ALTERNATE FORMS OF RELATIVE ATTITUDE KINEMATICS AND DYNAMICS EQUATIONS*

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ABSTRACT

In this paper the alternate forms of the relative attitude kinematics and relative dynamics equations are presented. These developments are different from the earlier developments that have been presented in other publications. The current forms of equations have the advantage of being simpler than earlier ones. These equations are applied in developing the necessary kinematics and dynamics for relative navigation in formation flying and virtual platforms. These equations also have application in the implementation of nonlinear full state feedback and nonlinear output feedback control for large attitude angle acquisition and tracking. This paper presents simulations from such a full state feedback control application.

INTRODUCTION

Navigation and control for spacecraft flying in formation requires the relative attitude information. Since the attitude rotation matrices and their kinematics relationships are generally defined in the inertial reference frame, Ref.1 was the first attempt at completely defining the kinematics and relative attitude dynamics when the attitude matrix is defined in a non-inertial frame. While the concept of relative attitude had earlier been addressed in Refs.2-4, Ref.1 was the first publication to provide the complete development that is necessary for formation flying control. This paper continues these developments, providing the relative attitude kinematics and dynamics that are alternative to the forms developed in Ref.1.

The attitude of a rigid-body with respect to the inertial frame is determined by a rotation transformation matrix from the inertial frame to body frame. This rotation matrix is referred to as the attitude matrix. In practical design, as noted in Ref.1, the attitude matrix is parameterized to be 4-dimension parameters such as axis/angle variables and quaternion, and 3-dimension parameters such as Euler angles, Rodrigues (Gibbs vector) and modified Rodrigues. Similarly, rotation matrix of the body-frame with respect to a non-inertial frame can be defined as the relative attitude matrix.

In developing the kinematics and dynamics of the relative attitude, Ref.1 addressed the following:

i. Kinematics equations for relative attitude matrix
ii. Kinematics equations for relative attitude parameters
iii. Relative attitude dynamics equations

This paper proceeds along the same line, providing alternate solutions to the above items. The primary difference in this approach results from a different definition of the relative angular velocity. This alternate form provides simpler relative kinematics and dynamics equations than those in Ref.1.

SYMBOLS

[dp1, dp2, dp3]	the three components of relative modified Rodrigues parameters
[dw1, dw2, dw3]	the three components of the relative angular velocity (rad/sec)
g	Rodrigues parameter
p	modified Rodrigues parameter

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q	quaternion parameter
R_{BD}	relative attitude matrix (transformation matrix) of reference frame $\mathscr{F}_B$ with respect to $\mathscr{F}_D$.
ω	angular rate
$d(.)/dt_B$	vector time derivative in $\mathscr{F}_B$..
$d(.)/dt_D$	vector derivative in $\mathscr{F}_D$
$\mathscr{F}_I$	vectrice of the inertial reference system
$\mathscr{F}_T$	vectrice of the local horizontal reference system for target satellite
$\mathscr{F}_S$	vectrice of the local horizontal reference system of the chasing spacecraft
$\mathscr{F}_B$	vectrice of the body-fixed reference system of the chasing spacecraft
$\mathscr{F}_D$	vectrice of the body-fixed reference system of the target attitude
⊗	nonlinear operator

REPRESENTATION OF RELATIVE ATTITUDE

Relative Attitude Matrix

It is assumed that any reference frame can be denoted by a vectrix[7]. The following series of reference frames can then be defined.

$$\mathscr{F}_I = \begin{bmatrix} \underline{i}_I \\ \underline{j}_I \\ \underline{k}_I \end{bmatrix} \quad \mathscr{F}_T = \begin{bmatrix} \underline{i}_T \\ \underline{j}_T \\ \underline{k}_T \end{bmatrix} \quad \mathscr{F}_S = \begin{bmatrix} \underline{i}_S \\ \underline{j}_S \\ \underline{k}_S \end{bmatrix} \quad \mathscr{F}_B = \begin{bmatrix} \underline{i}_B \\ \underline{j}_B \\ \underline{k}_B \end{bmatrix} \quad \mathscr{F}_D = \begin{bmatrix} \underline{i}_D \\ \underline{j}_D \\ \underline{k}_D \end{bmatrix} \tag{1}$$

The attitude of a rigid spacecraft is the orientation of the reference frame with respect to another frame. The most convenient reference frame is a dextral, orthogonal triad which is fixed with the rigid body of spacecraft. The other reference frame can be an inertial reference frame, or it can be a moveable reference frame which is fixed to another body. The attitude with respect to the inertial reference frame is the absolute attitude. The attitude with respect to a movable rotting reference frame is the relative attitude. which is respect to a movable rotating reference frame is named the relative attitude. For example in order to study the orientation relationship between two rotating reference frames, the relative attitude can be defined as a transformation matrix between two rotating reference frames.

