Statistical ensemble of large eddy simulations

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A statistical ensemble of large eddy simulations is run simultaneously for the same flow. The information provided by the different large-scale velocity fields is used in an ensemble-averaged version of the dynamic model. This produces local model parameters that only depend on the statistical properties of the flow. An important property of the ensemble-averaged dynamic procedure is that it does not require any spatial averaging and can thus be used in fully inhomogeneous flows. Also, the ensemble of LES's provides statistics of the large-scale velocity that can be used for building new models for the subgrid-scale stress tensor. The ensemble-averaged dynamic procedure has been implemented with various models for three flows: decaying isotropic turbulence, forced isotropic turbulence, and the time-developping plane wake. It is found that the results are almost independent of the number of LES's in the statistical ensemble provided that the ensemble contains at least 16 realisations.

1. Introduction

The number of degrees of freedom needed to characterize a velocity field u_i that corresponds to a turbulent flow is known to increase as $Re^{9/4}$ (Re is the Reynolds number) in three dimensional turbulent systems. Direct numerical simulations (DNS) of the Navier-Stokes equations governing the evolution of such systems are thus limited to moderately small Reynolds numbers. There is thus an interest in developping techniques in which only a fraction of the total number of degrees of freedom is actually simulated. Among these techniques, Large Eddy Simulation (LES) and Reynolds Averaged Navier-Stokes (RANS) simulation have attracted much interest in the past few decades. In LES, the number of degrees of freedom is reduced by using a spatial filtering:

$$\overline{u}_i(\mathbf{x}) = \int d\mathbf{y} G(\mathbf{x}-\mathbf{y}) u_i(\mathbf{y}), \tag{1.1}$$

where G is the filter kernel and $\overline{u}_i$ is the LES field. In RANS, an ensemble averaging is used to define the RANS field U_i:

$$U_i = \langle u_i \rangle, \tag{1.2}$$

In both cases, the equations for $\overline{u}_i$ or for U_i contain an unknown stress term that requires modelling. The purpose of the approach developed here is to combine concepts from the two methods to produce a statistical version of LES.

The present approach is motivated by the fact that, in both LES and RANS, models for the degrees of freedom that have been eliminated are inspired from statistical theories of turbulence. It is thus implicitly assumed that the filtering and ensemble averaging can both be regarded as projective operations that associate a number of different velocity realisations with a single LES or RANS field. There is, however, an important difference between ensemble averaging and spatial filtering. The ensemble averaging operation reduces the number of degrees of freedom by so much that almost no useful information on the fluctuations $\delta u_i \equiv u_i - U_i$ can be deduced from the knowledge of U_i alone. On the other hand, in LES the statistics of the unresolved scales $u_i' = u_i - \overline{u}_i$ must be closely related to the statistics of the resolved scales $\overline{u}_i$ since there is no clear scale separation between them. Hence, knowledge of the statistical properties of the LES fields $\overline{u}_i$ should be helpful in developing LES models. The advantage of studying a statistical ensemble of LES's is the ability to extract statistical information for building models for the unresolved scales. This will be explored in Section 3.

The definition of equivalent and independent LES fields is not necessarily obvious and should probably depend on the motivation for the simulation. We propose in Section 2 some conditions under which two simulations of a turbulent flow will be supposed to be independent and equivalent. In Section 3, we will show that the knowledge of an ensemble of LES's yields a good framework for developing a local version of the dynamic procedure in which model parameters are computed using statistical quantities. The application of this approach to isotropic turbulence is presented in Section 4. Application to the wake flow is presented in Section 5. In this last case, it is shown that the knowledge of an ensemble of realisations can be used to develop new models that explicitly incorporate averaged quantities made available through the ensemble.

