



AIAA 2001-1317

**FREE VIBRATION OF UNCERTAIN
UNSYMMETRICALLY LAMINATED BEAMS**

Rakesh K. Kapania and Vijay K. Goyal

Department of Aerospace and Ocean
Engineering

Virginia Polytechnic Institute and State
University

Blacksburg, VA 24061-0203

**42nd AIAA/ASME/ASCE/AHS/ASC
Structures, Structural Dynamics, and
Materials Conference and Exhibit
April 16-19, 2001/Seattle, Washington**

FREE VIBRATION OF UNCERTAIN UNSYMMETRICALLY LAMINATED BEAMS

Rakesh K. Kapania* and Vijay K. Goyal†

Department of Aerospace and Ocean Engineering

Virginia Polytechnic Institute and State University

Blacksburg, VA 24061-0203

Monte Carlo Simulation and Stochastic FEA are used to predict randomness in the free vibration response of thin unsymmetrically laminated beams. For the present study, it is assumed that randomness in the response is only caused by uncertainties in the ply orientations. The ply orientations may become random or uncertain during the manufacturing process. A new 16-dof beam element, based on the first-order shear deformation beam theory, is used to study the stochastic nature of the natural frequencies. Using variational principles, the element stiffness matrix and mass matrix are obtained through analytical integration. Using a random sequence a large data set is generated, containing possible random ply-orientations. This data is assumed to be symmetric. The stochastic-based finite element model for free vibrations predicts the relation between the randomness in fundamental natural frequencies and the randomness in ply-orientation. The sensitivity derivatives are calculated numerically through an exact formulation. The squared fundamental natural frequencies are expressed in terms of deterministic and probabilistic quantities, allowing to determine how sensitive they are to variations in ply angles. The predicted mean-valued fundamental natural frequency squared and the variance of the present model are in good agreement with Monte Carlo Simulation. Results, also, show that variations between $\pm 5^\circ$ in ply-angles can affect free vibration response of unsymmetrically and symmetrically laminated beams.

INTRODUCTION

In recent years, there has been an increasing demand for laminated composite materials in aircraft structures. The main reasons are that the composites possess the following characteristics: lightweight, cost-effective, and can handle different strengths in different directions¹. However, these materials offer quite a few challenges to structural engineers. Because of their inherent complexity, laminated structures can be difficult to manufacture according to their exact design specifications, resulting in un-

wanted uncertainties.

The design and analysis using conventional materials is easier than those using composites because, for conventional materials, both material and most geometric properties have either little or well known variation from their nominal value. On the other hand, the same cannot be said for the design of structures using laminated composite materials. The understanding of uncertainties in the laminate composite structures is highly important for an accurate design and analysis of aerospace and other structures using composite materials.

These uncertainties are defined as randomness from non-cognitive sources involving physical stochastic likelihood and human factors². This randomness can occur in each layer and involve quan-

*Professor, Associate Fellow AIAA, rkapania@vt.edu

†Graduate Research Assistant, Member AIAA, vgoyal@vt.edu

Copyright © 2001 by Rakesh K. Kapania and Vijay K. Goyal. Published by the American Institute of Aeronautics and Astronautics, Inc. with permission.

ties such as: ply-orientations, thickness, density, and material properties among others. Such variations can affect the behavior of the structure.

What has intrigued many engineers to study structures with laminated composites is the complexity of these materials. Since it is costly to analyze a composite structure as a three-dimensional solid, analysis of many composite structures can be performed using laminated one and two dimensional theories, such as beam theory. A review of various available theories for analyzing laminated beams is given by Raciti and Kapania³. Earlier, a 12-dof element was developed and formulated for deterministic symmetric laminated beams to study their static and dynamic behaviors⁴. Later a 20-dof element (Kapania-Raciti Element) was developed to study static, free vibration, buckling, and nonlinear vibrational analysis of unsymmetrically laminated beams^{5,6}. In both works the effects of uncertainties were not considered.

In considering uncertainties in a system, one can consider three different approaches: (i) probabilistic methods, (ii) fuzzy set or possibility-based methods, and (iii) antioptimization⁷. Here, however, only the probabilistic approach is considered. The probabilistic analysis can be performed using either an analytical or a computational approach. An analytical approach would be most accurate although cumbersome and impractical except for very simple systems. However, with the availability of extremely fast computers, the finite element method has become a widely used technique for design and analysis in engineering. Therefore, a finite element approach using Monte Carlo Simulation is developed to take into account the stochastic nature of the ply-orientations.

Work done by Vinckenroy⁸ presents a new technique to analyze these structures by combining the stochastic analysis and the finite element method in structural design. Agrawal et al.⁹ used a wavelet-based stochastic analysis to analyze isotropic beam structures.

Librescu et al.¹⁰ studied the free vibration and reliability of cantilever composite beams featuring structural uncertainties. They used a Stochastic Rayleigh-Ritz formulation. However, to the best of the authors' knowledge, no work has been found regarding the effect of uncertainties incurring in the ply-orientations on the natural frequencies of thin unsymmetrically laminated beams using Stochastic Finite Element Analysis.

The primary goal of this investigation is to predict the dynamic behavior of thin unsymmetrically

laminated beams with uncertainties. In order to study the stochastic nature of the dynamic response of such beams, a new 16-dof element is developed using first-order shear deformation beam theory to account for uncertainties. Only an overview of this element is presented here, and a more detailed analysis of this new 16-dof element will be presented elsewhere. Using this element the free vibration analysis is performed for only those uncertainties associated with layer-wise ply-orientations.

