A Study of Green's Function Methods Applied to Space Radiation Protection

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Introduction

The purpose of this research was to study the propagation of glactic ions through various materials. Galactic light ions result from the break up of heavy ion particles and their propagation through materials is modeled using the one-dimensional Boltzmann equation. When ions enter materials there can occur (i) the interaction of ions with orbital electrons which causes ionization within the material and (ii) ions collide with atoms causing production of secondary particles which penetrate deeper within the material. These processes are modeled by a continuum model. The basic idea is to place a control volume within the material and examine the change in ion flux across this control volume. In this way on can derive the basic equations for the transport of light and heavy ions in matter. Green's function perturbation methods can then be employed to solve the resulting equations using energy dependent nuclear cross sections.

Throughout this report we will use the following symbols and notation:

$\phi_j(\vec{x}, \vec{\Omega}, E)$ is the flux of ions of type j moving in direction $\vec{\Omega}$

having units $(\#\text{particles}/\text{cm}^2 - \sec - \text{sr} - \text{Mev}/\text{amu})$

E	is the ion energy. (Mev/amu)
A_j	is the atomic mass of the jth type ion (amu)
$\sigma_j(E)$	is the macroscopic cross section. (cm^{-1})
$S_j(E)$	is the average energy loss per unit length or stopping power
	or linear energy transfer $\frac{dE}{dx}$. (Mev/cm).
$R_j(E)$	is the slowing down range for type j ions. (cm) $R_j(E) = \int_0^E \frac{A_j \, dE'}{S_j(E')}$
j	is the ion type.
$\vec{\Omega}$	is a unit vector in the direction of propagation.
σ_{jk}	is production cross section of type j ions with energy E and direction $\vec{\Omega}$
	by collision with type k ions of energy E' and direction $\vec{\Omega}'$
	having units of $(\text{cm} - \text{sr} - \text{Mev}/\text{amu})$
$\hat{n}$	is the outward directed unit normal to boundary.
$\vec{\Gamma}$	is vector to boundary point. (cm)
$\vec{x}$	is the position vector to arbitrary point in region (cm) $\vec{x} = \rho\vec{\Omega} + \vec{x}_n$
ρ	is the projection of $\vec{x}$ on $\vec{\Omega}$ (cm)
$\vec{x}_n$	is the component of $\vec{x}$ perpendicular to $\vec{\Omega}$ direction.
Z_j	charge of jth type ion.
ν_j	a parameter defined as $\nu_j = \frac{Z_j^2}{A_j}$.

The basic Boltzmann equation results from examination of a control volume placed within the material. We find that

$$\begin{pmatrix} \text{Change in ion flux} \\ \text{within a volume element} \end{pmatrix} = \begin{pmatrix} \text{Gains within the} \\ \text{volume element} \end{pmatrix} - \begin{pmatrix} \text{Losses due to any} \\ \text{nuclear collisions} \end{pmatrix}$$

This gives the Boltzmann equation

$$\left[\vec{\Omega} \cdot \nabla - \frac{1}{A_j} \frac{\partial}{\partial E} S_j(E) + \sigma_j(E) \right] \phi_j(\vec{x}, \vec{\Omega}, E) = \sum_{k>j} \int_E^\infty dE' \int d\vec{\Omega}' \sigma_{jk} \phi_k(\vec{x}, \vec{\Omega}', E') \quad (1)$$

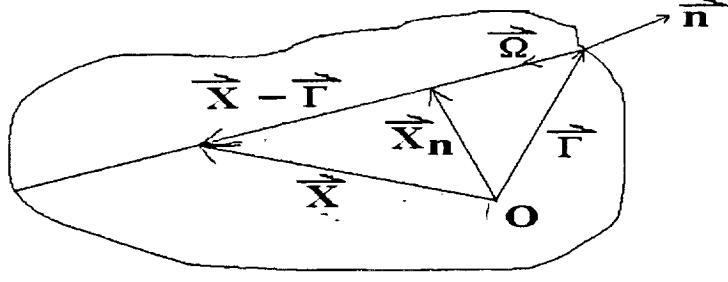The equation (1) is to be associated with the geometry of figure 1.

Figure 1. General Geometry for Boltzmann's equation

Multiply the equation (1) by $S_j(E)$ and define the quantities

$$\overline{\phi}_j(\vec{x}, \vec{\Omega}, E) = S_j(E)\phi_j(\vec{x}, \vec{\Omega}, E) \tag{2}$$

$$\overline{G}_j(\vec{x}, \vec{\Omega}, E) = S_j(E) \sum_{k>j} \int_E^\infty dE' \int d\vec{\Omega}' \, \sigma_{jk}\phi_k(\vec{x}, \vec{\Omega}', E') \tag{3}$$

to obtain

$$\left[\vec{\Omega} \cdot \nabla - \frac{S_j(E)}{A_j} \frac{\partial}{\partial E} + \sigma_j(E) \right] \overline{\phi}_j(\vec{x}, \vec{\Omega}, E) = \overline{G}_j(\vec{x}, \vec{\Omega}, E). \tag{4}$$

Note that $\vec{\Omega} \cdot \nabla \overline{\phi}_j = \frac{\partial \overline{\phi}_j}{\partial \rho}$ is the directional derivative in the direction $\vec{\Omega}$ and that

$$\frac{\partial \overline{\phi}_j}{\partial E} = \frac{\partial \overline{\phi}_j}{\partial R_j} \frac{\partial R_j}{\partial E} = \frac{\partial \overline{\phi}_j}{\partial R_j} \frac{A_j}{S_j(E)}$$

so that the equation (4) can be written as

$$\left[\frac{\partial}{\partial \rho} - \frac{\partial}{\partial R_j} + \sigma_j(E) \right] \overline{\phi}_j(\vec{x}, \vec{\Omega}, E) = \overline{G}_j(\vec{x}, \vec{\Omega}, E). \tag{5}$$