2. Statistical ensemble of LES's

The equation for large eddy simulation (LES) is obtained by applying a spatial filter to the Navier–Stokes equations. The LES equation thus describes the evolution of a filtered velocity field $\overline{u}_i$, which explicitly depends on the small scales through the subgrid scale stress $\tau_{ij} = \overline{u_i u_j} - \overline{u}_i\,\overline{u}_j$:

$$\partial_t \overline{u}_i + \partial_j \overline{u}_j \overline{u}_i = -\partial_i \overline{p} + \nu_0 \nabla^2 \overline{u}_i - \partial_j \tau_{ij} \,. \tag{2.1}$$

For simplicity, we only consider incompressible flows, in which p, the pressure divided by the density, is determined by the incompressibility condition. The unknown tensor τ_{ij} appears in the equation for the large-scale velocity $\overline{u}_i$ but it depends on the small-scale velocity field. The purpose of this study is to explore the advantages of simultaneously running several statistically equivalent and independent LES's for the same flow. In practice, we thus replace the equation (2.1) by the following set of equations for R large-scale velocity fields $\overline{u}_i^r$:

$$\partial_t \overline{u}_i^r + \partial_j \overline{u}_j^r \overline{u}_i^r = -\partial_i \overline{p}^r + \nu_0 \nabla^2 \overline{u}_i^r - \partial_j \tau_{ij}^r , \tag{2.2}$$

where $r = 1, \ldots, R$.

It is worth mentioning that the use of an ensemble of LES's is not per se much more expensive than the use of a single realisation. To show this, let us consider a stationary LES and denote by t_t the time of the transient period between the beginning of the simulation and the time at which the turbulence becomes fully developed. Let us also denote by t_s the time (beyond t_t) required to converge the statistics. Then, the CPU time required for obtaining converged statistics with a single LES is $t_t + t_s$. With an ensemble of realisations, statistics are accumulated over both the ensemble and time. Thus, for equivalent sample, the ensemble only needs to be advanced in time by the amount t_s/R. The total CPU cost for the ensemble is thus $R(t_t + t_s/R)$, which amounts in an overhead of $(R-1)t_t$ over a single realisation. If the ratio between the transient phase and the time needed to converge statistics is small, then the additional cost will be moderate. In the examples treated below, this additional cost is totally negligible. Moreover, if the LES is not stationary and if there is no direction of homogeneity, the ensemble-averaged approach is presumably the only way to obtain statistics.

2.1. Statistically equivalent and independent LES's

The knowledge of an ensemble of LES's can only be useful if the LES fields $\overline{u}_i^r$ are all independent. Yet all these fields have to correspond to the same experimental situation if some meaningful statistics are to be extracted from the ensemble. We therefore must define what will be considered statistically equivalent but independent LES fields. Although a proof of existence and uniqueness of solutions for the Navier–Stokes equations is not yet available, from a practical point of view a flow described by the Navier–Stokes equations or by an LES equation is assumed to be fully defined by the knowledge of

1. The domain $\mathcal{D}$ in which the flow is considered.

2. The conditions on the boundary $\partial\mathcal{D}$ of this domain $u_i(\partial\mathcal{D}, t) = b_i(t)$ where the functions $b_i(t)$ are given.

3. The initial conditions $u_i(\mathbf{x},0) = u_i^0(\mathbf{x})\ \forall \mathbf{x} \in \mathcal{D}$.

However, in a simulation of a turbulent flow only the domain and the boundary conditions are rigorously fixed. Indeed, because of the lack of sensitivity to initial conditions in turbulence, different simulations with different initial conditions sharing some properties are considered to characterize the same flow. Thus, the requirement that the initial conditions are known is somewhat relaxed for turbulent flows and the point (3) is thus replaced by a weaker constraint:

3′. The initial condition $u_i(\mathbf{x},0) = u_i^0(\mathbf{x};w_l)$ is generated using random numbers w_l and satisfies a certain number of constraints: $P_s[u_i^0] = p_s,\ \ s = 1,\dots S$.

For example, in homogeneous turbulence, the most important constraint will be on the spectrum $E(k)$ of u_i^0:

$$k^2 \int d\Omega\ \ |\hat{u}_i^0(\mathbf{k},0)|^2 = E(k) \tag{2.3}$$

where $\hat{u}_i^0$ is the Fourier transform of u_i^0 and Ω is the solid angle in the wavenumber $d^3\mathbf{k} = k^2\, d\Omega\, dk$. For channel flow, one could impose the plane-averaged profile of both the velocity $U(y)$ and the Reynolds stress $R_{ij}(y)$:

$$\langle u_i^0 \rangle_{x,z} = U(y)\delta_{i,1} \tag{2.4}$$

$$\langle (u_i^0 - U(y)\delta_{i,1})^2 \rangle_{x,z} = R_{ij}(y) \tag{2.5}$$

where x, y and z are respectively the longitudinal, the wall normal, and the transverse directions and $\langle\cdots\rangle_{x,z}$ represents the average in planes parallel to the wall. We will not discuss in detail the minimal constraints that must be imposed on the initial conditions in order to have a reasonable simulation. In fact, this minimal set of constraints will probably depend on the type of flow as well as on the quantities that are measured in the simulation. Here we only suppose that these constraints do exist in order to give a precise definition of equivalent LES's:

i) Two LES's are statistically equivalent if the domain of the flow and the boundary conditions are the same and if the initial conditions satisfy the same set of constraints.

Carrying an ensemble of equivalent LES's can be computationally effective only if the different members in the set of LES's are independent. Here again, the definition of independent LES's might depend on the flow as well as on the quantities that are measured in the simulation. Along the same line as for the definition of statistically equivalent LES's, we propose the following definition:

ii) Two LES's are statistically independent if their initial conditions are generated with uncorrelated random numbers w_l.

We remark that for a stationary flow, such equivalent and independent initial conditions can be obtained by running a single LES and recording several velocity fields separated by at least one large-eddy turnover time when turbulence is fully developed.

2.2. Universal model parameter in LES

Classical closure strategies in LES amount to modelling τ_{ij} in terms of the resolved velocity field:

$$\tau_{ij} = C\, m^{\tau}_{ij}[\overline{u}_l, \quad], \tag{2.6}$$

where is the filter width. The tensor m^{τ}_{ij} is supposed to characterize the dependence on both the filter width and the specific realisation of the large-scale flow $\overline{u}_l$. On the other hand, we will assume that the parameter C depends only on the type of flow and on the filter shape and should not depend on any particular realisation of the large-scale velocity field.† In the following, we will refer to this assumed property as the universality of the model parameters in LES: For a given geometry and for a given Reynolds number, the model parameters should be the same in all equivalent LES's. This concept of universality does not imply that the model parameters are constant in space and time. Clearly, $C = C(\mathbf{x}, t)$ can be a field quantity that needs to be adapted both in space and time to the local conditions of the flow. However, in our approach the variations of C are not supposed to take their origin in possible fluctuations in the large-scale flow. Rather, C is expected to depend only on the averaged properties of the flow, and in that sense it shares many properties with RANS quantities.

The assumption that the model parameters are universal has a direct influence on the formulation of models in an ensemble of statistically equivalent LES's. In the equations (2.2), these models should have the following structure:

$$\tau^{r}_{ij} = C\, m^{\tau}_{ij}[\overline{u}^{r}_l, \quad], \tag{2.7}$$

where C is now independent of the realisation index r.

It must be noted, however, that the developpment of the dynamic procedure in some ways challenges this viewpoint. In the dynamic procedure, information from the small scales of $\overline{u}_i$ is used for estimating the model parameters. This procedure is known to produce highly fluctuating model parameters. Such a property is sometimes regarded as a proof of the capability of the dynamic procedure to produce model parameters that account for the local conditions of the flow. However, these fluctuations in C are responsible for instabilities, and some averaging procedures are used to avoid this difficulty.

† Of course, more sophisticated models with more than one term have also been proposed, but the specific roles of the model parameters and of the model tensors m^{τ}_{ij} remain the same.

We propose in Section 3 an approach that reconciles the dynamic procedure with the concept of a universal model parameter. In this sense, it is fairly different from other procedures in which the concept of a universal parameter has not been adopted, such as the local dynamic procedure developped by Ghosal et al. (1995), the Lagrangian dynamic procedure proposed by Meneveau et al. (1996), or the time lagging procedure proposed by Piomelli and Liu (1995).

2.3. New modelling concepts

The knowledge of an ensemble of LES fields opens new possibilities in the modelling of the τ_{ij}^r. Indeed, it is now conceivable to introduce an explicit dependence on ensemble-averaged quantities into the models for τ_{ij}^r.

