



# A “Kane’s Dynamics” Model for the Active Rack Isolation System

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## LIST OF ACRONYMS AND ABBREVIATIONS

ARIS	Active Rack Isolation System
CAD	Computer-Aided Design
DOF	Degree of Freedom
ISPR	International Standard Payload Rack
<i>ISS</i>	<i>International Space Station</i>
MIM	Microgravity Isolation Mount
STABLE	Suppression of Transient Accelerations by Levitation
TM	Technical Memorandum



## NOMENCLATURE

### Operators and Diacritical Marks

Underline $\underline{\quad}$	Vector
Overdot $\dot{\quad}$	First time derivative
Overhat $\hat{\quad}$	Unit length
Circumflex $\frown$	Reference frame
Tilde $\sim$	Dynamical system
Presuperscript $_l$	Linearized quantity
Postsuperscript $^*$	Rigid-body center of mass
Postsuperscript $^T$	Matrix transpose
$\times$	Vector cross product
$\frac{d}{dt}$	Ordinary (or total) derivative with respect to time $t$
$\frac{\partial}{\partial u}$	Partial derivative with respect to scalar $u$
$\text{tr } A$	Trace of arbitrary matrix $A$

### Scalars

#### Lowercase

$a_j^i$	Geometric length for $i$ th actuator assembly (eq. 18)
$c_i$	Umbilical damping in the $\hat{\underline{\xi}}_i$ direction
$c_j^i$	Cosine of angle $q_j$ for $i$ th actuator assembly
$\gamma_i$	Torsional umbilical damping about the $\hat{\underline{\xi}}_i$ axis
$f_j^i$	Geometric length for $i$ th actuator assembly (eq. 19)

## NOMENCLATURE (Continued)

$f_{jk}^i$	Rotation matrix element (eq. 4)
$g_i$	Direction cosines for $\hat{n}_\phi$ in the $\hat{F}_1$ coordinate system
$k_i$	Umbilical spring stiffness in the $\hat{\xi}_i$ direction
$\kappa_i$	Torsional umbilical spring stiffness about the $\hat{\xi}_i$ axis
$l_j^i$	Geometric length for $i$ th actuator assembly (eq. 15)
$m_A$	Mass of arbitrary rigid body $A$
$p_j^i$	Geometric length for $i$ th actuator assembly (eq. 17)
$q_j^i$	$j$ th generalized coordinate for $i$ th actuator
$r_{jk}$	Rotation matrix element (eq. 121)
$s_j^i$	Sine of angle $q_j$ for $i$ th actuator assembly
$u_j^i$	$j$ th generalized speed for $i$ th actuator
$v_j^i$	Geometric length for $i$ th actuator assembly (eqs. 23–25)
$x_{Fu}, y_{Fu}, z_{Fu}$	Geometric lengths (eq. 118)
$x_{Su}, y_{Su}, z_{Su}$	Geometric lengths (eq. 119)
$x_i$	Umbilical elongation in the $\hat{\xi}_i$ direction (eq. 82)
$x_0, y_0, z_0$	Geometric lengths (eq. 120)
$\phi$	Flotor angle of twist relative to stator
$\phi_i$	Angle-of-twist component in the $\hat{\xi}_i$ direction

### Upper case

$A_{rs}$	Constraint-equation scalar (eq. 195)
$A_1^i$	Intersection point on $i$ th actuator arm (fig. 2)
$A_2^i$	Point locating $i$ th upper stinger (fig. 2)
$A_3^i$	Point locating $i$ th Lorentz coil (fig. 2)
$B_r$	Constraint-equation scalar (eq. 195)

## NOMENCLATURE (Continued)

$C$	System damping matrix
$E$	System disturbance input matrix
$F^{C_i}$	Magnitude of $i$ th Lorentz coil force vector $\underline{F}^{C_i}$
$F_i$	Point locating $i$ th cross-flexure (fig. 2)
$F_r$	System holonomic generalized active force, for the $r$ th generalized speed
$F_r^*$	System holonomic generalized inertia force, for the $r$ th generalized speed
$\tilde{F}_r$	System nonholonomic generalized active force, for the $r$ th generalized speed
$\tilde{F}_r^*$	System nonholonomic generalized inertia force, for the $r$ th generalized speed
$F_u$	Umbilical attachment point at the flotor end
$F_{uh}$	Stator-fixed home position of $F_u$
$I$	Identity matrix
$I_{jk}^{A/A^*}$	Central inertia scalar of rigid body $A$ , for body-fixed coordinate directions $\hat{a}_j$ and $\hat{a}_k$
$K$	System stiffness matrix
$M$	System mass matrix
$O$	Zero matrix
$Q$	Rotation matrix from the $\hat{S}_i^1$ coordinate system to the $\hat{F}_i^1$ coordinate system
$Q_{ij}$	$ij$ th element of rotation matrix $Q$
$Q_r^B$	Contribution to the system holonomic generalized active force for the $r$ th generalized speed, due to rigid body $B$
$(Q_r^*)^B$	Contribution to the system holonomic generalized inertia force for the $r$ th generalized speed, due to rigid body $B$

## NOMENCLATURE (Continued)

$S_i$	Point locating $i$ th lower stinger (fig. 2)
$S_u$	Umbilical attachment point at the stator end
$\tilde{S}$	Total dynamical system (stator, flotor, actuators, umbilical)
$X_{Fu}, Y_{Fu}, Z_{Fu}$	Geometric lengths (eq. 45)

### Vectors

#### Lowercase

${}^A \underline{a}^B$	Acceleration of arbitrary point $B$ , with arbitrary reference frame $A$ assumed fixed
$\underline{a}^I$	Translational acceleration of stator (due to g-jitter)
${}^A \underline{\alpha}^B$	Angular acceleration of arbitrary reference frame $B$ with respect to arbitrary reference frame $A$
$\underline{d}$	System disturbance vector
$\underline{i}$	System control current vector
$\hat{n}_\phi$	Unit vector in direction of rotation axis for stator-to-flotor rotation $\phi \hat{n}_\phi$
$\underline{q}$	Vector of generalized coordinates
$\underline{r}^{AB}$	Position vector from arbitrary point $A$ to arbitrary point $B$
$\underline{u}$	Vector of generalized speeds
$\underline{u}^I$	Vector of independent generalized speeds
${}^A \underline{v}^B$	Velocity of arbitrary point $B$ , with arbitrary reference frame $A$ assumed fixed
${}^A \underline{\omega}^B$	Angular velocity of arbitrary reference frame $B$ with respect to arbitrary reference frame $A$

## NOMENCLATURE (Continued)

### Uppercase

$\underline{F}_b$	Umbilical bias force in the home position
$\underline{F}^{C_i}$	Force exerted on the $i$ th Lorentz coil by the flotor
$\underline{F}^D$	Unknown disturbance force acting directly on the flotor
$\underline{F}^{F_i}$	Force exerted on the flotor by the $i$ th actuator arm
$\underline{F}^U$	Force exerted on the flotor by the umbilical
$\underline{H}^{A/A^*}$	Angular momentum of arbitrary rigid body $A$ with respect to its mass center $A^*$
$\underline{M}^{A_i}$	Moment exerted on the $i$ th actuator arm through the upper stinger, due to the $i$ th push-rod
$\underline{M}_b$	Umbilical bias moment in the home position
$\underline{M}^{C_i}$	Moment exerted on the $i$ th Lorentz coil by the flotor, with $\underline{F}^{C_i}$ assumed to act at the $i$ th cross-flexure
$\underline{M}^D$	Unknown disturbance moment acting directly on the flotor, with $\underline{F}^D$ assumed to act at the flotor mass center
$\underline{M}^{F_i}$	Moment exerted on the flotor by the $i$ th actuator, with $\underline{F}^{F_i}$ assumed to act at the $i$ th cross-flexure

## TECHNICAL MEMORANDUM

### A “KANE’S DYNAMICS” MODEL FOR THE ACTIVE RACK ISOLATION SYSTEM

#### 1. INTRODUCTION

The vibratory acceleration levels currently achievable, without isolation on manned space structures, exceed those required by many space science experiments.<sup>1-4</sup> Various active isolation devices have been built to address this need. The first in space was called Suppression of Transient Accelerations by Levitation (STABLE), which uses six independently controlled Lorentz actuators to levitate and isolate at the experiment or subexperiment level.<sup>5</sup> It was successfully flight tested on STS-73 (United States Microgravity Laboratory-02 (USML-02)) in October 1995. Building on the technology developed for STABLE, Marshall Space Flight Center is developing a second-generation experiment-level isolation system: GLovebox Integrated Microgravity Isolation Technology (g-LIMIT).<sup>6</sup> This compact system will isolate microgravity payloads in the Microgravity Science Glovebox.

A second experiment-level isolation system, the Microgravity Isolation Mount (MIM), was launched in the Priroda laboratory module, which docked with *Mir* in April 1996.<sup>7</sup> MIM uses eight Lorentz actuators with centralized control. It has supported several materials science experiments since its implementation in May 1996. A modified version of MIM (MIM II) supported additional experiments on STS-85 in August 1997.

Boeing’s Active Rack Isolation System (ARIS), in contrast to the above payload isolation systems, has been designed to isolate at the rack level. An entire international standard payload rack (ISPR) will be isolated by each copy of ARIS on the *International Space Station (ISS)*. The risk mitigation experiment for ARIS was conducted in September 1996 on STS-79.<sup>8</sup> Each of ARIS’s eight electromechanical actuators requires a two rigid-body model. When the ISPR (“flotor”) is included, the total isolation system model contains 17 rigid bodies.

In order to provide effective model-based isolation, the task of controller design requires prior development of an adequate dynamic (i.e., mathematical), model of the isolation system. This technical memorandum presents a dynamic model of ARIS in a state-space framework intended to facilitate the design of an optimal controller. The chosen approach is the method of Thomas R. Kane (Kane’s method).<sup>9</sup> The result is a state-space, analytical (algebraic) set of linearized equations of motion for ARIS. Interim versions of this work appeared as references 10 and 11.

## 2. THE CHOICE OF KANE'S METHOD

There are fundamentally two avenues for deriving system dynamical equations of motion: vector methods and energy methods. Both avenues lead to scalar equations, but they have different starting points. Vector methods begin with vector equations proceeding from Newton's laws of motion and energy methods begin with scalar energy expressions. The former category uses approaches built around (1) momentum principles, (2) D'Alembert's Principle, or (3) Kane's Method; and the latter uses approaches built around (1) Hamilton's Canonical Equations, (2) the Boltzmann-Hamel Equations, (3) the Gibbs Equations, or (4) Lagrange's Equations.

Although some problems might lend themselves better to solution by other approaches, Kane's method appears in general to be distinctly advantageous for complex problems. As a rule, of the above approaches, those that lead to the simplest and most intuitive dynamical equations are the Gibbs Equations and Kane's Equations. Of those two approaches, the latter is the more systematic and requires less labor. The reduction of labor is particularly evident when one seeks linearized equations of motion, as proved to be necessary in the present case (due to the otherwise excessive algebraic burden).

An overview of Kane's approach to developing linearized equations of motion is presented in reference 10, along with a summary of the relative advantages of the method. See Kane and Levinson for more extended treatments.<sup>9,12</sup>

### 3. DESCRIPTION OF ARIS

The total dynamical system  $\tilde{S}$  consists of the stator  $S$  (ISS and the integral frame, from the motion of which ARIS isolates the ISPR), the flotor  $F$  (the ISPR), eight electromechanical actuator assemblies, and the umbilicals (see fig. 1). The flotor is connected to the stator by the eight actuator assemblies and by a variable number of umbilicals. The actuator assemblies also (and fundamentally) act as the vibration isolation devices.

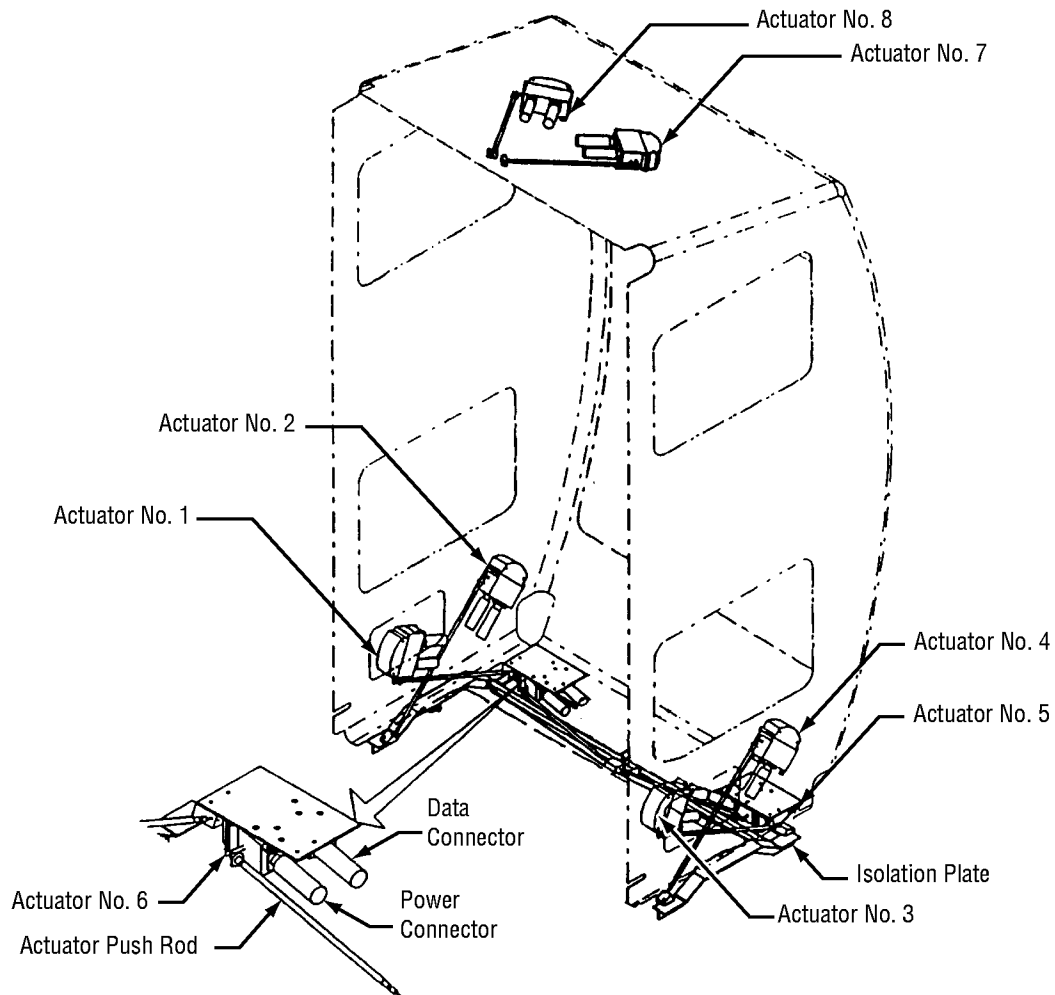


Figure 1. ARIS control assembly.



