Ref. 3

Glovebox Integrated Microgravity Isolation Technology (g-LIMIT):
A Linearized State-Space Model

R. David Hampton[1]
University of Alabama in Huntsville
Philip C. Calhoun[2]
NASA Langley Research Center
Mark S. Whorton[3]
NASA Marshall Space Flight Center

Abstract

Vibration acceleration levels on large space platforms exceed the requirements of many space experiments. The Glovebox Integrated Microgravity Isolation Technology (g-LIMIT) is being built by the NASA Marshall Space Flight Center to attenuate these disturbances to acceptable levels. G-LIMIT uses Lorentz (voice-coil) magnetic actuators to levitate and isolate payloads at the individual experiment/sub-experiment (versus rack) level. Payload acceleration, relative position, and relative orientation measurements are fed to a state-space controller. The controller, in turn, determines the actuator currents needed for effective experiment isolation. This paper presents the development of an algebraic, state-space model of g-LIMIT, in a form suitable for optimal controller design. The equations are first derived using Newton's Second Law directly, then simplified to a linear form for the purpose of controller design.

[1] Associate Professor, Mechanical & Aerospace Engineering Department, University of Alabama in Huntsville, Huntsville, Alabama, USA 35899

[2] Control Systems Engineer, NASA Langley Research Center (MS 365), Hampton, Virginia, USA 23681

[3] Senior Research Engineer, NASA Marshall Space Flight Center (TD55), Huntsville, Alabama, USA 35812

Introduction

Acceleration measurements on the U.S. Space Shuttle and the Russian Mir Space Station have shown that on-orbit acceleration environments are noisier than once expected [1]. The acceleration environment on the International Space Station (ISS) likewise is not as clean as originally anticipated; the ISS will not meet its microgravity requirements without the use of isolation systems [1], [2]. While the quasi-static acceleration levels due to such factors as atmospheric drag, gravity gradient, and spacecraft rotations are on the order of several micro-g, the vibration levels above 0.01 Hz exceed 300 micro-g rms, with peaks typically reaching milli-g levels [3]. These acceleration levels are sufficient to cause significant disturbances to many experiments that have fluid or vapor phases, including a large class of materials science experiments [4].

The Glovebox Integrated Microgravity Isolation Technology (g-LIMIT) is designed to isolate experiments from the high frequency (>0.01 Hz) vibrations on the Space Shuttle and the ISS, while passing the quasi-static (<0.01 Hz) accelerations to the experiment [5]. The acceleration-attenuation capability of g-LIMIT is limited primarily by two factors: (1) the character of the umbilical required between the g-LIMIT base (stator) and the g-LIMIT experiment platform (flotor), and (2) the allowed stator-to-flotor rattlespace. A primary goal in g-LIMIT design was to isolate at the individual experiment, rather than entire rack, level; ideally g-LIMIT isolates only the sensitive elements of an experiment. This typically results in a stator-to-flotor umbilical that can be greatly reduced in size and in the services it must provide. In the current design, g-LIMIT employs three umbilicals to provide experiments with power, and with data-acquisition and control services [6].

In order to design controllers for g-LIMIT it was necessary to develop an appropriate dynamic model of the system. The present paper presents an algebraic, state-space model of g-LIMIT, in a form appropriate for optimal controller design.

Problem Statement

The dynamic modeling and microgravity vibration isolation of a tethered, one-dimensional experiment platform has been studied extensively by Hampton, et al. [5, 7, 8]. It has been found that optimal control techniques can be effectively employed using a state-space system model, with relative-position, relative-velocity, and acceleration states. In these studies the experiment platform was assumed to be subject to Lorentz (voice-coil) electromagnetic actuation, and to indirect (umbilical-induced) and direct translational disturbances.

The task of the research presented below was to develop a corresponding state-space model for g-LIMIT. Translational and rotational relative-position and relative-velocity, and translational acceleration states, were to be included in the system model. The g-LIMIT dynamic model must incorporate indirect and direct translational and rotational disturbances.

System Model

A schematic of g-LIMIT is depicted in Figure 1. The stator, fixed in reference frame Ⓢ, is rigidly mounted to the ISS. The flotor, frame Ⓕ, is magnetically levitated above the stator by six Lorentz actuators (two shown), each consisting of a flat racetrack-shaped electrical coil, with an active linear (straight) region positioned between a set of Nd-Fe-Bo supermagnets. For more information on the basic actuator design see Reference [9]. The coils and the supermagnets are fixed to the stator and flotor, respectively. Control currents passing through the coils interact with their respective supermagnet flux fields to produce control forces used for flotor isolation and disturbance attenuation.

The flotor has mass center F^* and a dextral coordinate system with unit vectors $\underline{\hat{f}}_1$, $\underline{\hat{f}}_2$, and $\underline{\hat{f}}_3$, and origin F_0. The stator (actually, stator-plus-ISS) has mass center S^* and a dextral coordinate system with unit vectors $\underline{\hat{s}}_1$, $\underline{\hat{s}}_2$, and $\underline{\hat{s}}_3$, and origin S_0. The inertial reference frame Ⓝ is similarly defined by $\underline{\hat{n}}_1$, $\underline{\hat{n}}_2$, and $\underline{\hat{n}}_3$, and origin N_0. The umbilical is attached to the stator at S_u, and to the flotor at F_u. When the flotor is centered in its rattlespace (the "home" position), F^* and F_u are located at stator-fixed points F_h^*, and F_{uh}, respectively.

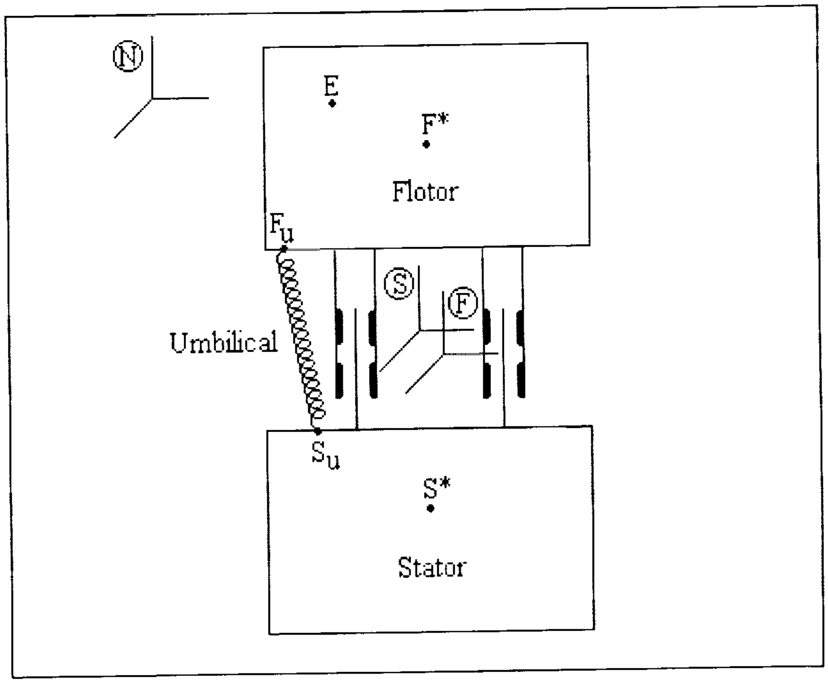

Figure 1. Schematic of g-LIMIT

State Equations of Motion

Preliminaries

Let E be some flotor-fixed point of interest for which the acceleration is to be determined. If E has inertial position $\underline{r}_{N_0 E}$, then its inertial velocity and acceleration are $\dot{\underline{r}}_{N_0 E} = \frac{{}^N d}{dt}\left(\underline{r}_{N_0 E}\right)$ and $\ddot{\underline{r}}_{N_0 E} = \frac{{}^N d}{dt}\left[\frac{{}^N d}{dt}\left(\underline{r}_{N_0 E}\right)\right]$, respectively. (The pre-superscript indicates the reference frame assumed fixed, for purpose of differentiation. The first and second subscripts indicate the vector origin and terminus, respectively.) The angular velocity and angular acceleration of the flotor with respect to the inertial frame are represented by ${}^N\underline{\omega}^F$ and ${}^N\underline{\alpha}^F$, respectively, where ${}^N\underline{\alpha}^F = \frac{{}^N d}{dt}\left({}^N\underline{\omega}^F\right)$.

Let $\underline{F}$ be the resultant of all external forces acting on the flotor; $\underline{M}^{F/F^*}$ (or simply $\underline{M}$), the moment resultant of these forces about F^*; m, the flotor mass; and $\underline{\underline{I}}^{F/F^*}$ (or $\underline{\underline{I}}$), the central inertia dyadic of the flotor for $\hat{\underline{f}}_1$, $\hat{\underline{f}}_2$, and $\hat{\underline{f}}_3$. Then Newton's Second Law for the flotor can be expressed in the following two forms:

$$\underline{F} = m\ddot{\underline{r}}_{N_0 F^*} \qquad \text{(Eq. 1)}$$

and

$$\underline{M} = \underline{\underline{I}} \cdot {}^N\underline{\alpha}^F + {}^N\underline{\omega}^F \times \left(\underline{\underline{I}} \cdot {}^N\underline{\omega}^F\right). \qquad \text{(Eq. 2)}$$

From Equation (2),

$${}^N\underline{\alpha}^F = \underline{\underline{I}}^{-1} \cdot \left[\underline{M} - {}^N\underline{\omega}^F \times \left(\underline{\underline{I}} \cdot {}^N\underline{\omega}^F\right)\right]. \qquad \text{(Eq. 3)}$$

Equation (1) presents the translational equations of motion for the system (in vector form) in terms of the acceleration of flotor mass center F^*. However, the primary objective of the control system design is to attenuate the absolute acceleration of point E, not of point F^*. Consequently, it will be useful to find an expression for $\ddot{\underline{r}}_{N_0 F^*}$ in terms of $\ddot{\underline{r}}_{N_0 E}$. The three $\hat{\underline{s}}_i$ measure numbers of $\ddot{\underline{r}}_{N_0 E}$ will be used as states in the state equations ultimately to be used for controller design.

In addition to the above objective, the controller must prevent the flotor from exceeding its rattlespace limits, i.e., from bumping into the stator. It is desirable, then, that relative-position and relative-velocity information (both translational and rotational) also be included in the equations of motion for controller design. The $\underline{\hat{s}}_i$ measure numbers of the relative position and relative velocity for F^*, from its home position F_h^*, will provide six translational states in the system state equations. Six corresponding rotational states will be included as well. The remainder of this paper will be concerned with expressing the system equations of motion [Equations (1) and (2)] in terms of the selected fifteen states.

Translational Equations of Motion

To develop the translational equations, in terms of the chosen states, begin with the following:

$$\underline{r}_{F_h^* F^*} = \underline{r}_{N_0 E} + \underline{r}_{EF^*} - \underline{r}_{N_0 S_0} - \underline{r}_{S_0 F_h^*}. \quad \text{(Eq. 4)}$$

Differentiation of Equation (4) yields

$$\dot{\underline{r}}_{F_h^* F^*} = \dot{\underline{r}}_{N_0 E} + {}^N\underline{\omega}^F \times \underline{r}_{EF^*} - \dot{\underline{r}}_{N_0 S_0} - {}^N\underline{\omega}^S \times \underline{r}_{S_o F_h^*}. \quad \text{(Eq. 5)}$$

A second differentiation gives

$$\begin{aligned}\ddot{\underline{r}}_{F_h^* F^*} = {} & \ddot{\underline{r}}_{N_0 E} + {}^N\underline{\alpha}^F \times \underline{r}_{EF^*} + {}^N\underline{\omega}^F \times \left({}^N\underline{\omega}^F \times \underline{r}_{EF^*} \right) - \ddot{\underline{r}}_{N_0 S_0} - {}^N\underline{\alpha}^S \times \underline{r}_{S_0 F_h^*} \\ & - {}^N\underline{\omega}^S \times \left({}^N\underline{\omega}^S \times \underline{r}_{S_0 F_h^*} \right).\end{aligned} \quad \text{(Eq. 6)}$$

Substitution for ${}^N\underline{\alpha}^F$ from Equation (3) into Equation (6) yields

$$\begin{aligned}\ddot{\underline{r}}_{F_h^* F^*} = {} & \ddot{\underline{r}}_{N_0 E} + \left\{ \underline{\underline{I}}^{-1} \cdot \left[\underline{M} - {}^N\underline{\omega}^F \times \left(\underline{\underline{I}} \cdot {}^N\underline{\omega}^F \right) \right] \right\} \times \underline{r}_{EF^*} - \ddot{\underline{r}}_{N_0 S_0} - {}^N\underline{\alpha}^S \times \underline{r}_{S_0 F_h^*} \\ & - {}^N\underline{\omega}^S \times \left({}^N\underline{\omega}^S \times \underline{r}_{S_0 F_h^*} \right) + {}^N\underline{\omega}^F \times \left({}^N\underline{\omega}^F \times \underline{r}_{EF^*} \right).\end{aligned} \quad \text{(Eq. 7)}$$

In these equations, from the addition theorem for angular velocities,

$$ {}^{N}\underline{\omega}^{F} = {}^{N}\underline{\omega}^{S} + {}^{S}\underline{\omega}^{F} . \tag{Eq. 8} $$

Under the assumptions that ${}^{N}\underline{\omega}^{S}$ and ${}^{N}\underline{\alpha}^{S}$ are negligibly small and, therefore, that

$$ \ddot{\underline{r}}_{N_0 S_0} \approx \ddot{\underline{r}}_{N_0 S_u} , \tag{Eq. 9} $$

Equation (7) reduces to

$$ \ddot{\underline{r}}_{F_h^* F^*} = \ddot{\underline{r}}_{N_0 E} + \left\{ \underline{\underline{I}}^{-1} \cdot \left[\underline{M} - {}^{S}\underline{\omega}^{F} \times \left(\underline{\underline{I}} \cdot {}^{S}\underline{\omega}^{F} \right) \right] \right\} \times \underline{r}_{EF^*} - \ddot{\underline{r}}_{N_0 S_u} + {}^{S}\underline{\omega}^{F} \times \left({}^{S}\underline{\omega}^{F} \times \underline{r}_{EF^*} \right). \tag{Eq. 10} $$

Linearization about ${}^{S}\underline{\omega}^{F} = \underline{0}$ yields the following result:

$$ \ddot{\underline{r}}_{F_h^* F^*} = \ddot{\underline{r}}_{N_0 E} + \left\{ \underline{\underline{I}}^{-1} \cdot \underline{M} \right\} \times \underline{r}_{EF^*} - \ddot{\underline{r}}_{N_0 S_u} . \tag{Eq. 11} $$

(Note that assuming ${}^{N}\underline{\omega}^{S}$ to be negligible does not imply that reference frames Ⓢ and Ⓝ are identical; it means rather that Ⓢ can be treated as if it is in pure translation relative to Ⓝ, for the frequencies of interest.)

Equation (11) describes $\ddot{\underline{r}}_{F_h^* F^*}$ in terms of the acceleration of an arbitrary flotor-fixed point E. For E located at F^*, Equation (11) can be used straightforwardly with Equation (1) to yield

$$ \ddot{\underline{r}}_{F_h^* F^*} = \left(\frac{1}{m} \right) \underline{F} - \ddot{\underline{r}}_{N_0 S_u} , \tag{Eq. 12} $$

where the second term on the right-hand-side is the indirect disturbance acceleration. Define now an (unknown) indirect translational acceleration disturbance input to the flotor, applied at the stator end of the umbilical:

$$ \underline{a}_{in} = \ddot{\underline{r}}_{N_0 S_u} . \tag{Eq. 13} $$

Substitution from Equation (13) into Equation (12) yields the following:

$$ \ddot{\underline{r}}_{F_h^* F^*} = \left(\frac{1}{m} \right) \underline{F} - \underline{a}_{in} . \tag{Eq. 14} $$

Turn next to $\ddot{\underline{r}}_{N_0E}$. Substituting from Equation (14) into Equation (11), using Equation (13), and solving for $\ddot{\underline{r}}_{N_0E}$, one obtains the following:

$$\ddot{\underline{r}}_{N_0E} = \left(\frac{1}{m}\right)\underline{F} + \underline{\underline{I}}^{-1} \cdot \underline{M} \times \underline{r}_{F^*E}. \qquad \text{Eq. 15)}$$

Appropriate expressions for $\underline{F}$ and $\underline{M}$ will now be determined in terms of the position and velocity of the flotor, relative to the stator. These expressions will be substituted into Equations (14) and (15) to obtain more useful representations of the equations of motion.