2.3.1. Model based on the fluctuating strain tensor

The first model we propose is based on the fluctuating part of the rate-of-strain tensor:

$$\tau_{ij}^r = -2\nu_e \left(\overline{S}_{ij}^r - \langle \overline{S}_{ij} \rangle \right) \equiv -2\nu_e \delta S_{ij}^r , \tag{2.8}$$

where ν_e is the eddy viscosity. This formulation has some nice properties. The averaged total dissipation is given by

$$\mathcal{E} = \langle \nu_T \delta S_{ij}^r \delta S_{ij}^r \rangle + \nu_0 \langle S_{ij}^r S_{ij}^r \rangle, \tag{2.9}$$

and consequently the turbulent dissipation originates only from the fluctuating part of the strain tensor. The mean part contributes only to the molecular dissipation. This property ensures that the model will not produce dissipation in a laminar region. In addition, while this model is dissipative on average (provided the eddy viscosity is positive), individual realisations can have negative dissipation, thus representing the inverse transfers of energy from the small unresolved scales to the large ones (backscatter) (Leith 1990; Mason & Thomson 1992; Carati et al. 1995a). It is generally believed that backscatter originates from fluctuation phenomena in the subgrid scales, and representation of this effect through fluctuations in the strain tensor is thus very reasonable.

Results using this model for the wake flow are given in sections below. It has already been used in the channel flow, where the plane of homogeneity is used to compute the average (Schumann 1975). However, the ensemble of LES's allows the use of such models even in fully inhomogeneous flows. Of course, many other models might be considered along the same lines, and the fluctuating strain rate is not the only quantity that could enter the model. In this paper, we will restrict our investigations to the model (2.8) in the study of the wake flow. However, we mention hereafter another possible use of the knowledge of an ensemble of LES's in the case of anisotropic flows.

2.3.2. Anisotropic model

Anisotropic effects are almost universally observed in turbulence. However, anisotropy usually originates from complex interactions between flow direction, solid boundaries and external constraints like pressure gradient or global rotation. It is thus quite difficult to predict a priori the main consequences of this anisotropy. In the context of statistical averaged LES, we have access at any instant to mean quantities that will display the anisotropic structure of the turbulence even for fully inhomogeneous flows. A model that would directly take advantage of the ensemble of LES's could be:

$$\tau_{ij}^r \approx \mu \gamma_{ik} \gamma_{jl} S_{kl}^r , \tag{2.10}$$

where the factor μ plays the role of an eddy viscosity but through an anisotropic relation between the subgrid scale stress and the strain tensor. The tensor γ_{ij} should be a measure of the anisotropy. It could be constructed from the velocity fluctuations:

$$\gamma_{ij} = \frac{3\langle \delta u_i^r \, \delta u_j^r \rangle}{\langle \delta u_k^r \, \delta u_k^r \rangle} . \tag{2.11}$$

This model reduces to the classical eddy viscosity model for isotropic turbulence ($\gamma_{ij} = \delta_{ij}$). The sign of the dissipation depends only on the sign of μ since the product of τ_{ij}^r and the strain tensor is given by

$$\tau_{ij} S_{ij} = \mu S_{ij} \gamma_{ik} \gamma_{jl} S_{kl} = \mu \, (S_{ij} \gamma_{ik})^2 . \tag{2.12}$$

Moreover, if there is no turbulence in one direction ($\delta u_a = 0$), the model has the property that the components $\tau_{ia} = \tau_{aj} = 0$. This is an expected property that is missed by isotropic eddy viscosity models.

3. Coupling the dynamic procedure and the ensemble of LES's

3.1. Classial dynamic procedures

The dynamic procedure is based on an identity (Germano 1992) that relates the unknown stress generated by different filters:

$$L_{ij} + \widehat{\tau}_{ij} - T_{ij} = 0, \tag{3.1}$$

where $T_{ij} = \widehat{\overline{u}_i \, \overline{u}_j} - \widehat{\overline{u}}_i \, \widehat{\overline{u}}_j$ is the subgrid scale stress generated by the successive application to the velocity field of two filters that are respectively denoted by $\overline{\ }$ and by $\widehat{\ }$. The Leonard tensor is given by $L_{ij} = \widehat{\overline{u}_i \, \overline{u}_j} - \widehat{\overline{u}}_i \, \widehat{\overline{u}}_j$. It depends only on $\overline{u}_i$ so that it does not require any modelling. This identity (3.1) is of course only valid for the exact and unknown subgrid scale stresses. When models are used, $\tau_{ij} \approx C m_{ij}^r[\overline{v}_l]$ and $T_{ij} \approx C m_{ij}^T[\widehat{\overline{v}}_l]$, the difference $E_{ij} = L_{ij} + \widehat{C m_{ij}^r} - C m_{ij}^T$ between the right hand side and the left hand side