Each actuator assembly consists of a Lorentz (voice-coil) actuator, an arm, an upper stinger, a push-rod, a lower stinger, and a position sensor. (See fig. 2 for a kinematic diagram and fig. 3 for a computer-aided design (CAD) drawing of a single actuator.) One end of each actuator arm is connected to the flotor through a cross-flexure, which allows the flotor a single rotational degree of freedom (DOF) with respect to the stator. The other end of the arm is connected to one end of the push-rod through the upper stinger, a wire of very high torsional stiffness. Each upper stinger provides two rotational DOF's in bending. The opposite end of the push-rod is connected to the stator through the lower stinger, another short wire that allows three rotational DOF's (two in bending, one in torsion) with respect to  $S$ .

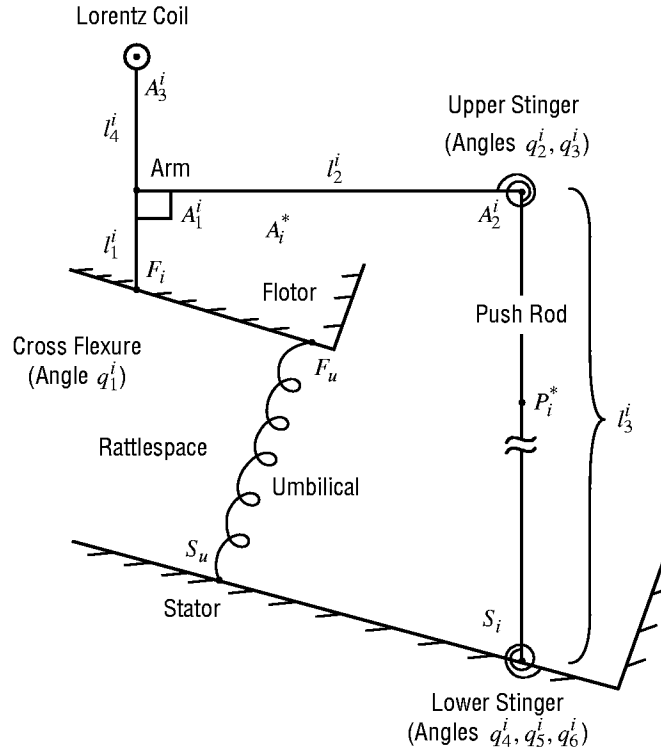


Figure 2. Kinematic diagram, including the  $i$ th actuator assembly and the umbilical.

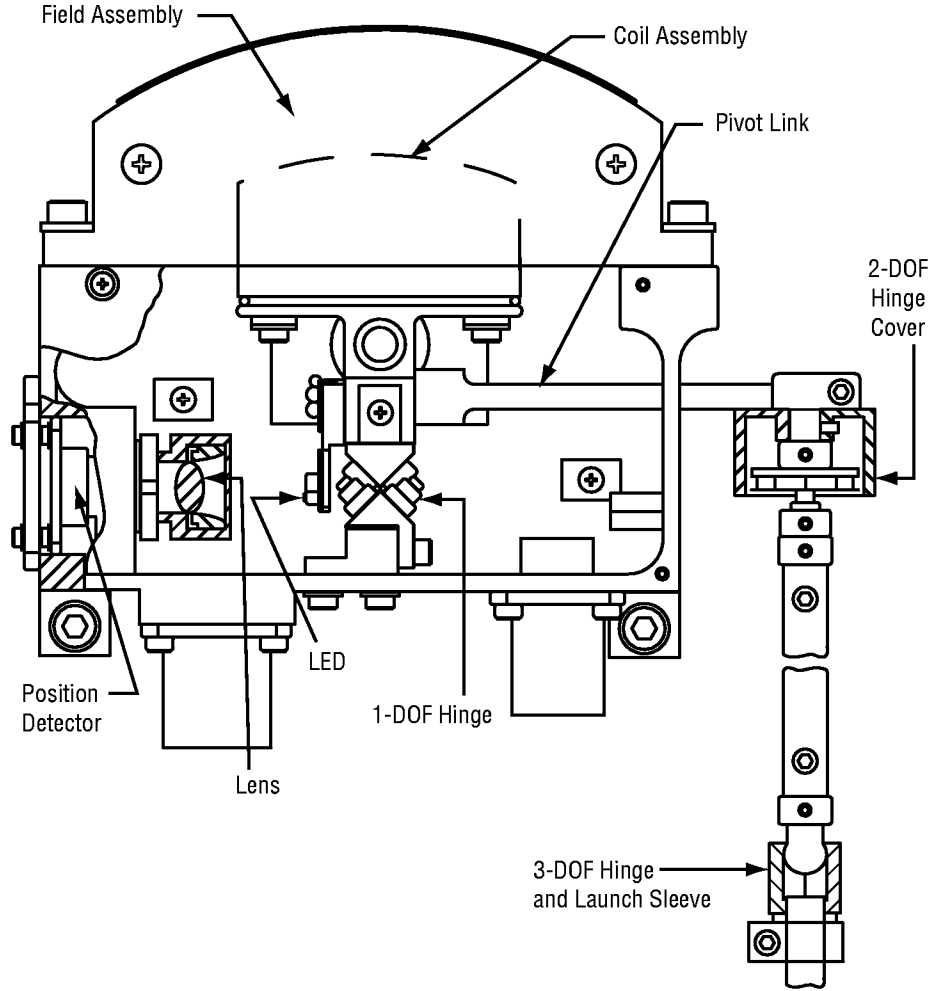


Figure 3. A single ARIS actuator.

Each stinger is modeled as a massless spring. The umbilicals are also considered to be massless; they are modeled together as a single, parallel spring-and-damper arrangement, attached at opposite ends to stator and flotor, at effective umbilical attachment points  $S_u$  and  $F_u$ , respectively. This effective umbilical applies both a force and a moment to the flotor. The force is assumed to act at point  $F_u$ .

The stator, the flotor, and each actuator arm and push-rod are considered to be rigid bodies with mass centers at points  $S^*$ ,  $F^*$ ,  $A_i^*$ , and  $P_i^*$ , respectively. The superscript  $*$  indicates the mass center of the indicated rigid body; the subscript  $i$  corresponds to the  $i$ th actuator ( $i = 1, \dots, 8$ ). All springs (cross flexures and stingers) are assumed to be relaxed when the ISPR is centered in its rattlespace (the “home position”).

#### 4. COORDINATE SYSTEMS

With the ISPR in the home position, fix eight right-handed, orthogonal coordinate systems in the flotor, one at each of the cross-flexure centers. Let the  $i$ th coordinate system ( $i = 1, \dots, 8$ ) have origin  $F_i$  located at the center of the  $i$ th cross flexure, with axis directions determined by an orthonormal set of unit vectors  $\underline{\hat{f}}_j^i$  ( $j = 1, 2, 3$ ). (The overhat indicates unit length, the index  $i$  corresponds to the  $i$ th actuator assembly, and the index  $j$  distinguishes the three vectors.) Orient the unit vectors so that  $\underline{\hat{f}}_2^i$  is along the  $i$ th arm, toward the  $i$ th voice coil;  $\underline{\hat{f}}_1^i$  is directed parallel to the other segment of the  $i$ th arm and toward the upper stinger (which is located at  $A_2^i$ ); and  $\underline{\hat{f}}_3^i$  is in the direction  $\underline{\hat{f}}_1^i \times \underline{\hat{f}}_2^i$  (along the intersection of the two cross-pieces of the  $i$ th cross-flexure).

Fix a similar right-handed coordinate system  $\underline{\hat{a}}_j^i$  ( $j = 1, 2, 3$ ) in the arm of each actuator. Locate each system  $\underline{\hat{a}}_j^i$  such that it is coincident with the corresponding flotor-fixed coordinate system  $\underline{\hat{f}}_j^i$  when the flotor is in the home position.

At the respective lower stingers (points  $S_i$ ), place eight push-rod-fixed coordinate systems  $\underline{\hat{p}}_j^i$  and eight stator-fixed coordinate systems  $\underline{\hat{s}}_j^i$ . Orient these 24 coordinate systems so that when the stingers are relaxed (i.e., with the ISPR in the home position), the coordinate directions  $\underline{\hat{p}}_j^i$  and  $\underline{\hat{s}}_j^i$  are coaligned for the  $i$ th actuator with  $\underline{\hat{p}}_2^i$  (along with  $\underline{\hat{s}}_2^i$ , in the home position) directed from  $S_i$  toward  $A_2^i$ .

Finally, define a primary, central, flotor-fixed, reference coordinate system with coordinate directions  $\underline{\hat{f}}_j$ . All other flotor-fixed coordinate systems  $\underline{\hat{f}}_j^i$  are assumed capable of being referenced (e.g., by known direction cosine angles) to this system (see eq. (4)).

#### 5. ROTATION MATRICES

Let the  $\underline{\hat{a}}_j^i$  coordinate system rotate, relative to the  $\underline{\hat{f}}_j^i$  coordinate system, through positive angle  $q_1^i$  about the  $\underline{\hat{f}}_3^i$  axis. Similarly, let the orientation of the  $\underline{\hat{a}}_j^i$  coordinate system, relative to the  $\underline{\hat{p}}_j^i$  coordinate system, be described by consecutive positive rotations  $q_2^i$  (about the  $\underline{\hat{p}}_1^i$  axis) and  $q_3^i$  (about the moved three-axis). Let the orientation of the  $\underline{\hat{p}}_j^i$  coordinate system, relative to the  $\underline{\hat{s}}_j^i$  coordinate system, be described by consecutive positive rotations  $q_4^i$  (about the  $\underline{\hat{s}}_3^i$  axis),  $q_5^i$  (about the moved two-axis), and  $q_6^i$  (about the moved one-axis).

Let  $c_j^i$  and  $s_j^i$  represent the cosines and sines of the respective angles  $q_j^i$ . Then the rotation matrices among the several coordinate systems for the  $i$ th actuator assembly are as follows:

$$\begin{Bmatrix} \hat{\underline{p}}_1^i \\ \hat{\underline{p}}_2^i \\ \hat{\underline{p}}_3^i \end{Bmatrix} = \begin{bmatrix} c_4^i c_5^i & s_4^i c_5^i & -s_5^i \\ -s_4^i c_6^i + c_4^i s_5^i s_6^i & c_4^i c_6^i + s_4^i s_5^i s_6^i & c_5^i s_6^i \\ s_4^i s_6^i + c_4^i s_5^i c_6^i & -c_4^i s_6^i + s_4^i s_5^i c_6^i & c_5^i c_6^i \end{bmatrix} \begin{Bmatrix} \hat{\underline{s}}_1^i \\ \hat{\underline{s}}_2^i \\ \hat{\underline{s}}_3^i \end{Bmatrix}, \quad (1)$$

$$\begin{Bmatrix} \hat{\underline{a}}_1^i \\ \hat{\underline{a}}_2^i \\ \hat{\underline{a}}_3^i \end{Bmatrix} = \begin{bmatrix} c_3^i & c_2^i s_3^i & s_2^i s_3^i \\ -s_3^i & c_2^i c_3^i & s_2^i c_3^i \\ 0 & -s_2^i & c_2^i \end{bmatrix} \begin{Bmatrix} \hat{\underline{p}}_1^i \\ \hat{\underline{p}}_2^i \\ \hat{\underline{p}}_3^i \end{Bmatrix}, \quad (2)$$

and

$$\begin{Bmatrix} \hat{\underline{a}}_1^i \\ \hat{\underline{a}}_2^i \\ \hat{\underline{a}}_3^i \end{Bmatrix} = \begin{bmatrix} c_1^i & s_1^i & 0 \\ -s_1^i & c_1^i & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \hat{\underline{f}}_1^i \\ \hat{\underline{f}}_2^i \\ \hat{\underline{f}}_3^i \end{Bmatrix}. \quad (3)$$

Finally, define a rotation matrix between the eight flotor-fixed coordinate systems  $\hat{\underline{f}}_j^i$  and the single flotor-fixed reference coordinate system  $\hat{\underline{f}}_j$ :

$$\begin{Bmatrix} \hat{\underline{f}}_1^i \\ \hat{\underline{f}}_2^i \\ \hat{\underline{f}}_3^i \end{Bmatrix} = \begin{bmatrix} f_{11}^i & f_{12}^i & f_{13}^i \\ f_{21}^i & f_{22}^i & f_{23}^i \\ f_{31}^i & f_{32}^i & f_{33}^i \end{bmatrix} \begin{Bmatrix} \hat{\underline{f}}_1 \\ \hat{\underline{f}}_2 \\ \hat{\underline{f}}_3 \end{Bmatrix}. \quad (4)$$

## 6. GENERALIZED COORDINATES FOR $\tilde{S}$

The 48 angles  $q_j^i$  are the generalized coordinates of the system. For the  $i$ th actuator, the six associated generalized coordinates are as follows:  $q_1^i$  is the angle at the cross flexure of the  $i$ th actuator;  $q_2^i$  and  $q_3^i$  are the angles at the upper stinger; and  $q_4^i$ ,  $q_5^i$ , and  $q_6^i$  are the angles at the lower stinger.