The force resultant $\underline{F}$ is the vector sum of the six actuator (coil) forces $\underline{F}_c^i\,(i = 1,\ldots,6)$, with resultant $\underline{F}_c$; of the umbilical force $\underline{F}_u$, caused by umbilical extensions and rotations from the umbilical home position; of the (unknown) direct disturbance forces, with resultant $\underline{F}_d$; and of the gravitational force $\underline{F}_g$. Gravity may be neglected for a space vehicle in free-fall orbit. (Gravity gradient and fluctuations in the gravitational field are lumped together in $\underline{F}_d$). The moment resultant $\underline{M}$ is the vector sum of the various moments acting on the flotor. These moments are from four sources: the coil forces, the umbilical forces, the umbilical moment, and the direct disturbance moment. Let $\underline{M}_c$ represent the moment about F^*, due to the respective coil forces $\underline{F}_c$, assumed to act at their respective coil centers B_i. Represent by $\underline{M}_{uf}$ the moment about F^* due to the umbilical force $\underline{F}_u$. Let $\underline{M}_u$ represent the umbilical-transmitted moment, due (1) to translations of umbilical attachment point F_u from its home position F_{uh}, (2) to rotations about F_u, of the flotor relative to the stator, from the home orientation, and (3) to umbilical bias moment in the home position. Finally, let $\underline{M}_d$ represent the (unknown) moment due to the direct-disturbance forces and moments, with $\underline{F}_d$ assumed to act at F^*. In equation form,

$$\underline{F} = \underline{F}_c + \underline{F}_u + \underline{F}_d, \quad \text{(Eq. 16)}$$

where
$$\underline{F}_c = \sum_{i=1}^{6} \underline{F}_c^i; \quad \text{(Eq. 17)}$$

and
$$\underline{M} = \underline{M}_c + \underline{M}_{uf} + \underline{M}_u + \underline{M}_d, \quad \text{(Eq. 18)}$$

where
$$\underline{M}_c = \sum_{i=1}^{6} \underline{r}_{F^*B_i} \times \underline{F}_c^i, \quad \text{(Eq. 19)}$$

and
$$\underline{M}_{uf} = \underline{r}_{F^*F_u} \times \underline{F}_u. \quad \text{(Eq. 20)}$$

More explicit expressions for $\underline{F}_c^i$ and $\underline{F}_u$ will now be developed. If the actuator has coil current $I_i\hat{\underline{I}}_i$, length L_i, and magnetic flux density $B_i\hat{\underline{B}}_i$, then the associated actuator force becomes

$$\underline{F}_c^i = -I_i L_i B_i \hat{\underline{I}}_i \times \hat{\underline{B}}_i. \quad \text{(Eq. 21)}$$

Assume a translational stiffness $K_{tt}^{i,j}$ in the $\hat{\underline{s}}_j$ direction for an umbilical elongation in the $\hat{\underline{s}}_i$ direction, and a corresponding translational damping $C_{tt}^{i,j}$. Assume also a translational stiffness $K_{tr}^{i,j}$ in the $\hat{\underline{s}}_j$ direction for a positive umbilical rotation about the $\hat{\underline{s}}_i$ direction, and a corresponding translational damping $C_{tr}^{i,j}$. Let $\underline{F}_b$ represent the umbilical bias force, exerted by the umbilical on the flotor in the home position. Then the total force of the umbilical on the flotor becomes

$$\begin{aligned}\underline{F}_u = -\Bigg\{&\sum_{j=1}^{3}\sum_{i=1}^{3} K_{tt}^{i,j}\left[\underline{r}_{F_{uh}F_u}\cdot\hat{\underline{s}}_i\right]\hat{\underline{s}}_j + \sum_{j=1}^{3}\sum_{i=1}^{3} C_{tt}^{i,j}\left[\frac{{}^S d}{dt}\left(\underline{r}_{F_{uh}F_u}\right)\cdot\hat{\underline{s}}_i\right]\hat{\underline{s}}_j \\ &+ \sum_{j=1}^{3}\sum_{i=1}^{3} K_{tr}^{i,j}\left[{}^{F/S}\underline{\theta}\cdot\hat{\underline{s}}_i\right]\hat{\underline{s}}_j + \sum_{j=1}^{3}\sum_{i=1}^{3} C_{tr}^{i,j}\left[\frac{{}^S d}{dt}\left({}^{F/S}\underline{\theta}\cdot\hat{\underline{s}}_i\right)\cdot\hat{\underline{s}}_i\right]\hat{\underline{s}}_j\Bigg\} + \underline{F}_b,\end{aligned} \quad \text{(Eq. 22)}$$

where the vector ${}^{F/S}\underline{\theta}$ represents the rotation about F_u of the flotor, relative to the stator, from the home orientation. Equations (17), (21), and (22) now provide expressions for $\underline{F}_c$ and $\underline{F}_u$ that can be used in Equation (16) for substitution into Equations (14) and (15).

As with $\underline{F}_c^i$ and $\underline{F}_u$ above, $\underline{M}_u$ can also be expressed in more explicit form, in analogous fashion. Assume a rotational stiffness $K_{rr}^{i,j}$ and a rotational damping $C_{rr}^{i,j}$, about the $\hat{\underline{s}}_j$ direction, for umbilical twist about the $\hat{\underline{s}}_i$ direction. Assume also a rotational stiffness $K_{rt}^{i,j}$ and a rotational damping $C_{rt}^{i,j}$, about the $\hat{\underline{s}}_j$ direction, for umbilical extension in the $\hat{\underline{s}}_i$ direction. Let $\underline{M}_b$ represent the total umbilical bias moment, exerted about F^*, by the umbilical on the flotor, in the home position. Then the moment $\underline{M}_u$ can be expressed by the following:

$$\underline{M}_u = -\left\{ \sum_{j=1}^{3} \sum_{i=1}^{3} K_{rt}^{i,j} \left[\underline{r}_{F_{uh}F_u} \cdot \hat{\underline{s}}_i \right] \hat{\underline{s}}_j + \sum_{j=1}^{3} \sum_{i=1}^{3} C_{rt}^{i,j} \left[\frac{{}^S d}{dt} \left(\underline{r}_{F_{uh}F_u} \right) \cdot \hat{\underline{s}}_i \right] \hat{\underline{s}}_j \right.$$
$$\left. + \sum_{j=1}^{3} \sum_{i=1}^{3} K_{rr}^{i,j} \left[{}^{F/S}\underline{\theta} \cdot \hat{\underline{s}}_i \right] \hat{\underline{s}}_j + \sum_{j=1}^{3} \sum_{i=1}^{3} C_{rr}^{i,j} \left[\frac{{}^S d}{dt} \left({}^{F/S}\underline{\theta} \right) \cdot \hat{\underline{s}}_i \right] \hat{\underline{s}}_j \right\} + \underline{M}_b \,. \qquad \text{(Eq. 23)}$$

Substituting Equation (21) into (19), and Equation (22) into (20) produces expressions for $\underline{M}_c$ and $\underline{M}_{uf}$ respectively. These expressions, along with Equations (17), (21), (22), and (23) can be substituted into Equation (15) to produce a more useful expression for $\ddot{\underline{r}}_{N_0E}$.

In principle, one could select a set of umbilical elongations, elongation rates, rotations, and rotation rates as the system states. However, since the translational form of Newton's Second Law given by Equation (14) employs the acceleration $\ddot{\underline{r}}_{F_h^*F^*}$, it is more natural to choose states based on $\underline{r}_{F_h^*F^*}$ rather than $\underline{r}_{F_{uh}F_u}$. To accomplish this one must substitute appropriately for $\underline{r}_{F_{uh}F_u}$ and its derivative into Equations (22) and (23). The pertinent relationship is

$$\underline{r}_{F_{uh}F_u} = \underline{r}_{F_{uh}F_h^*} + \underline{r}_{F_h^*F^*} + \underline{r}_{F^*F_u} \quad . \tag{Eq. 24}$$

Note that $\underline{r}_{F_{uh}F_h^*}$ and $\underline{r}_{F^*F_u}$ are constant vectors of equal magnitude and opposite direction in the stator- and flotor-fixed reference frames, respectively; in particular,

$$\begin{Bmatrix} \underline{r}_{F_{uh}^*F_h^*} \cdot \hat{\underline{s}}_1 \\ \underline{r}_{F_{uh}^*F_h^*} \cdot \hat{\underline{s}}_2 \\ \underline{r}_{F_{uh}^*F_h^*} \cdot \hat{\underline{s}}_3 \end{Bmatrix} = \begin{Bmatrix} -\underline{r}_{F^*F_u} \cdot \hat{\underline{f}}_1 \\ -\underline{r}_{F^*F_u} \cdot \hat{\underline{f}}_2 \\ -\underline{r}_{F^*F_u} \cdot \hat{\underline{f}}_3 \end{Bmatrix} . \tag{Eq. 25}$$

It will be shown later that Equations (24) and (25) facilitate expressing the umbilical forces and moments in terms of the following twelve scalar quantities: $\underline{r}_{F_h^*F^*} \cdot \hat{\underline{s}}_i$, $\frac{{}^S d}{dt}\left(\underline{r}_{F_h^*F^*}\right) \cdot \hat{\underline{s}}_i$, ${}^{F/S}\underline{\theta} \cdot \hat{\underline{s}}_i$, and $\frac{{}^S d}{dt}\left({}^{F/S}\underline{\theta}\right) \cdot \hat{\underline{s}}_i$. These quantities will be chosen as system states, for the state-space description. It is also desired to include, as auxiliary states, the $\hat{\underline{s}}_i$ measure numbers of $\ddot{\underline{r}}_{N_0E}$ [see Eq. (15)], for control purposes. Then Equations (14) and (15) will yield the translational equations of motion in state-space form, provided that (as will be shown later) $\ddot{\underline{r}}_{F_h^*F^*}$ and $\ddot{\underline{r}}_{N_0E}$ can be expressed in terms of $\frac{{}^S d^2}{dt^2}\left(\underline{r}_{F_h^*F^*}\right)$ and $\frac{{}^S d^2}{dt^2}\left(\underline{r}_{N_0E}\right)$, respectively. To this end, recall that ${}^N\underline{\omega}^S$ and ${}^N\underline{\alpha}^S$ have been assumed to be negligible in the frequency range of interest; therefore, one has the following:

$$\dot{\underline{r}}_{F_h^*F^*} = \frac{{}^S d}{dt}\left(\underline{r}_{F_h^*F^*}\right) \tag{Eq. 26}$$

and

$$\ddot{\underline{r}}_{F_h^*F^*} = \frac{{}^S d^2}{dt^2}\left(\underline{r}_{F_h^*F^*}\right) . \tag{Eq. 27}$$

Similarly, $$\underline{\ddot{r}}_{N_0E} = \frac{^S d^2}{dt^2}\left(\underline{r}_{N_0E}\right).\qquad\text{(Eq. 28)}$$

Note that although the terms involving ${}^N\underline{\omega}^S$ and ${}^N\underline{\alpha}^S$ have been neglected in Equation (27) their effects in Equation (14) could alternatively be considered as lumped together with $\underline{\ddot{r}}_{N_0S_u}$ in $\underline{a}_{in}$. Including these higher-order terms, Equation (13) would be rewritten as follows:

$$\underline{a}_{in} = \underline{\ddot{r}}_{N_0S_u} - {}^N\underline{\alpha}^S \times \underline{r}_{F_h^*F^*} - 2\ {}^N\underline{\omega}^S \times \frac{^S d}{dt}\left(\underline{r}_{F_h^*F^*}\right) - {}^N\underline{\omega}^S \times \left({}^N\underline{\omega}^S \times \underline{r}_{F_h^*F^*}\right).\qquad\text{(Eq. 29)}$$

Equations (14), (16), (17), (21), (22), (26), and (27) provide the basis for a state-space form of the translational equations of motion, using as states the $\hat{\underline{s}}_i$ components of $\underline{r}_{F_h^*F^*}$, $\frac{^S d}{dt}\left(\underline{r}_{F_h^*F^*}\right)$, ${}^{F/S}\underline{\theta}$, and $\frac{^S d}{dt}\left({}^{F/S}\underline{\theta}\right)$. In addition, for control purposes, low-pass-filtered approximations of $\underline{\ddot{r}}_{N_0E}$ [beginning with Eqs. (15) and (28)] will be used as auxiliary states. This development will be presented later [see Equations (65) through (67)].

Rotational Equations of Motion

The rotational equations of motion are given by Equation (2) [or, equivalently, by Equation (3)], where $\underline{M}$ is defined by Equations (18), (19), (20), and (23). Consider the left-hand-side of Equation (3). It will be shown that ${}^N\underline{\alpha}^F = \frac{^S d^2}{dt^2}\left({}^{F/S}\underline{\theta}\right)$, under the assumption that ${}^N\underline{\omega}^S$ and ${}^N\underline{\alpha}^S$ are approximately zero.

Differentiating Equation (8), one can express the angular acceleration ${}^N\underline{\alpha}^F$, in terms of ${}^N\underline{\omega}^S$ and ${}^S\underline{\omega}^F$, as

$$ {}^{N}\underline{\alpha}^{F} = \frac{{}^{N}d}{dt}\left({}^{N}\underline{\omega}^{S}\right) + \frac{{}^{N}d}{dt}\left({}^{S}\underline{\omega}^{F}\right), \qquad \text{(Eq. 30)} $$

or, expanding, as

$$ {}^{N}\underline{\alpha}^{F} = \frac{{}^{N}d}{dt}\left({}^{N}\underline{\omega}^{S}\right) + \frac{{}^{S}d}{dt}\left({}^{S}\underline{\omega}^{F}\right) + {}^{N}\underline{\omega}^{S} \times {}^{S}\underline{\omega}^{F}. \qquad \text{(Eq. 31)} $$

Assuming, as before, that ${}^{N}\underline{\omega}^{S} \approx \underline{0}$ and ${}^{N}\underline{\alpha}^{S} \approx \underline{0}$, Equation (31) simplifies to:

$$ {}^{N}\underline{\alpha}^{F} = \frac{{}^{S}d}{dt}\left({}^{S}\underline{\omega}^{F}\right). \qquad \text{(Eq. 32)} $$

The rotation vector ${}^{F/S}\underline{\theta}$, defined previously, represents the rotation of the flotor, relative to the stator, from the home position. (Conceptually, ${}^{F/S}\underline{\theta}$ is a free vector $\phi\ {}^{F/S}\underline{\hat{n}}_{\phi}$, where ${}^{F/S}\underline{\hat{n}}_{\phi}$ indicates the positive direction of the rotation axis; and ϕ, the rotation angle about that axis.) The angular velocity vector ${}^{S}\underline{\omega}^{F}$ is related to the rotation vector as follows [10]:

$$ {}^{S}\underline{\omega}^{F} = \frac{{}^{S}d}{dt}\left({}^{F/S}\underline{\theta}\right). \qquad \text{(Eq. 33)} $$

Substitution from Equation (33) into Equation (32) yields the following:

$$ {}^{N}\underline{\alpha}^{F} = \frac{{}^{S}d^{2}}{dt^{2}}\left({}^{F/S}\underline{\theta}\right). \qquad \text{(Eq. 34)} $$

Substitution from Equation (34) into Equation (3) yields:

$$ \frac{{}^{S}d^{2}}{dt^{2}}\left({}^{F/S}\underline{\theta}\right) = \underline{\underline{I}}^{-1} \cdot \left[\underline{M} - {}^{N}\underline{\omega}^{F} \times \left(\underline{\underline{I}} \cdot {}^{N}\underline{\omega}^{F}\right)\right]. \qquad \text{(Eq. 35)} $$

