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High Order Discontinuous Galerkin Methods for Convection Dominated Problems with Applications to Aeroacoustics

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This project is about the investigation of the development of the discontinuous Galerkin finite element methods, for general geometry and triangulations, for solving convection dominated problems, with applications to aeroacoustics. On the analysis side, we have studied the efficient and stable discontinuous Galerkin framework for small second derivative terms, for example in Navier-Stokes equations, and also for related equations such as the Hamilton-Jacobi equations. This is a truly local discontinuous formulation where derivatives are considered as new variables. On the applied side, we have implemented and tested the efficiency of different approaches numerically. Related issues in high order ENO and WENO finite difference methods and spectral methods have also been investigated.

In [9], jointly with Hu, we have presented a discontinuous Galerkin finite element method for solving the nonlinear Hamilton-Jacobi equations. This method is based on the Runge-Kutta discontinuous Galerkin finite element method for solving conservation laws. The method has the flexibility of treating complicated geometry by using arbitrary triangulation, can achieve high order accuracy with a local, compact stencil, and are suited for efficient parallel implementation. One and two dimensional numerical examples are given to illustrate the capability of the method.

In [8], jointly with Hu, we have constructed third and fourth order WENO schemes on two dimensional unstructured meshes (triangles) in the finite volume formulation. The third order schemes are based on a combination of linear polynomials with nonlinear weights, and the fourth order schemes are based on combination of quadratic polynomials with nonlinear weights. We have addressed several difficult issues associated with high order WENO schemes on unstructured mesh, including the choice of linear and nonlinear weights, what to do with negative weights, etc. Numerical examples are shown to demonstrate the accuracies and robustness of the methods for shock calculations.

In [13], jointly with P. Montarnal, we have used a recently developed energy relaxation theory by Coquel and Perthame and high order weighted essentially non-oscillatory (WENO) schemes to simulate the Euler equations of real gas. The main idea is an energy decomposition under the form $\varepsilon = \varepsilon_1 + \varepsilon_2$, where ε_1 is associated with a simpler pressure law (γ-law in this paper) and the nonlinear deviation ε_2 is convected with the flow. A relaxation process is performed for each time step to ensure that the original pressure law is satisfied. The necessary characteristic decomposition for the high order WENO schemes is performed on the characteristic fields based on the ε_1 γ-law. The algorithm only calls for the original pressure law once per grid point per time step, without the need to compute its derivatives or any Riemann solvers. Both one and two dimensional numerical examples are shown to illustrate the effectiveness of this approach.

In [1], jointly with Balsara, we have developed a class of higher order (from 7th to 13th) WENO schemes. Numerical tests are given in assessing the accuracy and non-oscillatory properties of such schemes. These schemes might be useful in situations when really high order accuracy is needed.

In [10], jointly with Lepsky and Hu, a discontinuous Galerkin finite element method for nonlinear Hamilton-Jacobi equations, first proposed by Hu and Shu, is further investigated from theoretical and computational points of view. This method handles the complicated geometry by using arbitrary triangulation, achieves the high order accuracy in smooth regions and the high resolution of the derivatives discontinuities. Theoretical results on accuracy and stability properties of the method are proven for certain cases and related numerical examples are presented.

In [11], jointly with Liu, we introduce a high order discontinuous Galerkin method for two dimensional incompressible flow in vorticity streamfunction formulation. The momentum equation is treated explicitly, utilizing the efficiency of the discontinuous Galerkin method. The streamfunction is obtained by a standard Poisson solver using continuous finite elements. There is a natural matching between these two finite element spaces, since the normal component of the velocity field is *continuous* across element boundaries. This allows for a correct upwinding gluing in the discontinuous Galerkin framework, while still maintaining total energy conservation with no numerical dissipation and total enstrophy stability. The method is suitable for inviscid or high Reynolds number flows. Optimal error estimates are proven and verified by numerical experiments.

In [2], jointly with Cockburn and Karniadakis, we have given a survey on discontinuous Galerkin method, summarizing the history, current status and open problems for the development of discontinuous Galerkin methods.

In [3], jointly with Cockburn, Luskin and Suli, we discuss the preliminary result on a post-processing technique to increase the order of accuracy of the discontinuous Galerkin method. It is well known that the discontinuous Galerkin (DG) method for scalar linear conservation laws has an order of convergence of $k + 1/2$ when polynomials of degree k are used and the exact solution is sufficiently smooth. In this paper, we show that a suitable post-processing of the DG approximate solution is of order $2k + 1$ in $\mathrm{L}^2(\Omega_0)$ where Ω_0 is a domain on which the exact solution is smooth enough. The post-processing is a convolution with a kernel whose support has measure of order one if the meshes are arbitrary; if the meshes are translation invariant, the support of the kernel is a cube whose edges are of size of order Δx only. The post-processing has to be performed only once, at the final time level. This work has been continued in [4] with a complete theory and extensive numerical tests.