## 7. GENERALIZED SPEEDS FOR $\tilde{S}$

Define generalized speeds  $u_j^i$  for the system as the time rate of change of the generalized coordinates of  $\tilde{S}$  in the inertial reference frame:

$$u_j^i = \dot{q}_j^i \text{ (for } j = 1, \dots, 6; i = 1, \dots, 8) . \quad (5)$$

## 8. ANGULAR VELOCITIES OF REFERENCE FRAMES AND RIGID BODIES

Designate the reference frames corresponding to the stator, the  $i$ th push-rod, the  $i$ th arm, and the flotor, by the symbols  $\hat{S}$ ,  $\hat{P}_i$ ,  $\hat{A}_i$ , and  $\hat{F}$ , respectively. Let  $\hat{S}_i$  and  $\hat{F}_i$  represent, respectively, the coordinate systems in  $\hat{S}$  and  $\hat{F}$  defined respectively by  $\begin{bmatrix} \hat{s}_1^i & \hat{s}_2^i & \hat{s}_3^i \end{bmatrix}^T$  and  $\begin{bmatrix} \hat{f}_1^i & \hat{f}_2^i & \hat{f}_3^i \end{bmatrix}^T$ . Two intermediate reference frames were introduced previously to permit describing the angular velocity of each push-rod relative to the stator; designate those intermediate frames corresponding to the  $i$ th actuator assembly by  $\hat{R}_i$  and  $\hat{Q}_i$ . Another intermediate reference frame was previously introduced between frames  $\hat{P}_i$  and  $\hat{A}_i$ ; designate this by  $\hat{T}_i$ .

Let each intermediate reference frame have a frame-fixed, dextral set of unit vectors. Indicate the unit vectors for each of these frame-fixed coordinate systems by using the corresponding lowercase letter ( $\hat{r}_j^i$  corresponding to  $\hat{R}_i$ , etc.). The following, then, give the expressions for the angular velocities of the various reference frames and rigid bodies of  $\tilde{S}$ :

$$F_i \underline{\omega}^{A_i} = u_1^i \hat{f}_3^i , \quad (6)$$

$$P_i \underline{\omega}^{T_i} = u_2^i \hat{p}_1^i , \quad (7)$$

$$T_i \underline{\omega}^{A_i} = u_3^i \hat{t}_3^i , \quad (8)$$

$$S_i \underline{\omega}^{R_i} = u_4^i \hat{s}_3^i , \quad (9)$$

$$R_i \underline{\omega}^{Q_i} = u_5^i \hat{r}_2^i , \quad (10)$$

$$Q_i \underline{\omega}^{P_i} = u_6^i \hat{q}_1^i . \quad (11)$$

Using the addition theorem for angular velocities, the angular velocities of the rigid bodies of  $\tilde{S}$  are

$$S_i \underline{\omega}^{A_i} = u_2^i \hat{p}_1^i + u_3^i \hat{t}_3^i + u_4^i \hat{s}_3^i + u_5^i \hat{t}_2^i + u_6^i \hat{q}_1^i , \quad (12)$$

$$S_i \underline{\omega}^{P_i} = u_4^i \hat{s}_3^i + u_5^i \hat{t}_2^i + u_6^i \hat{q}_1^i , \quad (13)$$

and

$$S_i \underline{\omega}^{F_i} = u_2^i \hat{p}_1^i + u_3^i \hat{t}_3^i + u_4^i \hat{s}_3^i + u_5^i \hat{t}_2^i + u_6^i \hat{q}_1^i - u_1^i \hat{f}_3^i . \quad (14)$$

## 9. BASIC ASSUMPTIONS

In the subsequent development of the ARIS equations of motion, it is assumed that ARIS works as intended; i.e., that the ARIS controller prevents the ISPR from exceeding its rattlespace constraints. It is also assumed that the small-angle approximations hold for angles  $q_j^i$ . Angular velocities and angular accelerations are assumed to be small as well. This means that the use of first-order linear perturbations will permit the full nonlinear equations of motion to be approximated accurately by a set of first-order linear differential equations. Finally, it is assumed that the angular velocity of the stator is negligible and that the stator translational velocities and accelerations are small.

## 10. LINEARIZED VELOCITIES OF THE CENTERS OF MASS FOR THE RIGID BODIES OF $\tilde{S}$

Represent by  $\underline{r}^{AB}$  the position vector from arbitrary point  $A$  to arbitrary point  $B$ . Define the following position vectors using the indicated scalars:

$$\underline{r}^{F_i A_2^i} = l_1^i \hat{a}_2^i + l_2^i \hat{a}_1^i , \quad (15)$$

$$\underline{r}^{S_i A_2^i} = l_3^i \hat{p}_2^i , \quad (16)$$

$$\underline{r}^{S_i P_i^*} = p_2^i \hat{p}_2^i , \quad (17)$$

$$\underline{r}^{A_2^i A_i^*} = a_1^i \hat{\underline{a}}_1^i + a_2^i \hat{\underline{a}}_2^i , \quad (18)$$

and

$$\underline{r}^{F_i F^*} = f_1^i \hat{\underline{f}}_1^i + f_2^i \hat{\underline{f}}_2^i + f_3^i \hat{\underline{f}}_3^i . \quad (19)$$

First-time derivatives of the appropriate position vectors, under the stated assumptions, yield expressions for the velocities of the centers of mass for the 17 rigid bodies. The following expressions are the linearized velocities for those centers of mass. (The presubscript indicates that the expressions are linearized; the presuperscript indicates the reference frame assumed fixed for purposes of the differentiations.)

$$S_{l\underline{v}}^{P_i^*} = p_2^i \left( -u_4^i \hat{\underline{s}}_1^i + u_6^i \hat{\underline{s}}_3^i \right) (i = 1, \dots, 8) , \quad (20)$$

$$S_{l\underline{v}}^{A_i^*} = \left[ -a_2^i u_3^i - (a_2^i + l_3^i) u_4^i \right] \hat{\underline{p}}_1^i + \left( a_1^i u_3^i + a_1^i u_4^i \right) \hat{\underline{p}}_2^i + \left[ a_2^i u_2^i - a_1^i u_5^i + (a_2^i + l_3^i) u_6^i \right] \hat{\underline{p}}_3^i , \quad (21)$$

and

$$\begin{aligned} S_{l\underline{v}}^{F^*} = & \left[ f_2^1 u_1^1 - v_2^1 u_3^1 - (v_2^1 + l_3^1) u_4^1 + v_3^1 u_5^1 \right] \hat{\underline{p}}_1^1 \\ & + \left[ -f_1^1 u_1^1 - v_3^1 (u_2^1 + u_6^1) + v_1^1 (u_3^1 + u_4^1) \right] \hat{\underline{p}}_2^1 + \left[ v_2^1 u_2^1 - v_1^1 u_5^1 + (v_2^1 + l_3^1) u_6^1 \right] \hat{\underline{p}}_3^1 , \end{aligned} \quad (22)$$

where

$$v_1^i = f_1^i - l_2^i , \quad (23)$$

$$v_2^i = f_2^i - l_1^i , \quad (24)$$

and

$$v_3^i = f_3^i . \quad (25)$$

## 11. LINEARIZED ACCELERATIONS OF THE CENTERS OF MASS FOR THE RIGID BODIES OF $\tilde{S}$

Taking the time derivatives of the respective linearized velocity vectors yields expressions for the linearized accelerations of the centers of mass for each rigid body. Note that the linearized velocity vectors may be used in this step—the full nonlinear accelerations need not be determined.

This is a tremendous savings of effort, which would not be afforded if Newton's Second Law were applied directly instead of Kane's approach:

$$S_i^{\underline{a}^{P_i^*}} = p_2^i \left( -\dot{u}_4^i \hat{s}_1^i + \dot{u}_6^i \hat{s}_3^i \right) , \quad (26)$$

$$\begin{aligned} S_i^{\underline{a}^{A_i^*}} = & \left[ -a_2^i \dot{u}_3^i - (a_2^i + l_3^i) \dot{u}_4^i \right] \hat{p}_1^i + (a_1^i \dot{u}_3^i + a_1^i \dot{u}_4^i) \hat{p}_2^i \\ & + \left[ a_2^i \dot{u}_2^i - a_1^i \dot{u}_5^i + (a_2^i + l_3^i) \dot{u}_6^i \right] \hat{p}_3^i , \end{aligned} \quad (27)$$

and

$$\begin{aligned} S_1^{\underline{a}^{F^*}} = & \left[ f_2^1 \dot{u}_1^1 - v_2^1 \dot{u}_3^1 - (v_2^1 + l_3^1) \dot{u}_4^1 + v_3^1 \dot{u}_5^1 \right] \hat{p}_1^1 \\ & + \left[ -f_1^1 \dot{u}_1^1 - v_3^1 (\dot{u}_2^1 + \dot{u}_6^1) + v_1^1 (\dot{u}_3^1 + \dot{u}_4^1) \right] \hat{p}_2^1 \\ & + \left[ v_2^1 \dot{u}_2^1 - v_1^1 \dot{u}_5^1 + (v_2^1 + l_3^1) \dot{u}_6^1 \right] \hat{p}_3^1 . \end{aligned} \quad (28)$$

## 12. LINEARIZED PARTIAL VELOCITY VECTORS FOR THE POINTS OF $\tilde{S}$ AT WHICH THE CONTACT/DISTANCE FORCES ARE ASSUMED TO ACT

The partial velocities and partial angular velocities are formed by inspection of the relevant velocity vectors. These partial velocities are then (and the order here is crucial) linearized by neglecting higher order terms.

### 12.1 Linearized Partial Velocities of $P_i^*$

For the  $i$ th push-rod, the linearized partial velocities are

$$S_i^{\underline{v}_r^{P_i^*}} = \underline{0} \text{ (for } r = 1, 2, 3) , \quad (29)$$

$$S_i^{\underline{v}_4^{P_i^*}} = -p_2^i \left( \hat{s}_1^i + q_4^i \hat{s}_2^i \right) , \quad (30)$$

$$S_i^{\underline{v}_5^{P_i^*}} = p_2^i q_6^i \hat{s}_1^i , \quad (31)$$

and



$$S_{l\underline{v}_6^{P_i^*}} = p_2^i \left( q_5^i \hat{s}_1^i - q_6^i \hat{s}_2^i + \hat{s}_3^i \right) . \quad (32)$$

### 12.2 Linearized Partial Velocities of $A_i^*$

For the  $i$ th arm, the linearized partial velocities are

$$S_{l\underline{v}_1^{A_i^*}} = \underline{0} , \quad (33)$$

$$S_{l\underline{v}_2^{A_i^*}} = -a_2^i q_2^i \hat{p}_2^i + \left( a_2^i + a_1^i q_3^i \right) \hat{p}_3^i , \quad (34)$$

$$S_{l\underline{v}_3^{A_i^*}} = -\left( a_2^i + a_1^i q_3^i \right) \hat{p}_1^i + \left( a_1^i - a_2^i q_3^i \right) \hat{p}_2^i + a_1^i q_2^i \hat{p}_3^i , \quad (35)$$

$$S_{l\underline{v}_4^{A_i^*}} = -\left( a_2^i + a_1^i q_3^i + l_3^i \right) \hat{p}_1^i + \left( a_1^i - a_2^i q_3^i \right) \hat{p}_2^i - \left( a_2^i q_5^i + a_1^i q_6^i + l_3^i q_5^i \right) \hat{p}_3^i , \quad (36)$$

$$S_{l\underline{v}_5^{A_i^*}} = \left( a_2^i q_2^i + a_2^i q_6^i + l_3^i q_6^i \right) \hat{p}_1^i - a_1^i q_6^i \hat{p}_2^i + \left( a_2^i q_3^i - a_1^i \right) \hat{p}_3^i , \quad (37)$$

and

$$S_{l\underline{v}_6^{A_i^*}} = -a_2^i q_2^i \hat{p}_2^i + \left( a_2^i + a_1^i q_3^i + l_3^i \right) \hat{p}_3^i . \quad (38)$$