Assuming, as before, that ${}^{N}\underline{\omega}^{S} \approx \underline{0}$, one obtains:

$$ \frac{{}^{S}d^{2}}{dt^{2}}\left({}^{F/S}\underline{\theta}\right) = \underline{\underline{I}}^{-1} \cdot \left[\underline{M} - {}^{S}\underline{\omega}^{F} \times \left(\underline{\underline{I}} \cdot {}^{S}\underline{\omega}^{F}\right)\right]. \qquad \text{(Eq. 36)} $$

Linearization of Equation (36) about ${}^{S}\underline{\omega}^{F}=\underline{0}$ yields the following form of the rotational equations of motion:

$$\frac{{}^{S}d^{2}}{dt^{2}}\left({}^{F/S}\underline{\theta}\right)=\underline{\underline{I}}^{-1}\cdot\underline{M}. \tag{Eq. 37}$$

Substitution from Equation (18) into Equation (37) yields:

$$\frac{{}^{S}d^{2}}{dt^{2}}\left({}^{F/S}\underline{\theta}\right)=\underline{\underline{I}}^{-1}\cdot\left(\underline{M}_{c}+\underline{M}_{u}+\underline{M}_{uf}+\underline{M}_{d}\right), \tag{Eq. 38}$$

where (from before)

$$\underline{M}_{c}=\sum_{i=1}^{6}\underline{r}_{F^{*}B_{i}}\times\underline{F}_{c}^{i}, \tag{Eq. 39}$$

$$\begin{aligned}\underline{M}_{u}=-\Bigg\{&\sum_{j=1}^{3}\sum_{i=1}^{3}K_{rt}^{i,j}\left[\underline{r}_{F_{uh}F_{u}}\cdot\hat{\underline{s}}_{i}\right]\hat{\underline{s}}_{j}+\sum_{j=1}^{3}\sum_{i=1}^{3}C_{rt}^{i,j}\left[\frac{{}^{S}d}{dt}\left(\underline{r}_{F_{uh}F_{u}}\right)\cdot\hat{\underline{s}}_{i}\right]\hat{\underline{s}}_{j}\\&+\sum_{j=1}^{3}\sum_{i=1}^{3}K_{rr}^{i,j}\left[{}^{F/S}\underline{\theta}\cdot\hat{\underline{s}}_{i}\right]\hat{\underline{s}}_{j}+\sum_{j=1}^{3}\sum_{i=1}^{3}C_{rr}^{i,j}\left[\frac{{}^{S}d}{dt}\left({}^{F/S}\underline{\theta}\right)\cdot\hat{\underline{s}}_{i}\right]\hat{\underline{s}}_{j}\Bigg\}+\underline{M}_{b},\end{aligned} \tag{Eq. 40}$$

and

$$\underline{M}_{uf}=\underline{r}_{F^{*}F_{u}}\times\underline{F}_{u}. \tag{Eq. 41}$$

Equations (38) through (41), along with (21), (22), and (24) through (26), provide the basis for a state-space form of the rotational equations of motion. The states are those defined previously.

Equations of Motion in State-Space Form

Define the following relative-position states:

$$x_{ai}=\underline{r}_{F_{h}^{*}F^{*}}\cdot\hat{\underline{s}}_{i}, \tag{Eq. 42}$$

so that,

$$\underline{r}_{F_{h}^{*}F^{*}}=x_{a1}\hat{\underline{s}}_{1}+x_{a2}\hat{\underline{s}}_{2}+x_{a3}\hat{\underline{s}}_{3}. \tag{Eq. 43}$$

Define next the relative-velocity states, $x_{bi}=\dot{x}_{ai}$. (Eq. 44)

Differentiation of Equation (43), along with the use of Equations (44) and (26), leads to the following:

$$\underline{\dot{r}}_{F_h^* F^*} = x_{b1}\hat{\underline{s}}_1 + x_{b2}\hat{\underline{s}}_2 + x_{b3}\hat{\underline{s}}_3 . \qquad \text{(Eq. 45)}$$

A second differentiation, along with the use of Equation (27) yields:

$$\underline{\ddot{r}}_{F_h^* F^*} = \dot{x}_{b1}\hat{\underline{s}}_1 + \dot{x}_{b2}\hat{\underline{s}}_2 + \dot{x}_{b3}\hat{\underline{s}}_3 . \qquad \text{(Eq. 46)}$$

Introduce the use of a pre-superscript in parentheses to indicate the coordinate system used for componentiation of vectors. Then Equations (43), (45), and (46) take the following respective matric forms:

$$^{(S)}\underline{r}_{F_h^* F^*} = \begin{Bmatrix} x_{a1} \\ x_{a2} \\ x_{a3} \end{Bmatrix} = \underline{x}_a , \qquad \text{(Eq. 47)}$$

$$^{(S)}\underline{\dot{r}}_{F_h^* F^*} = \underline{\dot{x}}_a = \underline{x}_b , \qquad \text{(Eq. 48)}$$

and

$$^{(S)}\underline{\ddot{r}}_{F_h^* F^*} = \underline{\dot{x}}_b , \qquad \text{(Eq. 49)}$$

where $\underline{x}_a$ and $\underline{x}_b$ are defined as indicated.

Define the relative-angular-position states,

$$x_{di} = {}^{F/S}\underline{\theta} \cdot \hat{\underline{s}}_i , \qquad \text{(Eq. 50)}$$

so that

$$^{F/S}\underline{\theta} = x_{d1}\hat{\underline{s}}_1 + x_{d2}\hat{\underline{s}}_2 + x_{d3}\hat{\underline{s}}_3 . \qquad \text{(Eq. 51)}$$

Define next the relative-angular-velocity states,

$$x_{ei} = \dot{x}_{di} , \qquad \text{(Eq. 52)}$$

so that

$$\frac{^S d}{dt}\left({}^{F/S}\underline{\theta} \right) = x_{e1}\hat{\underline{s}}_1 + x_{e2}\hat{\underline{s}}_2 + x_{e3}\hat{\underline{s}}_3 , \qquad \text{(Eq. 53)}$$

and

$$ {}^{S}\frac{d^2}{dt^2}\left({}^{F/S}\underline{\theta}\right) = \dot{x}_{e1}\hat{\underline{s}}_1 + \dot{x}_{e2}\hat{\underline{s}}_2 + \dot{x}_{e3}\hat{\underline{s}}_3 . \qquad \text{(Eq. 54)} $$

The respective matric forms of Equations (51), (53), and (54) are then

$$ {}^{(S)}\,{}^{F/S}\underline{\theta} = \begin{Bmatrix} x_{d1} \\ x_{d2} \\ x_{d3} \end{Bmatrix} = \underline{x}_d , \qquad \text{(Eq. 55)} $$

$$ {}^{(S)}\,{}^{S}\frac{d}{dt}\left({}^{F/S}\underline{\theta}\right) = \dot{\underline{x}}_d = \underline{x}_e , \qquad \text{(Eq. 56)} $$

and

$$ {}^{(S)}\,{}^{S}\frac{d^2}{dt^2}\left({}^{F/S}\underline{\theta}\right) = \dot{\underline{x}}_e , \qquad \text{(Eq. 57)} $$

where $\underline{x}_d$ and $\underline{x}_e$ are defined as indicated.

Equations (14) and (46) can be used to develop a state-space equation for $\dot{\underline{x}}_b$. First, express Equation (46) in measure-number form:

$$ {}^{(S)}\ddot{\underline{r}}_{F_h^* F^*} = \begin{Bmatrix} \dot{x}_{b1} \\ \dot{x}_{b2} \\ \dot{x}_{b3} \end{Bmatrix} = \dot{\underline{x}}_b . \qquad \text{(Eq. 58)} $$

Next, represent by $\underline{a}_d$ the direct translational acceleration disturbance to the flotor, due to unknown direct-disturbance force $\underline{F}_d$. In particular,

$$ \underline{a}_d = \frac{1}{m}\underline{F}_d . \qquad \text{(Eq. 59)} $$

Then Equation (14) can be expressed as follows:

$$ \dot{\underline{x}}_b = \frac{1}{m}\left[{}^{(S)}\underline{F}_c + {}^{(S)}\underline{F}_u\right] - {}^{(S)}\underline{a}_{in} + {}^{(S)}\underline{a}_d . \qquad \text{(Eq. 60)} $$

Equations (38) and (54) can be used to develop a state-space equation for $\dot{\underline{x}}_e$. First express Equation (54) in measure number form:

$$^{S}\frac{d^2}{dt^2}\left({}^{F/S}\underline{\theta}\right)=\begin{Bmatrix}\dot{x}_{e1}\\ \dot{x}_{e2}\\ \dot{x}_{e3}\end{Bmatrix}=\underline{\dot{x}}_e\,. \qquad \text{(Eq. 61)}$$

Next, represent by $\underline{\alpha}_d$ the direct rotational acceleration disturbance to the flotor, due to unknown direct-disturbance moment $\underline{M}_d$. In particular,

$$\underline{\alpha}_d=\underline{\underline{I}}^{-1}\cdot\underline{M}_d\,. \qquad \text{(Eq. 62)}$$

Then Equation (38) can be expressed as follows:

$$\underline{\dot{x}}_e={}^{(S)}\left\{\underline{\underline{I}}^{-1}\cdot\left(\underline{M}_c+\underline{M}_u+\underline{M}_{uf}\right)+\underline{\alpha}_d\right\}. \qquad \text{(Eq. 63)}$$

Define a rotation matrix ${}^{S/F}Q$ that describes a coordinate transformation from the flotor-fixed to the stator-fixed frame, so that

$$\begin{Bmatrix}\underline{\hat{f}}_1\\ \underline{\hat{f}}_2\\ \underline{\hat{f}}_3\end{Bmatrix}=\left[{}^{S/F}Q\right]\begin{Bmatrix}\underline{\hat{s}}_1\\ \underline{\hat{s}}_2\\ \underline{\hat{s}}_3\end{Bmatrix}. \qquad \text{(Eq. 64)}$$

Then Equation (63) can be re-expressed as

$$\underline{\dot{x}}_e={}^{S/F}Q\left\{{}^{(F)}I^{-1}\left[{}^{(F)}\underline{M}_u+{}^{(F)}\underline{M}_{uf}+{}^{(F)}\underline{M}_c\right]+{}^{(F)}\underline{\alpha}_d\right\}, \qquad \text{(Eq. 65)}$$

where ${}^{(F)}I$ is a central inertia matrix of the flotor, for $\underline{\hat{f}}_1$, $\underline{\hat{f}}_2$, and $\underline{\hat{f}}_3$.

To approximate ${}^{(S)}\underline{\ddot{r}}_{N_0E}$ using states, define $\underline{x}_c$ by

$$\omega_h\,{}^{(S)}\underline{\ddot{r}}_{N_0E}=\underline{\dot{x}}_c+\omega_h\underline{x}_c\,, \qquad \text{(Eq. 66)}$$

for some high value of circular frequency ω_h. Taking the Laplace transform,

$$\mathcal{L}\left\{{}^{(S)}\underline{r}_{N_0E}\right\}=\left(\frac{s+\omega_h}{\omega_h s^2}\right)\mathcal{L}\left\{\underline{x}_c\right\}, \qquad \text{(Eq. 67)}$$

so that $$\underline{x}_c \approx {}^{(S)}\ddot{\underline{r}}_{N_0E} \, , \tag{Eq. 68}$$

for $\omega << \omega_h$.

Equation (15) can be expressed, in expanded form, as

$$\ddot{\underline{r}}_{N_0E} = \left(\frac{1}{m}\right)\left(\underline{F}_c + \underline{F}_u\right) + \underline{\underline{I}}^{-1} \cdot \left(\underline{M}_c + \underline{M}_u + \underline{M}_{uf}\right) \times \underline{r}_{F^*E} + \underline{\alpha}_d \times \underline{r}_{F^*E} + \underline{a}_d \, . \tag{Eq. 69}$$

Note that Equation (69) is a vector equation; to express it in measure-number (i.e., matric) form, one must employ a matric notation for the cross product. To this end, observe that, for arbitrary vectors

$$\underline{r}_1 = x_1 \hat{\underline{f}}_1 + y_1 \hat{\underline{f}}_2 + z_1 \hat{\underline{f}}_3 \tag{Eq. 70}$$

and $$\underline{r}_2 = x_2 \hat{\underline{f}}_1 + y_2 \hat{\underline{f}}_2 + z_2 \hat{\underline{f}}_3 \, , \tag{Eq. 71}$$

the cross product can be expressed, in measure-number form by

$${}^{(F)}\left(\underline{r}_1 \times \underline{r}_2\right) = \begin{bmatrix} 0 & -z_1 & y_1 \\ z_1 & 0 & -x_1 \\ -y_1 & x_1 & 0 \end{bmatrix} \begin{Bmatrix} x_2 \\ y_2 \\ z_2 \end{Bmatrix} . \tag{Eq. 72}$$

Represent the above skew-symmetric matrix by ${}^{(F)}\underline{r}_1^{\times}$ [11]. Using this notation, Equation (69) can be rewritten as follows:

$${}^{(S)}\ddot{\underline{r}}_{N_0E} = \frac{1}{m}\,{}^{(S)}\left(\underline{F}_c + \underline{F}_u\right) + {}^{S/F}Q\left\{\left[{}^{(F)}I^{-1}\;{}^{(F)}\left(\underline{M}_c + \underline{M}_u + \underline{M}_{uf}\right)\right]^{\times\,(F)}\underline{r}_{F^*E} + {}^{(F)}\left(\underline{\alpha}_d \times \underline{r}_{F^*E}\right)\right\} + {}^{(S)}\underline{a}_d \, . \tag{Eq. 73}$$

A state equation for $\underline{x}_c$ can now be formed by substituting from Equation (73) into Equation (66), and solving for $\dot{\underline{x}}_c$, to yield the following:

$$\dot{\underline{x}}_c = \omega_h \left\langle \frac{1}{m}\,{}^{(S)}\left(\underline{F}_c + \underline{F}_u\right) + {}^{S/F}Q\left\{\left[{}^{(F)}I^{-1}\;{}^{(F)}\left(\underline{M}_c + \underline{M}_u + \underline{M}_{uf}\right)\right]^{\times\,(F)}\underline{r}_{F^*E} + {}^{(F)}\left(\underline{\alpha}_d \times \underline{r}_{F^*E}\right)\right\} + {}^{(S)}\underline{a}_d \right\rangle - \omega_h \underline{x}_c \, . \tag{Eq. 74}$$

Six of the state equations for the system are given by Equations (48) and (56); nine more, by Equations (60), (65), and (74). The latter nine are written in terms of the various forces and moments acting on the system. The force loads are defined in vector form by Equations (17), (21), and (22); the moment loads, by Equations (39) through (41). In the following section these loads will be written in measure-number form and expressed in terms of the states, for substitution into Equations (60), (63), and (74).

Force Loads

In matric form, the i^{th} control force (Eq. (21)) can be expressed as

$$^{(S)}\underline{F}_c^i = \left[-L_i \, ^{(S)}\hat{\underline{I}}_i^{\times} \, ^{S/F}Q \, B_i \, ^{(F)}\hat{\underline{B}}_i \right] I_i \, . \qquad \text{(Eq. 75)}$$

Define the control input u_i, associated with the i^{th} control force, by

$$u_i = I_i \, . \qquad \text{(Eq. 76)}$$

The resultant control force becomes,

$$^{(S)}\underline{F}_c = \sum_{i=1}^{6} {}^{(S)}\underline{F}_c^i = F_c \underline{u} \, , \qquad \text{(Eq. 77)}$$

where F_c and $\underline{u}$ are a row vector and a column vector, respectively, defined as indicated.