of (3.1) can be considered as a measure of the performance of the model. The dynamic procedure uses this measure in order to prescribe the model parameter C by minimizing E_{ij}. When a homogeneous direction exists in the problem, the estimation for C is given by (Germano et al. 1991; Lilly 1992; Ghosal et al. 1995):

$$C \approx \frac{\langle L_{ij} M_{ij} \rangle_h}{\langle M_{ij} M_{ij} \rangle_h} , \tag{3.2}$$

where $M_{ij} = \widehat{m_{ij}^r} - m_{ij}^T$ and the average $\langle \cdots \rangle_h$ is supposed to be taken over the homogeneous direction(s). Obviously, this approach is restricted to special geometries with homogeneous direction(s). Complex geometries require an alternative treatment in which a local definition of the parameter C can be proposed. This is the case in the local dynamic procedure developed by Ghosal et al. (1995) as well as in the Lagrangian dynamic procedure proposed by Meneveau et al. (1996). In both cases, the model parameter is directly related to the large-scale field $\overline{u}_i$ through the tensor L_{ij} and M_{ij}. It will thus vary from one realisation to another, even if the underlying LES's are supposed to be equivalent. As already mentioned in Section 2.2, the dynamic procedure thus produces model parameters that are not universal. In the early stages of its development, the fact that the model parameters are directly related to the specific realisation of the flow was considered advantageous because this allowed the model to be more adaptative. However, this property proved to be problematic because it generates highly variable model parameters that cause numerical instabilities. Some of these practical problems have been resolved in the aforementioned local and Lagrangian versions of the dynamic procedure.

3.2. Ensemble-averaged dynamic procedure

The ensemble-averaged dynamic procedure (EADP) we propose here is conceptually very close to the volume-averaged or plane-averaged versions (3.2) of the dynamic procedure. The only difference comes from the nature of the average, which is now an ensemble average over the set of LES's. Considering that R LES's (2.2) are computed simultaneously, the model parameter is now given by:

$$C \approx \frac{\langle L_{ij} M_{ij} \rangle}{\langle M_{ij} M_{ij} \rangle} , \tag{3.3}$$

where $\langle \cdots \rangle$ now represents the ensemble average. The expression (3.3) is only valid if the parameter C is slowly dependent on space and can be taken out of the test filter $\widehat{\ }$. Such an assumption is not very restrictive, however, since the ensemble averaging is likely to smooth out the rapid variations in the field. In the next Section, it will be seen that the model coefficient does indeed become smoother and smoother as the ensemble size

is increased. The formulation (3.3) guarantees that the model parameters are universal since they depend only on the statistical properties of the large-scale velocity fields.

In a sense, the ensemble of LES's corresponds to an artificial direction of homogeneity, which always exists independent of the complexity of the flow. In the unexpected cases in which the model coefficient remains significantly variable in space for large ensemble sizes, the EADP could be coupled with other approaches like the local dynamic procedure developed by Ghosal et al. (1995), the Lagrangian dynamic procedure proposed by Meneveau et al. (1996), or the time lagging procedure proposed by Piomelli and Liu (1995). The coupling of the EADP with any of these methods would lead to a negligible cost since it would be used only once for the whole ensemble.