Introduce the characteristic variables (η_j, ξ_j) given by the transformation equations

$$\eta_j = \rho - R_j(E) \qquad \xi_j = \rho + R_j(E) \tag{6}$$

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where $\rho = \vec{\Omega} \cdot \vec{x}$. Also introduce the variables

$$\chi_j(\eta_j, \xi_j) = \overline{\phi}_j(\rho\vec{\Omega} + \vec{x}_n, \vec{\Omega}, E)$$

$$g_j(\eta_j, \xi_j) = \overline{G}_j(\rho\vec{\Omega} + \vec{x}_n, \vec{\Omega}, E) \tag{7}$$

$$\overline{\sigma}_j(\eta_j, \xi_j) = \sigma_j(E)$$
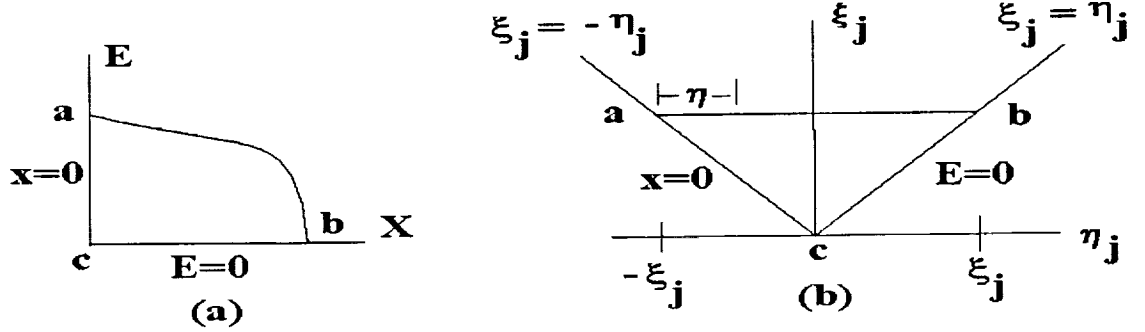

Figure 2. Geometry for characteristic variables

By the chain rule we have

$$\frac{\partial \overline{\phi}_j}{\partial \rho} = \frac{\partial \overline{\phi}}{\partial \eta_j} + \frac{\partial \overline{\phi}_j}{\partial \xi_j} \quad \text{and} \quad \frac{\partial \overline{\phi}_j}{\partial R_j} = \frac{\partial \overline{\phi}}{\partial \eta_j}(-1) + \frac{\partial \overline{\phi}_j}{\partial \xi_j}$$

so that the equation (5) simplifies to

$$\left(2\frac{\partial}{\partial \eta_j} + \overline{\sigma}_j\right) \chi_j(\eta_j, \xi_j) = g_j(\eta_j, \xi_j) \tag{8}$$

in terms of the new variables. This equation can be integrated using the integrating factor

$$\exp\left[\frac{1}{2} \int_a^{\eta_j} \overline{\sigma}_j(\eta', \xi_j) \, d\eta'\right] \tag{9}$$

to obtain

$$\chi_j(\eta_j, \xi_j) = \exp[-\frac{1}{2} \int_a^{\eta_j} \overline{\sigma}_j(\eta', \xi_j) d\eta']\chi_j(a, \xi_j)$$

$$+ \frac{1}{2} \int_a^{\eta_j} \exp[-\frac{1}{2} \int_{\eta'}^{\eta_j} \overline{\sigma}_j(\eta'', \xi_j) \, d\eta'']g_j(\eta', \xi_j) \, d\eta' \tag{10}$$

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where a is any real number. Consequently, the solution to the equation (4) can be written as

$$\overline{\phi}_j(\vec{x}, \vec{\Omega}, E) = f(a, \rho - R_j(E))\overline{\phi}_j\left(\frac{1}{2}(\xi_j + a)\vec{\Omega} + \vec{x}_n, \vec{\Omega}, R_j^{-1}(\frac{\xi_j - a}{2})\right)$$
$$+ \frac{1}{2}\int_a^{\rho - R_j(E)} f(\eta'', \rho - R_j(E))\overline{G}_j\left(\frac{1}{2}(\xi_j + \eta'')\vec{\Omega} + \vec{x}_n, \vec{\Omega}, R_j^{-1}(\frac{\xi_j - \eta''}{2})\right) d\eta'' \tag{11}$$

where

$$f(a, \rho - R_j(E)) = \exp\left[-\frac{1}{2}\int_a^{\rho - R_j(E)} \sigma_j(R_j^{-1}(\frac{\xi_j - \eta'}{2})) d\eta'\right] \tag{12}$$