In [7], jointly with Hu and Lepsky, we perform further investigation on the least square procedure used in the discontinuous Galerkin methods developed earlier by Hu and Shu for the two-dimensional Hamilton-Jacobi equations. The focus of this paper is upon the influence of this least square procedure to the accuracy and stability of the numerical results. We show through numerical examples that the procedure is crucial for the success of the discontinuous Galerkin methods developed earlier by Hu and Shu, especially for high order methods. New test cases using P^4 polynomials, which are at least fourth order and often fifth order accurate, are shown, in addition to the P^2 and P^3 cases presented earlier in Hu and Shu. This addition is non-trivial as the least square procedure plays a more significant role for the P^4 case.

In [12], jointly with Liu, we explore our recently introduced high order discontinuous Galerkin method for two dimensional incompressible flow in vorticity streamfunction formulation. In this method, the momentum equation is treated explicitly, utilizing the efficiency of the discontinuous Galerkin method. The streamfunction is obtained by a standard Poisson solver using continuous finite elements. There is a natural matching between these two finite element spaces, since the normal component of the velocity field is *continuous* across element boundaries. This allows for a correct upwinding gluing in the discontinuous Galerkin framework, while still maintaining total energy conservation with no numerical dissipation and total enstrophy stability. The method is suitable for inviscid or high Reynolds number flows. In our previous work, optimal error estimates are proven and verified by numerical experiments. In this paper we present one numerical example, the shear layer problem, in detail and from different angles to illustrate the resolution performance of the method.

In [5], jointly with Cockburn, we review the development of the discontinuous Galerkin method for solving partial differential equations. The main emphasis is on a review of the Runge-Kutta discontinuous Galerkin (RKDG) methods for non-linear convection-dominated problems. These robust and accurate methods have made their way into the main stream of computational fluid dynamics and are quickly finding use in a wide variety of applications. They combine a special class of Runge-Kutta time discretizations, that allows the method to be non-linearly stable regardless of its accuracy, with a finite element space discretization by discontinuous approximations, that incorporates the ideas of *numerical fluxes* and *slope limiters* coined during the remarkable development of the *high-resolution* finite difference and finite volume schemes. The resulting RKDG methods are stable, high-order accurate, and highly parallelizable schemes that can easily handle complicated geometries and boundary conditions. We review the theoretical and algorithmic aspects of these methods and show several applications including nonlinear conservation laws, the compressible and incompressible Navier-Stokes equations, and Hamilton-Jacobi-like equations.

In [6], jointly with Gottlieb and Tadmor, we review and further develop a class of strong stability preserving (SSP) high order time discretizations for semi-discrete method of lines approximations of partial differential equations. Termed TVD (total variation diminishing) time discretizations before, these high order time discretization methods preserve the strong stability properties of first order Euler time stepping and has proved very useful especially in solving hyperbolic partial differential equations. The new developments in this paper include the construction of optimal explicit SSP linear Runge-Kutta methods, their application to the strong stability of coercive approximations, a systematic study of explicit SSP multi-step methods for nonlinear problems, and the study of the strong stability preserving property of implicit Runge-Kutta and multi-step methods.

High order accurate weighted essentially non-oscillatory (WENO) schemes have recently been developed for finite difference and finite volume methods both in structured and in unstructured meshes. A key idea in WENO scheme is a linear combination of lower order fluxes or reconstructions to obtain a higher order approximation. The combination coefficients, also called linear weights, are determined by local geometry of the mesh and order of accuracy and may become negative. WENO procedures cannot be applied directly to obtain a stable scheme if negative linear weights are present. Previous strategy for handling this difficulty is by either regrouping of stencils or reducing the order of accuracy to get rid of the negative linear weights. In [15], jointly with Shi and Hu,, we present a simple and effective technique for handling negative linear weights without a need to get rid of them. Test cases are shown to illustrate the stability and accuracy of this approach.

In recent years high order numerical methods have been widely used in computational fluid dynamics (CFD), to effectively resolve complex flow features using meshes which are reasonable for today's computers. In [14] we review and compare three types of high order methods being used in CFD, namely the weighted essentially non-oscillatory (WENO) finite difference methods, the WENO finite volume methods, and the discontinuous Galerkin (DG) finite element methods. We summarize the main features of these methods, from a practical user's point of view, indicate their applicability and relative strength, and show a few selected numerical examples to demonstrate their performance on illustrative model CFD problems.

There are 15 papers published or submitted in this period acknowledging support from this NASA grant. These papers are listed in the References.

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