In some cases, the fact that the model parameter cannot be adapted to the specific realisation of the large scale flow might be considered as a drawback of the EADP. In particular, a greater adaptability might be desirable for very intermittent flows with, for instance, localized turbulent spots appearing inside a laminar sea (Henningson et al. 1987). Indeed, when the turbulent spots appear randomly in a statistically homogeneous domain, the model parameter predicted by the EADP is quasi-constant and is weakly affected by the turbulent spots. In fact, the EADP implicitly assumes that the model tensor $m_{ij}[\overline{u}_l^r]$ should, alone, take care of the turbulence activity. We realize however that, because the perfect model is not available, the assumption of a universal parameter might be sometimes inappropriate, depending on both the nature of the flow and the model adopted for $m_{ij}[\overline{u}_l^r]$. However, it must be noted that the same difficulty would be encountered for models that use parameters that are chosen a priori as well as for dynamic models that are based on volume or plane averaging. Moreover, in the test cases presented in the following sections, the predictions of the EADP are encouraging. More localized approaches (Ghosal et al. 1995; Meneveau et al. 1996; Piomelli & Liu 1995) for which the assumption of a universal model parameter is not adopted would probably respond more strongly to intermittent flows. It must be noted, however, that these generalized dynamic procedures are usually combined with models that are based on statistical approaches. In these cases, use of the localized dynamic model in a sense contradicts the underlying statistical assumptions used to build the eddy viscosity model.

Finally, we remark that the model parameter is the only coupling between the different LES's. The difference between DNS, LES, and an ensemble of LES's coupled through the EADP is illustrated in figure 1. As a consequence, the EADP is perfectly suited for distributed processing on parallel computers. The most natural implementation of this procedure amounts to running each member of the ensemble of LES on a separate

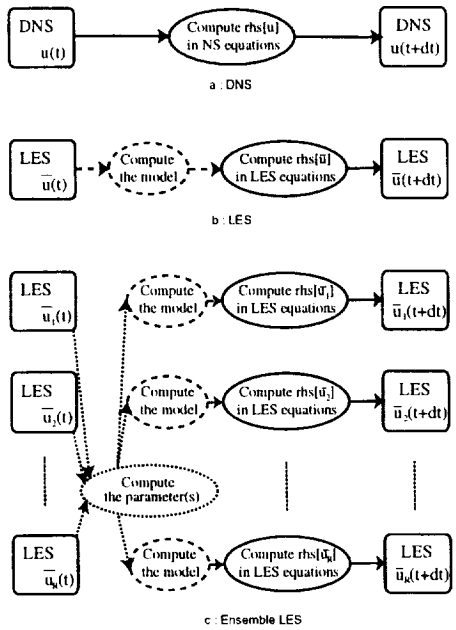

FIGURE 1. The differences between DNS, LES, and ensemble of LES's using the EADP are illustrated. In DNS (top), only the right hand side of the Navier–Stokes equations is needed for advancing the velocity field in time. In traditional LES (middle), an additional modelling term is needed. In the EADP (bottom), one substep, common for all the LES's, is added for computing the model parameters used in each of the simulations. This is the only point where information is required from the other fields.

node. Communication between the different processes is limited to the computation of the model coefficient. Otherwise each field $\overline{u}_i^r$ is advanced in time independently of the others. This property should guarantee very good scalability if large ensemble sizes are explored.

4. Tests in isotropic turbulence

4.1. Decaying turbulence

The EADP described in the previous section was tested in decaying isotropic turbulence for 32^3 LES's. The tensor m_{ij}^r was chosen to correspond to the Smagorinsky model:

$$\tau_{ij}^r \approx -2C|\overline{S}^r|\overline{S}_{ij}^r \tag{4.1}$$

A series of numerical experiments has determined (Carati et al. 1996) how large the ensemble of simultaneous LES's must be (*i.e.* how large R should be). The criteria used to determine the minimal size of the ensemble were focused on