$$\mathscr{F}_B = R_{BD}\mathscr{F}_D \tag{2}$$

Parametric Representation of the Relative Attitude Matrix

As with parametric representation of absolute attitude matrix (Refs.6-8), the relative attitude matrix can also be represented by the attitude parameters such as Axis/angle, Quaternion, Rodrigues parameters (Gibbs vector), modified Rodrigues parameters and Euler angles. The relationship between relative attitude matrix and absolute attitude matrix is

$$R_{BD}(p_{bd}) = R_{BI}(p_b)\, R_{DI}^{-1}(p_d) \tag{3}$$

It is obvious that the relative attitude parameters g_{bd} and absolute attitude parameters g_b , g_d have a nonlinear relationship. This can be represented by the following unified notation:

$$p_{bd} = p_b \otimes p_d^{-1} \tag{4}$$

where the rule of the nonlinear operator ⊗ is determined by composition rotation rule of the attitude parameters [6]. This operation will, therefore, be different for different attitude parameters.

For the modified Rodrigues parameters:

$$p_{bd} = p_b \otimes p_d^{-1} = \frac{p_d(p_b^T p_b - 1) + p_b(1 - p_d^T p_d) - 2[p_d^\times]p_b}{1 + p_d^T p_d p_b^T p_b + 2 p_d^T p_b} \tag{5}$$

For Rodigues parameters (Gibbs vector):

$$g_{bd} = g_b \otimes g_d^{-1} = \frac{g_b - g_d + [g_b^\times] g_d}{1 + g_b^T g_d} \tag{6}$$

For Quaternion parameters:

$$\bar{q}_{bd} = \bar{q}_b \otimes \bar{q}_d^{-1} = \begin{bmatrix} q_{d4} q_b - q_{b4} q_d + [q_b^\times] q_d \\ q_{b4} q_{d4} + q_b^T q_d \end{bmatrix} \tag{7}$$

where

$$\bar{q}_{bd} = \begin{bmatrix} q_{bd} \\ q_{bd4} \end{bmatrix} \quad \bar{q}_b = \begin{bmatrix} q_b \\ q_{b4} \end{bmatrix} \quad \bar{q}_d = \begin{bmatrix} q_d \\ q_{d4} \end{bmatrix} \quad a = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \quad [a^\times] = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \tag{8}$$

RELATIVE ATTITUDE KINEMATICS FOR ATTITUDE MATRIX

Equation (2) can be written as

$$\mathscr{F}_D^T = \mathscr{F}_B^T R_{BD} \tag{9}$$

Taking derivative with respect to the inertial reference frame for both sides,

$$\underline{\omega}_d \times \mathscr{F}_D^T = \frac{d(\mathscr{F}_B^T R_{BD})}{dt_B} + \underline{\omega}_b \times \mathscr{F}_B^T R_{BD} \tag{10}$$

Based on the differential rules for the vectrices (Ref.7), Eq.(10) can be developed as

$$\omega_d^T R_{DB} \mathscr{F}_B \times \mathscr{F}_B^T R_{BD} = \mathscr{F}_B^T \frac{dR_{BD}}{dt_B} + \omega_b^T \mathscr{F}_B \times \mathscr{F}_B^T R_{BD} \qquad \textbf{(11)}$$

In the body reference frame $\mathscr{F}_B$, the kinematics equation for relative attitude matrix R_{BD} will be

$$\frac{dR_{BD}}{dt_B} = [(R_{BD}\omega_d)^\times]R_{BD} - [\omega_b^\times]R_{BD} = [(R_{BD}\omega_d - \omega_b)^\times]R_{BD} \qquad \textbf{(12)}$$

If relative angular velocity is defined as

$$\underline{\omega}_{bd} = \underline{\omega}_b - \underline{\omega}_d = \mathscr{F}_B^T(\omega_b - R_{BD}\omega_d) \qquad \omega_{bd}^b = (\omega_b - R_{BD}\omega_d) \qquad \textbf{(13)}$$

the attitude kinematics equation for the relative attitude matrix (12) can be written the compact form in the body reference frame $\mathscr{F}_B$

$$\frac{dR_{BD}}{dt_B} = -[(\omega_{bd}^b)^\times]R_{BD} \qquad \textbf{(14)}$$

Considering

$$R_{BD}R_{DB} = I \qquad \textbf{(15)}$$

the relative kinematics equation for R_{DB} in Body reference frame $\mathscr{F}_B$ becomes