From equations (6) we find that

$$2\rho = \eta_j + \xi_j \quad \text{and} \quad 2R_j(E) = \xi_j - \eta_j \tag{13}$$

so that when $\eta' = a$ we will have $\rho = \frac{1}{2}(a + \xi_j)$. Observe from figure 2 that along the line of integration we will have $\xi_j = $ constant. The value of a is selected such that $\rho\vec{\Omega} + \vec{x}_n = \vec{\Gamma}$ is a point on the boundary. Thus, the vector $\left(\frac{a + \xi_j}{2}\right)\vec{\Omega} + \vec{x}_n = \vec{\Gamma}$ dotted with $\vec{\Omega}$ gives the value

$$a = 2\vec{\Omega} \cdot \vec{\Gamma} - \xi_j = 2d - \rho - R_j(E) \tag{14}$$

where $d = \vec{\Omega} \cdot \vec{\Gamma}$. Note that when $E = E'$ and $\eta_j = \eta'$ we have from equation (13) that

$$2R_j(E') = \xi_j - \eta' \tag{15}$$

or

$$E' = R_j^{-1}\left(\frac{\xi_j - \eta'}{2}\right) = R_j^{-1}\left(\frac{\rho + R_j(E) - \eta'}{2}\right) \quad \text{with} \quad dE' = \frac{-S_j(E')}{2A_j} d\eta' \tag{16}$$

and similarly by changing symbols when

$$E'' = R_j^{-1}\left(\frac{\rho + R_j(E) - \eta''}{2}\right) \quad \text{we have} \quad dE'' = \frac{-S_j(E'')}{2A_j} d\eta''. \tag{17}$$

We examine the limits of integration in equation (11) and observe that when $\eta' = a$ we have

$$2R_j(E') = \rho + R_j(E) - a \tag{18}$$

and from equation (14) we have

$$2d = \rho + R_j(E) + a. \tag{19}$$

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Adding the equations (18) and (19) we find

$$R_j(E') + d = \rho + R_j(E) \tag{20}$$

or

$$E' = R_j^{-1}(\rho - d + R_j(E)). \tag{21}$$

Next we examine the lower limit of integration and find that when $\eta' = \rho - R_j(E)$, then $2R_j(E') = \rho + R_j(E) - \rho + R_j(E)$ implies that $E' = E$. In the second term of equation (11) when $\eta'' = a$ we again find that $E'' = R_j^{-1}(\rho + R_j(E) - d)$ and when $\eta'' = \rho - R_j(E)$ then $E'' = E$. Also,

$$\frac{1}{2}(\xi_j + \eta'') = \frac{1}{2}(\xi_j + \xi_j - 2R_j(E'')) = \xi_j - R_j(E'') = \rho + R_j(E) - R_j(E'').$$

Consequently, the equation (11) can be written in the form

$$\begin{aligned}
\overline{\phi}_j(\vec{x}, \vec{\Omega}, E) &= F_j(E, R_j^{-1}(R_j(E) - d + \rho))\overline{\phi}_j(\vec{\Gamma}, \vec{\Omega}, R_j^{-1}(R_j(E) + \rho - d)) \\
&+ \int_E^{R_j^{-1}(R_j(E)+\rho-d)} F_j(E, E'')\overline{G}_j((\rho + R_j(E) - R_j(E''))\vec{\Omega} + \vec{x}_n, \vec{\Omega}, E'')\frac{A_j}{S_j(E'')}\, dE''
\end{aligned} \tag{22}$$

where

$$F_j(E1, E2) = \exp\left[-\int_{E1}^{E2} \frac{A_j\sigma_j(E')}{S_j(E')}\, dE'\right] \tag{23}$$

Define the nuclear survival probability (reference Wilson 1977) as

$$P_j(E) = \exp\left[-\int_0^E \frac{A_j\sigma_j(E')}{S_j(E')}\, dE'\right] \tag{24}$$

then the equation (23) can be written as

$$F_j(E_1, E_2) = \frac{P_j(E_2)}{P_j(E_1)}.$$

Then from equation (22) we can write the solution to equation (3) in the form

$$\begin{aligned}
\phi_j(\vec{x}, \vec{\Omega}, E) &= \frac{S_j(E_j)P_j(E_j)}{S_j(E)P_j(E)}\phi_j(\vec{\Gamma}, \vec{\Omega}, E_j) \\
&+ \sum_{k>j}\int_E^{E_j} dE' \frac{A_j P_j(E')}{S_j(E)P_j(E)}\int_{E'}^{\infty} dE'' \int d\vec{\Omega}'\sigma_{jk}(E', E'')\phi_k(\vec{x} + (R_j(E) - R_j(E'))\vec{\Omega}, \vec{\Omega}', E'')
\end{aligned} \tag{25}$$

where $E_j = R_j^{-1}(\rho + R_j(E) - d)$, $\vec{x} = \vec{x}_n + \rho\vec{\Omega}$ and E' and E'' have been interchanged.

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In the one-dimensional straight ahead approximation $\vec{\Omega}$ is a unit vector in the direction of $\vec{x}$ with $\rho = x$, $\vec{x}_n = \vec{0}$, $\eta_j = x - R_j(E)$, $\xi_j = x + R_j(E)$ and $\vec{\Gamma} = \vec{0}$. (i.e. the origin 0 moves to the boundary $x = 0$). The equation (25) then reduces to

$$
\begin{aligned}
\phi_j(x, E) &= \frac{S_j(E_j)P_j(E_j)}{S_j(E)P_j(E)}\, \phi_j(0, E_j) \\
&+ \sum_{k>j} \int_E^{E_j} dE'\, \frac{A_j P_j(E')}{S_j(E)P_j(E)} \int_{E'}^{\infty} dE''\, \sigma_{jk}(E', E'')\phi_k(x + R_j(E) - R_j(E'), E'')
\end{aligned}
\tag{26}
$$

where E_j is determined from x and E such that

$$
E_j = R_j^{-1}(x + R_j(E)).
\tag{27}
$$

The solution given by equation (26) can be expressed in terms of Green's function as