1. The spatial variability of C.
2. The percentage of negative C.

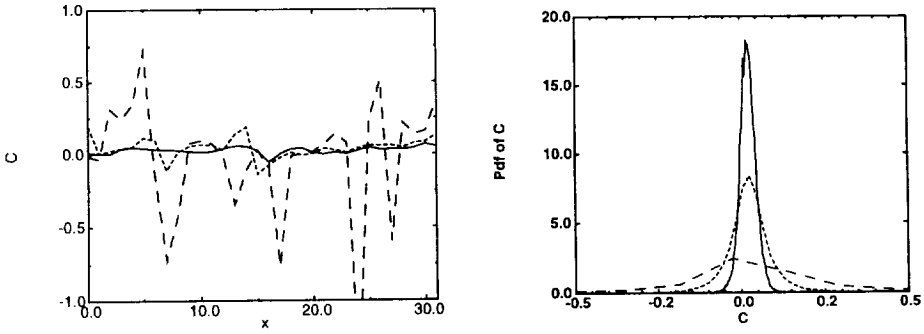

FIGURE 2. Typical one-dimensional profiles (left) and probability distribution function (right) of C in decaying isotropic turbulence for $R = 1$ (long-dashed line), $R = 4$ (short-dashed line) and $R = 16$ (solid line). The averaged value of $\langle C \rangle \approx 0.02$ before clipping is almost independent of the ensemble size. The expected "smoothing effect" of the ensemble averaging is reflected by a rapidly decreasing deviation $\sigma_C^2 = \langle (C - \langle C \rangle)^2 \rangle$ with R. For instance, $\sigma_C \approx 0.3$ for $R = 1$, $\sigma_C \approx 0.06$ for $R = 4$, and $\sigma_C \approx 0.02$ for $R = 16$.

3. Comparison with the volume-averaged dynamic model.

4. Comparison with direct numerical simulations.

The results are quite encouraging. It appears that with only 16 simultaneous LES's, the ensemble-averaged dynamic model performs as well as the volume-averaged model. The spatial variability of C decreases drastically when R increases. This is also reflected in the probability distribution function (PDF) of C (see figure 2). Some quantitative measurements of the spatial variability as a function of the ensemble size are given in Table 1. In particular, the fraction of negative C before clipping drops from 41% for $R = 1$ to 15% for $R = 16$. Hence, the fraction of points for which C has to be clipped is still significant even for $R = 16$. However, the consequences of this clipping are less and less significant because the clipped values of C have smaller magnitudes for increasing ensemble sizes. For instance, the ratio between the averaged values of C before and after the clipping is only 0.19 for $R = 1$ while it reaches 0.91 for $R = 16$. Hence, the effect of clipping on physical quantities like the energy dissipation becomes small for $R \geq 16$.

The comparison between a 512^3 DNS and the dynamic model shows good agreement both for the total resolved energy and for the spectra. The ensemble-averaged results for $R = 16$ are indistinguishable from the volume-averaged values and only the EADP results are plotted on figure 3. An ensemble of independent volume-averaged LES's was run to allow comparison of both the means and the standard deviations.

4.2. Forced turbulence

We have run an ensemble of 16 32^3 forced turbulence LES's with zero molecular viscosity. Figure 4 shows that the mean resolved energy and the standard deviation evolve simi-

R	Before clipping		After clipping	
	$\langle C \rangle$	σ_C	$\langle C \rangle$	σ_C
1	0.018	0.29	0.089	0.19
2	0.020	0.12	0.048	0.081
4	0.020	0.057	0.031	0.040
8	0.019	0.031	0.024	0.024
16	0.018	0.020	0.020	0.017
32	0.018	0.013	0.019	0.012

TABLE 1. Average and standard deviation of the model coefficient (before and after clipping) versus the ensemble size.

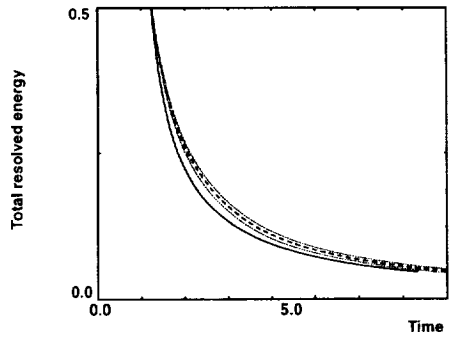

FIGURE 3. Comparison of the energy decay between the truncated DNS (solid line) and the averaged energy predicted by the set of LES's using EADP (dashed line). The dotted lines correspond the averaged energy ± one standard deviation as predicted by the set of LES's using EADP.

larly for both the volume- and ensemble-averaged models. This shows that the coupling induced by the computation of the model parameter through the EADP approach does not introduce spurious correlations between the different members of the ensemble. The standard deviations remain similar in the two approaches, indicating that the LES fields in the EADP remain nearly as independent as those in the ensemble of independent volume-averaged simulations. It is also interesting to compare the compensated energy spectrum $E(k) = E(k)k^{5/3}\epsilon^{-2/3}$, where $E(k)$ is the energy spectrum and ϵ is the dissipation rate. Of course, with 32^3 LES, we do not expect to observe a well developed inertial range or to obtain a very good estimate of the Kolmogorov constant. However, examination of the results in figure 4 indicates that the observed "Kolmogorov constant" is reasonable.

