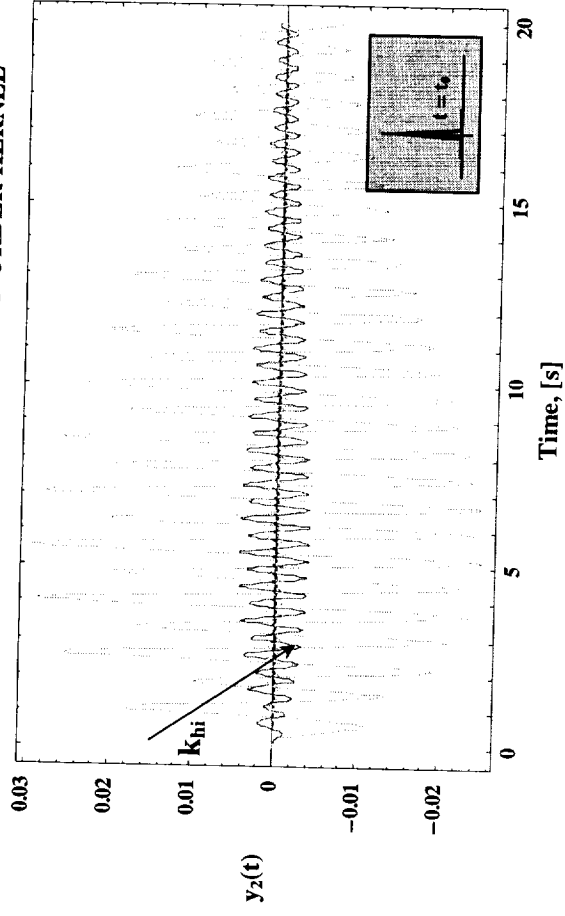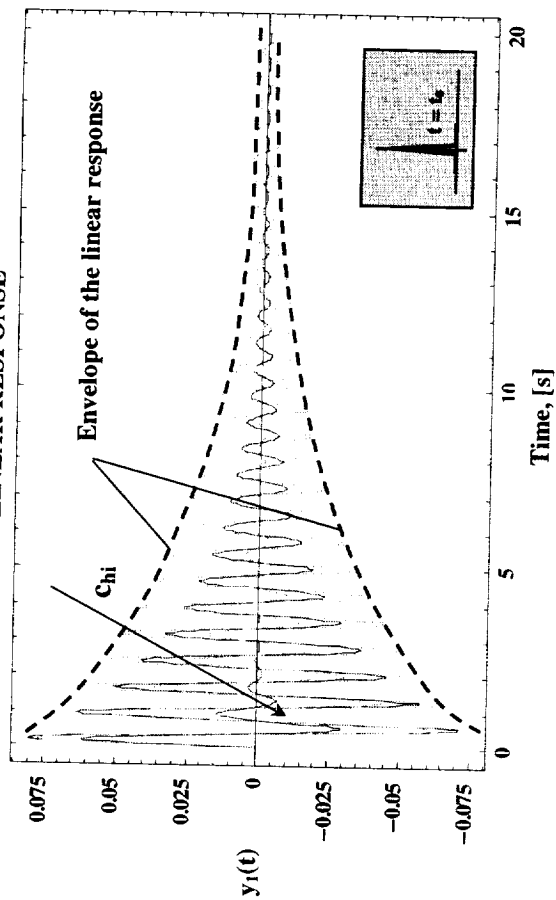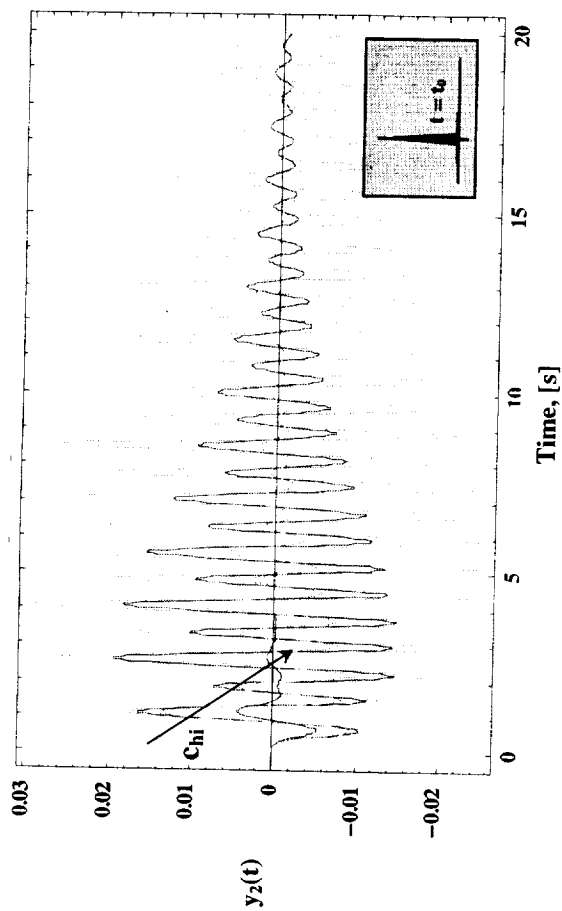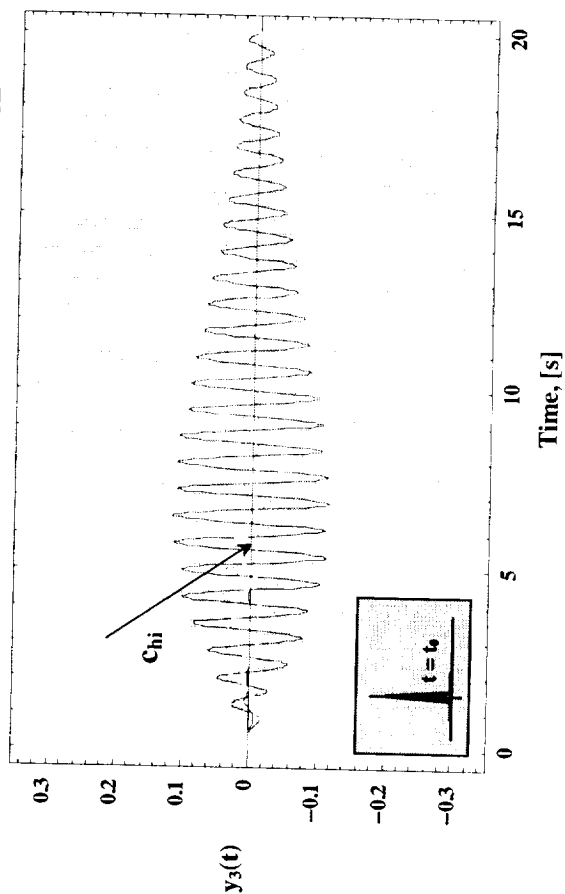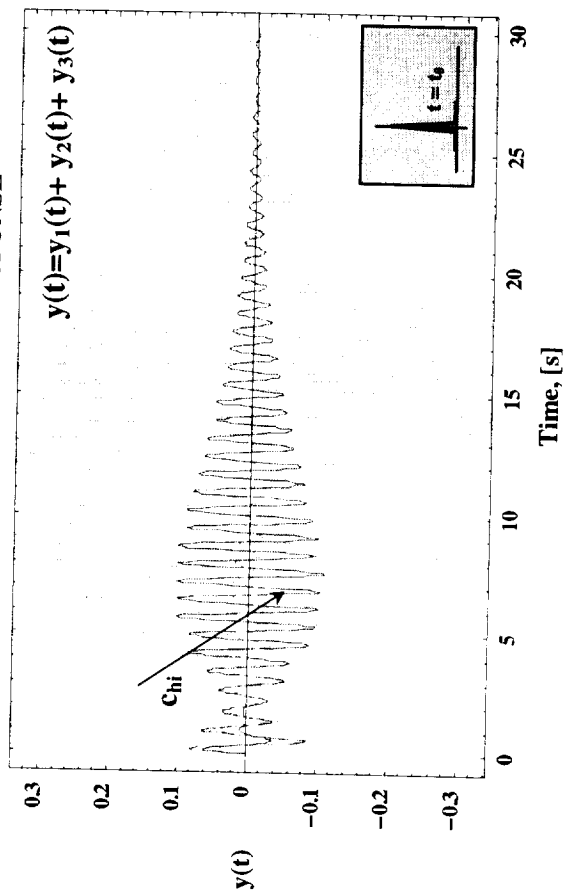
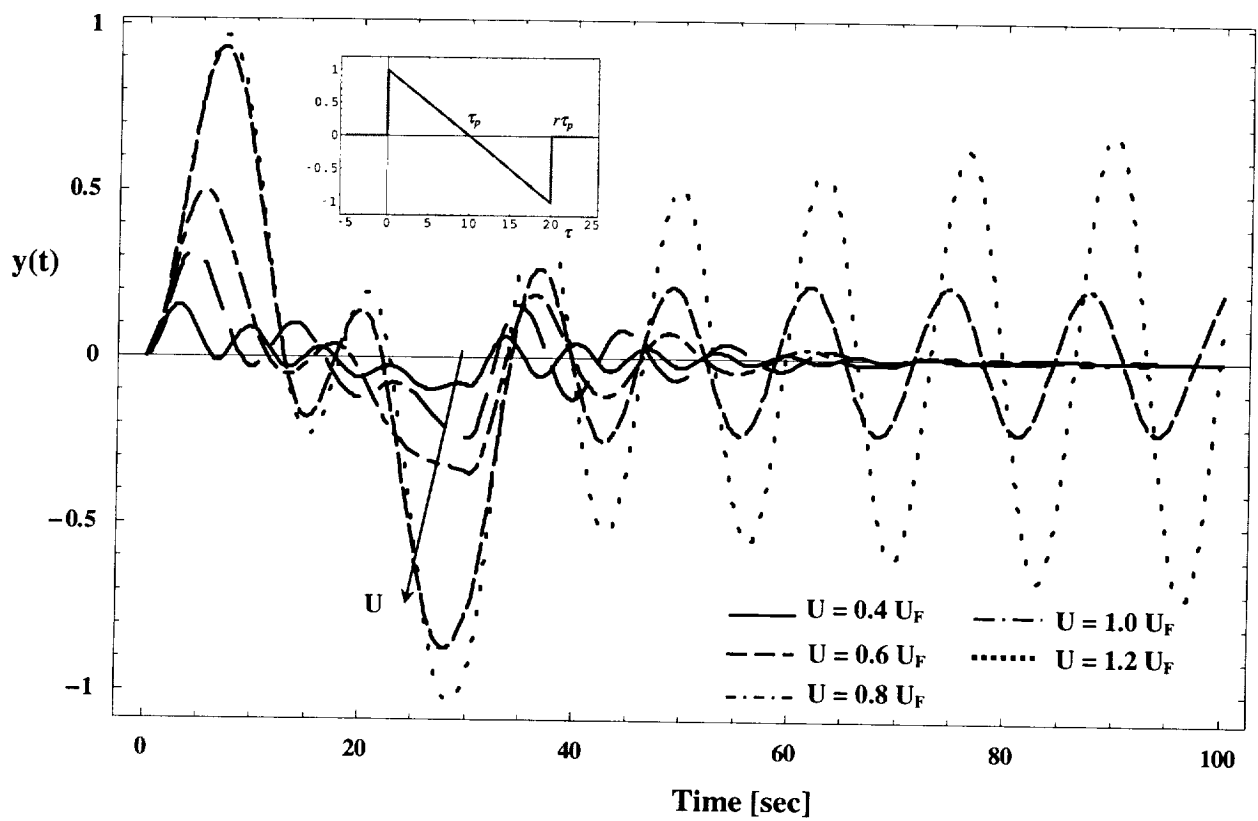## IMPULSE RESPONSE - THIRD ORDER KERNEL

NONLINEAR IMPULSE RESPONSE

$$y(t)=y_1(t)+ y_2(t)+ y_3(t)$$

NONLINEAR AEROELASTIC RESPONSE TIME-HISTORIES

IMPULSE RESPONSE - FIRST ORDER KERNEL: LINEAR RESPONSE

Envelope of the linear response

c_{hi}

$t = t_0$

$y_1(t)$

Time, [s]

IMPULSE RESPONSE - SECOND ORDER KERNEL

c_{hi}

$t = t_0$

$y_2(t)$

Time, [s]

IMPULSE RESPONSE - THIRD ORDER KERNEL

c_{hi}

$t = t_0$

$y_3(t)$

Time, [s]

NONLINEAR IMPULSE RESPONSE

$y(t) = y_1(t) + y_2(t) + y_3(t)$

c_{hi}

$t = t_0$

$y(t)$

Time, [s]

Fig. 13