16-DOF LAMINATED BEAM ELEMENT

An overview

The present 16-dof laminated beam element takes into account the existence of various coupling effects, which play a major roll in laminate composite materials. This element is valid for the analysis of both symmetric and unsymmetrically laminated beams. The motivation to develop a new beam element was to have a formulation consistent with the first-order shear deformation beam theory (FSDT), able to analyze unsymmetrically laminated beams, and that would account for most of the uncertainties involved in a thin laminated beam when modelled using FSDT. In the present work, the reference system of coordinate is such that the x - axis lies along the length of the beam and the z - axis is placed at the mid-surface measuring the transverse displacements. The present work assumes that the x - z plane divides the beam in two identical parts: in other words, material, geometry, and loading are symmetric about the x - z plane. When considering twisting and ignoring in-plane shear, the displacement field for the first-order shear deformation beam theory, in the defined reference system, can be expressed as follows

$$U(x, y, z) = u(x) + z\phi(x) \quad (1a)$$

$$V(x, y, z) = 0 \quad (1b)$$

$$W(x, y, z) = w(x) - y\tau(x) \quad (1c)$$

Therefore, the finite element formulation considers eight degrees of freedom at each node: axial displacement u , transverse deflection w , rotation of the transverse normals ϕ , twist angle τ , and their derivatives with respect to x . These nodal displacements are denoted as

$$\{q\} = \{u, u', w, w', \phi, \phi', \tau, \tau'\}^T \quad (2)$$

The deflection behavior of the beam element for the first-order theory including transverse twist effect is

described as follows

$$\begin{aligned} u(x) &= N_1 u_1 + N_2 u'_1 + N_3 u_2 + N_4 u'_2 \\ w(x) &= N_1 w_1 + N_2 w'_1 + N_3 w_2 + N_4 w'_2 \\ \phi(x) &= N_1 \phi_1 + N_2 \phi'_1 + N_3 \phi_2 + N_4 \phi'_2 \\ \tau(x) &= N_1 \tau_1 + N_2 \tau'_1 + N_3 \tau_2 + N_4 \tau'_2 \end{aligned} \quad (3)$$

where the shape functions $N_i(x)$ are the well known Hermitian polynomials.

Constitutive Material Law

In this investigation, all laminated composites are considered as orthotropic materials. Also, considering a state of plane stress and eliminating e_{zz} from the stress-strain relationship, the reduced material coefficients are expressed as \bar{Q}_{ij} . Since in laminated composites each ply may have different orientation, the stresses are expressed in terms of an arbitrary angle θ and the transformed stress-strain relationship takes the form of¹¹

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{Bmatrix} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} & 0 & 0 \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} & 0 & 0 \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} & 0 & 0 \\ 0 & 0 & 0 & \bar{Q}_{44} & \bar{Q}_{45} \\ 0 & 0 & 0 & \bar{Q}_{45} & \bar{Q}_{55} \end{bmatrix} \begin{Bmatrix} e_{xx} \\ e_{yy} \\ 2e_{xy} \\ 2e_{yz} \\ 2e_{zx} \end{Bmatrix} \quad (4)$$

where \bar{Q}_{ij} are the transformed stresses¹.

[A], [B], [D] Matrices

The extensional matrix [A], the extensional-bending coupling matrix [B], and the bending stiffness matrix [D] are of great importance in the present work and these are calculated as follows¹¹

$$A_{ij} = \sum_{k=1}^{N_{lam}} \bar{Q}_{ij} (z_{k+1} - z_k) \quad i, j = 1, 2, 6 \quad (5)$$

$$A_{ij} = K \sum_{k=1}^{N_{lam}} \bar{Q}_{ij} (z_{k+1} - z_k) \quad i, j = 4, 5 \quad (6)$$

$$B_{ij} = \sum_{k=1}^{N_{lam}} \bar{Q}_{ij} \left(\frac{z_{k+1}^2 - z_k^2}{2} \right) \quad i, j = 1, 2, 6 \quad (7)$$

$$D_{ij} = \sum_{k=1}^{N_{lam}} \bar{Q}_{ij} \left(\frac{z_{k+1}^3 - z_k^3}{3} \right) \quad i, j = 1, 2, 6 \quad (8)$$

where $K = \frac{5}{6}$, the shear correction factor¹², and N_{lam} is the total number of plies considered. When considering symmetrically laminated composites, [B]

is identically zero and the coupling between bending and stretching vanish. However, for unsymmetrically laminated beams the coupling cannot be ignored and it must be included in the analysis. In the presence of uncertainties, laminate composite structures are no longer symmetric and the analysis of unsymmetrically laminated structures is a more accurate one.

Laminate Constitutive Relations

The basic constitutive relation is

$$\{N\} = [D] \{\varepsilon\} \quad (9)$$

where $\{N\}^T$ is the stress resultant vector, $[D]$ is the bending-stiffness matrix, and $\{\varepsilon\}^T$ is the stretching and bending strain vector.

The displacement field, Eq. (1), suggests that

$$2e_{xy} = \gamma_{xy} + z\kappa_{xy} = 0 \quad (10)$$

Therefore, N_{xy} and M_{xy} are not considered because they vanish in the strain energy formulation. In addition, the present formulation assumes:

$$M_{yy} = N_{yy} = 0 \quad (11)$$

This leads to the following bending-stiffness matrix:

$$[D] = \begin{bmatrix} [D_{I,I}] & [D_{I,II}] \\ [D_{II,I}] & [D_{II,II}] \end{bmatrix} \quad (12)$$

where

$$[D_{I,I}] = \begin{bmatrix} A_{11} & B_{11} \\ B_{11} & D_{11} \end{bmatrix} \quad (13)$$

$$[D_{I,II}] = \begin{bmatrix} A_{12} & B_{12} \\ B_{12} & D_{12} \end{bmatrix} \quad (14)$$

$$[D_{II,I}] = \begin{bmatrix} A_{12} & B_{12} \\ B_{12} & D_{12} \end{bmatrix} \quad (15)$$

$$[D_{II,II}] = \begin{bmatrix} A_{22} & B_{22} \\ B_{22} & D_{22} \end{bmatrix} \quad (16)$$