### 12.3 Linearized Partial Velocities of $F^*$

For the flotor, the linearized partial velocities are

$$S_{l\underline{v}_1^{F^*}} = f_2^1 \hat{f}_1^1 - f_1^1 \hat{f}_2^1 , \quad (39)$$

$$S_{l\underline{v}_2^{F^*}} = -\left( v_2^1 q_2^1 + v_3^1 \right) \hat{p}_2^1 + \left( v_1^1 q_3^1 + v_2^1 - v_3^1 q_2^1 - f_1^1 q_1^1 \right) \hat{p}_3^1 , \quad (40)$$

$$S_{l\underline{v}_3^{F^*}} = \left( -v_1^1 q_3^1 - v_2^1 + f_1^1 q_1^1 \right) \hat{p}_1^1 + \left( v_1^1 - v_2^1 q_3^1 + f_2^1 q_1^1 \right) \hat{p}_2^1 + v_1^1 q_2^1 \hat{p}_3^1 , \quad (41)$$

$$\begin{aligned} S_{l\underline{v}_4^{F^*}} = & \left[ -v_1^1 q_3^1 - v_2^1 + v_3^1 \left( q_2^1 + q_6^1 \right) + f_1^1 q_1^1 - l_3^1 \right] \hat{p}_1^1 \\ & + \left( v_1^1 - v_2^1 q_3^1 + v_3^1 q_5^1 + f_2^1 q_1^1 \right) \hat{p}_2^1 - \left( v_1^1 q_6^1 + v_2^1 q_5^1 + l_3^1 q_5^1 \right) \hat{p}_3^1 , \end{aligned} \quad (42)$$

$$S_{l\psi_5}^{F*} = \left[ v_2^1(q_2^1 + q_6^1) + v_3^1 + l_3^1 q_6^1 \right] \hat{p}_1^1 - v_1^1 q_6^1 \hat{p}_2^1 - (v_1^1 - v_2^1 q_3^1 + f_2^1 q_1^1) \hat{p}_3^1, \quad (43)$$

and

$$S_{l\psi_6}^{F*} = -\left( v_2^1 q_2^1 + v_3^1 \right) \hat{p}_2^1 + \left( v_1^1 q_3^1 + v_2^1 - v_3^1 q_2^1 - f_1^1 q_1^1 + l_3^1 \right) \hat{p}_3^1. \quad (44)$$

## 12.4 Linearized Partial Velocities of $F_u$

Define measure numbers for  $\underline{r}^{F*F_u}$  as follows:

$$\underline{r}^{F*F_u} = X_{F_u} \hat{f}_1^1 + Y_{F_u} \hat{f}_2^1 + Z_{F_u} \hat{f}_3^1. \quad (45)$$

Since

$$S_{l\psi_r}^{F_u} = S_{l\psi_r}^{F*} + \left[ \frac{\partial}{\partial u_r} \left( S_1 \underline{\omega}^F \times \underline{r}^{F*F_u} \right) \right], \quad (46)$$

the linearized partial velocities for the umbilical attachment point  $F_u$  can be expressed as follows:

$$S_{l\psi_1}^{F_u} = S_{l\psi_1}^{F*} + Y_{F_u} \hat{f}_1^1 - X_{F_u} \hat{f}_2^1, \quad (47)$$

$$S_{l\psi_2}^{F_u} = S_{l\psi_2}^{F*} - \left( Z_{F_u} + Y_{F_u} q_2^1 \right) \hat{p}_2^1 + \left[ Y_{F_u} + X_{F_u} (q_3^1 - q_1^1) - Z_{F_u} q_2^1 \right] \hat{p}_3^1, \quad (48)$$

$$S_{l\psi_3}^{F_u} = S_{l\psi_3}^{F*} + \left[ X_{F_u} (q_1^1 - q_3^1) - Y_{F_u} \right] \hat{p}_1^1 + \left[ X_{F_u} + Y_{F_u} (q_1^1 - q_3^1) \right] \hat{p}_2^1 + X_{F_u} q_2^1 \hat{p}_3^1, \quad (49)$$

$$\begin{aligned} S_{l\psi_4}^{F_u} = & S_{l\psi_4}^{F*} + \left[ X_{F_u} (q_1^1 - q_3^1) + Z_{F_u} (q_2^1 + q_6^1) - Y_{F_u} \right] \hat{p}_1^1 \\ & + \left[ X_{F_u} + Y_{F_u} (q_1^1 - q_3^1) + Z_{F_u} q_5^1 \right] \hat{p}_2^1 - \left[ X_{F_u} q_6^1 + Y_{F_u} q_5^1 \right] \hat{p}_3^1, \end{aligned} \quad (50)$$

$$S_{l\psi_5}^{F_u} = S_{l\psi_5}^{F*} + \left[ Z_{F_u} + Y_{F_u} (q_2^1 + q_6^1) \right] \hat{p}_1^1 - \left( X_{F_u} q_6^1 \right) \hat{p}_2^1 - \left[ Y_{F_u} (q_1^1 - q_3^1) + X_{F_u} \right] \hat{p}_3^1, \quad (51)$$

and

$$S_{l\psi_6}^{F_u} = S_{l\psi_6}^{F*} - \left( Z_{F_u} + Y_{F_u} q_2^1 \right) \hat{p}_2^1 + \left[ Y_{F_u} + X_{F_u} (q_3^1 - q_1^1) - Z_{F_u} q_2^1 \right] \hat{p}_3^1. \quad (52)$$

## 12.5 Linearized Partial Velocities of $F_i$

The linearized partial velocities of  $F_i$  are as follows:

$$S_{l\underline{v}_1}^{F_i} = \underline{0} , \quad (53)$$

$$S_{l\underline{v}_2}^{F_i} = -(l_1^i + l_2^i q_3^i) \hat{\underline{a}}_3^i , \quad (54)$$

$$S_{l\underline{v}_3}^{F_i} = l_1^i \hat{\underline{a}}_1^i - l_2^i \hat{\underline{a}}_2^i , \quad (55)$$

$$S_{l\underline{v}_4}^{F_i} = (l_1^i - l_3^i) \hat{\underline{a}}_1^i + (l_3^i q_3^i - l_2^i) \hat{\underline{a}}_2^i + [l_1^i q_5^i + l_2^i (q_2^i + q_6^i) - l_3^i q_5^i] \hat{\underline{a}}_3^i , \quad (56)$$

$$S_{l\underline{v}_5}^{F_i} = [-l_1^i (q_2^i + q_6^i) + l_3^i q_6^i] \hat{\underline{a}}_1^i + l_2^i (q_2^i + q_6^i) \hat{\underline{a}}_2^i + (l_2^i - l_1^i q_3^i) \hat{\underline{a}}_3^i , \quad (57)$$

and

$$S_{l\underline{v}_6}^{F_i} = l_3^i q_2^i \hat{\underline{a}}_2^i - (l_1^i + l_2^i q_3^i - l_3^i) \hat{\underline{a}}_3^i . \quad (58)$$

## 13. LINEARIZED PARTIAL ANGULAR VELOCITIES FOR THE RIGID BODIES OF $\tilde{S}$

The following are the linearized partial angular velocities for the system:

For the  $i$ th push-rod:

$$S_{l\underline{\omega}_r}^{P_i} = \underline{0} \quad (\text{for } r = 1, 2, 3) , \quad (59)$$

$$S_{l\underline{\omega}_4}^{P_i} = \hat{\underline{s}}_3^i , \quad (60)$$

$$S_{l\underline{\omega}_5}^{P_i} = \hat{\underline{e}}_2^i , \quad (61)$$

and

$$S_{l\underline{\omega}_6}^{P_i} = \hat{\underline{q}}_1^i ; \quad (62)$$

for the  $i$ th actuator arm:

$$S_i \omega_1^{A_i} = \underline{0} \ , \quad (63)$$

$$S_i \omega_2^{A_i} = \hat{p}_1^i \ , \quad (64)$$

$$S_i \omega_3^{A_i} = \hat{t}_3^i \ , \quad (65)$$

$$S_i \omega_4^{A_i} = \hat{s}_3^i \ , \quad (66)$$

$$S_i \omega_5^{A_i} = \hat{t}_2^i \ , \quad (67)$$

and

$$S_i \omega_6^{A_i} = \hat{q}_1^i \ ; \quad (68)$$

for the flotor:

$$S_1 \omega_1^F = -\underline{a}_3^1 \ , \quad (69)$$

$$S_1 \omega_2^F = \hat{p}_1^1 \ , \quad (70)$$

$$S_1 \omega_3^F = \hat{t}_3^1 \ , \quad (71)$$

$$S_1 \omega_4^F = \hat{s}_3^1 \ , \quad (72)$$

$$S_1 \omega_5^F = \hat{t}_2^1 \ , \quad (73)$$

$$S_1 \omega_6^F = \hat{q}_1^1 \ , \quad (74)$$

and

$$S_i \omega_r^F = \underline{0} \ (r = 1, \dots, 6; i = 2, \dots, 8) \ . \quad (75)$$

## 14. LINEARIZED ANGULAR ACCELERATIONS FOR THE RIGID BODIES OF $\tilde{S}$

### 14.1 Linearized Angular Acceleration of Actuator Push-Rod $\hat{P}_i$

$$S_i \underline{\alpha}^P = \dot{u}_6^i \hat{\underline{p}}_1^i + \dot{u}_5^i \hat{\underline{p}}_2^i + \dot{u}_4^i \hat{\underline{p}}_3^i . \quad (76)$$

### 14.2 Linearized Angular Acceleration of Actuator Arm $\hat{A}_i$

$$S_i \underline{\alpha}^A = (\dot{u}_2^i + \dot{u}_6^i) \hat{\underline{a}}_1^i + \dot{u}_5^i \hat{\underline{a}}_2^i + (\dot{u}_3^i + \dot{u}_4^i) \hat{\underline{a}}_3^i . \quad (77)$$

### 14.3 Linearized Angular Acceleration of the Flotor $\hat{F}$

$$S_i \underline{\alpha}^F = (\dot{u}_2^i + \dot{u}_6^i) \hat{\underline{f}}_1^i + \dot{u}_5^i \hat{\underline{f}}_2^i + (-\dot{u}_1^i + \dot{u}_3^i + \dot{u}_4^i) \hat{\underline{f}}_3^i . \quad (78)$$

## 15. CONTRIBUTIONS TO THE SET OF GENERALIZED ACTIVE FORCES DUE TO THE RIGID BODIES OF $\tilde{S}$

### 15.1 Contributions Due to the Flotor

On orbit (i.e., neglecting the effects of gravity), the flotor is acted upon by forces and moments due to each Lorentz coil, actuator arm, and umbilical, and by direct disturbances.

Let  $-\underline{F}^{C_i}$  and  $-\underline{M}^{C_i}$  represent, respectively, the force and moment exerted by the  $i$ th Lorentz coil (located at  $A_3^i$ ) on the flotor,

where

$$\underline{F}^{C_i} = F^{C_i} \hat{\underline{a}}_1^i , \quad (79)$$

assumed to act at point  $F_i$ , and

$$\underline{M}^{C_i} = \underline{r}^{F_i A_3^i} \times \underline{F}^{C_i} = -F^{C_i} (l_1^i + l_4^i) \hat{\underline{a}}_3^i . \quad (80)$$

Let  $\underline{F}^{F_i}$  and  $\underline{M}^{F_i}$  represent, respectively, the force and moment exerted by the  $i$ th actuator arm on the flotor, at the  $i$ th cross-flexure. Since  $\underline{F}^{F_i}$  is a noncontributing force, it can be ignored in the analysis. The total moment  $\underline{M}^F$  due to the eight cross-flexure springs has value

$$\underline{M}^F = \sum_{i=1}^8 k_1^i q_1^i \hat{f}_3^i, \quad (81)$$

where  $k_1^i$  is the  $i$ th cross-flexure spring stiffness.

Let  $\underline{F}^U$  and  $\underline{M}^U$  represent, respectively, the force and moment applied to the flotor by the umbilical, where the force is assumed to act at flotor-fixed point  $F_u$ . Umbilical force  $\underline{F}^U$  is given by the equation

$$\begin{aligned} \underline{F}^U = & (-k_1 x_1 - c_1 \dot{x}_1) \hat{\underline{\xi}}_1 + (-k_2 x_2 - c_2 \dot{x}_2) \hat{\underline{\xi}}_2 \\ & + (-k_3 x_3 - c_3 \dot{x}_3) \hat{\underline{\xi}}_3 + \underline{F}_b; \end{aligned} \quad (82)$$

where  $\hat{\underline{\xi}}_i$  is some appropriate stator-fixed coordinate system;  $x_1$ ,  $x_2$ , and  $x_3$  are the umbilical elongations in the respective  $\hat{\underline{\xi}}_i$  ( $i = 1, 2, 3$ ) directions;  $\underline{F}_b$  is the umbilical bias force in the home position;  $k_1$ ,  $k_2$ , and  $k_3$  are umbilical spring stiffnesses; and  $c_1$ ,  $c_2$ , and  $c_3$  are umbilical damping constants. Umbilical moment  $\underline{M}^U$  is given by

$$\underline{M}^U = (-\kappa_1 \phi_1 - \gamma_1 \dot{\phi}_1) \hat{\underline{\xi}}_1 + (-\kappa_2 \phi_2 - \gamma_2 \dot{\phi}_2) \hat{\underline{\xi}}_2 + (-\kappa_3 \phi_3 - \gamma_3 \dot{\phi}_3) \hat{\underline{\xi}}_3 + \underline{M}_b, \quad (83)$$

where  $\phi_1$ ,  $\phi_2$ , and  $\phi_3$  are components of the umbilical angle of twist  $\phi$  in the respective  $\hat{\underline{\xi}}_i$  ( $i = 1, 2, 3$ ) directions;  $\underline{M}_b$  is the umbilical bias moment in the home position;  $\kappa_1$ ,  $\kappa_2$ , and  $\kappa_3$  are torsional umbilical spring stiffnesses; and  $\gamma_1$ ,  $\gamma_2$ , and  $\gamma_3$  are torsional umbilical damping constants.