Introduce the following skew-symmetric, small-angle representation for the coordinate transformation matrix [10, (p. 352)]:

$$^{S/F}Q = \left(I + {}^{(S)} \, {}^{F/S}\underline{\theta}^{\times} \right), \qquad \text{(Eq. 78)}$$

where I is the identity matrix. Then, substituting Equation (78) into Equation (75), one has

$$^{(S)}\underline{F}_c^i = \left[-L_i \, ^{(S)}\hat{\underline{I}}_i^{\times} \left(I + {}^{(S)} \, {}^{F/S}\underline{\theta}^{\times} \right) B_i \, ^{(F)}\hat{\underline{B}}_i \right] I_i \, . \qquad \text{(Eq. 79)}$$

Expanding,

$$ {}^{(S)}\underline{F}_c^i = -L_i\, B_i\, I_i \left[{}^{(S)}\hat{\underline{I}}_i^{\times}\; {}^{(F)}\hat{\underline{B}}_i - {}^{(S)}\hat{\underline{I}}_i^{\times}\; {}^{(F)}\hat{\underline{B}}_i^{\times}\; {}^{(S)\,F/S}\underline{\theta} \right]. \qquad \text{(Eq. 80)} $$

Next, substitute into Equation (80) from Equation (55), and represent the current I_i as the sum of a bias value I_{B_i} and a fluctuating value I_{F_i}. This yields the following expression for the control force:

$$ {}^{(S)}\underline{F}_c^i = -L_i\, B_i\, I_i\; {}^{(S)}\hat{\underline{I}}_i^{\times}\; {}^{(F)}\hat{\underline{B}}_i + L_i\, B_i \left(I_{B_i} + I_{F_i}\right) {}^{(S)}\hat{\underline{I}}_i^{\times}\; {}^{(F)}\hat{\underline{B}}_i^{\times}\, \underline{x}_d\,. \qquad \text{(Eq. 81)} $$

The bias current is the current portion necessary to counteract the bias force and moment required to hold the flotor at its home position, and the fluctuating current is the additional portion necessary for control. Assuming that the fluctuating current and the rotational states are small, Equation (81) linearizes to the following:

$$ {}^{(S)}\underline{F}_c^i = \left(-L_i\, B_i\; {}^{(S)}\hat{\underline{I}}_i^{\times}\; {}^{(F)}\hat{\underline{B}}_i \right) I_i + \left(L_i\, B_i\, I_{B_i}\; {}^{(S)}\hat{\underline{I}}_i^{\times}\; {}^{(F)}\hat{\underline{B}}_i^{\times} \right) \underline{x}_d\,. \qquad \text{(Eq. 82)} $$

The resultant control force can now be expressed as follows:

$$ {}^{(S)}\underline{F}_c = \sum_{i=1}^{8} {}^{(S)}\underline{F}_c^i = F_{cu}\,\underline{u} + F_{cd}\,\underline{x}_d\,, \qquad \text{(Eq. 83)} $$

where (as before)

$$ u_i = I_i\,, \qquad \text{(Eq. 84)} $$

and F_{cu}, and F_{cd} are defined as indicated, from Equation (82).

Turn next to the umbilical force. Equations (22) and (23) can be expanded into matric form, in terms of the selected state variables as follows. Recall from Equation (25) that

$$ {}^{(S)}\underline{r}_{F_{uh}F_h^*} = -{}^{(F)}\underline{r}_{F^*F_u}\,. \qquad \text{(Eq. 85)} $$

Then Equation (24) can be represented as follows:

$$ {}^{(S)}\underline{r}_{F_{uh}F_u} = {}^{(S)}\underline{r}_{F_h^* F^*} + \left({}^{S/F}Q - I \right) {}^{(F)}\underline{r}_{F^* F_u} \,. \qquad \text{(Eq. 86)} $$

Assuming small angles Equation (86) can be re-expressed as follows:

$$ {}^{(S)}\underline{r}_{F_{uh}F_u} = {}^{(S)}\underline{r}_{F_h^* F^*} + {}^{(S)\ F/S}\underline{\theta}^{\times}\ {}^{(F)}\underline{r}_{F^* F_u} \,, \qquad \text{(Eq. 87)} $$

or, in terms of the states,

$$ {}^{(S)}\underline{r}_{F_{uh}F_u} = \underline{x}_a - {}^{(F)}\underline{r}^{\times}_{F^* F_u}\ \underline{x}_d \,. \qquad \text{(Eq. 88)} $$

One can also represent $\frac{{}^S d}{dt}\left(\underline{r}_{F_{uh}F_u}\right)$ in terms of the states, as follows. First differentiate Equation (24), with the stator frame assumed fixed:

$$ \frac{{}^S d}{dt}\left(\underline{r}_{F_{uh}F_u}\right) = \frac{{}^S d}{dt}\left(\underline{r}_{F_h^* F^*}\right) + {}^S\underline{\omega}^F \times \underline{r}_{F^* F_u} \,. \qquad \text{(Eq. 89)} $$

Substituting from Equation (34) into Equation (89) yields:

$$ \frac{{}^S d}{dt}\left(\underline{r}_{F_{uh}F_u}\right) = \frac{{}^S d}{dt}\left(\underline{r}_{F_h^* F^*}\right) + \frac{{}^S d}{dt}\left({}^{F/S}\underline{\theta}\right) \times \underline{r}_{F^* F_u} \,, \qquad \text{(Eq. 90)} $$

or, in terms of stator-fixed coordinates,

$$ {}^{(S)\,S}\frac{d}{dt}\left(\underline{r}_{F_{uh}F_u}\right) = {}^{(S)\,S}\frac{d}{dt}\left(\underline{r}_{F_h^* F^*}\right) + \left[{}^{(S)\,S}\frac{d}{dt}\left({}^{F/S}\underline{\theta}\right) \right]^{\times} \left({}^{S/F}Q\ {}^{(F)}\underline{r}_{F^* F_u} \right) . \qquad \text{(Eq. 91)} $$

Using Equation (78), Equation (91) can be re-expressed as

$$ {}^{(S)\,S}\frac{d}{dt}\left(\underline{r}_{F_{uh}F_u}\right) = {}^{(S)\,S}\frac{d}{dt}\left(\underline{r}_{F_h^* F^*}\right) + \left[{}^{(S)\,S}\frac{d}{dt}\left({}^{F/S}\underline{\theta}\right) \right]^{\times} \left(I + {}^{(S)\ F/S}\underline{\theta}^{\times} \right) {}^{(F)}\underline{r}_{F^* F_u} \,. \qquad \text{(Eq. 92)} $$

Substituting with state variables,

$$ {}^{(S)\,S}\frac{d}{dt}\left(\underline{r}_{F_{uh}F_u}\right) = \underline{x}_b + \underline{x}_e^{\times}\left(I + \underline{x}_d^{\times} \right) {}^{(F)}\underline{r}_{F^* F_u} \,; \qquad \text{(Eq. 93)} $$

or, assuming the states are small,

$$ {}^{(S)}\,{}^{S}\frac{d}{dt}\left(\underline{r}_{F_{uh}F_u}\right) = \underline{x}_b - {}^{(F)}\underline{r}^{\times}_{F^*F_u}\,\underline{x}_e \; . \qquad \text{(Eq. 94)} $$

Finally, the umbilical force, Equation (22), can be expressed in terms of the states by substituting from Equations (88) and (94):

$$ {}^{(S)}\underline{F}_u = -K_{tt}\left\{\underline{x}_a - {}^{(F)}\underline{r}^{\times}_{F^*F_u}\,\underline{x}_d\right\} - C_{tt}\left\{\underline{x}_b - {}^{(F)}\underline{r}^{\times}_{F^*F_u}\,\underline{x}_e\right\} - K_{tr}\,\underline{x}_d - C_{tr}\,\underline{x}_e + {}^{(S)}\underline{F}_b\,, \qquad \text{(Eq. 95)} $$

where

$$ K_{tt} = \begin{bmatrix} K_{tt}^{1,1} & K_{tt}^{1,2} & K_{tt}^{1,3} \\ K_{tt}^{2,1} & K_{tt}^{2,2} & K_{tt}^{2,3} \\ K_{tt}^{3,1} & K_{tt}^{3,2} & K_{tt}^{3,3} \end{bmatrix}, \qquad \text{(Eq. 96)} $$

$$ C_{tt} = \begin{bmatrix} C_{tt}^{1,1} & C_{tt}^{1,2} & C_{tt}^{1,3} \\ C_{tt}^{2,1} & C_{tt}^{2,2} & C_{tt}^{2,3} \\ C_{tt}^{3,1} & C_{tt}^{3,2} & C_{tt}^{3,3} \end{bmatrix}, \qquad \text{(Eq. 97)} $$

$$ K_{tr} = \begin{bmatrix} K_{tr}^{1,1} & K_{tr}^{1,2} & K_{tr}^{1,3} \\ K_{tr}^{2,1} & K_{tr}^{2,2} & K_{tr}^{2,3} \\ K_{tr}^{3,1} & K_{tr}^{3,2} & K_{tr}^{3,3} \end{bmatrix}, \qquad \text{(Eq. 98)} $$

and,

$$ C_{tr} = \begin{bmatrix} C_{tr}^{1,1} & C_{tr}^{1,2} & C_{tr}^{1,3} \\ C_{tr}^{2,1} & C_{tr}^{2,2} & C_{tr}^{2,3} \\ C_{tr}^{3,1} & C_{tr}^{3,2} & C_{tr}^{3,3} \end{bmatrix}. \qquad \text{(Eq. 99)} $$

Collecting terms, one obtains the following:

$$ {}^{(S)}\underline{F}_u = -K_{tt}\,\underline{x}_a - C_{tt}\,\underline{x}_b - \left\{K_{tr} - K_{tt}\,{}^{(F)}\underline{r}^{\times}_{F^*F_u}\right\}\underline{x}_d - \left\{C_{tr} - C_{tt}\,{}^{(F)}\underline{r}^{\times}_{F^*F_u}\right\}\underline{x}_e + {}^{(S)}\underline{F}_b\,, \qquad \text{(Eq. 100)} $$

or

$$ {}^{(S)}\underline{F}_u = F_{uta}\,\underline{x}_a + F_{utb}\,\underline{x}_b + F_{utd}\,\underline{x}_d + F_{ute}\,\underline{x}_e + {}^{(S)}\underline{F}_b\,, \qquad \text{(Eq. 101)} $$

where F_{uta}, F_{utb}, F_{utd}, and F_{ute} are constant matrices defined as indicated.

Moment Loads

The i^{th} control force, $\underline{F}_c^i$, exerts on the flotor a control moment,

$$\underline{M}_c^i = \underline{r}_{F^*B_i} \times \underline{F}_c^i . \qquad \text{(Eq. 102)}$$

Substituting from Equation (21),

$$\underline{M}_c^i = \underline{r}_{F^*B_i} \times \left(-I_i L_i B_i \hat{\underline{I}}_i \times \hat{\underline{B}}_i\right); \qquad \text{(Eq. 103)}$$

or, in measure-number form,

$$^{(F)}\underline{M}_c^i = \left(-I_i L_i B_i\right) {}^{(F)}\underline{r}_{F^*B_i}^{\times} \left[\left({}^{F/S}Q^{(S)} \hat{\underline{I}}_i \right)^{\times} {}^{(F)}\hat{\underline{B}}_i \right]. \qquad \text{(Eq. 104)}$$

Reversing the order of the second cross product above, and using the small-angle approximation to ${}^{S/F}Q$ [Eq. (78)], along with its orthogonality property, one obtains the following:

$$^{(F)}\underline{M}_c^i = \left(I_i L_i B_i\right) {}^{(F)}\underline{r}_{F^*B_i}^{\times} \left\{ {}^{(F)}\hat{\underline{B}}_i^{\times} \left[\left(I - {}^{(S)\,F/S}\underline{\theta}^{\times} \right) {}^{(S)}\hat{\underline{I}}_i \right] \right\}. \qquad \text{(Eq. 105)}$$

Expanding, and reversing the order of the cross product ${}^{(S)\,F/S}\underline{\theta}^{\times}\,{}^{(S)}\hat{\underline{I}}_i$, one has

$$^{(F)}\underline{M}_c^i = \left(I_i L_i B_i\right) {}^{(F)}\underline{r}_{F^*B_i}^{\times} \left({}^{(F)}\hat{\underline{B}}_i^{\times}\, {}^{(S)}\hat{\underline{I}}_i \right) + \left(I_i L_i B_i\right) {}^{(F)}\underline{r}_{F^*B_i}^{\times} \left[{}^{(F)}\hat{\underline{B}}_i^{\times} \left({}^{(S)}\hat{\underline{I}}_i^{\times}\, \underline{x}_d \right) \right]. \qquad \text{(Eq. 106)}$$

Since
$$I_i = I_{B_i} + I_{F_i}, \qquad \text{(Eq. 107)}$$

and since I_{F_i} and x_{di} have been assumed small, Equation (106) reduces to

$$^{(F)}\underline{M}_c^i = \left(I_i L_i B_i\right) {}^{(F)}\underline{r}_{F^*B_i}^{\times} \left({}^{(F)}\hat{\underline{B}}_i^{\times}\, {}^{(S)}\hat{\underline{I}}_i \right) + \left(I_{B_i} L_i B_i\right) {}^{(F)}\underline{r}_{F^*B_i}^{\times} \left[{}^{(F)}\hat{\underline{B}}_i^{\times} \left({}^{(S)}\hat{\underline{I}}_i^{\times}\, \underline{x}_d \right) \right], \qquad \text{(Eq. 108)}$$

or
$$^{(F)}\underline{M}_c^i = \left[L_i B_i\, {}^{(F)}\underline{r}_{F^*B_i}^{\times}\, {}^{(F)}\hat{\underline{B}}_i^{\times}\, {}^{(S)}\hat{\underline{I}}_i \right] I_i + \left[I_{B_i} L_i B_i\, {}^{(F)}\underline{r}_{F^*B_i}^{\times}\, {}^{(F)}\hat{\underline{B}}_i^{\times}\, {}^{(S)}\hat{\underline{I}}_i^{\times} \right] \underline{x}_d . \qquad \text{(Eq. 109)}$$

Equation (109) can be written alternatively as

$$^{(F)}\underline{M}_c^i = M_{cu}^i\, I_i + M_{cd}^i\, \underline{x}_d , \qquad \text{(Eq. 110)}$$

for M^i_{ci} and M^i_{cd} appropriately defined. Then the resultant moment due to the coil forces can be expressed as

$$^{(F)}\underline{M}_c = \sum_{i=1}^{8} M^i_{cu} I_i + M^i_{cd}\, \underline{x}_d = M_{cu}\, \underline{u} + M_{cd}\, \underline{x}_d , \qquad \text{(Eq. 111)}$$

where (as before),

$$u_i = I_i , \qquad \text{(Eq. 112)}$$

and M_{cu} and M_{cd} are defined as indicated.