The reduced form of the bending-stiffness matrix is calculated as

$$[D_R] = [D_{I,I}] - [D_{I,II}] [D_{II,II}]^{-1} [D_{II,I}] \quad (17)$$

Using the above expression, an equivalent bending-stiffness matrix $[D_c]$ for a thin unsymmetrically laminated beam is found:

$$\begin{Bmatrix} N_{xx} \\ M_{xx} \\ Q_x \\ Q_y \end{Bmatrix} = \begin{bmatrix} D_{c11} & D_{c12} & 0 & 0 \\ D_{c12} & D_{c22} & 0 & 0 \\ 0 & 0 & D_{c33} & D_{c34} \\ 0 & 0 & D_{c34} & D_{c44} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \kappa_{xx} \\ \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix} \quad (18)$$

where,

$$\Delta = B_{22}^2 - A_{22} D_{22}$$

$$D_{c11} = (A_{22} B_{12}^2 - 2 A_{12} B_{12} B_{22} + A_{11} B_{22}^2 + A_{12}^2 D_{22} - A_{11} A_{22} D_{22}) / \Delta$$

$$D_{c12} = (B_{12}^2 B_{22} + B_{11} B_{22}^2 + A_{22} B_{12} D_{12} - A_{12} B_{22} D_{12} - A_{22} B_{11} D_{22} + A_{12} B_{12} D_{22}) / \Delta$$

$$D_{c22} = (B_{22}^2 D_{11} - 2 B_{12} B_{22} D_{12} + A_{22} D_{12}^2 + B_{12}^2 D_{22} - A_{22} D_{11} D_{22}) / \Delta$$

$$D_{c33} = A_{44}$$

$$D_{c34} = A_{45}$$

$$D_{c44} = A_{55}$$

Note that the availability of symbolic manipulator like MATHEMATICA® Version 4.0[†] has made it possible to determine the above matrix analytically. By obtaining this matrix analytically, the CPU time is saved, a great help in the Monte Carlo Simulation.

Strain-Displacement Relationship

The strains in the above formulation are related to the displacements as follows:

$$\{\epsilon\} = \begin{Bmatrix} \epsilon_{xx} \\ \kappa_{xx} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial^2 w}{\partial x^2} \\ -\tau \\ \frac{\partial w}{\partial x} - y \frac{\partial \tau}{\partial x} + \phi \end{Bmatrix} \quad (19)$$

Using Eqs. (2), (3) and (19) the strain-displacement relation can be expressed as

$$\{\epsilon\} = [B_{sd}] \{q\} \quad (20)$$

where $[B_{sd}]$ is the strain-displacement matrix.

Element Stiffness Matrix

Using Eqs. (9) and (20), and noting that matrix $[D_c]$ has been integrated throughout the thickness, the strain energy for this new 16-dof beam element becomes

$$U = \frac{1}{2} \{q\}^T \left[\iint_A [B_{sd}]^T [D_c] [B_{sd}] dA \right] \{q\} \quad (21)$$

[†]Mathematica is a registered trademark of Wolfram Research, Inc.

The minimization of Eq. (21) results in the element stiffness matrix for the 16-dof laminated beam element:

$$[K^e] = \int_0^{\ell_e} \int_{-\frac{b_0}{2}}^{\frac{b_0}{2}} [B_{sd}]^T [D_c] [B_{sd}] dy dx \quad (22)$$

where b_0 is the width of the beam, ℓ_e is the length of the beam element, and $[D_c]$ is the equivalent bending-stiffness matrix.

Element Mass Matrix

The kinetic energy for this 16-dof beam element is

$$T = \frac{1}{2} \iiint_V \rho [\dot{U}^2 + \dot{V}^2 + \dot{W}^2] dV \quad (23)$$

where ρ is the mass density. The mass matrix is obtained by substituting Eq. (1) into Eq. (23) and then taking the first variation of the kinetic energy:

$$\delta T = \frac{\partial T}{\partial u} \delta u + \frac{\partial T}{\partial w} \delta w + \frac{\partial T}{\partial \phi} \delta \phi + \frac{\partial T}{\partial \tau} \delta \tau \quad (24)$$

The mass matrix coefficients are obtained as follows

$$\delta u_j : \left[I_0 \int_0^{\ell_e} N_i(x) N_j(x) dx \right] u_i + \left[I_1 \int_0^{\ell_e} N_i(x) N_j(x) dx \right] \phi_i \quad (25)$$

$$\delta w_j : \left[I_0 \int_0^{\ell_e} N_i(x) N_j(x) dx \right] w_i \quad (26)$$

$$\delta \phi_j : \left[I_1 \int_0^{\ell_e} N_i(x) N_j(x) dx \right] u_i + \left[I_2 \int_0^{\ell_e} N_i(x) N_j(x) dx \right] \phi_i \quad (27)$$

$$\delta \tau_j : \left[J_x \int_0^{\ell_e} N_i(x) N_j(x) dx \right] \tau_i \quad (28)$$

where

$$I_0 = b_o \sum_{k=1}^{N_{lam}} \rho^k (z_{k+1} - z_k) \quad (29)$$

$$I_1 = b_o \sum_{k=1}^{N_{lam}} \rho^k \left(\frac{z_{k+1}^2 - z_k^2}{2} \right) \quad (30)$$

$$I_2 = b_o \sum_{k=1}^{N_{lam}} \rho^k \left(\frac{z_{k+1}^3 - z_k^3}{3} \right) \quad (31)$$

$$J_x = \frac{b_o^3}{12} \sum_{k=1}^{N_{lam}} \rho^k (z_{k+1} - z_k) \quad (32)$$

In the present work, analytical expressions for coefficients of both mass and stiffness matrices are obtained. Once again, this greatly saves CPU time for the Monte Carlo Simulation as one no longer has to perform numerical integration for each case.