$$\frac{dR_{DB}}{dt_B} = R_{DB}[(\omega_{bd}^b)^\times] \qquad \textbf{(16)}$$

In the similar way, the kinematics equation for R_{DB} in reference frame $\mathscr{F}_D$ can be obtained:

$$\frac{dR_{DB}}{dt_D} = -[(\omega_{db}^d)^\times]R_{DB} \qquad \frac{dR_{BD}}{dt_D} = R_{BD}[(\omega_{db}^d)^\times] \qquad \textbf{(17)}$$

where

$$\omega_{db}^d = \omega_d - R_{DB}\omega_b \qquad \textbf{(18)}$$

Since

$$\frac{dR_{BD}}{dt_B}=\frac{dR_{BD}}{dt_D}=\frac{dR_{BD}}{dt}=\dot{R}_{BD} \qquad \frac{dR_{DB}}{dt_B}=\frac{dR_{DB}}{dt_D}=\frac{dR_{DB}}{dt}=\dot{R}_{DB} \tag{19}$$

these kinematics equations for relative attitude above can be represented in form of a theorem of relative attitude kinematics:

Theorem of Relative Attitude Kinematics

The angular velocities of $\mathcal{F}_B$, $\mathcal{F}_D$ with respect to the inertial reference frame are are ω_b ,ω_d , and

$$\underline{\omega}_b = \mathcal{F}_B^T\omega_b \qquad \underline{\omega}_d = \mathcal{F}_D^T\omega_d \tag{20}$$

The relative attitude between the reference frames $\mathcal{F}_I$, $\mathcal{F}_B$, $\mathcal{F}_D$ are:

$$\mathcal{F}_B = R_{BI}\mathcal{F}_I \qquad \mathcal{F}_D = R_{DI}\mathcal{F}_I \qquad \mathcal{F}_B = R_{BD}\mathcal{F}_D \tag{21}$$

The kinematics equations for R_{BD} and R_{DB} are:

$$\dot{R}_{BD} = -[(\omega_{bd}^b)^\times]R_{BD}=[(\omega_{db}^b)^\times]R_{BD}=R_{BD}[(\omega_{db}^d)^\times]=-R_{BD}[(\omega_{bd}^d)^\times] \tag{22}$$

$$\dot{R}_{DB} = -R_{DB}[(\omega_{db}^b)^\times]=R_{DB}[(\omega_{bd}^b)^\times]=R_{DB}[(\omega_{bd}^d)^\times]=-R_{DB}[(\omega_{db}^d)^\times] \tag{23}$$

where

$$\omega_{bd}^b=(\omega_b-R_{BD}\omega_d)=-\omega_{db}^b=-R_{BD}\omega_{db}^d=R_{BD}\omega_{bd}^d \tag{24}$$

RELATIVE ATTITUDE KINEMATICS FOR ATTITUDE PARAMETERS

The First Kind of Relative Attitude Kinematics Equations

Based on the definitions of the relative attitude and relative angular velocity given above the relative attitude kinematics equations for various attitude representation have been developed in Ref.1. These equations are referred to as the first kind of relative attitude kinematics equations, and are summarized as follows:

Quaternion

Relative kinematics equation

$$\dot{q}_{bd} = \frac{1}{2}([q_{bd}^\times]+q_{bd4}I)\omega_{bd}^b \qquad \dot{q}_{bd4} = -\frac{1}{2}(\omega_{bd}^b)^T q_{bd} \tag{25}$$

Relative velocity

$$\omega_{bd}^{b} = 2(q_{bd4}\dot{q}_{bd} - \dot{q}_{bd4}q_{bd} - q_{bd} \times \dot{q}_{bd}) \quad \textbf{(26)}$$

Attitude transformation matrix

$$R_{BD} = (q_{bd4}^{2} - q_{bd}^{T}q_{bd})I + 2q_{bd}q_{bd}^{T} - 2q_{bd4}[q_{bd}\times] \quad \textbf{(27)}$$

<u>Rodrigues Parameters</u>

Relative attitude kinematics equations

$$\dot{g}_{bd} = \frac{1}{2}([g_{bd}^{\times}] + g_{bd}g_{bd}^{T} + I)\omega_{bd}^{b} \quad \textbf{(28)}$$

Relative velocity

$$\omega_{bd}^{b} = \frac{2}{(1 + g_{bd}^{T}g_{bd})}(I - [g_{bd}^{\times}])\dot{g}_{bd} \quad \textbf{(29)}$$