The umbilical force $\underline{F}_u$ exerts a moment $\underline{M}_{uf}$ on the flotor, as defined by Equation (20). An expression for $^{(F)}\underline{M}_{uf}$ is obtained by appropriately incorporating $^{(S)}\underline{F}_u$ from Equation (101) into Equation (20):

$$^{(F)}\underline{M}_{uf} = {}^{(F)}\underline{r}^{\times}_{F^*F_u}\ {}^{F/S}Q\left[F_{uta}\,\underline{x}_a + F_{utb}\,\underline{x}_b + F_{utd}\,\underline{x}_d + F_{ute}\,\underline{x}_e\right] . \qquad \text{(Eq. 113)}$$

The moment $\underline{r}_{F^*F_u} \times \underline{F}_b$, due to the bias force $\underline{F}_b$, has been omitted from Equation (113) since it is considered part of the bias moment $\underline{M}_b$ in Equation (23). Upon linearizing $^{F/S}Q$ as before, one can rewrite Equation (113) as

$$^{(F)}\underline{M}_{uf} = {}^{(F)}\underline{r}^{\times}_{F^*F_u} \left(I - {}^{(S)\,F/S}\underline{\theta}^{\times}\right) \left[F_{uta}\,\underline{x}_a + F_{utb}\,\underline{x}_b + F_{utd}\,\underline{x}_d + F_{ute}\,\underline{x}_e\right] . \qquad \text{(Eq. 114)}$$

Equation (114) can be simplified under the assumption that the states are small quantities:

$$^{(F)}\underline{M}_{uf} = {}^{(F)}\underline{r}^{\times}_{F^*F_u} \left[F_{uta}\,\underline{x}_a + F_{utb}\,\underline{x}_b + F_{utd}\,\underline{x}_d + F_{ute}\,\underline{x}_e\right]. \qquad \text{(Eq. 115)}$$

Substitution of the definitions for F_{uta}, F_{utb}, F_{utd}, and F_{ute}, given by Equations (100) and (101), into Equation (115), yields

$$^{(F)}\underline{M}_{uf} = {}^{(F)}\underline{r}^{\times}_{F^*F_u}\left[-K_{tt}\,\underline{x}_a - \left(K_{tr} - K_{tt}\ {}^{(F)}\underline{r}^{\times}_{F^*F_u} \right) \underline{x}_d - C_{tt}\,\underline{x}_b - \left(C_{tr} - C_{tt}\ {}^{(F)}\underline{r}^{\times}_{F^*F_u} \right) \underline{x}_e \right] , \qquad \text{(Eq. 116)}$$

or, expanding,

$$
\begin{aligned}
{}^{(F)}\underline{M}_{uf} = &\left[-{}^{(F)}\underline{r}^{\times}_{F^*F_u} K_{tt}\right]\underline{x}_a + \left[-{}^{(F)}\underline{r}^{\times}_{F^*F_u} K_{tr} - {}^{(F)}\underline{r}^{\times}_{F^*F_u} K_{tt}\, {}^{(F)}\underline{r}^{\times}_{F^*F_u}\right]\underline{x}_d \\
&\left[-{}^{(F)}\underline{r}^{\times}_{F^*F_u} C_{tt}\right]\underline{x}_b + \left[-{}^{(F)}\underline{r}^{\times}_{F^*F_u} C_{tr} - {}^{(F)}\underline{r}^{\times}_{F^*F_u} C_{tt}\, {}^{(F)}\underline{r}^{\times}_{F^*F_u}\right]\underline{x}_e \;.
\end{aligned}
\qquad \text{(Eq. 117)}
$$

The vector $\underline{M}_u$ can also be expanded into matric form as a function of the state variables by substituting into Equation (23) from Equations (88) and (94), to yield

$$
{}^{(F)}\underline{M}_u = {}^{F/S}Q\left[-K_{rr}\underline{x}_d - C_{rr}\underline{x}_e - K_{rt}\left(\underline{x}_a - {}^{(F)}\underline{r}^{\times}_{F^*F_u}\underline{x}_d\right) - C_{rt}\left(\underline{x}_b - {}^{(F)}\underline{r}^{\times}_{F^*F_u}\underline{x}_e\right) + {}^{(S)}\underline{M}_b\right], \qquad \text{(Eq. 118)}
$$

where

$$
K_{rr} = \begin{bmatrix} K^{1,2}_{rr} & K^{1,2}_{rr} & K^{1,3}_{rr} \\ K^{2,1}_{rr} & K^{2,2}_{rr} & K^{2,3}_{rr} \\ K^{3,1}_{rr} & K^{3,2}_{rr} & K^{3,3}_{rr} \end{bmatrix}, \qquad \text{(Eq. 119)}
$$

$$
C_{rr} = \begin{bmatrix} C^{1,2}_{rr} & C^{1,2}_{rr} & C^{1,3}_{rr} \\ C^{2,1}_{rr} & C^{2,2}_{rr} & C^{2,3}_{rr} \\ C^{3,1}_{rr} & C^{3,2}_{rr} & C^{3,3}_{rr} \end{bmatrix}, \qquad \text{(Eq. 120)}
$$

$$
K_{rt} = \begin{bmatrix} K^{1,2}_{rt} & K^{1,2}_{rt} & K^{1,3}_{rt} \\ K^{2,1}_{rt} & K^{2,2}_{rt} & K^{2,3}_{rt} \\ K^{3,1}_{rt} & K^{3,2}_{rt} & K^{3,3}_{rt} \end{bmatrix}, \qquad \text{(Eq. 121)}
$$

and

$$
C_{rt} = \begin{bmatrix} C^{1,2}_{rt} & C^{1,2}_{rt} & C^{1,3}_{rt} \\ C^{2,1}_{rt} & C^{2,2}_{rt} & C^{2,3}_{rt} \\ C^{3,1}_{rt} & C^{3,2}_{rt} & C^{3,3}_{rt} \end{bmatrix}. \qquad \text{(Eq. 122)}
$$

Use of Equation (78) with Equation (118) yields

$$
{}^{(F)}\underline{M}_u = \left(I - {}^{(S)\,F/S}\underline{\theta}^{\times}\right)\left[-K_{rr}\underline{x}_d - C_{rr}\underline{x}_e - K_{rt}\left(\underline{x}_a - {}^{(F)}\underline{r}^{\times}_{F^*F_u}\underline{x}_d\right) - C_{rt}\left(\underline{x}_b - {}^{(F)}\underline{r}^{\times}_{F^*F_u}\underline{x}_e\right) + {}^{(S)}\underline{M}_b\right]. \qquad \text{(Eq. 123)}
$$

Assuming that the each of the state variables is small,

$${}^{(F)}\underline{M}_u = -K_{rr}\underline{x}_d - C_{rr}\underline{x}_e - K_{rt}\left(\underline{x}_a - {}^{(F)}\underline{r}^{\times}_{F^*F_u}\underline{x}_d\right) - C_{rt}\left(\underline{x}_b - {}^{(F)}\underline{r}^{\times}_{F^*F_u}\underline{x}_e\right) + {}^{(S)}\underline{M}_b + {}^{(S)}\underline{M}_b^{\times}\underline{x}_d\,. \quad \text{(Eq.124)}$$

Combining Equations (117) and (124), one obtains

$$\begin{aligned}
{}^{(F)}\underline{M}_u + {}^{(F)}\underline{M}_{uf} &= \left(-K_{rt} - {}^{(F)}\underline{r}^{\times}_{F^*F_u}K_{tt}\right)\underline{x}_a + \left(-C_{rt} - {}^{(F)}\underline{r}^{\times}_{F^*F_u}C_{tt}\right)\underline{x}_b \\
&+ \left(K_{rt}\,{}^{(F)}\underline{r}^{\times}_{F^*F_u} - K_{rr} - {}^{(F)}\underline{r}^{\times}_{F^*F_u}K_{tr} + {}^{(F)}\underline{r}^{\times}_{F^*F_u}K_{tt}\,{}^{(F)}\underline{r}^{\times}_{F^*F_u} + {}^{(S)}\underline{M}_b^{\times}\right)\underline{x}_d \quad \text{(Eq. 125)} \\
&+ \left(C_{rt}\,{}^{(F)}\underline{r}^{\times}_{F^*F_u} - C_{rr} - {}^{(F)}\underline{r}^{\times}_{F^*F_u}C_{tr} + {}^{(F)}\underline{r}^{\times}_{F^*F_u}C_{tt}\,{}^{(F)}\underline{r}^{\times}_{F^*F_u}\right)\underline{x}_e \\
&+ {}^{(S)}\underline{M}_b.
\end{aligned}$$

Or,

$${}^{(F)}\underline{M}_u + {}^{(F)}\underline{M}_{uf} = M_{uta}\underline{x}_a + M_{utb}\underline{x}_b + M_{urd}\underline{x}_d + M_{ure}\underline{x}_e + {}^{(S)}\underline{M}_b\,, \quad \text{(Eq. 126)}$$

for $M_{uta}, M_{utb}, M_{urd}$ and M_{ure} appropriately defined.

State Space Equations

The force loads ($\underline{F}_c$ and $\underline{F}_u$) and the moment loads ($\underline{M}_c$, $\underline{M}_u$, and $\underline{M}_{uf}$) have now been expressed in measure number form [see Eqs. (83), (101), (111), and (126)] in terms of the state variables. These expressions can now be used to obtain the final, expanded form of the linearized, state-space equations of motion.

Beginning with the state equation for $\underline{x}_b$, substitution from Equations (83) and (101) into Equation (60) yields

$$\dot{\underline{x}}_b = \left(\frac{1}{m}F_{uta}\right)\underline{x}_a + \left(\frac{1}{m}F_{utb}\right)\underline{x}_b + \left[\frac{1}{m}\left(F_{utd}+F_{cd}\right)\right]\underline{x}_d + \left(\frac{1}{m}F_{ute}\right)\underline{x}_e + \left(\frac{1}{m}F_{cu}\right)\underline{u} - {}^{(S)}\underline{a}_{in} + {}^{(S)}\underline{a}_d + \frac{1}{m}\,{}^{(S)}\underline{F}_b\,. \quad \text{(Eq. 127)}$$

Recall that the control, $\underline{u}$, is a column vector of actuator currents I_i, each of which is the sum of a bias value I_{B_i} and a fluctuating value I_{F_i}. Then one can define a new control $\underline{u}^*$ such that,

$$\underline{u} = \underline{I}_B + \underline{u}^*, \qquad \text{(Eq. 128)}$$

where $\underline{I}_B$ is a column vector of actuator bias currents and $\underline{u}^*$ is a column vector of fluctuating currents. Since the bias current is defined as the current necessary to counteract the umbilical bias loads, substitution from Equation (128) into Equation (127) yields the following linear state equation for $\underline{x}_b$:

$$\dot{\underline{x}}_b = \left[\frac{1}{m}F_{uta}\right]\underline{x}_a + \left[\frac{1}{m}F_{utb}\right]\underline{x}_b + \left[\frac{1}{m}(F_{utd}+F_{cd})\right]\underline{x}_d + \left[\frac{1}{m}F_{ute}\right]\underline{x}_e + \left[\frac{1}{m}F_{cu}\right]\underline{u}^* - {}^{(S)}\underline{a}_{in} + {}^{(S)}\underline{a}_d \ . \qquad \text{(Eq. 129)}$$

Turning now to the state equation for $\underline{x}_e$, substitution of Equations (111) and (126) into Equation (65) results in the following expression for $\dot{\underline{x}}_e$:

$$\dot{\underline{x}}_e = {}^{S/F}Q\,{}^{(F)}I^{-1}\left[M_{uta}\underline{x}_a + M_{utb}\underline{x}_b + (M_{urd}+M_{cd})\underline{x}_d + M_{ure}\underline{x}_e + {}^{(S)}\underline{M}_b + M_{cu}\underline{u}\right] + {}^{S/F}Q\,{}^{(F)}\underline{\alpha}_d \ , \qquad \text{(Eq. 130)}$$

or, in terms of the new control $\underline{u}^*$,

$$\dot{\underline{x}}_e = {}^{S/F}Q\,{}^{(F)}I^{-1}\left[M_{uta}\underline{x}_a + M_{utb}\underline{x}_b + (M_{urd}+M_{cd})\underline{x}_d + M_{ure}\underline{x}_e + M_{cu}\underline{u}^*\right] + {}^{S/F}Q\,{}^{(F)}\underline{\alpha}_d \ . \qquad \text{(Eq. 131)}$$

Using Equation (78), and combining terms, one obtains

$$\begin{aligned}\dot{\underline{x}}_e = &\left(I + {}^{(S)F/S}\underline{\theta}^{\times}\right){}^{(F)}I^{-1}\left[M_{uta}\underline{x}_a + M_{utb}\underline{x}_b + (M_{urd}+M_{cd})\underline{x}_d + M_{ure}\underline{x}_e + M_{cu}\underline{u}^*\right] \\ &+ \left(I + {}^{(S)F/S}\underline{\theta}^{\times}\right){}^{(F)}\underline{\alpha}_d \ .\end{aligned} \qquad \text{(Eq. 132)}$$

Assuming small state variables, small control, and small direct angular acceleration disturbances, Equation (132) reduces to the following linear state equation for $\underline{x}_e$:

$$\dot{\underline{x}}_e = \left[{}^{(F)}I^{-1}M_{uta}\right]\underline{x}_a + \left[{}^{(F)}I^{-1}M_{utb}\right]\underline{x}_b + \left[{}^{(F)}I^{-1}(M_{urd}+M_{cd})\right]\underline{x}_d + \left[{}^{(F)}I^{-1}M_{ure}\right]\underline{x}_e + \left[{}^{(F)}I^{-1}M_{cu}\right]\underline{u}^* + {}^{(F)}\underline{\alpha}_d \ . \qquad \text{(Eq. 133)}$$

Turning now to the state equation for $\underline{x}_c$, substitution from Equations (83), (101), (111) and (126) into Equation (74) results in the following expression for $\dot{\underline{x}}_c$:

$$\begin{aligned}\underline{\dot{x}}_c &= \omega_h\left(\frac{1}{m}F_{uta} - {}^{S/F}Q\,{}^{(F)}\underline{r}^{\times}_{F^*E}\,{}^{(F)}I^{-1}M_{uta}\right)\underline{x}_a + \omega_h\left(\frac{1}{m}F_{utb} - {}^{S/F}Q\,{}^{(F)}\underline{r}^{\times}_{F^*E}\,{}^{(F)}I^{-1}M_{utb}\right)\underline{x}_b - \omega_h\underline{x}_c \\ &+ \omega_h\left[\frac{1}{m}(F_{utd}+F_{cd}) - {}^{S/F}Q\,{}^{(F)}\underline{r}^{\times}_{F^*E}\,{}^{(F)}I^{-1}(M_{urd}+M_{cd})\right]\underline{x}_d + \omega_h\left(\frac{1}{m}F_{ute} - {}^{S/F}Q\,{}^{(F)}\underline{r}^{\times}_{F^*E}\,{}^{(F)}I^{-1}M_{ure}\right)\underline{x}_e \\ &+ \omega_h\left(\frac{1}{m}F_{cu} - {}^{S/F}Q\,{}^{(F)}\underline{r}^{\times}_{F^*E}\,{}^{(F)}I^{-1}M_{cu}\right)\underline{u} \\ &+ \omega_h\,{}^{(S)}\underline{a}_d + \omega_h\left(\frac{1}{m}\right){}^{(S)}\underline{F}_b - \omega_h\left({}^{S/F}Q\,{}^{(F)}\underline{r}^{\times}_{F^*E}\right){}^{(F)}\underline{\alpha}_d - \omega_h\left({}^{S/F}Q\,{}^{(F)}\underline{r}^{\times}_{F^*E}\,{}^{(F)}I^{-1}\right){}^{(S)}\underline{M}_b,\end{aligned} \qquad \text{(Eq. 134)}$$

or, in terms of the new control $\underline{u}^*$,

$$\begin{aligned}\underline{\dot{x}}_c &= \omega_h\left(\frac{1}{m}F_{uta} - {}^{S/F}Q\,{}^{(F)}\underline{r}^{\times}_{F^*E}\,{}^{(F)}I^{-1}M_{uta}\right)\underline{x}_a + \omega_h\left(\frac{1}{m}F_{utb} - {}^{S/F}Q\,{}^{(F)}\underline{r}^{\times}_{F^*E}\,{}^{(F)}I^{-1}M_{utb}\right)\underline{x}_b - \omega_h\underline{x}_c \\ &+ \omega_h\left[\frac{1}{m}(F_{utd}+F_{cd}) - {}^{S/F}Q\,{}^{(F)}\underline{r}^{\times}_{F^*E}\,{}^{(F)}I^{-1}(M_{urd}+M_{cd})\right]\underline{x}_d + \omega_h\left(\frac{1}{m}F_{ute} - {}^{S/F}Q\,{}^{(F)}\underline{r}^{\times}_{F^*E}\,{}^{(F)}I^{-1}M_{ure}\right)\underline{x}_e \\ &+ \omega_h\left(\frac{1}{m}F_{cu} - {}^{S/F}Q\,{}^{(F)}\underline{r}^{\times}_{F^*E}\,{}^{(F)}I^{-1}M_{cu}\right)\underline{u}^* \\ &+ \omega_h\,{}^{(S)}\underline{a}_d - \omega_h\left({}^{S/F}Q\,{}^{(F)}\underline{r}^{\times}_{F^*E}\right){}^{(F)}\underline{\alpha}_d.\end{aligned} \qquad \text{(Eq. 135)}$$