FREE VIBRATION ANALYSIS

The Hamilton's principle is used to study the dynamic nature of the structure. The principle uses the Lagrangian which is defined as

$$\mathcal{L} = \mathcal{T} - \mathcal{U} - \mathcal{V} \quad (33)$$

where,

\mathcal{T} is the total kinetic energy defined by Eq. (23)

\mathcal{U} is the total strain energy defined by Eq. (21)

\mathcal{V} is the potential of the applied loads

The Hamilton's Principle can be represented as¹³

$$\delta \mathcal{H} = \int_{t_1}^{t_2} \{ \delta \mathcal{T} - \delta \mathcal{U} - \delta \mathcal{V} \} dt = 0 \quad (34)$$

The free vibration analysis is obtained by setting $\delta \mathcal{V} = 0$ in Eq. (34). This leads to the free-vibration equations of motion for the defined finite element problem:

$$[M] \{\ddot{q}\} + [K] \{q\} = 0 \quad (35)$$

where $[M]$ is the mass matrix and $[K]$ is the stiffness matrix. Further assuming a harmonic response, the solution of these equations results in an eigenvalue problem:

$$[[K] - \omega^2 [M]] \{q\} = 0 \quad (36)$$

A PROBABILISTIC APPROACH

An overview

The present analysis assumes that the random data is symmetrically distributed. Therefore, it assumes normal distribution¹⁴:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp \left[-\frac{1}{2} \left(\frac{x - \mu}{\sigma} \right)^2 \right] \quad (37)$$

where σ^2 is the variance of the random variable; and μ is the mean value of the random variable. In the present work, the random variables are considered as independent and are denoted as

$$\mathbf{x} = \{\theta_1, \theta_2, \dots, \theta_n\} \quad (38)$$

where θ_i 's are the ply-angles.

Various methods exist to analyze an uncertain unsymmetrically laminated beam by integrating stochastic aspects into the finite element modelling: perturbation techniques, Taylor Series, and Monte Carlo Simulation. These techniques have been used in the past two decades in fields involving randomness. Especially, in the last decade there has been a growing interest in applying these methods to better understand laminated composite structures. Moreover, with the finite element technique becoming so popular, there has been a new interest in integrating the stochastic nature of the structure in the finite element analysis¹⁵.

Monte Carlo Simulation

Monte Carlo Simulation, although computationally expensive, methods are quite versatile techniques capable of handling situations where other methods fail to succeed. The MCS is also used to verify the results obtained from other methods. Monte Carlo Simulation methods are based on the use of random numbers and probability statistics to investigate problems. For purposes of the present work a random number generator is used to generate possible angle-variations between -5.0° and 5.0° .

A large sample is generated and then using PDF's one evaluates the probability of having such values. The larger the number of simulations more the confidence in the probability distribution of the results obtained. Therefore, for the present analysis at least ten thousand realizations of the uncertain beam are performed, increasing the accuracy of the ply-angle distribution fitted to the sample data.

Stochastic Eigenvalue Analysis

The stochastic eigenvalues problem is expressed as

$$[K - \lambda M] \{\phi\} = 0 \quad (39)$$

where K , M , λ , and ϕ are the stochastic stiffness matrix, mass matrix, eigenvalues and eigenvectors, respectively.

The presence of uncertainties in ply angle orientations results in certain randomness in the extensional matrix $[A]$, bending-stretching coupling matrix $[B]$, and bending matrix $[D]$. Since the coefficients of these matrices are present in the equivalent bending stiffness matrix $[D_c]$, Eq. (18), the matrix $[D_c]$

will have certain randomness associated as well and these uncertainties are expressed as

$$[D_c] = D_c^0 + D_c^s \quad (40)$$

where D_c^0 is the equivalent bending stiffness matrix evaluated at the ply mean values and D_c^s is the equivalent bending stiffness matrix taking into account the random nature of the problem. As a result, these uncertainties affect the stiffness matrix. However, the mass matrix is not affected when only considering those uncertainties involving ply-orientations. Therefore, the mass matrix M is evaluated at the given deterministic values.

In problems where uncertainties are considered, there exists no density function describing the random nature of the system. The information is limited to only the mean values of the random variables. In such cases, perturbations techniques are suggested among other existent techniques^{15,16}. In the free vibrational analysis of the present problem, the random nature of the stiffness matrix, eigenvalues, and eigenvectors are studied using a Taylor series expansion up to second order about the mean of each random variable. The present formulation follows the approach followed by Librescu et al.¹⁰ Opposed to the work done by Librescu et al., the present formulation can use results provided by commercial finite element codes, greatly reducing the number of calculations, and calculates the sensitivity of the eigenvalues up to the second-order approximation.

Therefore, the random nature of the elastic stiffness, K , is expanded in terms of the mean-centered zeroth-, first-, and second-order rates of change with respect to the random variables as

$$\begin{aligned} K(x_1, x_2, \dots, x_n) = & \quad (41) \\ K^0 + \sum_{i=1}^n K_i^I \varepsilon_i + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n K_{ij}^{II} \varepsilon_i \varepsilon_j \\ K^0 = & K|_{\mathbf{x}=(\mu_1, \mu_2, \dots, \mu_n)} \\ K_i^I = & \frac{\partial K}{\partial x_i} \Big|_{\mathbf{x}=(\mu_1, \mu_2, \dots, \mu_n)} \\ K_{ij}^{II} = & \frac{\partial^2 K}{\partial x_i \partial x_j} \Big|_{\mathbf{x}=(\mu_1, \mu_2, \dots, \mu_n)} \\ \varepsilon_i = & x_i - \mu_i \end{aligned}$$

The eigenvalues and eigenvectors are also affected by uncertainties in the ply angles. The perturbed eigenvalues and eigenvectors are expressed in terms

of their mean-centered zeroth-, first-, and second-order rates of change with respect to the random variables as

Eigenvalues

$$\lambda(x_1, x_2, \dots, x_n) = \quad (42)$$

$$\lambda^0 + \sum_{i=1}^n \lambda_i^I \varepsilon_i + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \lambda_{ij}^{II} \varepsilon_i \varepsilon_j$$

$$\lambda^0 = \lambda \Big|_{\mathbf{x}=(\mu_1, \mu_2, \dots, \mu_n)}$$

$$\lambda_i^I = \frac{\partial \lambda}{\partial x_i} \Big|_{\mathbf{x}=(\mu_1, \mu_2, \dots, \mu_n)}$$

$$\lambda_{ij}^{II} = \frac{\partial^2 \lambda}{\partial x_i \partial x_j} \Big|_{\mathbf{x}=(\mu_1, \mu_2, \dots, \mu_n)}$$