Let  $\underline{F}^D$  and  $\underline{M}^D$  represent, respectively, the unknown disturbance force and moment acting on the flotor. Assume  $\underline{F}^D$  to act through the flotor mass center  $F^*$ . Define  $F_i^D$  and  $M_i^D$  to be the  $i$ th components, respectively, of  $\underline{F}^D$  and  $\underline{M}^D$ , componentiated in  $\hat{F}_1$ .

In terms of the above, the flotor's contribution to the set of generalized active forces, for the  $r$ th generalized speed, is

$$\begin{aligned}
{}_l Q_r^F = & S_1 {}_l \underline{v}_r^{F_u} \cdot \underline{F}^U + S_1 {}_l \underline{v}_r^{F^*} \cdot \underline{F}^D + S_1 {}_l \underline{\omega}_r^F \cdot \left( \underline{M}^U + \underline{M}^D + \sum_{i=1}^8 k_1^i q_1^i \hat{f}_3^i \right) \\
& - \sum_{i=1}^8 \left( S_1 {}_l \underline{v}_r^{F_i} \cdot \underline{F}^{C_i} + S_1 {}_l \underline{\omega}_r^F \cdot \underline{M}^{C_i} \right) .
\end{aligned} \tag{84}$$

The umbilical contributions to the  ${}_l Q_r^F$ 's, namely,  $S_1 {}_l \underline{v}_r^{F_u} \cdot \underline{F}^U + S_1 {}_l \underline{\omega}_r^F \cdot \underline{M}^U$ , are addressed in the following two sections. The remaining terms of the  ${}_l Q_r^F$ 's are as follows:

$$S_1 {}_l \underline{v}_1^{F^*} \cdot \underline{F}^D = f_2^1 F_1^D - f_1^1 F_2^D , \tag{85}$$

$$S_1 {}_l \underline{\omega}_1^F \cdot \underline{M}^D = -M_3^D , \tag{86}$$

$$S_1 {}_l \underline{v}_1^{F_i} \cdot (-\underline{F}^{C_i}) = 0 , \tag{87}$$

$$S_1 {}_l \underline{\omega}_1^F \cdot (-\underline{M}^{C_i}) = -F^{C_i} (l_1^i + l_4^i) , \tag{88}$$

$$S_1 {}_l \underline{\omega}_1^F \cdot \left( \sum_{i=1}^8 k_1^i q_1^i \hat{f}_3^i \right) = -k_1^1 q_1^1 - k_1^2 q_1^2 + k_1^3 q_1^3 + k_1^4 q_1^4 - k_1^5 q_1^5 , \tag{89}$$

$$S_1 {}_l \underline{v}_2^{F^*} \cdot \underline{F}^D = v_3^1 (q_1^1 - q_3^1) F_1^D - v_3^1 F_2^D + F_3^D (v_1^1 q_3^1 + v_2^1 - f_1^1 q_1^1) , \tag{90}$$

$$S_1 {}_l \underline{\omega}_2^F \cdot \underline{M}^D = M_1^D + (q_1^1 - q_3^1) M_2^D , \tag{91}$$

$$S_1 {}_l \underline{v}_2^{F_i} \cdot (-\underline{F}^{C_i}) = 0 , \tag{92}$$

$$S_1 {}_l \underline{\omega}_2^F \cdot (-\underline{M}^{C_i}) = 0 , \tag{93}$$

$$S_1 {}_l \underline{\omega}_2^F \cdot \left( \sum_{i=1}^8 k_1^i q_1^i \hat{f}_3^i \right) = 0 , \tag{94}$$

$$S_1 {}_l \underline{v}_3^{F^*} \cdot \underline{F}^D = (-v_2^1 + l_2^1 q_1^1) F_1^D + (v_1^1 + l_1^1 q_1^1) F_2^D , \tag{95}$$

$$S_1 {}_l \underline{\omega}_3^F \cdot \underline{M}^D = M_3^D , \tag{96}$$

$$S_1 \underline{v}_3^{F_i} \cdot (-\underline{F}^{C_i}) = -F^{C_i} l_1^i , \quad (97)$$

$$S_1 \underline{\omega}_3^F \cdot (-\underline{M}^{C_i}) = F^{C_i} (l_1^i + l_4^i) , \quad (98)$$

$$S_1 \underline{\omega}_3^F \cdot \left( \sum_{i=1}^8 k_1^i q_1^i \hat{f}_3^i \right) = -k_1^1 q_1^1 - k_1^2 q_1^2 + k_1^3 q_1^3 + k_1^4 q_1^4 - k_1^5 q_1^5 , \quad (99)$$

$$\begin{aligned} S_1 \underline{v}_4^{F^*} \cdot \underline{F}^D &= \left[ -l_3^1 - v_2^1 + v_3^1 (q_2^1 + q_6^1) + l_2^1 q_1^1 \right] F_1^D \\ &\quad + \left[ l_3^1 (q_3^1 - q_1^1) + v_1^1 + l_1^1 q_1^1 + v_3^1 q_5^1 \right] F_2^D \\ &\quad + \left[ -v_1^1 (q_2^1 + q_6^1) + (v_2^1 - l_3^1) q_5^1 \right] F_3^D , \end{aligned} \quad (100)$$

$$S_1 \underline{\omega}_4^F \cdot \underline{M}^D = (q_2^1 + q_6^1) M_2^D - q_5^1 M_1^D + M_3^D , \quad (101)$$

$$S_1 \underline{v}_4^{F_i} \cdot (-\underline{F}^{C_i}) = -F^{C_i} (l_1^i - l_3^i) , \quad (102)$$

$$S_1 \underline{\omega}_4^F \cdot (-\underline{M}^{C_i}) = S_1 \underline{\omega}_3^F \cdot (-\underline{M}^{C_i}) , \quad (103)$$

$$S_1 \underline{\omega}_4^F \cdot \left( \sum_{i=1}^8 k_1^i q_1^i \hat{f}_3^i \right) = S_1 \underline{\omega}_3^F \cdot \left( \sum_{i=1}^8 k_1^i q_1^i \hat{f}_3^i \right) , \quad (104)$$

$$\begin{aligned} S_1 \underline{v}_5^{F^*} \cdot \underline{F}^D &= \left[ l_3^1 q_6^1 + v_2^1 (q_2^1 + q_6^1) + v_3^1 \right] F_1^D \\ &\quad + \left[ -v_1^1 (q_2^1 + q_6^1) + v_3^1 (q_1^1 - q_3^1) \right] F_2^D \\ &\quad + \left[ -v_1^1 + v_2^1 q_3^1 - f_2^1 q_1^1 \right] F_3^D , \end{aligned} \quad (105)$$

$$S_1 \underline{\omega}_5^F \cdot \underline{M}^D = -(q_1^1 - q_3^1) M_1^D + M_2^D - (q_2^1 + q_6^1) M_3^D , \quad (106)$$

$$S_1 \underline{v}_5^{F_i} \cdot (-\underline{F}^{C_i}) = F^{C_i} \left[ l_1^i q_2^i + (l_1^i - l_3^i) q_6^i \right] , \quad (107)$$

$$S_1 \underline{\omega}_5^F \cdot (-\underline{M}^{C_i}) = -F^{C_i} (l_1^i + l_4^i) (q_2^i + q_6^i) , \quad (108)$$



$$S_1 \underline{\omega}_5^F \cdot \left( \sum_{i=1}^8 k_1^i q_1^i \hat{f}_3^i \right) = 0 , \quad (109)$$

$$S_1 \underline{v}_6^{F*} \cdot \underline{F}^D = v_3^1 (q_1^1 - q_3^1) F_1^D + (l_3^1 q_2^1 - v_3^1) F_2^D + (l_3^1 + v_1^1 q_3^1 + v_2^1 - f_1^1 q_1^1) F_3^D , \quad (110)$$

$$S_1 \underline{\omega}_6^F \cdot \underline{M}^D = S_1 \underline{\omega}_2^F \cdot \underline{M}^D , \quad (111)$$

$$S_1 \underline{v}_6^{F_i} \cdot (-\underline{F}^{C_i}) = 0 , \quad (112)$$

$$S_1 \underline{\omega}_6^F \cdot (-\underline{M}^{C_i}) = 0 , \quad (113)$$

and

$$S_1 \underline{\omega}_6^F \cdot \left( \sum_{i=1}^8 k_1^i q_1^i \hat{f}_3^i \right) = S_1 \underline{\omega}_2^F \cdot \left( \sum_{i=1}^8 k_1^i q_1^i \hat{f}_3^i \right) . \quad (114)$$

Notice the coupling between the unknown-disturbance measure numbers and the generalized coordinates. This coupling will make the disturbance input matrix  $E$  (in eq. (203)) time-varying.

## 15.2 Umbilical Force $\underline{F}^U$

Equation (82) expresses umbilical force  $\underline{F}^U$  in terms of umbilical-elongation components  $x_1, x_2$ , and  $x_3$ , and their time derivatives. These items must be reexpressed in terms of the generalized coordinates and generalized speeds.

If the umbilical attachment point  $F_u$  is at  $F_{uh}$  in the home position, then

$$x_i = \left( \underline{r}^{S_u F_u} - \underline{r}^{S_u F_{uh}} \right) \cdot \hat{\underline{x}}_i , \quad (i = 1, 2, 3) . \quad (115)$$

But

$$\underline{r}^{S_u F_u} - \underline{r}^{S_u F_{uh}} = \underline{r}^{F_{uh} F_u} = \underline{r}^{S_1 F_1} + \underline{r}^{F_1 F_u} - \underline{r}^{S_1 S_u} - \underline{r}^{S_u F_{uh}} , \quad (116)$$

where the right-hand-side terms can be expressed by

$$\underline{r}^{S_1 F_1} = l_3^1 \hat{\underline{p}}_2^1 - l_2^1 \hat{\underline{a}}_1^1 - l_1^1 \hat{\underline{a}}_2^1 , \quad (117)$$

$$\underline{r}^{F_1 F_u} = x_{Fu} \hat{\underline{f}}_1^1 + y_{Fu} \hat{\underline{f}}_2^1 + z_{Fu} \hat{\underline{f}}_3^1 , \quad (118)$$

$$\underline{r}^{S_1 S_u} = x_{Su} \hat{\underline{s}}_1^1 + y_{Su} \hat{\underline{s}}_2^1 + z_{Su} \hat{\underline{s}}_3^1 , \quad (119)$$

and

$$\underline{r}^{S_u F_{uh}} = x_0 \hat{\underline{s}}_1^1 + y_0 \hat{\underline{s}}_2^1 + z_0 \hat{\underline{s}}_3^1 , \quad (120)$$

for appropriately defined coefficients.

Now, define the following rotation matrix:

$$\begin{Bmatrix} \hat{\underline{\xi}}_1 \\ \hat{\underline{\xi}}_2 \\ \hat{\underline{\xi}}_3 \end{Bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \begin{Bmatrix} \hat{\underline{s}}_1^1 \\ \hat{\underline{s}}_2^1 \\ \hat{\underline{s}}_3^1 \end{Bmatrix} . \quad (121)$$

In terms of the  $\hat{\underline{\xi}}_i$  coordinate system, equation (116) can now be written in linearized form as

$$\underline{r}^{F_{uh} F_u} = x_1 \hat{\underline{\xi}}_1 + x_2 \hat{\underline{\xi}}_2 + x_3 \hat{\underline{\xi}}_3 , \quad (122)$$

where

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \cdot \begin{Bmatrix} C_1 + y_{Fu} (q_1^1 - q_3^1 - q_4^1) + z_{Fu} q_5^1 - l_3^1 q_4^1 + l_1^1 (q_3^1 + q_4^1) \\ C_2 - z_{Fu} (q_2^1 + q_6^1) - x_{Fu} (q_1^1 - q_3^1 - q_4^1) - l_2^1 (q_3^1 + q_4^1) \\ C_3 + (l_2^1 - x_{Fu}) q_5^1 + (y_{Fu} - l_1^1) (q_2^1 + q_6^1) + l_3^1 q_6^1 \end{Bmatrix} \quad (123)$$

for

$$C_1 = x_{Fu} - x_{Su} - l_2^1 - x_0 , \quad (124)$$

$$C_2 = y_{Fu} - y_{Su} + l_3^1 - l_1^1 - y_0 , \quad (125)$$

and

$$C_3 = z_{Fu} - z_{Su} - z_0 . \quad (126)$$

Differentiating,

$$\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{Bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \cdot \begin{Bmatrix} y_{F_u} (u_1^1 - u_3^1 - u_4^1) + z_{F_u} u_5^1 - l_3^1 u_4^1 + l_1^1 (u_3^1 + u_4^1) \\ -z_{F_u} (u_2^1 + u_6^1) - x_{F_u} (u_1^1 - u_3^1 - u_4^1) - l_2^1 (u_3^1 + u_4^1) \\ (l_2^1 - x_{F_u}) u_5^1 + (y_{F_u} - l_1^1) (u_2^1 + u_6^1) + l_3^1 u_6^1 \end{Bmatrix}. \quad (127)$$

Using equations (82) and (123)–(127), a linearized expression could now be written straightforwardly for the umbilical force  $\underline{F}^U$ .