Using Equation (78), one obtains

$$\begin{aligned}\underline{\dot{x}}_c &= \omega_h\left[\frac{1}{m}F_{uta} - \left(I + {}^{(S)\,F/S}\underline{\theta}^{\times}\right){}^{(F)}\underline{r}^{\times}_{F^*E}\,{}^{(F)}I^{-1}M_{uta}\right]\underline{x}_a + \omega_h\left[\frac{1}{m}F_{utb} - \left(I + {}^{(S)\,F/S}\underline{\theta}^{\times}\right){}^{(F)}\underline{r}^{\times}_{F^*E}\,{}^{(F)}I^{-1}M_{utb}\right]\underline{x}_b - \omega_h\underline{x}_c \\ &+ \omega_h\left[\frac{1}{m}(F_{utd}+F_{cd}) - \left(I + {}^{(S)\,F/S}\underline{\theta}^{\times}\right){}^{(F)}\underline{r}^{\times}_{F^*E}\,{}^{(F)}I^{-1}(M_{urd}+M_{cd})\right]\underline{x}_d + \omega_h\left[\frac{1}{m}F_{ute} - \left(I + {}^{(S)\,F/S}\underline{\theta}^{\times}\right){}^{(F)}\underline{r}^{\times}_{F^*E}\,{}^{(F)}I^{-1}M_{ure}\right]\underline{x}_e \\ &+ \omega_h\left[\frac{1}{m}F_{cu} - \left(I + {}^{(S)\,F/S}\underline{\theta}^{\times}\right){}^{(F)}\underline{r}^{\times}_{F^*E}\,{}^{(F)}I^{-1}M_{cu}\right]\underline{u}^* \\ &+ \omega_h\,{}^{(S)}\underline{a}_d - \omega_h\left(I + {}^{(S)\,F/S}\underline{\theta}^{\times}\right){}^{(F)}\underline{r}^{\times}_{F^*E}\,{}^{(F)}\underline{\alpha}_d.\end{aligned} \qquad \text{(Eq. 136)}$$

Assuming small states, small direct disturbances, and small controls, Equation (136) reduces to the following linear state equation for $\underline{x}_c$:

$$\underline{\dot{x}}_c = \omega_h\left[\frac{1}{m}F_{uta} - {}^{(F)}\underline{r}^{\times}_{F^*E}\,{}^{(F)}I^{-1}M_{uta}\right]\underline{x}_a + \omega_h\left[\frac{1}{m}F_{utb} - {}^{(F)}\underline{r}^{\times}_{F^*E}\,{}^{(F)}I^{-1}M_{utb}\right]\underline{x}_b - \omega_h\underline{x}_c$$
$$+\omega_h\left[\frac{1}{m}\left(F_{utd}+F_{cd}\right) - {}^{(F)}\underline{r}^{\times}_{F^*E}\,{}^{(F)}I^{-1}\left(M_{urd}+M_{cd}\right)\right]\underline{x}_d \qquad \text{(Eq. 137)}$$
$$+\omega_h\left[\frac{1}{m}F_{ute} - {}^{(F)}\underline{r}^{\times}_{F^*E}\,{}^{(F)}I^{-1}M_{ure}\right]\underline{x}_e + \omega_h\left[\frac{1}{m}F_{cu} - {}^{(F)}\underline{r}^{\times}_{F^*E}\,{}^{(F)}I^{-1}M_{cu}\right]\underline{u}^* + \omega_h\,{}^{(S)}\underline{a}_d - \omega_h\,{}^{(F)}\underline{r}^{\times}_{F^*E}\,{}^{(F)}\underline{\alpha}_d\,.$$

A state-space representation of the system is given by Equations (48), (129), (137), (56), and (133), for state vector

$$\underline{x} = \left[\underline{x}_a^T \quad \underline{x}_b^T \quad \underline{x}_c^T \quad \underline{x}_d^T \quad \underline{x}_e^T\right]^T, \qquad \text{(Eq. 138)}$$

and control vector

$$\underline{u}^* = \left[I_{F_1} \quad I_{F_2} \quad I_{F_3} \quad I_{F_4} \quad I_{F_5} \quad I_{F_6}\right]^T. \qquad \text{(Eq. 139)}$$

In summary, the final form of the linearized state-space equations of motion are as follows:

$$\underline{\dot{x}}_a = \underline{x}_b, \qquad \text{(Eq. 140)}$$

$$\underline{\dot{x}}_b = \left[\frac{1}{m}F_{uta}\right]\underline{x}_a + \left[\frac{1}{m}F_{utb}\right]\underline{x}_b + \left[\frac{1}{m}\left(F_{utd}+F_{cd}\right)\right]\underline{x}_d + \left[\frac{1}{m}F_{ute}\right]\underline{x}_e + \left[\frac{1}{m}F_{cu}\right]\underline{u}^* - {}^{(S)}\underline{a}_{in} + {}^{(S)}\underline{a}_d\,, \qquad \text{(Eq. 141)}$$

$$\underline{\dot{x}}_c = \omega_h\left[\frac{1}{m}F_{uta} - {}^{(F)}\underline{r}^{\times}_{F^*E}\,{}^{(F)}I^{-1}M_{uta}\right]\underline{x}_a + \omega_h\left[\frac{1}{m}F_{utb} - {}^{(F)}\underline{r}^{\times}_{F^*E}\,{}^{(F)}I^{-1}M_{utb}\right]\underline{x}_b - \omega_h\underline{x}_c$$
$$+\omega_h\left[\frac{1}{m}\left(F_{utd}+F_{cd}\right) - {}^{(F)}\underline{r}^{\times}_{F^*E}\,{}^{(F)}I^{-1}\left(M_{urd}+M_{cd}\right)\right]\underline{x}_d \qquad \text{(Eq. 142)}$$
$$+\omega_h\left[\frac{1}{m}F_{ute} - {}^{(F)}\underline{r}^{\times}_{F^*E}\,{}^{(F)}I^{-1}M_{ure}\right]\underline{x}_e + \omega_h\left[\frac{1}{m}F_{cu} - {}^{(F)}\underline{r}^{\times}_{F^*E}\,{}^{(F)}I^{-1}M_{cu}\right]\underline{u}^* + \omega_h\,{}^{(S)}\underline{a}_d - \omega_h\,{}^{(F)}\underline{r}^{\times}_{F^*E}\,{}^{(F)}\underline{\alpha}_d,$$

$$\underline{\dot{x}}_d = \underline{x}_e\,, \qquad \text{(Eq. 143)}$$

and

$$\underline{\dot{x}}_e = \left[{}^{(F)}I^{-1}M_{uta}\right]\underline{x}_a + \left[{}^{(F)}I^{-1}M_{utb}\right]\underline{x}_b + \left[{}^{(F)}I^{-1}\left(M_{urd}+M_{cd}\right)\right]\underline{x}_d + \left[{}^{(F)}I^{-1}M_{ure}\right]\underline{x}_e + \left[{}^{(F)}I^{-1}M_{cu}\right]\underline{u}^* + {}^{(F)}\underline{\alpha}_d\,. \qquad \text{(Eq. 144)}$$

Verification of State-Space Equations

The linearized state-space equations of motion for g-LIMIT, given by Equations (140) through (144), were verified by comparing the linearized-system open-loop response to the open-loop response of a nonlinear "truth" model, for various inputs. Two input scenarios were used for these comparisons. In the first scenario (Scenario # 1), the flotor was excited via an actuator by an arbitrary force-plus-torque pulse combination, to result in a fully coupled six-degree-of-freedom response. In the second scenario, the flotor was excited via an actuator by a force pulse in the $+Z$ stator-fixed direction. For each scenario the motion of the linearized system was then compared with the response of the nonlinear model. It was found that the nonlinear and linearized models exhibited essentially identical responses. For each scenario, the potential and kinetic energies were plotted as functions of time. When the systems were made conservative (i.e., with the umbilical damping removed) it was shown that the total energy was constant, as expected.

The nonlinear model was constructed from the nonlinear form of the equations of motion, Equations (1) and (3). Employing the quaternion representation ${}^{F/S}\underline{q}$ of the transformation from stator-fixed to flotor-fixed coordinates, the following kinematic equation was used propagate the rotational motion of the flotor [12]:

$$\frac{d}{dt}\left({}^{F/S}\underline{q}\right)=\frac{1}{2}\;{}^{S}\Omega^{F}\;{}^{F/S}\underline{q}, \qquad \text{(Eq. 145)}$$

where

$${}^{S}\Omega^{F}=\begin{bmatrix} 0 & \omega_3 & -\omega_2 & \omega_1 \\ -\omega_3 & 0 & \omega_1 & \omega_2 \\ \omega_2 & -\omega_1 & 0 & \omega_3 \\ -\omega_1 & -\omega_2 & -\omega_3 & 0 \end{bmatrix}, \qquad \text{(Eq. 146)}$$

and

$$\omega_i={}^{S}\underline{\omega}^{F}\cdot\underline{\hat{f}}_i. \qquad \text{(Eq. 147)}$$

Equations (16) through (23) were used, in their respective nonlinear vector forms, to compute the forces and moments in the nonlinear model. The small-angle approximation

$$^{F/S}\underline{\theta}\cdot\hat{\underline{s}}_i = 2q_i q_4 \; (i=1,2,3) \qquad \text{(Eq. 147a)}$$

was used in the third summation terms of Equations (22) and (23), where the factors q_i are as defined by

$$^{F/S}\underline{q} = [q_1 \;\; q_2 \;\; q_3 \;\; q_4]^T. \qquad \text{(Eq. 147b)}$$

(This is a reasonable approximation for the expected range of angular motions of the flotor relative to the stator.)

Representative flotor and umbilical parameters, shown in Table 1, were used for the simulations. There are three umbilicals included in this model of g-LIMIT. The translational and rotational stiffness matrices for each umbilical were assumed to be diagonal along an umbilical-fixed set of coordinate directions. Similarity transformations of these diagonal matrices were performed assuming a coordinate transformation from each local umbilical-fixed reference frame to the stator-fixed frame. First, a coordinate rotation about the stator-fixed +Z axis of 120 deg and 240 deg was performed to align umbilicals #2 and #3, in their respective home locations. Then, for each umbilical, a 20-deg rotation about each coordinate axis was used to represent an arbitrary misalignment of the diagonal-stiffness directions to the stator-fixed directions. The translational and rotational damping matrices were assumed to be proportional to the stiffness matrices with a damping ratio of 3% used for all of the vibrational modes. The resulting umbilical stiffness and damping matricies, given below by Equations (148) through (159), were used in the simulation study. The superscript-in-parentheses notation denotes the umbilical identification number for each of the three g-LIMIT umbilicals. All stiffness and damping translation/rotational cross-terms, i.e., $K_{tr}^{(i)}$, $K_{rt}^{(i)}$, $C_{tr}^{(i)}$, and $C_{rt}^{(i)}$, were considered to be zero.

$$K_{tt}^{(1)} = \begin{bmatrix} 28.49 & -2.46 & 8.30 \\ -2.46 & 26.73 & -5.85 \\ 8.30 & -5.85 & 44.77 \end{bmatrix} \text{N / m}, \quad \text{(Eq. 148)}$$

$$K_{tt}^{(2)} = \begin{bmatrix} 25.04 & 0.47 & 0.91 \\ 0.47 & 30.18 & 10.12 \\ 0.91 & 10.12 & 44.77 \end{bmatrix} \text{N / m}, \quad \text{(Eq. 149)}$$

$$K_{tt}^{(3)} = \begin{bmatrix} 29.30 & 1.99 & -9.22 \\ 1.99 & 25.92 & -4.27 \\ -9.22 & -4.27 & 44.77 \end{bmatrix} \text{N / m}, \quad \text{(Eq. 150)}$$

$$C_{tt}^{(1)} = \begin{bmatrix} 1.26 & -0.04 & 0.16 \\ -0.04 & 1.22 & -0.11 \\ 0.16 & -0.11 & 1.58 \end{bmatrix} \text{N / m/sec}, \quad \text{(Eq. 151)}$$

$$C_{tt}^{(2)} = \begin{bmatrix} 1.19 & 0.00 & 0.01 \\ 0.00 & 1.29 & 0.20 \\ 0.01 & 0.20 & 1.58 \end{bmatrix} \text{N / m/sec}, \quad \text{(Eq. 152)}$$

$$C_{tt}^{(3)} = \begin{bmatrix} 1.27 & 0.03 & -0.18 \\ 0.03 & 1.21 & -0.08 \\ -0.18 & -0.08 & 1.58 \end{bmatrix} \text{N / m/sec}, \quad \text{(Eq. 153)}$$

$$K_{rr}^{(1)} = \begin{bmatrix} 1.86 & 0.09 & -0.33 \\ 0.09 & 1.93 & 0.23 \\ -0.33 & 0.23 & 1.20 \end{bmatrix} \text{N-m / rad}, \quad \text{(Eq. 154)}$$

$$K_{rr}^{(2)} = \begin{bmatrix} 1.99 & -0.01 & -0.03 \\ -0.01 & 1.79 & -0.40 \\ -0.03 & -0.40 & 1.20 \end{bmatrix} \text{N-m / rad}, \quad \text{(Eq. 155)}$$

$$K_{rr}^{(3)} = \begin{bmatrix} 1.82 & -0.07 & 0.36 \\ -0.07 & 1.96 & 0.17 \\ 0.36 & 0.17 & 1.20 \end{bmatrix} \text{N-m / rad,} \qquad \text{(Eq. 156)}$$

$$C_{rr}^{(1)} = \begin{bmatrix} 0.055 & 0.001 & -0.012 \\ 0.001 & 0.063 & 0.010 \\ -0.012 & 0.010 & 0.033 \end{bmatrix} \text{N-m / rad/sec,} \qquad \text{(Eq. 157)}$$

$$C_{rr}^{(2)} = \begin{bmatrix} 0.062 & 0.002 & -0.002 \\ 0.002 & 0.056 & -0.015 \\ -0.002 & -0.015 & 0.033 \end{bmatrix} \text{N-m / rad/sec,} \qquad \text{(Eq. 158)}$$

and

$$C_{rr}^{(3)} = \begin{bmatrix} 0.059 & -0.004 & 0.014 \\ -0.004 & 0.059 & 0.005 \\ 0.014 & 0.005 & 0.033 \end{bmatrix} \text{N / m/sec.} \qquad \text{(Eq. 159)}$$

In addition to the parameters listed in Table #1 the actuator currents were set to initial bias values. These bias currents were required to produce a bias force and moment to move the flotor from its assumed relaxed position to the home location. The flotor relaxed-position was assumed to be 2 mm from the home-position, and to be misaligned by approximately 2 deg about each stator-fixed coordinate axis. This resulted in the following set of bias current values: $I_{B_2} = -0.264$ A, $I_{B_4} = -0.1593$ A, and $I_{B_6} = 0.123$ A.

In Scenario #1 the six actuator current pulses were chosen to approximate combined impulsive force loads in the +*Z* stator-fixed direction, and moment loads about the +*X* and +*Y* stator-fixed directions. These loads were sized to produce a total energy output of 0.003 N-m equally divided among the three loads. The current pulses were shaped as one-minus-cosine functions to provide easily integrable inputs with no temporal discontinuities. The pulses were initiated at 1 sec and lasted for a 0.1-sec duration. Table #2 shows the integrated values of the actuator current pulses for Scenario #1.

Figures 2 through 6 show comparisons of the simulation results for the linear state-space model and the nonlinear "truth" model for Scenario #1. As shown in these figures the linear model responses match the truth model very well for all of the state variables. The state responses evolve in time with the expected frequency and damping. The errors between the linear responses and the truth-model responses are shown in Figures 7 and 8. These relatively small errors, assumed to be a result of the linearization process, diminish as the system response decays. Figure 9 shows a plot of the flotor kinetic and potential energy for Scenario #1. As expected, the impulsive loading results in an initial translational kinetic energy of 0.001 joules and a rotational energy of 0.002 joules. The kinetic and potential energies oscillate out-of-phase, and the total energy dampens out exponentially.

A second scenario (Scenario #2) was simulated to demonstrate that the model conserves energy when the damping matrices are all set equal to zero. In Scenario #2 the six actuator-current pulses were chosen to approximate an impulsive force load in the +*Z* stator-fixed direction only. The impulse moments used in Scenario #1 were eliminated from the actuator loading. As in Scenario #1 the loads were sized to produce a total energy output of 0.003 N-m. In this case the total energy input to the flotor was along the stator-fixed +*Z* direction. The current pulses were shaped as one-minus-cosine functions to provide easily integrable inputs with no temporal discontinuities. The pulses were initiated at 1 sec and lasted for a 0.1-sec duration. Table #2 shows the integrated values of the actuator current pulses for Scenario #2.