Attitude transformation matrix

$$R_{BD} = \frac{1}{(1 + g_{bd}{}^{T}g_{bd})}((1 - g_{bd}{}^{T}g_{bd})I + 2g_{bd}g_{bd}{}^{T} - 2[g_{bd}\times]) \quad \textbf{(30)}$$

<u>Modified Rodrigues Parameter</u>

Relative attitude kinematics equation

$$\dot{p}_{bd} = \frac{1}{4}[(1 - p_{bd}^{T}p_{bd})I + 2[p_{bd}^{\times}] + 2p_{bd}p_{bd}^{T}]\omega_{bd}^{b} \quad \textbf{(31)}$$

Relative velocity

$$\omega_{bd}^{b} = \frac{4}{(1 + p_{bd}^{T}p_{bd})^{2}}[(1 - p_{bd}^{T}p_{bd}) + 2p_{bd}p_{bd}^{T} - 2[p_{bd}^{\times}]]\ \dot{p}_{bd} \quad \textbf{(32)}$$

Attitude transformation matrix

$$R_{BD} = \frac{1}{(1 + p_{bd}^{T}p_{bd})^{2}}[(1 - 6p_{bd}^{T}p_{bd} + (p_{bd}^{T}p_{bd})^{2})I + 8p_{bd}p_{bd}^{T} - 4(1 - p_{bd}^{T}p_{bd})[p_{bd}^{\times}]] \quad \textbf{(33)}$$

The Second Kind of Relative Attitude Kinematics Equations

The relative kinematics equations are dictated by the definition of the relative angular velocity. Using the following (alternate) definition of angular velocity

$$\Delta\omega = \omega_b - \omega_d$$

an alternate set of relative attitude kinematics are developed. These may be defined as the alternate (second) form of relative attitude kinematics equations. It should be noted that in this case, the subscript of relative attitude are replaced by the use of Δ.

<u>Quaternion</u>

Using the new notation, the first kind of relative kinematics equation in Quaternion can be written as

$$\frac{d\Delta\bar{q}}{dt} = M\omega_{bd}^{b} \tag{35}$$

where

$$M = \frac{1}{2}\begin{bmatrix} [\Delta q^{\times}] + \Delta q_4 I \\ -\Delta q^T \end{bmatrix} \tag{36}$$

Since

$$\omega_{bd} = (\omega_b - R_{BD}\omega_d) = \Delta\omega + (I - R_{BD})\omega_d \tag{37}$$

the relative attitude kinematics Eq.(35) can be written as

$$\frac{d\Delta\bar{q}}{dt} = M\Delta\omega + M(I - R_{BD})\omega_d \tag{38}$$

where

$$R_{BD} = (\Delta q_4^2 - \Delta q^T\Delta q)I + 2\Delta q\Delta q^T - 2\Delta q_4[\Delta q\times] \tag{39}$$

Because

$$M(I - R_{BD}) = \frac{1}{2}\begin{bmatrix} [\Delta q^{\times}] + \Delta q_4 I \\ -\Delta q^T \end{bmatrix}[(1 - \Delta q_4^2 + \Delta q^T\Delta q)I - 2\Delta q\Delta q^T + 2\Delta q_4[\Delta q\times]] \tag{40}$$

Considering the following equality relationships for any 3 dimensional vector p:

$$[p^{\times}][p^{\times}]+p^{T}pI-pp^{T}=0_{3\times 3}\ , \qquad p^{T}ppp^{T}=pp^{T}pp^{T} \tag{41}$$

$$pp^{T}[p^{\times}]=0_{3\times 3}, \qquad [p^{\times}]pp^{T}=0_{3\times 3} \tag{42}$$

Eq.(40) can be simplified as follows:

$$M(I-R_{BD}) = \begin{bmatrix} [\Delta q^{\times}] \\ 0_{1\times 3} \end{bmatrix} \tag{43}$$

Finally Eq.(38) can be simplified to obtain the second kind of relative attitude kinematics equations in Quaternion :

$$\frac{d\Delta\bar{q}}{dt} = \frac{1}{2}\begin{bmatrix} [\Delta q^{\times}]+\Delta q_4 I \\ -\Delta q^{T} \end{bmatrix}\Delta\omega + \begin{bmatrix} [\Delta q^{\times}] \\ 0_{1\times 3} \end{bmatrix}\omega_d \tag{44}$$

Relative angular velocity:

$$\Delta\omega = 2(\Delta q_4\Delta\dot{q}-\Delta\dot{q}_4\Delta q-\Delta q\times\Delta\dot{q})+(R_{BD}(\Delta\bar{q})-I)\omega_d \tag{45}$$