$$
\phi_j(x, E) = \sum_{k>j} \int_0^{\infty} G_{jk}(x, E, E_0)\phi_k(0, E_0)\, dE_0
\tag{28}
$$

where $\phi_k(0, E_0) = f_k(E_0)$ are boundary conditions. Substituting the assumed solution given by equation (28) into equation (26) we obtain

$$
\begin{aligned}
\sum_{\ell} \int_0^{\infty} G_{j\ell}(x, E, E_0)\phi_\ell(0, E_0) dE_0 &= \sum_{\ell} \int_0^{\infty} \frac{S_j(E_j)P_j(E_j)}{S_j(E)P_j(E)} G_{j\ell}(0, E_j, E_0)\phi_\ell(0, E_0) dE_0 \\
&+ \sum_{k} \int_E^{E_j} dE'\, \frac{A_j P_j(E')}{S_j(E)P_j(E)} \int_{E'}^{\infty} dE'' \sigma_{jk}(E', E'') \sum_{\ell} \int_0^{\infty} G_{k\ell}(x + R_j(E) - R_j(E'), E'', E_0)\phi_\ell(0, E_0)\, dE_0.
\end{aligned}
$$

Note that when $\ell = K$ we can equate like coefficients and find that $G_{jm}(x, E, E_0)$ must satisfy the integral equation

$$
\begin{aligned}
G_{jm}(x, E, E_0) &= \frac{S_j(E_j)P_j(E_j)}{S_j(E)P_j(E)}\, G_{jm}(0, E_j, E_0) \\
&+ \sum_{k>j} \int_E^{E_j} dE'\, \frac{A_j P_j(E')}{S_j(E)P_j(E)} \int_{E'}^{\infty} dE''\, \sigma_{jk}(E', E'')G_{km}(x + R_j(E) - R_j(E'), E'', E_0)
\end{aligned}
\tag{29}
$$

subject to the boundary condition $G_{jm}(0, E, E_0) = \delta_{jm}\delta(E - E_0)$, where the value for E_j is determined from the inverse relation $E_j = R_j^{-1}(x + R_j(E))$. The G_{jm} terms are written using the Neumann expansion as a perturbation series

$$
G_{jm}(x, E, E_0) = \sum_{i=0}^{\infty} G_{jm}^{(i)}(x, E, E_0)
\tag{30}
$$

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with leading term

$$G_{jm}^{(0)}(x, E, E_0) = \frac{S_j(E_j)P_j(E_j)}{S_j(E)P_j(E)}\delta_{jm}\delta(E_j - E_0).$$ (31)

with $E_j = R_j^{-1}(x + R_j(E))$. Note that when $x = 0$ we have $E_j = E$ so that $G_{jm}^{(0)}(0, E, E_0)$ satisfies the above boundary condition. The higher order terms are determined from the recursive definition

$$G_{jm}^{(n+1)}(x, E, E_0) =$$
$$\sum_k \int_E^{E_j} dE' \frac{A_j P_j(E')}{S_j(E)P_j(E)} \int_{E'}^{\infty} dE'' \sigma_{jk}(E', E'') G_{km}^{(n)}(x + R_j(E) - R_j(E'), E'', E_0).$$ (32)

and must satisfy the boundary conditions $G_{jm}^{(n+1)}(x, E, E_0) = 0$ for $n = 0, 1, 2, \ldots$. In the special case $n = 0$ the equation (32) reduces to

$$G_{jm}^{(1)}(x, E, E_0) =$$
$$\sum_k \int_E^{E_j} dE' \frac{A_j P_j(E')}{S_j(E)P_j(E)} \int_{E'}^{\infty} dE'' \sigma_{jk}(E', E'') \frac{S_k(E_k')P_k(E_k')}{S_k(E'')P_k(E'')}\delta_{km}\delta(E_k' - E_0)$$ (33)

where $R_k(E_k') = x + R_j(E) - R_j(E') + R_k(E'')$. (i.e. treat $x + R_j(E) - R_j(E'')$ as an x^* value. See for example equation (27) .) Again we observe that when $x = 0$ we have $E_j = E$ and so the boundary condition at $x = 0$ is satisfied.

Cross Section assumption 1

For interactions dominated by peripheral processes we use

$$\sigma_{jm}(E', E'') = \sigma_{jm}(E'')\delta(E' - E'')$$ (34)

so that the equation (33) becomes

$$G_{jm}^{(1)}(x, E, E_0) =$$
$$\sum_k \int_E^{E_j} dE' \frac{A_j P_j(E')}{S_j(E)P_j(E)} \int_{E'}^{\infty} dE'' \sigma_{jk}(E'')\delta(E' - E'') \frac{S_k(E_k')P_k(E_k')}{S_k(E'')P_k(E'')}\delta_{km}\delta(E_k' - E_0)$$ (35)

where

$$E_k' = R_k^{-1}(x + R_j(E) - R_j(E') + R_k(E'')).$$ (36)

We integrate with respect to E'' and observe that the only nonzero term occurs when $E'' = E'$. This gives

$$G_{jm}^{(1)}(x, E, E_0) = \sum_k \int_E^{E_j} dE' \frac{A_j P_j(E')}{S_j(E)P_j(E)}\sigma_{jk}(E') \frac{S_k(E_k')P_k(E_k')}{S_k(E')P_k(E')}\delta(E_k' - E_0)\delta_{km}$$ (37)

where

$$R_k(E_k') = x + R_j(E) - R_j(E') + R_k(E') \tag{38}$$

$$R_j(E') - R_k(E') = x + R_j(E) - R_k(E_k').$$

We know that $\nu_j R_j(E') = \nu_k R_k(E')$ so that the above can be written as

$$\left(\frac{\nu_k}{\nu_j} - 1\right) R_k(E') = x + R_j(E) - R_k(E_k')$$

$$\text{or} \quad \left(1 - \frac{\nu_j}{\nu_k}\right) R_j(E') = x + R_j(E) - R_k(E_k')$$