Eigenvectors

$$\{\phi(x_1, x_2, \dots, x_n)\} = \quad (43)$$

$$\{\phi^0\} + \sum_{i=1}^n \{\phi_i^I\} \varepsilon_i + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \{\phi_{ij}^{II}\} \varepsilon_i \varepsilon_j$$

$$\{\phi^0\} = \{\phi\} \Big|_{\mathbf{x}=(\mu_1, \mu_2, \dots, \mu_n)}$$

$$\{\phi_i^I\} = \frac{\partial \{\phi\}}{\partial x_i} \Big|_{\mathbf{x}=(\mu_1, \mu_2, \dots, \mu_n)}$$

$$\{\phi_{ij}^{II}\} = \frac{\partial^2 \{\phi\}}{\partial x_i \partial x_j} \Big|_{\mathbf{x}=(\mu_1, \mu_2, \dots, \mu_n)}$$

After substituting Eqs. (42), (43), and (44) into Eq. (39), the stochastic eigenvalue problem is expressed as

$$\begin{aligned} & \left[K^0 + \sum_{i=1}^n K_i^I \varepsilon_i + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n K_{ij}^{II} \varepsilon_i \varepsilon_j \right] \\ & \times \left\{ \phi^0 + \sum_{i=1}^n \phi_i^I \varepsilon_i + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \phi_{ij}^{II} \varepsilon_i \varepsilon_j \right\} \\ & = \left(\lambda^0 + \sum_{i=1}^n \lambda_i^I \varepsilon_i + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \lambda_{ij}^{II} \varepsilon_i \varepsilon_j \right) [M^0] \\ & \times \left\{ \phi^0 + \sum_{i=1}^n \phi_i^I \varepsilon_i + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \phi_{ij}^{II} \varepsilon_i \varepsilon_j \right\} \quad (44) \end{aligned}$$

The uncertainties in the random variables are in general small. As a consequence, in the applied perturbation technique it is sufficient to only consider the first and second derivatives of eigenvectors and

eigenvalues with respect to the random variables. By equating the zeroth order terms of ε_i in Eq. (44), an eigenvalue problem for the mean-valued system is obtained. From which, the mean-centered zeroth derivative eigenvalues and associated eigenvectors are obtained

$$[K^0 - \lambda^0 M^0] \{\phi^0\} = 0 \quad (45)$$

Premultiplying the first and second order terms of ε_i in Eq. (44) by $\{\phi^0\}^T$ and simplifying, the following expressions are obtained

$$\{\phi^0\}^T [K_i^I - \lambda_i^I M^0] \{\phi^0\} = 0 \quad (46)$$

$$\{\phi^0\}^T [K_{ij}^{II} - \lambda_{ij}^{II} M^0] \{\phi^0\} = 0 \quad (47)$$

This results in expressions for the mean-centered first and second eigenvalue derivatives:

$$\lambda_i^I = \frac{\{\phi^0\}^T [K_i^I] \{\phi^0\}}{\{\phi^0\}^T [M^0] \{\phi^0\}} \quad (48)$$

$$\lambda_{ij}^{II} = \frac{\{\phi^0\}^T [K_{ij}^{II}] \{\phi^0\}}{\{\phi^0\}^T [M^0] \{\phi^0\}} \quad (49)$$

The advantage of this method is that the eigenvalue problem needs to be solved only once. The sensitivity analysis is done by using results from the mean-valued eigenvalue problem. This results in a great computational saving.

When studying the effect of uncertainties of random variables on the fundamental natural frequencies, it is convenient to study their squared value, i.e., eigenvalues. Ang¹⁶ gives a good description on statistical analysis.

The mean value of the eigenvalue for random variables is obtained by taking the expected value of Eq. (43)

$$\mu_\lambda = E[\lambda] = \lambda^0 + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \lambda_{ij}^{II} E[\varepsilon_i \varepsilon_j] \quad (50)$$

The variance of the eigenvalues is obtained as

$$Var[\lambda] = E[\lambda^2] - \mu_\lambda^2 \quad (51)$$

For symmetrically distributed independent random variables,

$$\begin{aligned} Var[\lambda] = & \sum_{i=1}^n \lambda_i^I \lambda_i^I E[\varepsilon_i^2] + \\ & \frac{1}{4} \sum_{i=1}^n \sum_{k=1}^n \lambda_{ii}^{II} \lambda_{kk}^{II} \{E[\varepsilon_i^2 \varepsilon_k^2] - E[\varepsilon_i^2] E[\varepsilon_k^2]\} \end{aligned} \quad (52)$$

where

$$E[\varepsilon_i^2] = \sum_{q=1}^{N_{sam}} \frac{(x_q - \mu_i)^2}{N_{sam} - 1}$$

$$E[\varepsilon_i^2 \varepsilon_k^2] = \sum_{q=1}^{N_{sam}} \frac{(x_q - \mu_i)^2 (x_q - \mu_k)^2}{N_{sam} - 1}$$

$$E[\varepsilon_i^2] E[\varepsilon_k^2] = \sum_{q=1}^{N_{sam}} \sum_{r=1}^{N_{sam}} \left[\frac{(x_q - \mu_i)(x_r - \mu_k)}{N_{sam} - 1} \right]^2$$

and N_{sam} is the number of samples, equal to ten thousand in this work.