Attitude transformation matrix:

$$R_{BD}(\Delta\bar{q}) = (\Delta q_4^2-\Delta q^{T}\Delta q)I+2\Delta q\Delta q^{T}-2\Delta q_4[\Delta q\times] \tag{46}$$

Rodrigues Parameters

Using new notation, the relative attitude kinematics equations can be written as

$$\frac{d\Delta g}{dt} = M\Delta\omega+M(I-R_{BD})\omega_d \tag{47}$$

where

$$R_{BD} = \frac{1}{(1+\Delta g^T \Delta g)}((1-\Delta g^T \Delta g)I + 2\Delta g \Delta g^T - 2[\Delta g^\times]) \quad \textbf{(48)}$$

Applying equality relationship Eq.(41-42) simplifies the equation as

$$\begin{aligned} M(I-R_{BD}) &= \frac{1}{2}([\Delta g^\times] + \Delta g \Delta g^T + I)(I - \frac{1}{(1+\Delta g^T \Delta g)}((1-\Delta g^T \Delta g)I + 2\Delta g \Delta g^T - 2[\Delta g^\times])) \\ &= [\Delta g^\times] \end{aligned} \quad \textbf{(49)}$$

Substituting Eq.(49) into Eq.(47), the second kind of relative attitude kinematics equation in Rodrigues parameters are obtained:

$$\frac{d\Delta g}{dt} = \frac{1}{2}([\Delta g^\times] + \Delta g \Delta g^T + I)\Delta\omega + [\Delta g^\times]\omega_d \quad \textbf{(50)}$$

Relative velocity:

$$\Delta\omega = \frac{2}{(1+\Delta g^T \Delta g)}(I - [\Delta g^\times])\Delta\dot{g} + (R_{BD}(\Delta g) - I)\omega_d \quad \textbf{(51)}$$

Attitude transformation matrix:

$$R_{BD}(\Delta g) = \frac{1}{(1+\Delta g^T \Delta g)}((1-\Delta g^T \Delta g)I + 2\Delta g \Delta g^T - 2[\Delta g\times]) \quad \textbf{(52)}$$

<u>Modified Rodrigues Parameter</u>

In a similar way, the second kind of relative attitude kinematics equations in modified Rodrigues parameters can be written as

$$\frac{d\Delta p}{dt} = \frac{1}{4}[(1-\Delta p^T \Delta p)I + 2[\Delta p^\times] + 2\Delta p \Delta p^T]\Delta\omega + [\Delta p^\times]\omega_d \quad \textbf{(53)}$$

Relative velocity:

$$\Delta\omega = \frac{4}{(1+\Delta p^T \Delta p)^2}[(1-\Delta p^T \Delta p) + 2\Delta p \Delta p^T - 2[\Delta p^\times]]\ \Delta\dot{p} + [R_{BD}(\Delta p) - I]\omega_d \quad \textbf{(54)}$$

Attitude transformation matrix:

$$R_{BD}(\Delta p) = \frac{1}{(1+\Delta p^{T}\Delta p)^{2}}[(1-6\Delta p^{T}\Delta p+(\Delta p^{T}\Delta p)^{2})I + 8\Delta p\Delta p^{T}-4(1-\Delta p^{T}\Delta p)[\Delta p^{\times}]] \quad \textbf{(55)}$$

RELATIVE ATTITUDE DYNAMICS EQUATIONS

The First Kind of Relative Attitude Dynamics Equations

It is assumed that $\underline{J}_b(t)$, $\underline{\omega}_b(t)$, $\underline{h}_b(t)$ and $\underline{L}_b$ are the inertial dyadics, angular velocity, angular momentum and the applied angular moment of the chase satellite. Similarly, $\underline{J}_d(t)$, $\underline{\omega}_d(t)$, $\underline{h}_d(t)$ and $\underline{L}_d$ are the corresponding items for the target satellite. The relative angular velocity of the chase satellite body reference frame $\mathscr{F}_B$ with respect to the target satellite reference frame $\mathscr{F}_D$ is defined as follows:

$$\underline{\omega}_{bd}=\underline{\omega}_b-\underline{\omega}_d = \mathscr{F}_B^T\omega_{bd}^b \qquad \omega_{db}^b=(\omega_d-R_{DB}\omega_b) \quad \textbf{(56)}$$