Thus, we can write

$$R_j(E') = \frac{\nu_k}{|\nu_k - \nu_j|}(x + R_j(E) - R_k(E_k'))$$

$$\text{or} \quad R_k(E') = \frac{\nu_j}{|\nu_k - \nu_j|}(x + R_j(E) - R_k(E_k')). \tag{39}$$

Differentiate the equation (39) with respect to E_k' to obtain

$$R_k'(E')dE' = \frac{\nu_j}{|\nu_k - \nu_j|}(-R_k'(E_k'))dE_k' \quad \text{or} \quad dE' = \frac{\nu_j}{|\nu_k - \nu_j|}\frac{\frac{A_k}{S_k(E_k')}}{\frac{A_k}{S_k(E')}}dE_k' \tag{40}$$

The equation (35) can then be written as

$$G_{jm}^{(1)}(x, E, E_0) = \sum_k \int_{E_{k1}'}^{E_{k2}'} dE_k' \frac{A_j P_j(E')}{S_j(E)P_j(E)}\sigma_{jk}(E')\frac{\nu_j}{|\nu_k - \nu_j|}\frac{P_k(E_k')}{P_k(E')}\delta(E_k' - E_0)\delta_{km} \tag{41}$$

The only nonzero contribution comes when $k = m$ and $E_k' = E_0$ and so equation (41) reduces to

$$G_{jm}^{(1)}(x, E, E_0) = \begin{cases} h_{jm}(x, E, E_0, E') & \text{if } \frac{\nu_m}{\nu_j}(R_m(E_0) - x) < R_j(E) < \frac{\nu_m}{\nu_j}R_m(E_0) - x \\ 0 & \text{otherwise} \end{cases} \tag{42}$$

where

$$h_{jm}(x, E, E_0, E') = \frac{A_j P_j(E')}{S_j(E)P_j(E)}\sigma_{jm}(E')\frac{\nu_j}{|\nu_m - \nu_j|}\frac{P_m(E_0)}{P_m(E')} \tag{43}$$

and

$$E' = R_j^{-1}\left(\frac{\nu_m}{|\nu_m - \nu_j|}[x + R_j(E) - R_m(E_0)]\right). \tag{44}$$

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That is, when $E'_k = E_0$ and $k = m$ we have from the equation (38) that

$$R_m(E_0) = x + R_j(E) - R_j(E') + R_m(E')$$

$$R_j(E') - R_m(E') = x + R_j(E) - R_m(E_0)$$

$$\left(1 - \frac{\nu_j}{\nu_m}\right) R_j(E') = x + R_j(E) - R_m(E_0)$$

$$R_j(E') = \frac{\nu_m}{|\nu_m - \nu_j|}(x + R_j(E) - R_m(E_0)).$$

Also from the transformation equations (13) the $\eta_k, \xi_k, \eta_j, \xi_j$ variables are related through the range scale factors ν_j and ν_k, where $\nu_j R_j = \nu_k R_k$. This produces the relations

$$\eta_k - \xi_k = -2R_k = -2\frac{\nu_j}{\nu_k}R_j = \frac{\nu_j}{\nu_k}(\eta_j - \xi_j).$$

Then from the equations

$$\xi_j + \eta_j = \xi_k + \eta_k = 2\rho \tag{45}$$

$$\eta_j - \xi_j = \frac{\nu_k}{\nu_j}(\eta_k - \xi_k) \tag{46}$$

we find that by adding the equations (45) and (46) that

$$2\eta_j = \left(1 + \frac{\nu_k}{\nu_j}\right)\eta_k + \left(1 - \frac{\nu_k}{\nu_j}\right)\xi_k \tag{47}$$

and subtracting (46) from (45) we obtain

$$2\xi_j = \left(1 - \frac{\nu_k}{\nu_j}\right)\eta_k + \left(1 + \frac{\nu_k}{\nu_j}\right)\xi_k. \tag{48}$$

Interchanging j and k in the equations (47) and (48) we find that

$$\eta_k = \left(\frac{\nu_j + \nu_k}{2\nu_k}\right)\eta_j + \left(\frac{\nu_k - \nu_j}{2\nu_k}\right)\xi_j$$

$$\xi_k = \left(\frac{\nu_k - \nu_j}{2\nu_k}\right)\eta_j + \left(\frac{\nu_k + \nu_j}{2\nu_k}\right)\xi_j.$$

Then when η_j is a value η' lying between the constants $-\xi_j$ and $+\xi_j$, (See Figure 2(b)), we will have

$$\eta_k = \left(\frac{\nu_j + \nu_k}{2\nu_k}\right)\eta' + \left(\frac{\nu_k - \nu_j}{2\nu_k}\right)\xi_j$$

$$\xi_k = \left(\frac{\nu_k - \nu_j}{2\nu_k}\right)\eta' + \left(\frac{\nu_k + \nu_j}{2\nu_k}\right)\xi_j.$$

Changing k to m we find

$$\xi_m = \left(\frac{\nu_m - \nu_j}{2\nu_m}\right)\eta' + \left(\frac{\nu_m + \nu_j}{2\nu_m}\right)\xi_j.$$