### 15.3 Umbilical Moment $\underline{M}^U$

Equation (83) expresses umbilical moment  $\underline{M}^U$  in terms of angle-of-twist components  $\phi_1$ ,  $\phi_2$ , and  $\phi_3$ , and their time derivatives. These items must be reexpressed in terms of the generalized coordinates and generalized speeds.

Let  $\phi \hat{n}_\phi$  represent the rotation of the flotor, relative to the stator, from the home position.  $\hat{n}_\phi$  is the rotation axis, and  $\phi$  is the angle of twist about that axis. Note that

$$\phi_i = \phi \hat{n}_\phi \cdot \hat{\underline{\xi}}_i \quad (i = 1, 2, 3). \quad (128)$$

Express  $\hat{n}_\phi$  as

$$\hat{n}_\phi = g_1 \hat{\underline{f}}_1^1 + g_2 \hat{\underline{f}}_2^1 + g_3 \hat{\underline{f}}_3^1. \quad (129)$$

Define rotation matrix  $Q$  by

$$\begin{Bmatrix} \hat{\underline{f}}_1^1 \\ \hat{\underline{f}}_2^1 \\ \hat{\underline{f}}_3^1 \end{Bmatrix} = [Q] \begin{Bmatrix} \hat{\underline{s}}_1^1 \\ \hat{\underline{s}}_2^1 \\ \hat{\underline{s}}_3^1 \end{Bmatrix}. \quad (130)$$

The linearized 3×3 rotation matrix  ${}_lQ$  has elements  ${}_lQ_{ij}$  defined as follows:

$$\begin{Bmatrix} \hat{f}_1^1 \\ \hat{f}_2^1 \\ \hat{f}_3^1 \end{Bmatrix} = \begin{bmatrix} 1 & -q_1^1 + q_3^1 + q_4^1 & -q_5^1 \\ q_1^1 - q_3^1 - q_4^1 & 1 & q_2^1 + q_6^1 \\ q_5^1 & -q_2^1 - q_6^1 & 1 \end{bmatrix} \begin{Bmatrix} \hat{s}_1^1 \\ \hat{s}_2^1 \\ \hat{s}_3^1 \end{Bmatrix}. \quad (131)$$

For small  $\phi$ , it can be shown<sup>13</sup> that

$$\phi \begin{bmatrix} 0 & g_3 & -g_2 \\ -g_3 & 0 & g_1 \\ g_2 & -g_1 & 0 \end{bmatrix} = \frac{{}_lQ - {}_lQ^T}{(1 + \text{tr } {}_lQ)^{1/2}}, \quad (132)$$

where the post-superscript  $T$  indicates matrix transposition and  $\text{tr } {}_lQ$  represents the trace of  ${}_lQ$ . Substitution from equation (131) into equation (132), and simplification, yields

$$g_1 = \frac{1}{\phi} \cdot (q_2^1 + q_6^1), \quad (133)$$

$$g_2 = \frac{1}{\phi} \cdot q_5^1, \quad (134)$$

and

$$g_3 = \frac{-1}{\phi} \cdot (q_1^1 - q_3^1 - q_4^1). \quad (135)$$

Substituting from equations (133)–(135) into equation (129), and transforming into the  $\hat{s}_i^1$  coordinate system by use of  ${}_lQ$ , one obtains the following expression for the spin axis:

$${}_l\hat{n}_\phi = \frac{1}{{}_l\phi} \left[ (q_2^1 + q_6^1) \hat{s}_1^1 + q_5^1 \hat{s}_2^1 - (q_1^1 - q_3^1 - q_4^1) \hat{s}_3^1 \right]. \quad (136)$$

Since  ${}_l\hat{n}_\phi$  has unit length,

$${}_l\phi = \left[ (q_2^1 + q_6^1)^2 + (q_5^1)^2 + (q_1^1 - q_3^1 - q_4^1)^2 \right]^{1/2}. \quad (137)$$

Use of equations (128) and (136) leads to the following linearized forms for angular position and rotation rate:

$$\begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{Bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \begin{Bmatrix} (q_2^1 + q_6^1) \\ q_5^1 \\ -(q_1^1 - q_3^1 - q_4^1) \end{Bmatrix}, \quad (138)$$

and

$$\begin{Bmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \\ \dot{\phi}_3 \end{Bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \begin{Bmatrix} (u_2^1 + u_6^1) \\ u_5^1 \\ -(u_1^1 - u_3^1 - u_4^1) \end{Bmatrix}. \quad (139)$$

From equations (83), (138), and (139), a linearized expression could now be written straightforwardly for the umbilical moment  $\underline{M}^U$ . The flotor's contribution to the set of generalized active forces for the  $r$ th generalized speed could then be found by substituting the expressions for  $\underline{F}^U$  (sec. 15.2) and  $\underline{M}^U$  into equation (84).

#### 15.4 Contributions Due to the Actuator Arms

The forces and moments acting on the  $i$ th actuator arm are due to the respective Lorentz coil (located at  $A_3^i$ ), the flotor (through the  $i$ th cross flexure), and the respective push-rod (through the upper stinger). The coil force  $\underline{F}^{C_i}$  is the only contributing force. The contributing loads, in the above indicated order, are as follows:

$$\underline{F}^{C_i} = F^{C_i} \hat{\underline{a}}_1^i, \text{ assumed to act at point } F_i, \quad (140)$$

$$\underline{M}^{C_i} = \underline{r}^{F_i A_3^i} \times \underline{F}^{C_i} = (l_1^i + l_4^i) \hat{\underline{a}}_2^i \times F^{C_i} \hat{\underline{a}}_1^i = -F^{C_i} (l_1^i + l_4^i) \hat{\underline{a}}_3^i, \quad (141)$$

$$-\underline{M}^{F_i} = -k_1^i q_1^i \hat{\underline{f}}_3^i, \quad (142)$$

and

$$\underline{M}^{A_i} = -k_2^i q_2^i \hat{\underline{p}}_1^i - k_3^i q_3^i \hat{\underline{t}}_3^i, \quad (143)$$

where  $l_2^i$  and  $l_4^i$  are pertinent geometric lengths, and  $k_2^i$  and  $k_3^i$  are pertinent upper-stinger spring stiffnesses.

In terms of the above, the contribution for the  $i$ th actuator arm to the set of generalized active forces for the  $r$ th generalized speed is

$${}_l Q_r^{A_i} = S_{l\underline{v}_r^{F_i}} \cdot \underline{F}^{C_i} + S_{l\underline{\omega}_r^{A_i}} \cdot (\underline{M}^{A_i} + \underline{M}^{C_i} - \underline{M}^{F_i}) . \quad (144)$$

The individual terms of the  ${}_l \tilde{Q}_r^{A_i}$ 's are as follows:

$$S_{l\underline{v}_1^{F_i}} \cdot \underline{F}^{C_i} = 0 , \quad (145)$$

$$S_{l\underline{\omega}_1^{A_i}} \cdot (\underline{M}^{A_i} + \underline{M}^{C_i} - \underline{M}^{F_i}) = 0 , \quad (146)$$

$$S_{l\underline{v}_2^{F_i}} \cdot \underline{F}^{C_i} = 0 , \quad (147)$$

$$S_{l\underline{\omega}_2^{A_i}} \cdot (\underline{M}^{A_i} + \underline{M}^{C_i} - \underline{M}^{F_i}) = -k_2^i q_2^i , \quad (148)$$

$$S_{l\underline{v}_3^{F_i}} \cdot \underline{F}^{C_i} = F^{C_i} l_1^i , \quad (149)$$

$$S_{l\underline{\omega}_3^{A_i}} \cdot (\underline{M}^{A_i} + \underline{M}^{C_i} - \underline{M}^{F_i}) = k_1^i q_1^i - k_3^i q_3^i - F^{C_i} (l_1^i + l_4^i) , \quad (150)$$

$$S_{l\underline{v}_4^{F_i}} \cdot \underline{F}^{C_i} = F^{C_i} (l_1^i - l_3^i) , \quad (151)$$

$$S_{l\underline{\omega}_4^{A_i}} \cdot (\underline{M}^{A_i} + \underline{M}^{C_i} - \underline{M}^{F_i}) = -k_1^i q_1^i - k_3^i q_3^i - F^{C_i} (l_1^i + l_4^i) , \quad (152)$$

$$S_{l\underline{v}_5^{F_i}} \cdot \underline{F}^{C_i} = -F^{C_i} [l_1^i (q_2^i + q_6^i) - l_3^i q_6^i] , \quad (153)$$

$$S_{l\underline{\omega}_5^{A_i}} \cdot (\underline{M}^{A_i} + \underline{M}^{C_i} - \underline{M}^{F_i}) = F^{C_i} (l_1^i + l_4^i) (q_2^i + q_6^i) , \quad (154)$$

$$S_{l\underline{v}_6^{F_i}} \cdot \underline{F}^{C_i} = 0 , \quad (155)$$

and

$$S_{l\underline{\omega}_6^{A_i}} \cdot (\underline{M}^{A_i} + \underline{M}^{C_i} - \underline{M}^{F_i}) = -k_2^i q_2^i . \quad (156)$$

Notice the coupling between the control inputs and the generalized coordinates. This coupling will make the disturbance input matrix  $B$  (in eq. (204)) time varying.

### 15.5 Contributions Due to the Push-Rods

The contributing loads on each push-rod are moments  $\underline{M}^{P_i}$  and  $-\underline{M}^{A_i}$ , where (using pertinent lower-stinger stiffnesses)

$$\underline{M}^{P_i} = -k_4^i q_4^i \hat{s}_3^i - k_5^i q_5^i \hat{t}_2^i - k_6^i q_6^i \hat{q}_1^i . \quad (157)$$

The contribution for the  $i$ th push-rod to the set of generalized active forces for the  $r$ th generalized speed is

$${}_l Q_r^{P_i} = {}^S_l \underline{\omega}_r^{P_i} \cdot (\underline{M}^{P_i} - \underline{M}^{A_i}) . \quad (158)$$

The individual terms of the  ${}_l Q_r^{P_i}$ 's are as follows:

$${}^S_l \underline{\omega}_r^{P_i} \cdot (\underline{M}^{P_i} - \underline{M}^{A_i}) = 0 \quad (r = 1, 2, 3) , \quad (159)$$

$${}^S_l \underline{\omega}_4^{P_i} \cdot (\underline{M}^{P_i} - \underline{M}^{A_i}) = k_3^i q_3^i - k_4^i q_4^i , \quad (160)$$

$${}^S_l \underline{\omega}_5^{P_i} \cdot (\underline{M}^{P_i} - \underline{M}^{A_i}) = -k_5^i q_5^i , \quad (161)$$

and

$${}^S_l \underline{\omega}_6^{P_i} \cdot (\underline{M}^{P_i} - \underline{M}^{A_i}) = k_2^i q_2^i - k_6^i q_6^i . \quad (162)$$

## 16. CONTRIBUTIONS TO THE SET OF GENERALIZED INERTIA FORCES DUE TO THE RIGID BODIES OF $\tilde{S}$

Represent (by  $I_{jk}^{A_i/A_i^*}$ ) the central moment/product of inertia of the  $i$ th actuator arm for the  $j$  and  $k$  body-fixed coordinate directions  $\hat{a}_j^i$  and  $\hat{a}_k^i$ . Define push-rod inertias  $I_j^{P_i/P_i^*}$  analogously, where the single subscript indicates that the axes are assumed to be principal axes. Let  $I_{jk}^{F/F^*}$  represent the central inertia scalar of the flotor for the flotor-fixed coordinate directions  $\hat{f}_j^1$  and  $\hat{f}_k^1$ . Use the symbol  $\underline{H}$  to represent an angular momentum vector. Associated post-superscripts on  $\underline{H}$  have the same meanings as for the inertias. The contributions to the generalized inertia forces for  $\tilde{S}$  can now be expressed.