The relative attitude dynamics equation in pursuer satellite body reference frame $\mathscr{F}_B$ developed in Ref.1 is

$$\begin{aligned} &J_b\dot{\omega}_{bd}^b+[(\omega_{bd}^b)^{\times}]J_b\omega_{bd}^b+[(R_{BD}\omega_d)^{\times}]J_b\omega_{bd}^b+[(\omega_{bd}^b)^{\times}]J_bR_{BD}\omega_d-J_b[(\omega_{bd}^b)^{\times}]R_{BD}\omega_d \\ &= L_b-R_{BD}L_d-R_{BD}(\Delta J_d\dot{\omega}_d+[\omega_d^{\times}]\Delta J_d\omega_d) \end{aligned} \quad \textbf{(57)}$$

where

$$J_d\dot{\omega}_d + [\omega_d^{\times}]J_d\omega_d = L_d \qquad \Delta J_d = R_{DB}J_bR_{BD}-J_d \quad \textbf{(58)}$$

The Second Kind of Relative Attitude Dynamics Equations

Using the alternate definition of relative angular velocity:

$$\Delta\omega=\omega_b-\omega_d \quad \textbf{(59)}$$

The relationship between ω_{bd} and $\Delta\omega$ (the two definitions of relative angular velocity) is

$$\omega_{bd}^b=\Delta\omega+(I-R_{Bd})\omega_d \quad \textbf{(60)}$$

Considering Eq.(59) and (60), the left hand terms of Eq.(57) can be developed as:

$$J_b\dot{\omega}_{bd}^b=J_b\Delta\dot{\omega}+J_b(I-R_{BD})\dot{\omega}_d+J_b[\Delta\omega^{\times}]R_{BD}\omega_d+J_b[\omega_d^{\times}]R_{BD}\omega_d \quad \textbf{(61)}$$

$$[(\omega_{bd}^{b})^{\times}]J_b\omega_{bd}=[\Delta\omega^{\times}]J_b\Delta\omega+[\omega_d^{\times}]J_b\Delta\omega-[(R_{BD}\omega_d)^{\times}]J_b\Delta\omega+[\Delta\omega^{\times}]J_b\omega_d-[\Delta\omega^{\times}]J_bR_{BD}\omega_d$$
$$+[\omega_d^{\times}]J_b\omega_d-[\omega_d^{\times}]J_bR_{BD}\omega_d-[(R_{BD}\omega_d)^{\times}]J_b\omega_d+[(R_{BD}\omega_d)^{\times}]J_bR_{BD}\omega_d \quad \textbf{(62)}$$

$$[(\omega_{bd}^{b})^{\times}]J_bR_{BD}\omega_d=[\Delta\omega^{\times}]J_bR_{BD}\omega_d+[\omega_d^{\times}]J_bR_{BD}\omega_d-[(R_{BD}\omega_d)^{\times}]J_bR_{BD}\omega_d \quad \textbf{(63)}$$

$$[(R_{BD}\omega_d)^{\times}]J_b\omega_{bd}^{b}=[(R_{BD}\omega_d)^{\times}]J_b\Delta\omega+[(R_{BD}\omega_d)^{\times}]J_b\omega_d-[(R_{BD}\omega_d)^{\times}]J_bR_{BD}\omega_d \quad \textbf{(64)}$$

$$J_b[(\omega_{bd}^{b})^{\times}]R_{BD}\omega_d=J_b[\Delta\omega^{\times}]R_{BD}\omega_d+J_b[\omega_d^{\times}]R_{BD}\omega_d-J_b[(R_{BD}\omega_d)^{\times}]R_{BD}\omega_d \quad \textbf{(65)}$$

Substituting Eqs. (61-65) into Eq.(57) and re-arranging

$$J_b\Delta\dot{\omega}+[\Delta\omega^{\times}]J_b\Delta\omega+[\Delta\omega^{\times}]J_b\omega_d=J_b(R_{BD}-I)\dot{\omega}_d-[\omega_d^{\times}]J_b\Delta\omega-[\omega_d^{\times}]J_b\omega_d$$
$$+\ [(R_{BD}\omega_d)^{\times}]J_bR_{BD}\omega_d+L_b-R_{BD}L_d-R_{BD}(\Delta J_d\dot{\omega}_d+[\omega_d^{\times}]\Delta J_d\omega_d) \quad \textbf{(66)}$$

where

$$J_d\dot{\omega}_d\ +\ [\omega_d^{\times}]J_d\omega_d\ =\ L_d \quad \textbf{(67)}$$

These are the second kind of relative attitude dynamics equation. The two equations may be summarized in terms of a relative attitude dynamics theorem:

Theorem of Relative Attitude Dynamics

If the relative angular velocity is defined as

$$\omega_{bd}^{b}=\omega_b-R_{BD}\omega_d \quad \textbf{(68)}$$

then the first kind of relative attitude dynamics equation in the chase satellite body reference frame $\mathscr{F}_B$ is

$$J_b\dot{\omega}_{bd}^{b}+[(\omega_{bd}^{b})^{\times}]J_b\omega_{bd}^{b}+[(\omega_{bd}^{b})^{\times}]J_bR_{BD}\omega_d$$
$$=J_b[(\omega_{bd}^{b})^{\times}]R_{BD}\omega_d-[(R_{BD}\omega_d)^{\times}]J_b\omega_{bd}^{b}+L_b-R_{BD}L_d-R_{BD}(\Delta J_d\dot{\omega}_d+[\omega_d^{\times}]\Delta J_d\omega_d) \quad \textbf{(69)}$$

If the relative angular velocity is defined as

$$\Delta\omega = \omega_b - \omega_d \tag{70}$$

then the second kind of relative attitude dynamics equation in the $\mathscr{F}_B$ frame is

$$\begin{aligned} J_b\Delta\dot{\omega} + [\Delta\omega^\times]J_b\Delta\omega + [\Delta\omega^\times]J_b\omega_d = J_b(R_{BD} - I)\dot{\omega}_d - [\omega_d^\times]J_b\Delta\omega - [\omega_d^\times]J_b\omega_d \\ + [(R_{BD}\omega_d)^\times]J_b R_{BD}\omega_d + L_b - R_{BD}L_d - R_{BD}(\Delta J_d\dot{\omega}_d + [\omega_d^\times]\Delta J_d\omega_d) \end{aligned} \tag{71}$$

where

$$\Delta J_d = R_{DB}J_b R_{BD} - J_d \tag{72}$$

$$J_d\dot{\omega}_d + [\omega_d^\times]J_d\omega_d = L_d \tag{73}$$

APPLICATION TO LARGE ATTITUDE ANGLE ACQUISITION AND TRACKING CONTROL

An application of relative attitude kinematics and dynamics equations developed in this paper is the large angle acquisition and tracking control. This section provides the simulations of such an application. The advantage in using relative attitude is that the tracking control problem is converted into a regulator problem, simplifying the control system design.

It is assumed that the second kind of relative kinematics and dynamics equations in modified Rodrigues parameters are used for the acquisition and tracking control. The equations are as follows:

Relative kinematics equations:

$$\frac{d\Delta p}{dt} = M\Delta\omega + [\Delta p^\times]\omega_d \tag{74}$$

where

$$M = \frac{1}{4}[(1 - \Delta p^T\Delta p)I + 2[\Delta p^\times] + 2\Delta p\Delta p^T] \tag{75}$$

The relative dynamics equations:

$$J\Delta\dot{\omega} + \Delta\omega \times J\omega_d + \Delta\omega \times J\Delta\omega = L \tag{76}$$

where

$$L = J_b(R_{BD} - I)\dot{\omega}_d - [\omega_d^{\times}]J_b\Delta\omega - [\omega_d^{\times}]J_b\omega_d$$
$$+ [(R_{BD}\omega_d)^{\times}]J_b R_{BD}\omega_d + L_b - R_{BD}L_d - R_{BD}(\Delta J_d\dot{\omega}_d + [\omega_d^{\times}]\Delta J_d\omega_d) \tag{77}$$

The Lyapunov function V can be selected to be

$$V = K_p\Delta p^{T}\Delta p + \frac{1}{2}\Delta\omega^{T}J\Delta\omega \tag{78}$$

where K_p is a positive constant. The time-derivative of Lyapunov function V is

$$\dot{V} = 2K_p\Delta p^{T}\Delta\dot{p} + \Delta\omega^{T}J\Delta\dot{\omega} = \Delta\omega^{T}(2K_pM^{T}\Delta p + L) \tag{79}$$

If

$$L = -2K_pM^{T}\Delta p - K_dJ\Delta\omega \tag{80}$$

then

$$\dot{V} = -K_d\Delta\omega^{T}J\Delta\omega \leq 0 \tag{81}$$

where K_d is a positive constant. The closed loop system dynamics equation is

$$J\Delta\dot{\omega} + \Delta\omega\times J\omega_d + \Delta\omega\times J\Delta\omega = -K_dJ\Delta\omega - 2K_pM^{T}\Delta p \tag{82}$$