The standard deviation is calculated as

$$\sigma_\lambda = \sqrt{Var[\lambda]} \quad (53)$$

Calculating derivatives

The above formulation only requires the calculation of the derivatives of the stiffness matrix. These derivatives are obtained by taking the derivatives of the equivalent bending-stiffness matrix, eq. (18). However, the derivative for the first two rows and columns (reduced bending-stiffness matrix) are more involved. Various numerical schemes exist to evaluate these derivatives. When using some of these numerical schemes, ill-conditioning could be a problem. This problem can be avoided by the following formulation which allows the derivatives to be obtained exactly by numerical multiplication. The technique consists in taking derivatives of eq. (17)

$$\begin{aligned} [D_R]_{,x_i} = & [D_{I,I}]_{,x_i} - \\ & [D_{I,II}]_{,x_i} [D_{II,II}]^{-1} [D_{II,I}] - \\ & [D_{I,II}] [D_{II,II}]^{-1}_{,x_i} [D_{II,I}] - \\ & [D_{I,II}] [D_{II,II}]^{-1} [D_{II,I}]_{,x_i} \end{aligned} \quad (54)$$

$$\begin{aligned} [D_R]_{,x_i x_j} = & [D_{I,I}]_{,x_i x_j} - \\ & [D_{I,II}]_{,x_i x_j} [D_{II,II}]^{-1} [D_{II,I}] - \\ & [D_{I,II}]_{,x_i} [D_{II,II}]^{-1}_{,x_j} [D_{II,I}] - \\ & [D_{I,II}]_{,x_i} [D_{II,II}]^{-1} [D_{II,I}]_{,x_j} - \\ & [D_{I,II}]_{,x_j} [D_{II,II}]^{-1}_{,x_i} [D_{II,I}] - \\ & [D_{I,II}] [D_{II,II}]^{-1}_{,x_i x_j} [D_{II,I}] - \\ & [D_{I,II}] [D_{II,II}]^{-1}_{,x_i} [D_{II,I}]_{,x_j} - \\ & [D_{I,II}]_{,x_j} [D_{II,II}]^{-1} [D_{II,I}]_{,x_i} - \\ & [D_{I,II}] [D_{II,II}]^{-1}_{,x_j} [D_{II,I}]_{,x_i} - \\ & [D_{I,II}] [D_{II,II}]^{-1} [D_{II,I}]_{,x_i x_j} \end{aligned} \quad (55)$$

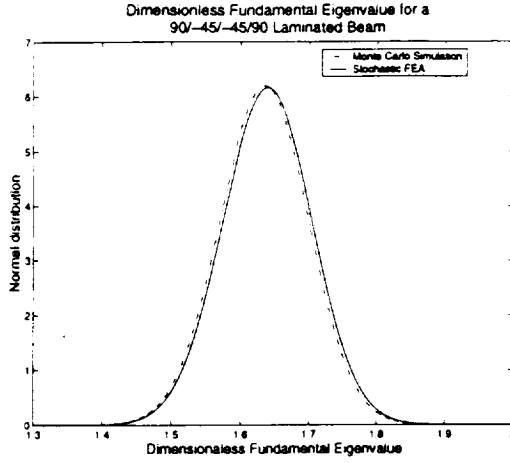


Figure 1: Dimensionless Eigenvalue distribution for MCS and SFEA for a thin symmetrically 4-ply laminated beams

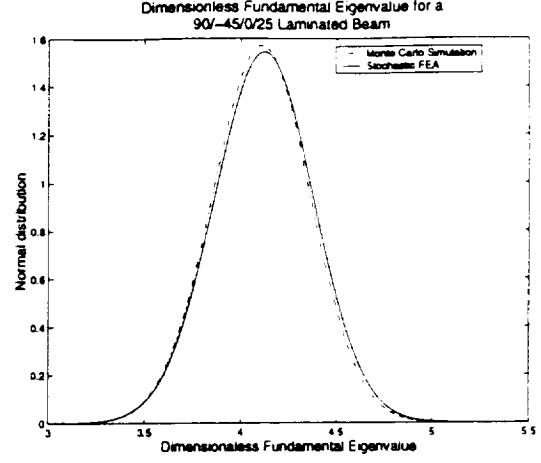


Figure 2: Dimensionless Eigenvalue distribution for MCS and SFEA for a thin unsymmetrically 4-ply laminated beams

The derivatives of $[D_{II,II}]^{-1}$ are calculated using the following matrix definition

$$[D_{II,II}]^{-1} [D_{II,II}] = [I] \quad (56)$$

and these derivatives being

$$[D_{II,II}]^{-1}_{,x_i} = -[D_{II,II}]^{-1} [D_{II,II}]_{,x_i} [D_{II,II}]^{-1} \quad (57)$$

$$[D_{II,II}]^{-1}_{,x_i, x_j} = -[D_{II,II}]^{-1}_{,x_i} [D_{II,II}]_{,x_j} [D_{II,II}]^{-1} - [D_{II,II}]^{-1}_{,x_j} [D_{II,II}]_{,x_i} [D_{II,II}]^{-1} - [D_{II,II}]^{-1} [D_{II,II}]_{,x_i, x_j} [D_{II,II}]^{-1} \quad (58)$$

RESULTS

The numerical results are obtained for a cantilevered thin laminated beam. The plies are assumed to be made of Graphite-Epoxy. The beam's material and geometrical properties used in the analysis are[§]:

Material density:

$$\rho = 1.449 \times 10^{-4} \frac{\text{slugs}}{\text{in}^3}$$

Major in-plane Poisson's ratio:

$$\nu_{xy} = 0.30$$

[§]All properties are assumed to be uniform throughout the beam

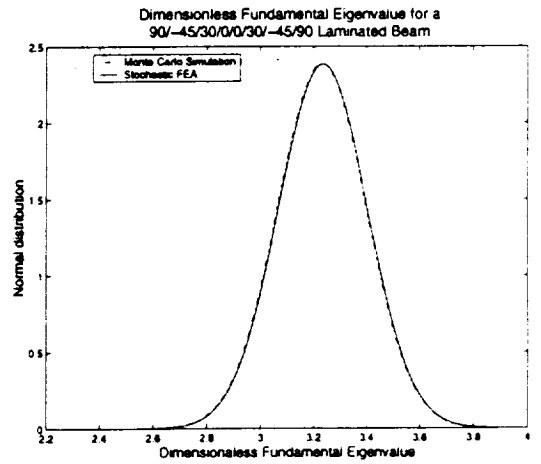


Figure 3: Dimensionless Eigenvalue distribution for MCS and SFEA for a thin symmetrically 8-ply laminated beams