Figures 10 and 11 show a comparison of the flotor output energy responses for Scenarios #1 and #2, respectively. The potential and kinetic energy responses differ between the two scenario, due to the difference in the impulsive loading. However, the total flotor energy remains constant at the expected value of 0.003 joules for each of the scenarios, after application of its respective impulsive loading, at 1 sec. This demonstrates that the model conserves total energy at the appropriate level for an arbitrarily chosen actuator loading.

Table 1 – g-LIMIT Parameters

Parameter	Symbol	Value
Flotor Mass	m	15.12 kg
Flotor Moments of Inertia	I_{xx} I_{yy} I_{zz}	0.50 kg m^2 0.62 kg m^2 0.18 kg m^2
Flotor Products of Inertia	I_{xy} I_{xz} I_{yz}	1e-4 kg m^2 -1e-4 kg m^2 -8e-4 kg m^2
Umbilical Locations (F) (3 Umbilicals)	${}^{(F)}\underline{r}_{F^*F_u}$	[0.0 -0.12 -0.032] m [0.1 0.06 -0.032] m [-0.1 0.06 -0.032] m
Actuator Current Vectors (S) (6 Actuator Coils)	${}^{(S)}\hat{\underline{I}}_i$	[0.0 0.0 1.0] [-1.0 0.0 0.0] [0.0 0.0 1.0] [0.5 0.866 0.0] [0.0 0.0 1.0] [0.5 -0.866 0.0]
Actuator Magnet B-Field Vectors (F) (3 Actuator Magnets)	${}^{(F)}\hat{\underline{B}}_i$	[0.0 1.0 0.0] [0.0 1.0 0.0] [0.866 -0.5 0.0] [0.866 -0.5 0.0] [-0.866 -0.5 0.0] [-0.866 -0.5 0.0]
Actuator Constant	$(L_i B_i)$	1.0 N/Amp

Table 2 – Actuator Current Pulse Values

	Scenario #1 [Amp-sec]	Scenario #2 [Amp-sec]
Actuator #1 Current Pulse	0.0	0.0
Actuator #2 Current Pulse	-0.231	-0.10
Actuator #3 Current Pulse	0.0	0.0
Actuator #4 Current Pulse	-0.139	-0.10
Actuator #5 Current Pulse	0.0	0.0
Actuator #6 Current Pulse	0.193	-0.10