### 16.1 Contributions Due to the Push-Rods

The contributions  $\left({}_l Q_r^*\right)^{P_i}$  to the generalized inertia forces due to the  $i$ th push-rod are as follows:

$$\left({}_l Q_r^*\right)^{P_i} = S_{l v_r}^{P_i^*} \cdot \left(-m_{P_i} S_{l \underline{a}}^{P_i^*}\right) + S_{l \underline{\omega}_r}^{P_i} \cdot \left(-l \underline{\dot{H}}^{P_i/P_i^*}\right) = \underline{0} \quad (r = 1, 2, 3; i = 1, \dots, 8), \quad (163)$$

$$\begin{aligned} \left({}_l Q_4^*\right)^{P_i} &= S_{l v_4}^{P_i^*} \cdot \left(-m_{P_i} S_{l \underline{a}}^{P_i^*}\right) + S_{l \underline{\omega}_4}^{P_i} \cdot \left(-l \underline{\dot{H}}^{P_i/P_i^*}\right) \\ &= -\left[m_{P_i} \left(p_2^i\right)^2 + I_3^{P_i/P_i^*}\right] \dot{u}_4^i, \end{aligned} \quad (164)$$

$$\left({}_l Q_5^*\right)^{P_i} = S_{l v_5}^{P_i^*} \cdot \left(-m_{P_i} S_{l \underline{a}}^{P_i^*}\right) + S_{l \underline{\omega}_5}^{P_i} \cdot \left(-l \underline{\dot{H}}^{P_i/P_i^*}\right) = -I_2^{P_i/P_i^*} \dot{u}_5^i, \quad (165)$$

$$\begin{aligned} \left({}_l Q_6^*\right)^{P_i} &= S_{l v_6}^{P_i^*} \cdot \left(-m_{P_i} S_{l \underline{a}}^{P_i^*}\right) + S_{l \underline{\omega}_6}^{P_i} \cdot \left(-l \underline{\dot{H}}^{P_i/P_i^*}\right) \\ &= -\left[m_{P_i} \left(p_2^i\right)^2 + I_1^{P_i/P_i^*}\right] \dot{u}_6^i. \end{aligned} \quad (166)$$

### 16.2 Contributions Due to the Actuator Arms

The contribution  $\left({}_l Q_r^*\right)^{A_i}$  to the generalized inertia forces due to the  $i$ th actuator arm for the  $r$ th generalized speed is

$$\left({}_l Q_r^*\right)^{A_i} = S_{l v_r}^{A_i^*} \cdot \left(-m_{A_i} S_{l \underline{a}}^{A_i^*}\right) + S_{l \underline{\omega}_r}^{A_i} \cdot \left(-l \underline{\dot{H}}^{A_i/A_i^*}\right). \quad (167)$$

Then the individual, nonzero terms of the  $\left({}_l Q_r^*\right)^{A_i}$ 's are as follows:

$$S_{l v_2}^{A_i^*} \cdot \left(-m_{A_i} S_{l \underline{a}}^{A_i^*}\right) = -m_{A_i} \left\{ \left(a_2^i\right)^2 \dot{u}_2^i + \left[a_2^i \left(a_2^i + l_3^i\right)\right] \dot{u}_6^i - a_1^i a_2^i \dot{u}_5^i \right\}, \quad (168)$$



$$S_l \underline{\omega}_2^{A_i} \cdot \left( -l \underline{\dot{H}}^{A_i/A_i^*} \right) = -I_{11}^{A_i/A_i^*} (\dot{u}_2^i + \dot{u}_6^i) - I_{12}^{A_i/A_i^*} \dot{u}_5^i - I_{13}^{A_i/A_i^*} (\dot{u}_3^i + \dot{u}_4^i) , \quad (169)$$

$$S_l \underline{\nu}_3^{A_i^*} \cdot \left( -m_{A_i} S_l \underline{a}^{A_i^*} \right) = -m_{A_i} \left\{ \left[ (a_1^i)^2 + (a_2^i)^2 \right] \dot{u}_3^i + \left[ (a_1^i)^2 + a_2^i (a_2^i + l_3^i) \right] \dot{u}_4^i \right\} , \quad (170)$$

$$S_l \underline{\omega}_3^{A_i} \cdot \left( -l \underline{\dot{H}}^{A_i/A_i^*} \right) = -I_{31}^{A_i/A_i^*} (\dot{u}_2^i + \dot{u}_6^i) - I_{32}^{A_i/A_i^*} \dot{u}_5^i - I_{33}^{A_i/A_i^*} (\dot{u}_3^i + \dot{u}_4^i) , \quad (171)$$

$$S_l \underline{\nu}_4^{A_i^*} \cdot \left( -m_{A_i} S_l \underline{a}^{A_i^*} \right) = -m_{A_i} \left\{ \left[ (a_1^i)^2 + a_2^i (a_2^i + l_3^i) \right] \dot{u}_3^i + \left[ (a_1^i)^2 + (a_2^i + l_3^i)^2 \right] \dot{u}_4^i \right\} , \quad (172)$$

$$S_l \underline{\omega}_4^{A_i} \cdot \left( -l \underline{\dot{H}}^{A_i/A_i^*} \right) = S_l \underline{\omega}_3^{A_i} \cdot \left( -l \underline{\dot{H}}^{A_i/A_i^*} \right) , \quad (173)$$

$$S_l \underline{\nu}_5^{A_i^*} \cdot \left( -m_{A_i} S_l \underline{a}^{A_i^*} \right) = -m_{A_i} \left\{ \left[ (a_1^i)^2 \dot{u}_5^i \right] - a_1^i a_2^i \dot{u}_2^i - a_1^i (a_2^i + l_3^i) \dot{u}_6^i \right\} , \quad (174)$$

$$S_l \underline{\omega}_5^{A_i} \cdot \left( -l \underline{\dot{H}}^{A_i/A_i^*} \right) = -I_{21}^{A_i/A_i^*} (\dot{u}_2^i + \dot{u}_6^i) - I_{22}^{A_i/A_i^*} \dot{u}_5^i - I_{23}^{A_i/A_i^*} (\dot{u}_3^i + \dot{u}_4^i) , \quad (175)$$

$$S_l \underline{\nu}_6^{A_i^*} \cdot \left( -m_{A_i} S_l \underline{a}^{A_i^*} \right) = -m_{A_i} \left\{ a_2^i (a_2^i + l_3^i) \dot{u}_2^i + (a_2^i + l_3^i)^2 \dot{u}_6^i - a_1^i (a_2^i + l_3^i) \dot{u}_5^i \right\} , \quad (176)$$

and

$$S_l \underline{\omega}_6^{A_i} \cdot \left( -l \underline{\dot{H}}^{A_i/A_i^*} \right) = S_l \underline{\omega}_2^{A_i} \cdot \left( -l \underline{\dot{H}}^{A_i/A_i^*} \right) . \quad (177)$$

### 16.3 Contributions Due to the Flotor

The contribution  $\left( {}_l Q_r^* \right)^F$  to the generalized inertia forces for the  $r$ th generalized speed is

$$\left( {}_l Q_r^* \right)^F = S_l \underline{\nu}_r^{F^*} \cdot \left( -m_F S_l \underline{a}^{F^*} \right) + S_l \underline{\omega}_r^F \cdot \left( -l \underline{\dot{H}}^{F/F^*} \right) . \quad (178)$$

Then the individual terms of the  $({}_l Q_r^*)^F$ 's are as follows:

$$\begin{aligned} S_{l\underline{v}_1^{F^*}} \cdot (-m_F S_{l\underline{a}^{F^*}}) = & -m_F \left\{ \left[ (f_2^1)^2 + (f_1^1)^2 \right] \dot{u}_1^1 + f_1^1 v_3^1 \dot{u}_2^1 \right. \\ & \left. + \left[ -f_1^1 v_1^1 - f_2^1 v_2^1 \right] \dot{u}_3^1 - \left[ f_2^1 (v_2^1 + l_3^1) + f_1^1 v_1^1 \right] \dot{u}_4^1 + f_2^1 v_3^1 \dot{u}_5^1 + f_1^1 v_3^1 \dot{u}_6^1 \right\}, \quad (179) \end{aligned}$$

$$S_{l\underline{\omega}_1^F} \cdot (-{}_l \underline{\dot{H}}^{F/F^*}) = I_{31}^{F/F^*} (\dot{u}_2^i + \dot{u}_6^i) + I_{32}^{F/F^*} \dot{u}_5^i + I_{33}^{F/F^*} (\dot{u}_3^i + \dot{u}_4^i - \dot{u}_1^i), \quad (180)$$

$$\begin{aligned} S_{l\underline{v}_2^{F^*}} \cdot (-m_F S_{l\underline{a}^{F^*}}) = & -m_F \left\{ v_3^1 f_1^1 \dot{u}_1^1 + \left[ (v_2^1)^2 + (v_3^1)^2 \right] \dot{u}_2^1 - v_1^1 v_3^1 \dot{u}_3^1 - v_1^1 v_3^1 \dot{u}_4^1 \right. \\ & \left. - v_1^1 v_2^1 \dot{u}_5^1 + \left[ (v_3^1)^2 + v_2^1 (v_2^1 + l_3^1) \right] \dot{u}_6^1 \right\}, \quad (181) \end{aligned}$$

$$S_{l\underline{\omega}_2^F} \cdot (-{}_l \underline{\dot{H}}^{F/F^*}) = -I_{11}^{F/F^*} (\dot{u}_2^i + \dot{u}_6^i) - I_{12}^{F/F^*} \dot{u}_5^i - I_{13}^{F/F^*} (\dot{u}_3^i + \dot{u}_4^i - \dot{u}_1^i), \quad (182)$$

$$\begin{aligned} S_{l\underline{v}_3^{F^*}} \cdot (-m_F S_{l\underline{a}^{F^*}}) = & -m_F \left\{ \left[ -f_2^1 v_2^1 - f_1^1 v_1^1 \right] \dot{u}_1^1 - v_1^1 v_3^1 \dot{u}_2^1 + \left[ (v_1^1)^2 + (v_2^1)^2 \right] \dot{u}_3^1 \right. \\ & \left. + \left[ (v_1^1)^2 + v_2^1 (v_2^1 + l_3^1) \right] \dot{u}_4^1 - v_2^1 v_3^1 \dot{u}_5^1 - v_1^1 v_3^1 \dot{u}_6^1 \right\}, \quad (183) \end{aligned}$$

$$S_{l\underline{\omega}_3^F} \cdot (-{}_l \underline{\dot{H}}^{F/F^*}) = -I_{31}^{F/F^*} (\dot{u}_2^i + \dot{u}_6^i) - I_{32}^{F/F^*} \dot{u}_5^i - I_{33}^{F/F^*} (\dot{u}_3^i + \dot{u}_4^i - \dot{u}_1^i), \quad (184)$$

$$\begin{aligned} S_{l\underline{v}_4^{F^*}} \cdot (-m_F S_{l\underline{a}^{F^*}}) = & -m_F \left\{ -f_2^1 (l_3^1 + v_2^1) \dot{u}_1^1 \right. \\ & \left. - f_1^1 v_1^1 \dot{u}_1^1 - v_1^1 v_3^1 \dot{u}_2^1 + \left[ (v_1^1)^2 + v_2^1 (v_2^1 + l_3^1) \right] \dot{u}_3^1 \right. \\ & \left. + \left[ (v_1^1)^2 + (v_2^1 + l_3^1)^2 \right] \dot{u}_4^1 - v_3^1 (v_2^1 + l_3^1) \dot{u}_5^1 - v_1^1 v_3^1 \dot{u}_6^1 \right\}, \quad (185) \end{aligned}$$

$$S_l \underline{\omega}_4^F \cdot (-l \underline{\dot{H}}^{F/F*}) = S_l \underline{\omega}_3^F \cdot (-l \underline{\dot{H}}^{F/F*}) , \quad (186)$$

$$\begin{aligned} S_l \underline{v}_5^{F*} \cdot (-m_F S_l \underline{a}^{F*}) = & -m_F \left\{ f_2^1 v_3^1 \dot{u}_1^1 - v_1^1 v_2^1 \dot{u}_2^1 \right. \\ & \left. - v_2^1 v_3^1 \dot{u}_3^1 - v_3^1 (v_2^1 + l_3^1) \dot{u}_4^1 + \left[ (v_1^1)^2 + (v_3^1)^2 \right] \dot{u}_5^1 - v_1^1 (v_2^1 + l_3^1) \dot{u}_6^1 \right\} , \end{aligned} \quad (187)$$

$$S_l \underline{\omega}_5^F \cdot (-l \underline{\dot{H}}^{F/F*}) = -I_{21}^{F/F*} (\dot{u}_2^i + \dot{u}_6^i) - I_{22}^{F/F*} \dot{u}_5^i - I_{23}^{F/F*} (\dot{u}_3^i + \dot{u}_4^i - \dot{u}_1^i) , \quad (188)$$

$$\begin{aligned} S_l \underline{v}_6^{F*} \cdot (-m_F S_l \underline{a}^{F*}) = & -m_F \left\{ f_1^1 v_3^1 \dot{u}_1^1 + (v_3^1)^2 \dot{u}_2^1 \right. \\ & + v_2^1 (v_2^1 + l_3^1) \dot{u}_2^1 - v_1^1 v_3^1 \dot{u}_3^1 - v_1^1 v_3^1 \dot{u}_4^1 \\ & \left. - v_1^1 (v_2^1 + l_3^1) \dot{u}_5^1 + \left[ (v_2^1 + l_3^1)^2 + (v_3^1)^2 \right] \dot{u}_6^1 \right\} , \end{aligned} \quad (189)$$

and

$$S_l \underline{\omega}_6^F \cdot (-l \underline{\dot{H}}^{F/F*}) = -I_{11}^{F/F*} (\dot{u}_2^i + \dot{u}_6^i) - I_{12}^{F/F*} \dot{u}_5^i - I_{13}^{F/F*} (\dot{u}_3^i + \dot{u}_4^i - \dot{u}_1^i) . \quad (190)$$

## 17. EQUATIONS OF MOTION FOR THE SYSTEM

### 17.1 Kinematical Equations

There are 48 kinematical equations for the system, one for each generalized speed:

$$u_j^i = \dot{q}_j^i, \quad (j = 1, \dots, 6; \quad i = 1, \dots, 8) . \quad (191)$$

### 17.2 Dynamical Equations

Six dynamical equations are obtained using the following process. First, add the respective contributions of the 17 rigid bodies to the set of holonomic generalized active and holonomic generalized inertia forces, for each generalized speed (i.e.,  $r = 1, \dots, 48$ ). The holonomic generalized active force for the  $r$ th generalized speed is

$$F_r = \sum_{j=1}^{17} (F_r)^j , \quad (192)$$

where  $(F_r)^j$  is the contribution to the set of holonomic generalized active forces due to the  $j$ th rigid body. That is,

$$F_r = {}_l Q_r^F + \sum_{i=1}^8 {}_l Q_r^{A_i} + \sum_{i=1}^8 {}_l Q_r^{P_i} . \quad (193)$$

Likewise, the contribution to the set of holonomic generalized inertia forces is

$$F_r^* = \sum_{j=1}^{17} (F_r^*)^j = ({}_l Q_r^*)^F + \sum_{i=1}^8 ({}_l Q_r^*)^{A_i} + \sum_{i=1}^8 ({}_l Q_r^*)^{P_i} . \quad (194)$$

Second, develop the relationship between the dependent and the independent generalized speeds in the form

$$u_j^i = \sum_{s=1}^6 A_{rs} u_s^1 + B_r \quad (i = 2, \dots, 8; j = 1, \dots, 6; r = 7, \dots, 48) , \quad (195)$$

where the factors  $u_s^1$  are the six independent generalized speeds, and the terms  $u_j^i$  are the 42 dependent generalized speeds.  $A_{rs}$  and  $B_r$  are scalars, derived from the nonholonomic constraint equations (see sec. 17.3). The nonholonomic and holonomic generalized active forces are related to each other as follows:

$$\tilde{F}_r = F_r + \sum_{s=7}^{48} F_s A_{sr} \quad (r = 1, \dots, 6) . \quad (196)$$

Similarly, the nonholonomic and holonomic generalized inertia forces are related to each other as follows:

$$\tilde{F}_r^* = F_r^* + \sum_{s=7}^{48} F_s^* A_{sr} \quad (r = 1, \dots, 6) . \quad (197)$$

Kane's dynamical equations,\* then, are

$$\tilde{F}_r + \tilde{F}_r^* = 0 \quad (r = 1, \dots, 6) . \quad (198)$$

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\* This step was erroneously omitted in references 10 and 11, resulting in incorrect incorporation of system constraints. The authors discovered and corrected the error during the course of model verification.