Young's Modulus:

$$E_{yy} = 1.36 \times 10^6 \text{ psi}$$

$$E_{xx} = 13.75 E_{yy}$$

Shear Modulus:

$$G_{xy} = 0.55 E_{yy}$$

$$G_{yz} = 0.25 E_{yy}$$

$$G_{zx} = 0.25 E_{yy}$$

Table 1: Statistics analysis

Ply-orientations	μ_λ		$\sigma_\lambda \times 10^{-2}$	
	MSC	SFEA	MSC	SFEA
$[90/-45]_s$	1.63468	1.63925	0.41519	0.41928
$[90/-45/0/25]$	4.10664	4.12107	6.44410	6.67425
$[90/-45/30/0]_s$	3.23110	3.23559	2.79163	2.80235
$[90/-45/30/0/25/45/-90/-30]$	4.53781	4.54736	9.51499	9.69171

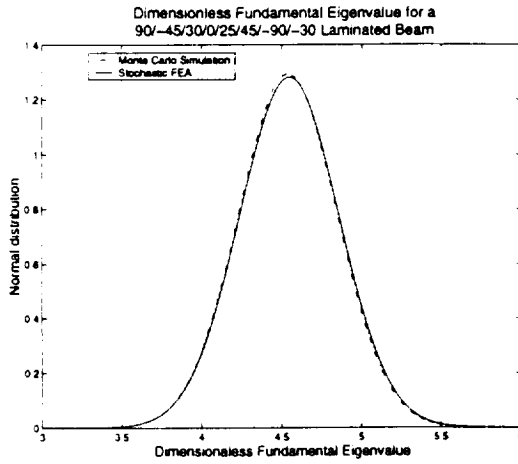


Figure 4: Dimensionless Eigenvalue distribution for MCS and SFEA for a thin unsymmetrically 8-ply laminated beams

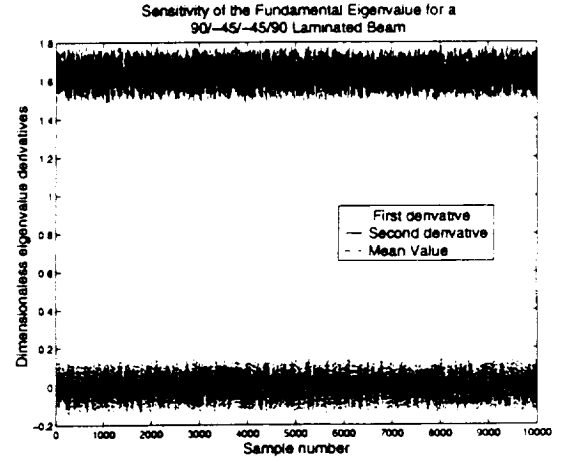


Figure 5: Sensitivity of the dimensionless eigenvalue for a thin symmetrically 4-ply laminated beams

Beam's dimensions

$$\begin{aligned}
 \text{Width : } b_o &= 1.0 \text{ in} \\
 \text{Thickness : } h_o &= 0.5 b_o \\
 \text{Length : } L_{beam} &= 30.0 h_o
 \end{aligned}$$

For both the stochastic finite element analysis and Monte Carlo Simulation ten beam elements were used and four different laminated beams are considered: (i) symmetrically laminated cantilever beam with four plies, (ii) and unsymmetrically lam-

inated cantilever beam with four plies, (iii) symmetrically laminated cantilever beam with eight plies, (iv) and unsymmetrically laminated cantilever beam with eight plies.

The natural frequencies are non-dimensionalized with respect to the deterministic quantities as follows

$$\hat{\omega}_n = \frac{\omega_n L_{beam}^2}{h_o} \sqrt{\frac{\rho}{E_{yy}}} \quad (59)$$

In general, the ply-angle uncertainties are between $\pm 2^\circ$. However, results show the effect in considering twice the uncertainty in ply orientations, i.e., from -5° to 5° . The statistical analysis is shown in Table

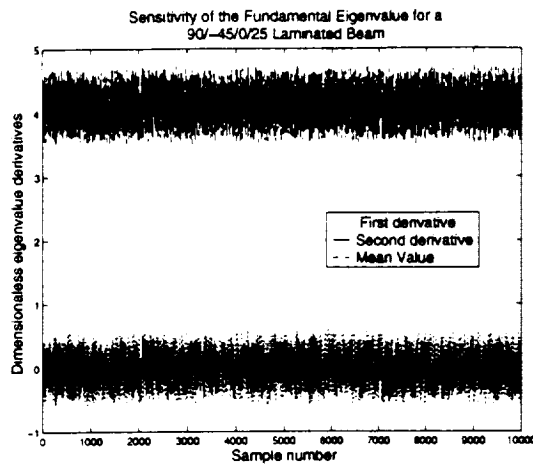


Figure 6: Sensitivity of the dimensionless eigenvalue for a thin unsymmetrically 4-ply laminated beams

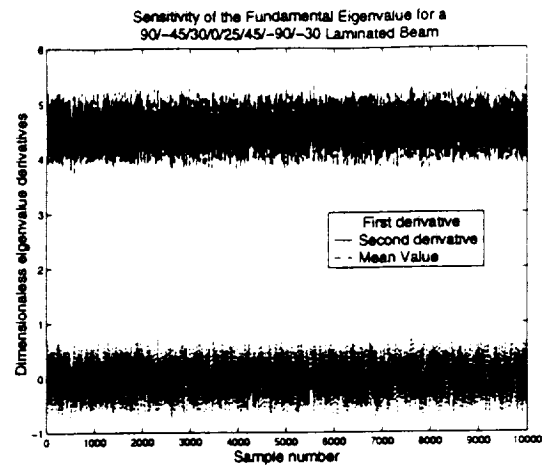


Figure 8: Sensitivity of the dimensionless eigenvalue for a thin unsymmetrically 8-ply laminated beams

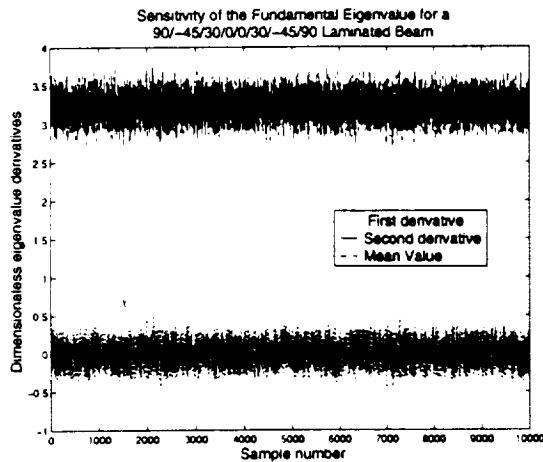


Figure 7: Sensitivity of the dimensionless eigenvalue for a thin symmetrically 8-ply laminated beams

1.