Note that the boundary condition $G_{jm}(0, E, E_0) = \delta_{jm}\delta(E - E_0)$ can be written in the form

$$G_{jm}(0, E, E_0) = \delta_{jm}\delta(R_j^{-1}(\xi_j) - E_0) = \delta_{jm}\delta(\xi_j - R_j(E_0)) = \delta(\xi_m - R_m(E_0))$$

so that when $\xi_m = R_m(E_0)$ we have

$$\eta' = \frac{2\nu_m}{\nu_m - \nu_j}R_m(E_0) - \left(\frac{\nu_m + \nu_j}{\nu_m - \nu_j}\right)\xi_j. \tag{49}$$

Using the equations (6) and (49) we now calculate the inequality which occurs in the equation (42) . From the equation (10) , with $a = -\xi$, we have the inequality $-\xi_j < \eta' < \eta_j$ which implies

$$-\xi_j < \eta' < \eta_j$$

$$-x - R_j(E) < \frac{2\nu_m}{\nu_m - \nu_j}R_m(E_0) - \left(\frac{\nu_m + \nu_j}{\nu_m - \nu_j}\right)(x + R_j(E)) < x - R_j(E)$$

$$-x < \frac{2\nu_m}{\nu_m - \nu_j}R_m(E_0) + R_j(E) - \left(\frac{\nu_m + \nu_j}{\nu_m - \nu_j}\right)(x + R_j(E)) < x$$

$$\left(\frac{\nu_m + \nu_j}{\nu_m - \nu_j}\right)x - x < \frac{2\nu_m R_m(E_0)}{\nu_m - \nu_j} + \left(1 - \frac{\nu_m + \nu_j}{\nu_m - \nu_j}\right)R_j(E) < x + \left(\frac{\nu_m + \nu_j}{\nu_m - \nu_j}\right)x$$

$$\nu_j x < \nu_m R_m(E_0) - \nu_j R_j(E) < \nu_m x$$

$$-\nu_j x > \nu_j R_j(E) - \nu_m R_m(E_0) > -\nu_m x$$

$$\nu_m R_m(E_0) - \nu_j x > \nu_j R_j(E) > \nu_m R_m(E_0) - \nu_m x$$

$$\frac{\nu_m}{\nu_j}R_m(E_0) - x > R_j(E) > \frac{\nu_m}{\nu_j}(R_m(E_0) - x)$$

$$\frac{\nu_m}{\nu_j}(R_m(E_0) - x) < R_j(E) < \frac{\nu_m}{\nu_j}R_m(E_0) - x$$

Cross Section assumption 2

We start with equation (33) and assume $\sigma_{jm}(E', E'')$ has a Gaussian distribution of the form

$$\sigma_{jm}(E', E'') = \bar{\sigma}_{jm}(E'')\frac{1}{\Delta_{jm}\sqrt{2\pi}}\exp\left[-\frac{(E' - E'' - \epsilon_{jm})^2}{2\Delta_{jm}^2}\right]$$

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we can then write the equation (33) in the form

$$G_{jm}^{(1)}(x, E, E_0) = \int_E^{E_j} dE' \int_{E'}^{\infty} dE'' F_{jk} \delta_{km} \tag{50}$$

where

$$F_{jk} = \frac{A_j P_j(E')}{S_j(E) P_j(E)} \sigma_{jk}(E', E'') \frac{S_k(E_k') P_k(E_k')}{S_k(E'') P_k(E'')} \delta(E_k' - E_0) \tag{51}$$

with

$$E_k' = R_k^{-1}(x + R_j(E) - R_j(E') + R_k(E'')) \tag{52}$$

The integration of (50) is over the region illustrated in the figure 3 in the limit as $T \to \infty$. In expanded form the equation (50) has the form

$$G_{jm}^{(1)}(x, E, E_0) =$$
$$\int_E^{E_j} dE' \int_{E'}^{\infty} dE'' \frac{A_j P_j(E')}{S_j(E) P_j(E)} \sigma_{jk} \frac{S_k(E_k') P_k(E_k')}{S_k(E'') P_k(E'')} \delta_{km} \delta(E_k' - E_0). \tag{53}$$

where

$$\sigma_{jk} = \bar{\sigma}_{jk}(E'') \frac{1}{\Delta_{jk}\sqrt{2\pi}} \exp\left[-\frac{(E' - E'' - \epsilon_{jk})^2}{2\Delta_{jk}^2}\right]$$
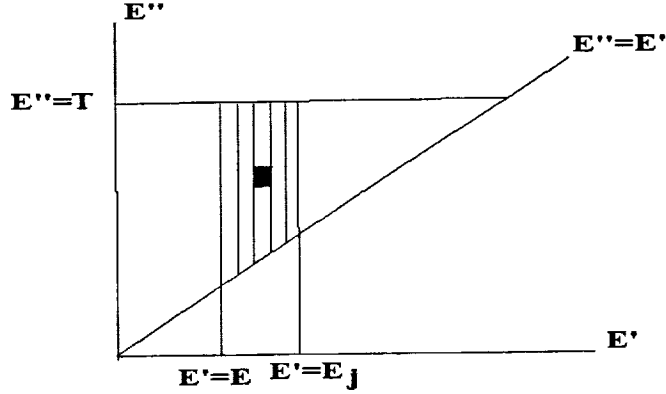

Figure 3. Limits of integration for Green's function term.