Equation (81) implies that $V(t) \leq V(0)$, and therefore, that Δp and $\Delta\omega$ are bounded. In addition, from Eq.(81)

$$\ddot{V} = -2K_d\Delta\omega J\Delta\dot{\omega} \tag{83}$$

so it can be seen that d^2V/dt^2 is bounded. Hence dV/dt is uniformly continuous [9]. Application of Barbalat's lemma [9] then indicates that $\Delta\omega \to 0$ as $t \to \infty$. Considering the closed loop equation (82), it can be obtained that $\Delta p \to 0$ as $t \to \infty$. Therefore, we have that $(\Delta p, \Delta\omega) \to (0,0)$ as $t \to \infty$. It means that the nonlinear control law given by Eq.(84)

$$L_b = R_{BD}L_d - J_b(R_{BD} - I)\dot{\omega}_d + [\omega_d^{\times}]J_b\Delta\omega + [\omega_d^{\times}]J_b\omega_d$$
$$-[(R_{BD}\omega_d)^{\times}]J_bR_{BD}\omega_d + R_{BD}(\Delta J_d\dot{\omega}_d + [\omega_d^{\times}]\Delta J_d\omega_d) - 2K_pM^{T}\Delta p - K_dJ\Delta\omega \tag{84}$$

is a global asymptotically stable control law for the system given by Eqs.(74-77).

SIMULATIONS

The Lyapunov nonlinear attitude control law has been used for large attitude angle acquisition and tracking control for the EO-1/LandSat 7 formation. For this simulation the control is implemented using full state feedback only. The simulations using measurement output feedback will be presented in future papers.

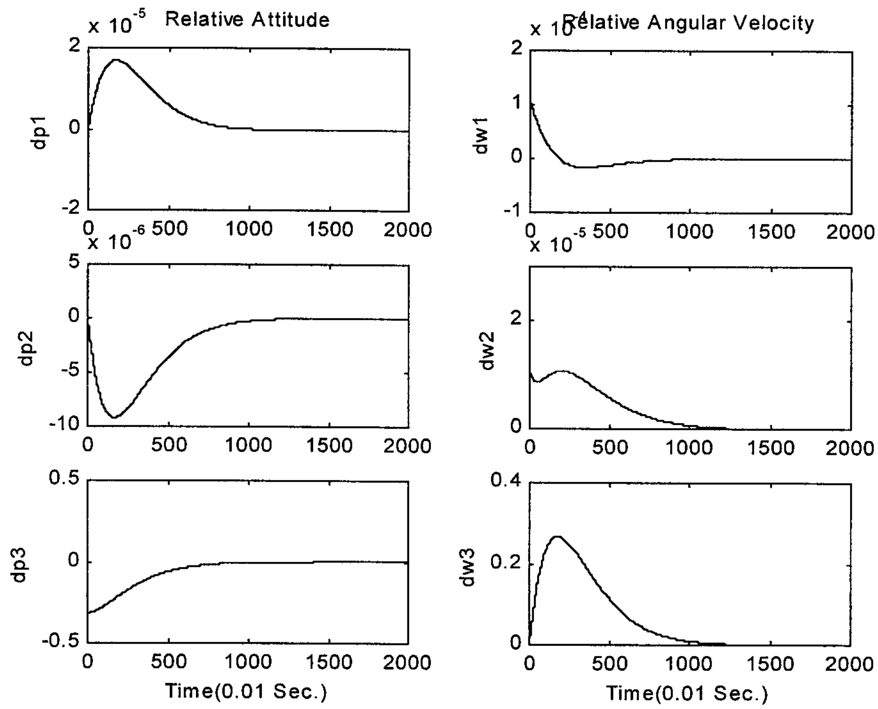

Figure 1. Large Attitude Angle Acquisition and Tracking Control for EO-1/LandSat7 Formation using Full State Feedback

The initial conditions for this full state feedback simulations are as follows:

Maneuver angle = 70 deg.
Target satellite (Landsat 7) angular velocity = $(0\ 0\ \omega_T)^T$
Pursuer satellite (EO-1) angular velocity = $(0.1\omega_T\ \ 0.01\omega_T\ \ 2\omega_T)^T$
ω_T =angular velocity of the 705 km circular orbit

[dp1, dp2, dp3] in Fig 1 are the three components of the relative attitude represented in the relative modified Rodrigues parameters, and [dw1, dw2, dw3] are the three components of the relative angular velocity (rad/sec). In this simulation, it is assumed that the attitude state vector is given and there no attitude measurement information available for feedback control.

CONCLUSIONS

In this paper the alternate relative attitude kinematics and dynamics equations are developed for the various attitude parametric representations. Compared with the first kind of relative attitude kinematics and dynamics equations (Ref.1), this has the advantage of being simpler. These developments will find ready application in the problems of relative attitude determination and control, and will be very useful for spacecraft formation control and relative navigation. As an example of such application, the Lyapunov nonlinear control law for large attitude angle acquisition and tracking has been developed and simulated for the EO-1/LandSat 7 formation. This simulation implemented the full state feedback control.

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