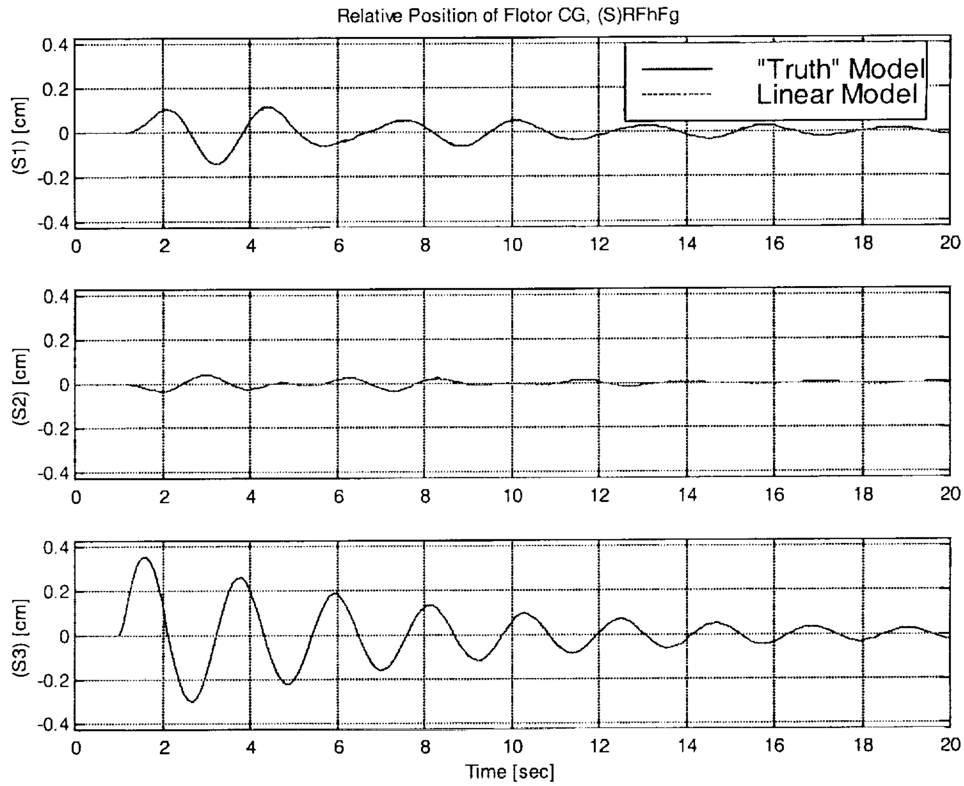

Figure 2. Flotor Position for Scenario #1

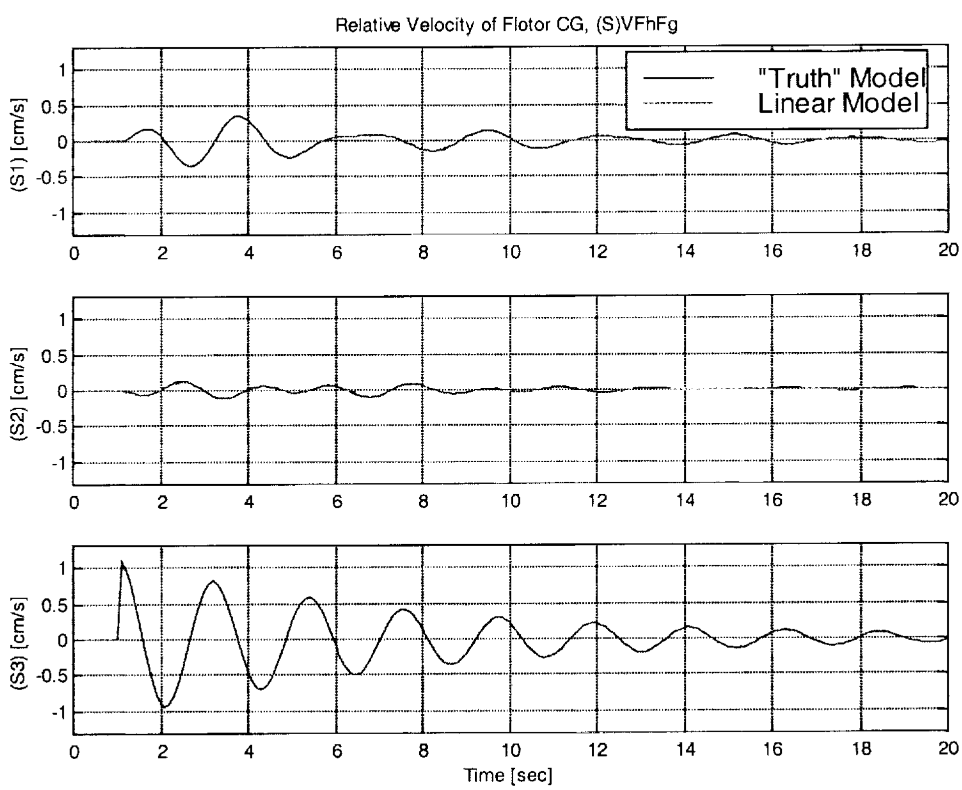

Figure 3. Flotor Velocity for Scenario #1

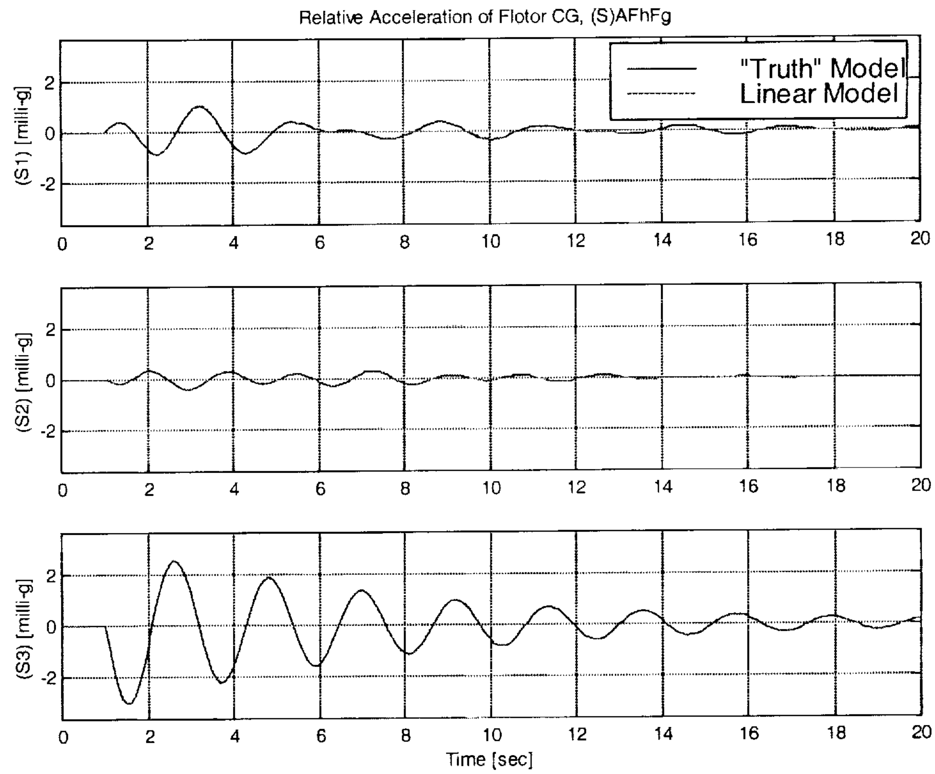

Figure 4. Flotor Acceleration for Scenario #1

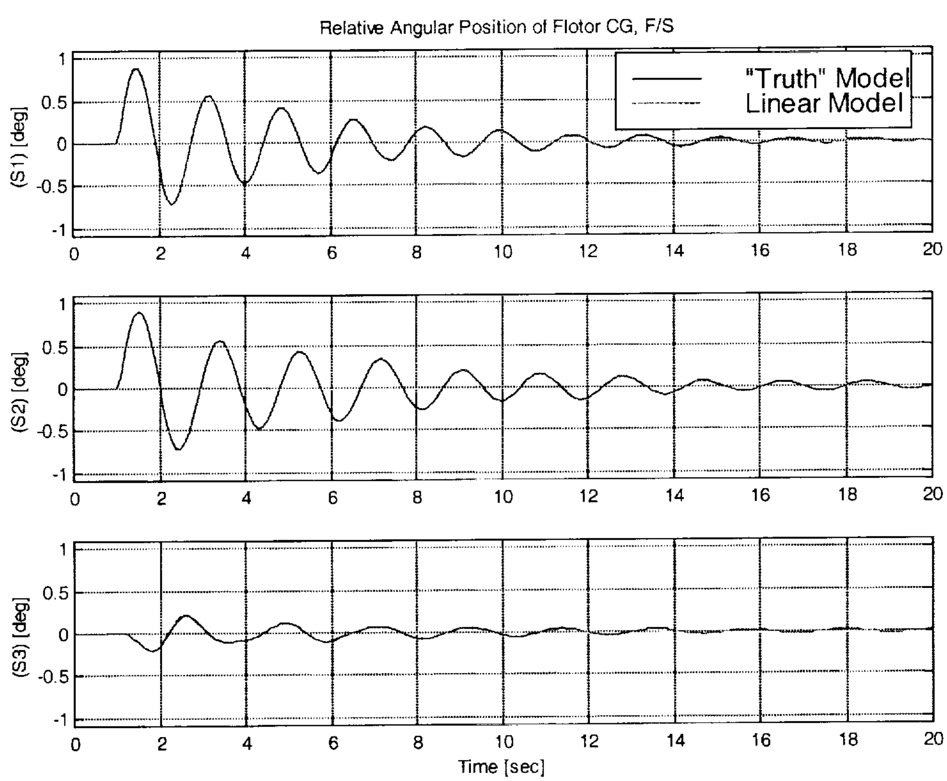

Figure 5. Flotor Angular Displacement for Scenario #1

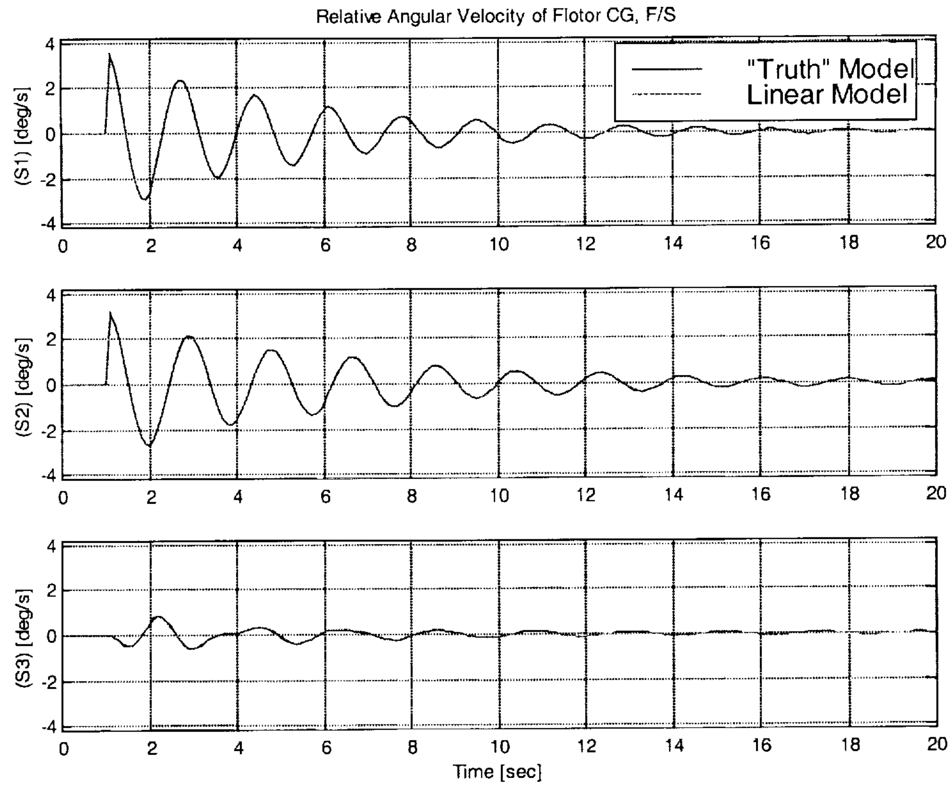

Figure 6. Flotor Angular Velocity for Scenario #1

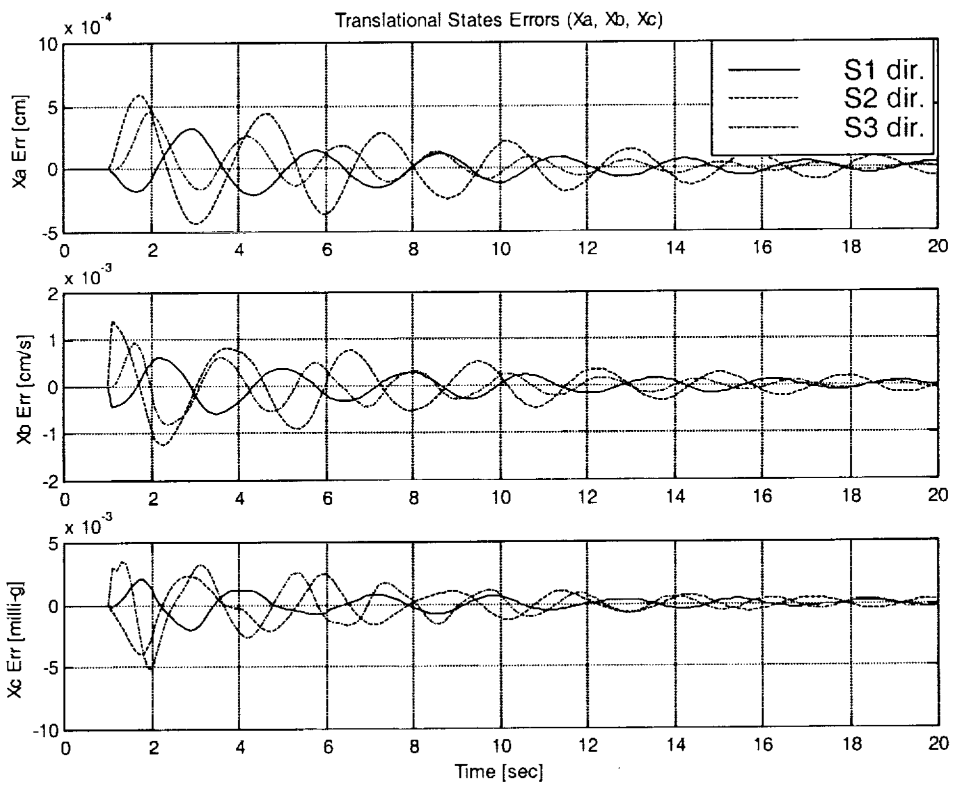

Figure 7. Xa, Xb, and Xc State Errors for Scenario #1

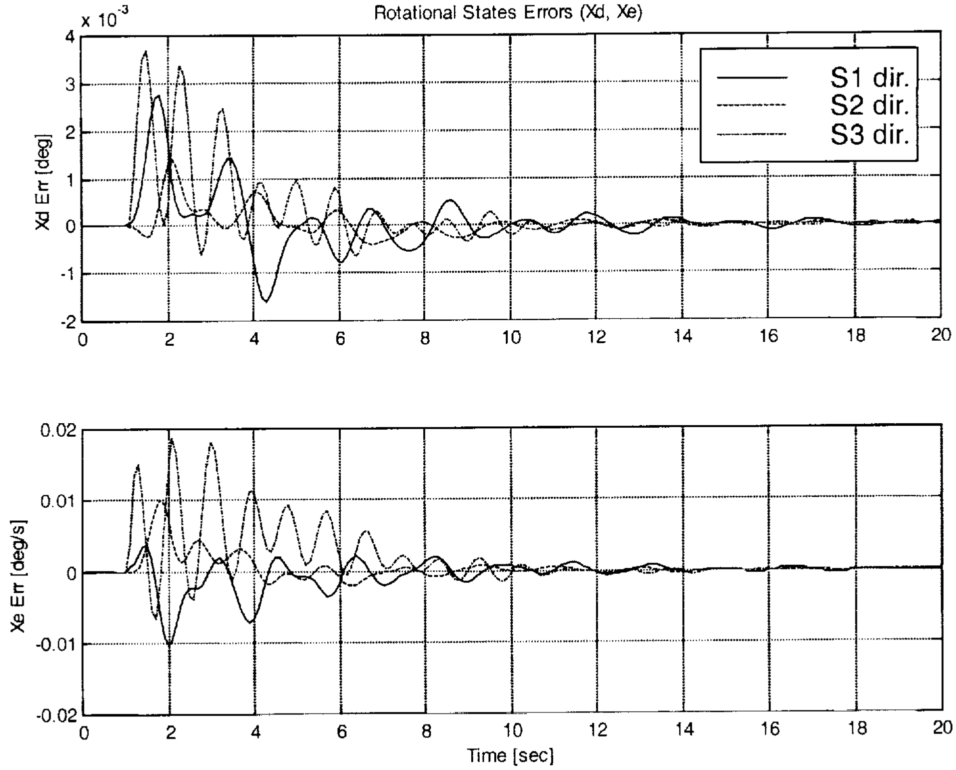

Figure 8. Xa, Xb, and Xc State Errors for Scenario #1

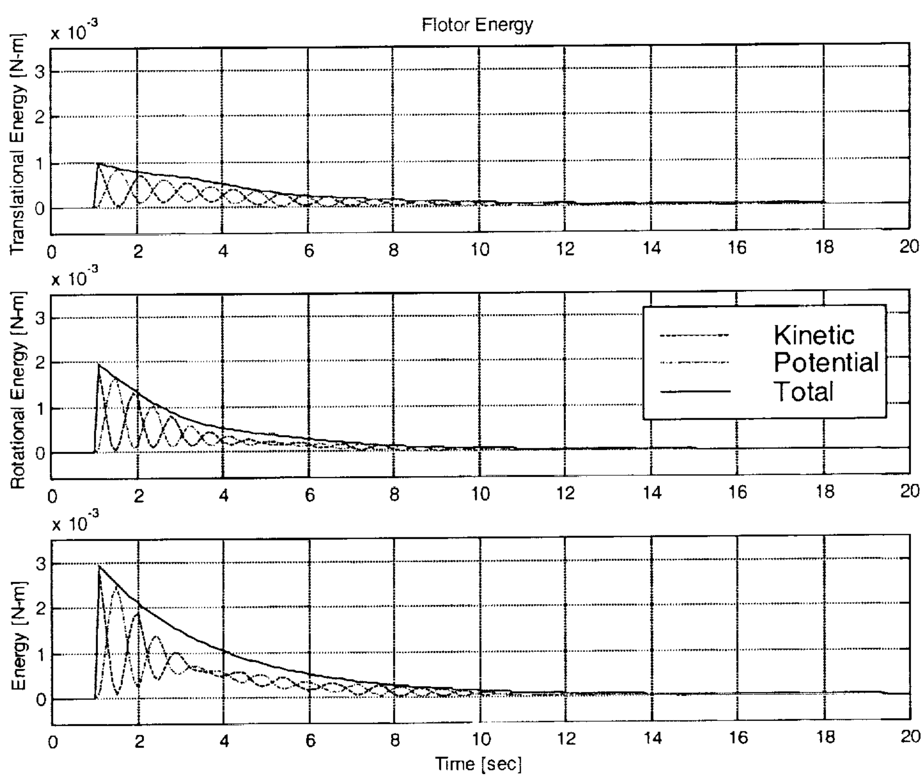

Figure 9. Flotor Energy for Scenario #1

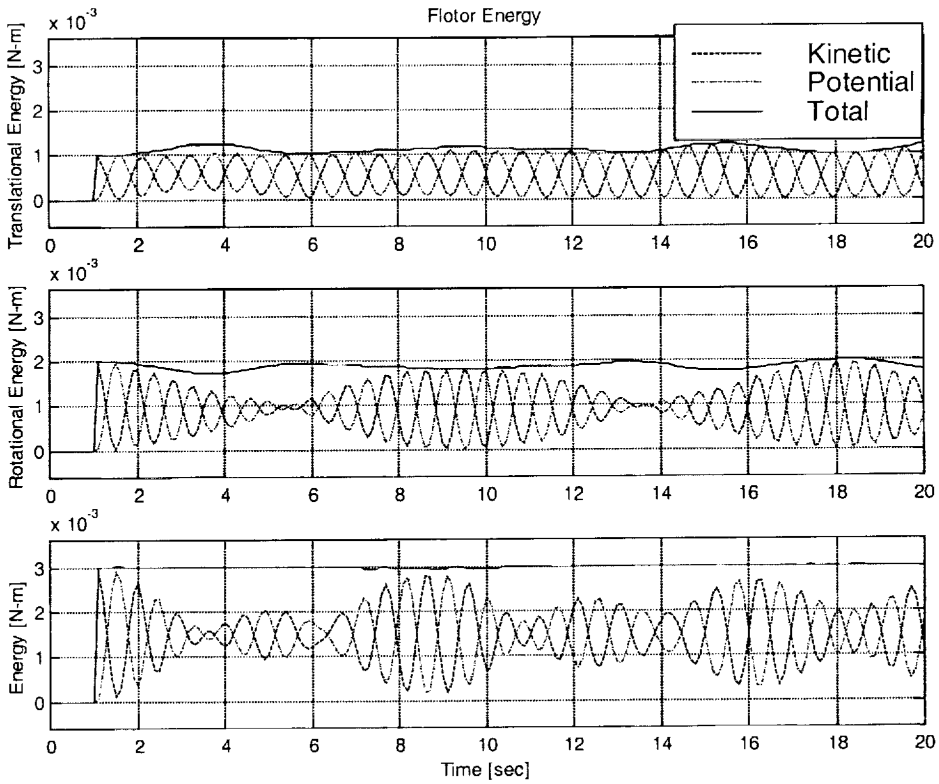

Figure 10. Flotor Energy for Scenario #1 with No Damping

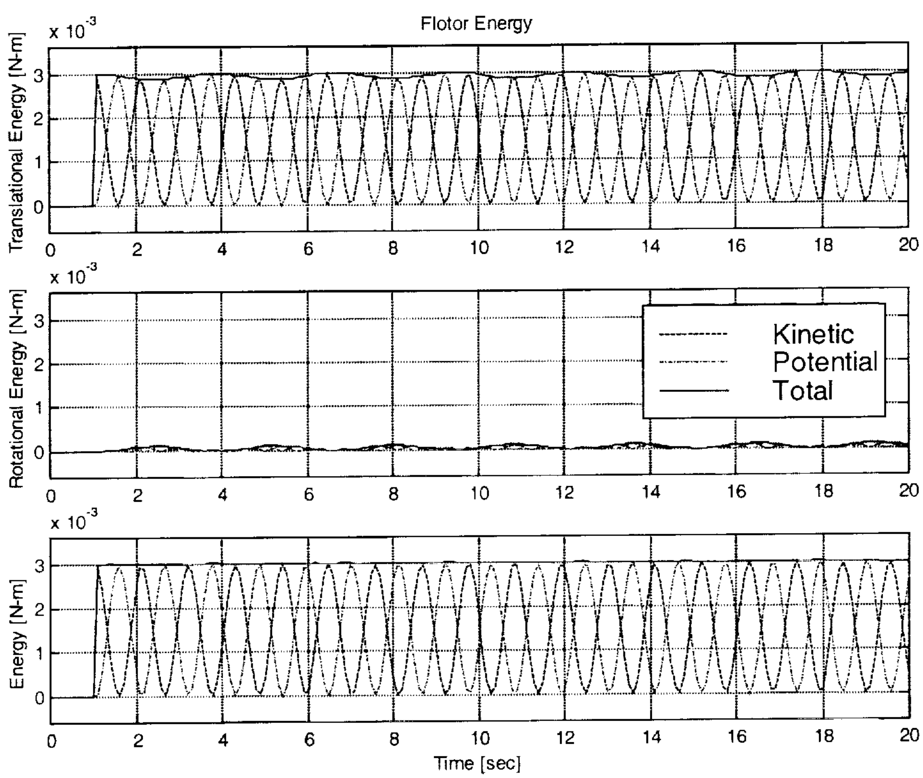

Figure 11. Flotor Energy for Scenario #2 with No Damping

Concluding Remarks

This paper has presented the derivation of algebraic, state-space equations for the Glovebox Integrated Microgravity Isolation Technology (g-LIMIT). The states employed include payload relative translational position $(\underline{x}_a)$ and velocity $(\underline{x}_b)$, payload relative rotation $(\underline{x}_d)$ and rotation rate $(\underline{x}_e)$, and payload translational acceleration $(\underline{x}_c)$. Feedback of $\underline{x}_a$ corresponds to a change in effective umbilical translational stiffness, where the effective umbilical is assumed to be attached at the flotor center of mass. Similarly, feedback of $\underline{x}_b$, $\underline{x}_d$, or $\underline{x}_e$ corresponds, respectively, to a change in effective umbilical translational damping, rotational stiffness, or rotational damping. Likewise, feedback of payload translational acceleration causes a change in effective payload mass. Thus, a cost functional which penalizes these states produces intuitive effects on system effective stiffness, damping, and inertia values.

The acceleration states can be selected to pertain to any arbitrary point E on the flotor. This allows an optimal controller to be developed which penalizes directly the acceleration of any significant point of interest, such as the location of a crystal in a crystal-growth experiment.

The equations have been put into state-space form so that the powerful controller-design methods of optimal control theory (e.g., H_2 synthesis, H_∞ synthesis, μ synthesis, mixed-μ synthesis, and μ analysis) can be used. References [7], [8], [13] and [14] detail the H_2 optimal controller design approach used for g-LIMIT, and Reference [15] describes the insights gained from a single-degree-of-freedom case study.

The linearized state-space equations have been verified against a nonlinear "truth" model in a simulation study. The state responses from the model were shown to agree very well with the "truth" model for a set of parameters representative of the g-LIMIT configuration. Additionally, it was shown that flotor kinetic and potential energy responses were as expected, demonstrating that the model conserves total energy for an arbitrarily chosen impulsive actuator loading.

Acknowledgments

The authors are grateful to Marshall Space Flight Center and the National Aeronautics and Space Administration for their partial funding of this work, through the Full-time Study Program.

References

[1]DeLombard, R., Bushnell, G. S., Edberg, D., Karchmer, A. M., and Tryggvason, B. V., "Microgravity Environment Countermeasures Panel Discussion," *AIAA 97-0351*, January 1997.

[2]"System Specification for the International Space Station," *Specification Number SSP41000*, Rev. D, Nov. 1, 1995, NASA Johnson Space Center.

[3]DelBasso, S., "The International Space Station Microgravity Environment," *AIAA-96-0402*, January 1996.

[4]Nelson, E. S., "An Examination of Anticipated g-Jitter on Space Station and Its Effects on Materials Processes," *NASA TM-103775*, April 1991.

[5]Hampton, R. D., *Controller Design for Microgravity Vibration Isolation Systems*, Ph.D. dissertation, University of Virginia, Charlottesville, Virginia, January 1993.

[6]Whorton, M. S., et al., "Design Definition Document (DDD) For Glovebox Integrated Microgravity Isolation Technology (g-LIMIT) Characterization Test," g-LIMIT-DOC-0001, NASA MSFC, September 30, 1999.

[7]Hampton, R. David and Whorton, Mark S., "Frequency-Weighting Filter Selection, for H_2 Control of Microgravity Isolation Systems: A Consideration of the 'Implicit Frequency

Weighting' Problem," IEEE 99-9184, *IEEE Transactions on Instrumentation and Measurement*, April 2000.

[8]Hampton, R. D., Whorton, M. S., "An Indirect Mixed-Sensitivity Approach To Microgravity Vibration Isolation: The Exploitation of Kinematic Coupling In Frequency-Weighting Design-Filter Selections" ACC00-AIAA0036, June 2000.

[9]Edberg, D., Boucher, R., Schenck, D., Nurre, G., Whorton, M., Kim, Y., and Alhorn, D., 1996, "Results of the STABLE Microgravity Vibration Isolation Flight Experiment," Guidance and Control, Vol. 92, pp. 567-581.

[10]Greenwood, D. T., *Principles of Dynamics,* Prentice-Hall Inc., Englewood Cliffs, New Jersey, 1988.

[11]Kane, T. R. and Levinson, D. A., *Dynamics: Theory and Applications*, McGraw-Hill, Inc., New York, 1985.

[12]Wertz, J. R., *Spacecraft Attitude Determination and Control,* Kluwer Academic Publishers, AH Dordrecht, The Netherlands, 1978.

[13]Hampton, R. D., Knospe, C. R., Allaire, P. E., and Grodsinsky, C. M., "Microgravity Isolation System Design: A Modern Control Synthesis Framework," *Journal of Spacecraft and Rockets*, Vol. 33, No. 1, January-February 1996, pp. 101-109.

[14]Hampton, R. D., Knospe, C. R., Allaire, P. E., and Grodsinsky, C. M., "Microgravity Isolation System Design: A Case Study," *Journal of Spacecraft and Rockets*, Vol. 33, No. 1, January-February 1996, pp. 120-125.

[15]Hampton, R. David, and Whorton, Mark S., "Design-Filter Selection for H2 Control of Microgravity Isolation Systems: A Single-Degree-of-Freedom Case Study," AIAA-2000-3955, August 2000. Submitted to the Journal of Guidance, Control, and Dynamics, September 2000.