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Note that for the first integration in the E'' direction we have E' is a constant. Consequently, we let

$$r = \frac{E'' + \epsilon_{jk} - E'}{\sqrt{2}\Delta_{jk}} \quad \text{with} \quad dr = \frac{dE''}{\sqrt{2}\Delta_{jk}} \tag{54}$$

The equation (53) can then be written in the form

$$G_{jm}^{(1)}(x, E, E_0) = \int_E^{E_j} dE' \int_{\frac{\epsilon_{jk}}{\sqrt{2}\Delta_{jk}}}^{\infty} dr \frac{A_j P_j(E')}{\sqrt{\pi}S_j(E)P_j(E)} \bar{\sigma}_{jk}(\bar{r}) e^{-r^2} \frac{S_k(E_k')P_k(E_k')}{S_k(\bar{r})P_k(\bar{r})} \delta_{km}\delta(E_k' - E_0). \tag{55}$$

where $\bar{r} = \sqrt{2}\Delta_{jk}r - \epsilon_{jk} + E'$ and

$$E_k' = R_k^{-1}(x + R_j(E) - R_j(E') - R_k(\bar{r})). \tag{56}$$

This integral can be simplified by using one of the mean value theorems for integrals and written as

$$G_{jm}^{(1)}(x, E, E_0) = \int_E^{E_j} dE' \frac{A_j P_j(E')}{2S_j(E)P_j(E)} \bar{\sigma}_{jk}(\bar{r}_*) \frac{S_k(E_{k_*}')P_k(E_{k_*}')}{S_k(\bar{r}_*)P_k(\bar{r}_*)} \delta_{km}\delta(E_{k_*}' - E_0) \frac{2}{\sqrt{\pi}} \int_{\frac{\epsilon_{jk}}{\sqrt{2}\Delta_{jk}}}^{\infty} e^{-r^2} dr. \tag{57}$$

with $\bar{r}_* = \sqrt{2}\Delta_{jk}r^* - \epsilon_{jk} + E'$ and

$$E_{k_*}' = R_k^{-1}(x + R_j(E) - R_j(E') - R_k(\bar{r}_*)) \tag{58}$$

where r^* is some mean value in the interval $(\frac{\epsilon_{jk}}{\sqrt{2}\Delta_{jk}}, \infty)$ and when $E_{k_*}' = E_0$, then E' is a solution of the nonlinear equation

$$R_k(E_0) = x + R_j(E) - R_j(E') - R_k(\sqrt{2}\Delta_{jk}r^* - \epsilon_{jk} + E') \tag{59}$$

provided $E < E' < E_j$. Consequently, we can write

$$G_{jm}^{(1)}(x, E, E_0) = \begin{cases} \frac{1}{2} \frac{A_j P_j(E')}{S_j(E)P_j(E)} \bar{\sigma}_{jm}(\bar{r}_*) \frac{S_m(E_0)P_m(E_0)}{S_m(\bar{r}_*)P_m(\bar{r}_*)} \operatorname{erfc}\left(\frac{\epsilon_{jm}}{\sqrt{2}\Delta_{jm}}\right) & \text{if } E < E' < E_j \\ 0 & \text{otherwise} \end{cases} \tag{60}$$

where E' is a solution of the nonlinear equation (59) , r^* is some mean value and erfc is the complimentary error function.
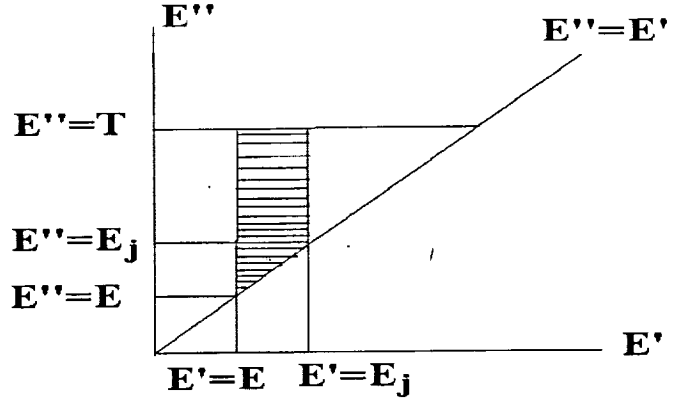
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Figure 4. New limits of integration for Green's function term.

Another viewpoint

By interchanging the order of integration in equation (33) we obtain the limits of integration illustrated in the figure 4 and the equation (33) can be written as

$$G_{jm}^{(1)}(x, E, E_0) = \int_E^{E_j} dE'' \int_E^{E''} dE' F_{jk}\delta_{km} + \lim_{T \to \infty} \int_{E_j}^T dE'' \int_E^{E_j} dE' F_{jk}\delta_{km}. \qquad (61)$$

Observe that along the line E''=constant, we have from equation (52) that

$$\frac{dR_k(E_k')}{dE_k'}\frac{dE_k'}{dE'} = -\frac{dR_j(E')}{dE'}$$

$$\frac{A_k}{S_k(E_k')}\frac{dE_k'}{dE'} = -\frac{A_j}{S_j(E')}$$

$$\text{or} \qquad dE' = -\frac{A_m S_j(E')}{A_j S_k(E_k')}\, dE_k'.$$

Hence, when $k = m$ and $\delta_{km} = 1$, the equation (61) reduces to

$$G_{jm}^{(1)}(x, E, E_0) =$$
$$\int_{E_j}^E dE'' \int_{E_{m3}}^{E_{m1}} F_{jm}\frac{A_m S_j(E')}{A_j S_m(E_m')}\, dE_m' + \lim_{T \to \infty} \int_{E_j}^T dE'' \int_{E''}^{E_{m1}} F_{jm}\frac{A_m S_j(E')}{A_j S_m(E_m')}\, dE_m' \qquad (62)$$