### 17.3 Constraint Equations

The kinematical and dynamical equations together are 54 in number: 48 kinematical and 6 dynamical. Since the complete set of equations for an  $n$ -degree-of-freedom system numbers  $2n$ , and since the system  $\tilde{S}$  has 48 degrees of freedom, 42 more equations are needed to describe completely the motion of the system. These missing equations are the holonomic constraint equations (in nonholonomic form) for the dependent generalized speeds  $u_j^i$  ( $i = 2, \dots, 8$ ;  $j = 1, \dots, 6$ ).

Since the velocity and angular velocity of the flotor center of mass  $F^*$  is the same irrespective of the actuator path chosen for describing its position, a set of constraint equations can be written in vector form using the following:

$$S_i \underline{v}^{F^*} = \left( \frac{S_i d \underline{r}_{S_i F^*}}{dt} \right) = S_j \underline{v}^{F^*} \quad (i = 1, j = 2, \dots, 8) , \quad (199)$$

and

$$S_i \underline{\omega}^{F^*} = S_j \underline{\omega}^{F^*} \quad (i = 1, j = 2, \dots, 8) . \quad (200)$$

If one expands equation (199) and resolves them into a common coordinate system (here, the  $\hat{\underline{f}}_i$  coordinate system), one obtains the following 21 (motion) constraint equations:

$$\begin{bmatrix} f_2^i f_{1j}^i - f_1^i f_{2j}^i \\ -v_3^i f_{2j}^i + v_2^i f_{3j}^i \\ -v_2^i f_{1j}^i + v_1^i f_{2j}^i \\ -(v_2^i - l_3^i) f_{1j}^i + v_1^i f_{2j}^i \\ v_3^i f_{1j}^i - v_1^i f_{3j}^i \\ -v_3^i f_{2j}^i + (v_2^i + l_3^i) f_{3j}^i \end{bmatrix}^T \begin{bmatrix} u_1^i \\ u_2^i \\ u_3^i \\ u_4^i \\ u_5^i \\ u_6^i \end{bmatrix} = \begin{bmatrix} f_2^1 f_{1j}^1 - f_1^1 f_{2j}^1 \\ -v_3^1 f_{2j}^1 + v_2^1 f_{3j}^1 \\ -v_2^1 f_{1j}^1 + v_1^1 f_{2j}^1 \\ -(v_2^1 - l_3^1) f_{1j}^1 + v_1^1 f_{2j}^1 \\ v_3^1 f_{1j}^1 - v_1^1 f_{3j}^1 \\ -v_3^1 f_{2j}^1 + (v_2^1 + l_3^1) f_{3j}^1 \end{bmatrix}^T \begin{bmatrix} u_1^1 \\ u_2^1 \\ u_3^1 \\ u_4^1 \\ u_5^1 \\ u_6^1 \end{bmatrix} \quad (i = 2, \dots, 8; j = 1, 2, 3) , \quad (201)$$

Similarly, if one expands equation (200) and resolves them into a common coordinate system (here again, the  $\hat{\underline{f}}_i$  coordinate system), one obtains the remaining 21 (motion) constraint equations:

$$\begin{bmatrix} -f_{3j}^i \\ f_{1j}^i \\ f_{3j}^i \\ f_{3j}^i \\ f_{2j}^i \\ f_{1j}^i \end{bmatrix}^T \begin{bmatrix} u_1^i \\ u_2^i \\ u_3^i \\ u_4^i \\ u_5^i \\ u_6^i \end{bmatrix} = \begin{bmatrix} -f_{3j}^1 \\ f_{1j}^1 \\ f_{3j}^1 \\ f_{3j}^1 \\ f_{2j}^1 \\ f_{1j}^1 \end{bmatrix}^T \begin{bmatrix} u_1^1 \\ u_2^1 \\ u_3^1 \\ u_4^1 \\ u_5^1 \\ u_6^1 \end{bmatrix}, (i = 2, \dots, 8; j = 1, 2, 3) . \quad (202)$$

(It should be noted here that, although equations (201) and (202) are in nonholonomic form, the constraints they represent are actually geometric.)

## 18. STATE-SPACE FORM OF THE EQUATIONS OF MOTION

The time derivatives of the constraint equations can be used with the constraint equations themselves, to write the kinematical and dynamical equations (eqs. (191) and (198), respectively) in the following descriptor form:

$$\begin{bmatrix} I & O \\ O & M \end{bmatrix} \begin{Bmatrix} \dot{\underline{q}} \\ \dot{\underline{u}}^I \end{Bmatrix} = \begin{bmatrix} O & N \\ K & C \end{bmatrix} \begin{Bmatrix} \underline{q} \\ \underline{u}^I \end{Bmatrix} + \begin{bmatrix} O \\ B \end{bmatrix} \{\dot{i}\} + \begin{bmatrix} O \\ E \end{bmatrix} \{\underline{d}\} . \quad (203)$$

The state vector consists of the 48 coordinates  $\underline{q}$  and the six independent generalized speeds  $\underline{u}^I$ , where

$$\underline{u}^I = [u_1^1 \quad u_2^1 \quad u_3^1 \quad u_4^1 \quad u_5^1 \quad u_6^1]^T . \quad (204)$$

The constant matrices  $M$ ,  $K$ , and  $C$  are system mass, stiffness, and damping matrices, respectively. The symbols  $I$  and  $O$  represent, respectively, an identity matrix and a zero matrix of appropriate dimensions; vector  $\dot{i}$  contains the eight control currents to the Lorentz coils; and vector  $\underline{d}$  is the disturbance vector. The input matrices  $B$  and  $E$  are time-varying matrix functions of the coordinates.  $N$  is a constant matrix that incorporates the kinematical equations;  $N$  and  $C$  together incorporate the holonomic constraints.

The disturbance term  $[E] \{\underline{d}\}$  accounts for the umbilical bias force  $\underline{F}_b$  and moment  $\underline{M}_b$ , and the unknown direct disturbance force  $\underline{F}^D$  and moment  $\underline{M}^D$ . Recall that in the development of the foregoing equations the angular acceleration of the stator was assumed to be negligible. However, the translational acceleration of the stator, although presumably unknown, cannot be neglected. In fact, that acceleration

is the source of the umbilical contribution to flotor g-jitter. To include this indirect disturbance contribution, one must add an (unknown) indirect acceleration disturbance term  $\underline{a}^I$  to each of equations (26)–(28). Along with the other disturbances, this indirect disturbance will appear in the final term of equation (203).

## 19. MODEL VERIFICATION

AUTOLEV™ software, marketed by Online Dynamics, Inc., was used to create a full nonlinear model of ARIS including the actuator (rigid-body) dynamics.<sup>14</sup> AUTOLEV was then used to develop and verify the linearized equations presented in this paper.

An independent model was developed using the DENEb Envision software, with current CAD models of an ARIS-outfitted ISPR.<sup>14</sup> This model was used as an independent (static) check of the actuator kinematics.

For dynamic validation, various loads were applied to the flotor, and the nonlinear AUTOLEV model was allowed to move in simulated response. Motions were permitted which greatly exceeded the rattle space constraints (with angles permitted up to  $\approx 90^\circ$ ), to test thoroughly the nonlinear model. The six angles for a single actuator (AUTOLEV model) were inserted into the Envision model, and the remaining 42 actuator angles were compared between the two nonlinear models. The respective angles were in consistent agreement, even with these large motions, to within less than half a degree.

## 20. FUTURE WORK

The next tasks will be the addition of umbilical forces to the AUTOLEV equations, the implementation of a linearized model in MATLAB®, and MATLAB model verification. The verification procedure will first entail comparing the eight position vectors from a common point on the stator to the flotor center of mass, as traced through the eight actuators, with the flotor centered in its home position. Then the procedure will be repeated with the flotor moved statically from its home position, in six degrees of freedom.

For dynamic verification, various loads will be applied to the flotor to verify that the eight position vectors track for motion inside the rattlespace (i.e., small angles). Simulations of the nonlinear AUTOLEV model will be compared with simulations of linearized system models (one with and one without actuator dynamics) to determine the simplest model suitable for controller design.

System dynamics will be incorporated into the Envision model, along with the capability of state-space, discrete time control. The MATLAB and Envision models will then be available, respectively, for centralized, state-space/optimal controller design and for closed-loop system simulation.

## REFERENCES

1. DeLombard, R.; Bushnell, G.S.; and Edberg, D.: "Microgravity Environment Countermeasures Panel Discussion," *AIAA 97-0351*, January 1997.
2. "System Specification for the *International Space Station*," *Specification SSP41000*, Rev. D, , NASA Johnson Space Center, Nov. 1, 1995.
3. DelBasso, S.: "The *International Space Station* Microgravity Environment," *AIAA 96-0402*, January 1996.
4. Nelson, E.S.: "An Examination of Anticipated g-Jitter on Space Station and Its Effects on Materials Processes," *NASA TM-103775*, April 1991.
5. Edberg, D.; et al.: "Results of the STABLE Microgravity Vibration Isolation Flight Experiment," *Guidance and Control*, Vol. 92, pp. 567-581, 1996.
6. Whorton, M.S.: "g-LIMIT: A Vibration Isolation System for the Microgravity Science Glovebox," presented to the 17th Microgravity Measurements Group Meeting, March 1998.
7. Hampton, R.D.; et al.: "The Microgravity Vibration Isolation Mount: A Dynamic Model for Optimal Controller Design," *Proceedings of the MAG '97 Industrial Conference and Exhibition*, pp. 220-230, August 1997.
8. Bushnell, G.: "STS-79 Final Report, RME-1313/ARIS," *Boeing Document SK683-61855-1*, December 1996.
9. Kane, T.R.; and Levinson, D.A.: *Dynamics: Theory and Applications*, McGraw-Hill, Inc., New York, 1985.
10. Hampton, R.D.; et al.: "A High-Fidelity Dynamic Model for the Active Rack Isolation System," *AIAA 98-0458*, January 1998.
11. Hampton, R.D.; and Beech, G.S.; "A 'Kane's Dynamics' Model for the Active Rack Isolation System," *IMECE DSC-7B-5*, November 1999.
12. Kane, T.R.; and Levinson, D.A.: "Formulation of Equations of Motion for Complex Aircraft," *Journal of Guidance and Control*, Vol. 3, No. 2, pp. 99-112, April 1979.
13. Greenwood, D.T.: *Principles of Dynamics*, 2nd ed., Prentice Hall, Englewood Cliffs, New Jersey, 1988.
14. Beech, G.; and Hampton, R.D.: "Validation of a 'Kane's Dynamics' Model for the Active Rack Isolation System," *AIAA 2000-0682*, January 2000.



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13. ABSTRACT (Maximum 200 words)  Many microgravity space science experiments require vibratory acceleration levels unachievable without active isolation. The Boeing Corporation's Active Rack Isolation System (ARIS) employs a novel combination of magnetic actuation and mechanical linkages to address these isolation requirements on the <i>International Space Station (ISS)</i> . ARIS provides isolation at the rack (International Standard Payload Rack (ISPR)) level.  Effective model-based vibration isolation requires (1) an appropriate isolation device, (2) an adequate dynamic (i.e., mathematical) model of that isolator, and (3) a suitable, corresponding controller. ARIS provides the <i>ISS</i> response to the first requirement. This paper presents one response to the second, in a state space framework intended to facilitate an optimal-controls approach to the third. The authors use "Kane's Dynamics" to develop a state-space, analytical (algebraic) set of linearized equations of motion for ARIS.				
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