The results in Figures (1), (2), (3), and (4) show that the present model has a good correlation when compared to MCS. These results are obtained for only 1000 samples opposed to ten thousand using MCS. As the number of plies is increased the model correlated even better with MCS. The mean-valued fundamental eigenvalues are accurately obtained with one hundred samples using the present model opposed to ten thousand Monte Carlo Simulations.

Figures (5), (6), (7), and (8) show that the eigen-

values are sensitive to the first derivatives and the second derivatives are not influential. These figures show that the sensitivity analysis is significant, therefore the ply angle uncertainties can play an important roll in affecting free vibrations of symmetrically and unsymmetrically laminated thin beams.

CONCLUSIONS

Monte Carlo Simulation has been applied to thin, symmetric and unsymmetrically laminated beams with randomness in ply orientation to study the free vibrations. At least ten thousand realizations of the Monte Carlo sampling have been performed to improve the accuracy of the analysis.

A second stochastic finite element approach has been developed using perturbation methods. Using Taylor Series expansion the eigenvalues has been expressed as mean-valued and probabilistic quantities. The accuracy of the results have been compared to those obtained by Monte Carlo Simulation.

An elegant way to obtain sensitivity derivatives is detailed. The present method is advantageous over other techniques because the eigenvalue problem needs to be solved only once. With only one hundred samples our model agrees with ten thousand MCS.

Based upon the results, this method results in a great computational saving when one is interested in predicting the statistics of the fundamental natural frequency of unsymmetric laminated beam in the presence of ply-angle uncertainties.

AKNOWLEDGMENTS

This work was performed under the grant NAG-1-2277, with Dr. Lucas Horta and Dr. Howard Adelman as the grant monitors, and the grant NAG-1-2286, with Dr. I. S. Raju as the grant monitor, from the NASA Langley Research Center. The authors gratefully acknowledge the technical discussions with all the grant monitors. We are also thankful for the tremendous computational resources provided by the Virginia Tech's College of Engineering.

REFERENCES

- ¹Jones, R. M., *Mechanics of Composite Materials*, Taylor & Francis, Philadelphia, PA, 2nd ed., 1999.
- ²Chen, L. and Mulkay, E., "A Unified Finite Element Method For Engineering Systems with Hybrid Uncertainties," *AIAA-98-1827*, 1998, pp. 1120-1130.
- ³Kapania, R. K. and Raciti, S., "Nonlinear Vibrations of Unsymmetrically Laminated Beams," *AIAA Journal*, Vol. 27, No. 2, Feb. 1989, pp. 201-210.
- ⁴Chen, A. T. and Yang, T. Y., "Static and Dynamic Formulation of a Symmetrically Laminated Beam Finite Element for a Microcomputer," *Journal of Composite Materials*, Vol. 19, Sept. 1985, pp. 459-475.
- ⁵Kapania, R. K. and Raciti, S., "Recent Advanced in Analysis of Laminated Beams and Plates Part I: Shear Effects and Buckling," *AIAA Journal*, Vol. 27, No. 7, 1989, pp. 923-934.
- ⁶Kapania, R. K. and Raciti, S., "Recent Advanced in Analysis of Laminated Beams and Plates Part II: Vibrations and Wave Propagation," *AIAA Journal*, Vol. 27, No. 7, 1989, pp. 935-946.
- ⁷Elishakoff, I., "Three Versions of the Finite Element Method Based on Concepts of Either Stochasticity, Fuzziness, or Antioptimization," *Proceedings of the Tenth International Modal Analysis Conference*, Vol. 51, 1998, pp. 209-218.
- ⁸Vinckenroy, G. V., de Wilde, W. P., and Vantomme, J., "Monte Carlo-based Stochastic Finite Element Method: A New Approach for Structural Design," *Vrije Universiteit Brussel*, 1995.
- ⁹Mei, H., Agrawal, O. P., and Pai, S. S., "Wavelet-Based Model for Stochastic Analysis of Beam Structures," *AIAA Journal*, Vol. 36, No. 3, 1998, pp. 465-470.
- ¹⁰Oh, D. H. and Librescu, L., "Free Vibration and Reliability of Composite Cantilevers Featuring Uncertain Properties," *Reliability Engineering and System Safety*, Vol. 56, 1997, pp. 265-272.
- ¹¹Reddy, J. N., *Mechanics of Laminated Composite Plates: Theory and Analysis*, CRC Press, New York, 1997.
- ¹²Reissner, E., "A Consistent Treatment of Transverse Shear Deformations in Laminated Anisotropic Plates," *AIAA Journal*, Vol. 10, 1972, pp. 716.
- ¹³Shames, I. H. and Dym, C. L., *Energy and Finite Element Methods in Structural Mechanics*, Taylor & Francis, Philadelphia, PA, 1985.
- ¹⁴Vinckenroy, G. V. and de Wilde, W. P., *Monte Carlo-based Stochastic Finite Element Method: An Application to Composite Structures*, Ph.D. thesis, Vrije Universiteit Brussel, Dec. 1994.
- ¹⁵Schuëller, G. I., "A State-of-the-Art Report on Computational Stochastic Mechanics," *Probabilistic Engineering Mechanics*, Vol. 12, No. 4, 1997, pp. 197-321, IASSAR report on Computational Stochastic Mechanics.

¹⁶Ang, A. H.-S. and Tang, W. H., *Probability Concepts in Engineering Planning and Design: Volume I-Basic Principles*, John Wiley & Sons, New York, 1975.