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The limits in the above equation are determined as follows. Observe that when $E' = E$ and $k = m$ the equation (52) gives

$$R_m(E'_m) = x + R_j(E) - R_j(E) + R_m(E'')$$
$$\text{or} \quad E'_m = R_m^{-1}(x + R_m(E'')) = E_{m1}$$
(63)

and when $E' = E_j$ and $k = m$ the equation (52) gives

$$R_m(E'_m) = x + R_j(E) - R_j(E_j) + R_m(E'')$$

. But $R_j(E_j) = x + R_j(E)$ so that $E'_m = E''$. Also when $E' = E''$ and $k = m$ we obtain from the equation (52) that

$$R_m(E'_m) = x + R_j(E) - R_j(E'') + R_m(E'')$$
$$R_m(E'_m) = x + R_j(E) - R_j(E'') + \frac{\nu_j R_j(E'')}{\nu_m}$$
$$R_m(E'_m) = x + R_j(E) + \left(\frac{\nu_j}{\nu_m} - 1\right) R_j(E'')$$
$$\text{or} \quad E'_m = R_m^{-1}(x + R_j(E) + \left(\frac{\nu_j}{\nu_m} - 1\right) R_j(E'')) = E_{m3}$$

with

$$E' = R_j^{-1}(x + R_j(E) - R_m(E'_m) + R_m(E'')).$$

Using the properties of the Dirac delta function we find that the only nonzero contribution to the integral dE'_m occurs when $E'_m = E_0$. In this case the integral given by equation (62) simplifies to

$$G_{jm}^{(1)}(x, E, E_0) =$$
$$\int_{E_j}^{E} dE'' \frac{A_m S_j(E') P_j(E')}{S_j(E) P_j(E)} \sigma_{jm}(E', E'') \frac{P_m(E_0)}{S_m(E'') P_m(E'')} f_1$$
$$+ \lim_{T \to \infty} \int_{E_j}^{T} dE'' \frac{A_m S_j(E') P_j(E')}{S_j(E) P_j(E)} \sigma_{jm}(E', E'') \frac{P_m(E_0)}{S_m(E'') P_m(E'')} f_2$$
(64)

where

$$E' = R_j^{-1}(x + R_j(E) - R_m(E_0) + R_m(E''))$$
(65)

and

$$f_1 = \begin{cases} 1 & \text{if } E_{m3} < E_0 < E_{m1} \\ 0 & \text{otherwise} \end{cases}$$
(66)

$$f_2 = \begin{cases} 1 & \text{if } E'' < E_0 < E_{m1} \\ 0 & \text{otherwise} \end{cases}$$
(67)

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In the solution

$$G_{jm}^{(1)}(x, E, E_0) =$$
$$\int_{E_j}^{E} dE'' \frac{A_m S_j(E')P_j(E')}{S_j(E)P_j(E)} \sigma_{jm}(E', E'') \frac{P_m(E_0)}{S_m(E'')P_m(E'')} f_1 \qquad (68)$$
$$+ \int_{E_j}^{\infty} dE'' \frac{A_m S_j(E')P_j(E')}{S_j(E)P_j(E)} \sigma_{jm}(E', E'') \frac{P_m(E_0)}{S_m(E'')P_m(E'')} f_2$$

we make the approximation that E' remains almost constant over the above intervals of integration. We define the quantity

$$E'_1 = R_j^{-1}(x + r_j(E) - R_m(E_0) + R_m(\frac{1}{2}(E_j + \frac{E + E_0}{2})))$$

and use a mean value theorem for integrals to write the solution of equation (68) in the form

$$G_{jm}^{(1)}(x, E, E_0) =$$
$$\frac{1}{2} \frac{A_m S_j(E'_1)P_j(E'_1)P_m(E_0)}{S_j(E)P_j(E)S_m(E_1^*)P_m(E_1^*)} f_1 \frac{2}{\sqrt{\pi}} \int_{r_1}^{r_2} e^{-r^2} dr$$
$$+ \frac{1}{2} \frac{A_m S_j(E'_1)P_j(E'_1)P_m(E_0)}{S_j(E)P_j(E)S_m(E_2^*)P_m(E_2^*)} f_2 \frac{2}{\sqrt{\pi}} \int_{r_1}^{\infty} e^{-r^2} dr$$

where

$$E_j < E_1^* < E$$
$$E_j < E_2^* < \infty$$
$$r_1 = \frac{E_j + \epsilon_{jk} - E'_1}{\sqrt{2}\Delta_{jk}}$$
$$r_2 = \frac{E + \epsilon_{jk} - E'_1}{\sqrt{2}\Delta_{jk}}$$

Using the error function

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-r^2} dr$$

and complimentary error function

$$\operatorname{erfc}(x) = \int_x^{\infty} e^{-r^2} dr$$

the above solution can be written in the form

$$G_{jm}^{(1)}(x, E, E_0) =$$
$$\frac{1}{2} \frac{A_m S_j(E'_1)P_j(E'_1)P_m(E_0)}{S_j(E)P_j(E)S_m(E_1^*)P_m(E_1^*)} f_1 (\operatorname{erf}(r_2) - \operatorname{erf}(r_1))$$
$$+ \frac{1}{2} \frac{A_m S_j(E'_1)P_j(E'_1)P_m(E_0)}{S_j(E)P_j(E)S_m(E_2^*)P_m(E_2^*)} f_2 \operatorname{erfc}(r_1)$$

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Publications resulting from research Grant NCC-1-320 are:

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