5. Tests in wake flows

The flow considered here is a time-evolving plane wake for which data from both direct numerical simulations (Moser & Rogers 1994; Moser et al. 1997) and large eddy

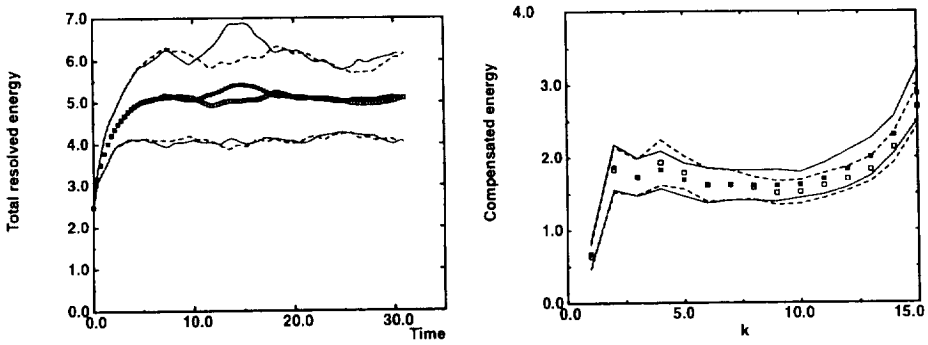

FIGURE 4. Resolved energy (left) and compensated energy spectrum (right) in forced isotropic turbulence: ensemble- (solid boxes) versus volume-averaged (open boxes) dynamic procedure. Dotted lines and dashed lines correspond to the mean ± one standard deviation in the EADP and in the volume-averaged dynamic model, respectively.

simulations (Ghosal & Rogers 1997) are available. This flow is both statistically non-stationary and inhomogeneous and should thus be a more demanding test of the EADP than the homogeneous flows studied in the previous section.

The pseudospectral direct numerical simulation of the plane wake considered here has been described in detail by Moser & Rogers (1994) and Moser, Rogers & Ewing (1997). The spatial dependence of the independent variables is represented in the periodic streamwise and spanwise directions by Fourier basis functions and the cross-stream dependence is represented by a class of Jacobi polynomials on a mapped infinite domain. Up to $512 \times 195 \times 128$ modes are required to accurately resolve the turbulence. The Reynolds number based on the integrated mass flux deficit,

$$\mu = -\int_{-\infty}^{+\infty} (U(y) - U_\infty)\, dy, \tag{5.1}$$

is $Re = \mu/\nu = 2000$. In a time-evolving plane wake, the integrated mass flux deficit is constant.

LES's of the same flow using the dynamic procedure, with a filtered DNS field as an initial condition, have been reported by Ghosal & Rogers (1997). The simulations were pseudospectral like the DNS, but the spatial dependence of the vorticity in the inhomogeneous cross-stream direction is represented in terms of Fourier modes on a finite domain. The appropriate non-periodic velocity field is then calculated using the method of Corral & Jimenez (1995). The number of modes used in the LES's was $64 \times 48 \times 16$ and the same number of modes and same numerical method have been adopted (Carati & Rogers 1998) for the EADP LES's examined here. Thus each LES requires up to 260 times fewer modes compared to the DNS.

5.1. The subgrid-scale models

In the present study, we have investigated three different models, all based on the eddy viscosity concept. The first one is the Smagorinsky model introduced in the previous section (4.1). In this model, the inertial range scaling for the eddy viscosity $\nu_t \sim C^{-4/3}\bar{\epsilon}^{1/3}$ is expressed in terms of the resolved strain-rate tensor by using the approximation for the dissipation rate $\bar{\epsilon} \sim \nu_t \overline{S}^r_{kl}\overline{S}^r_{kl}$. This approximation is required in traditional LES because a separate equation for the dissipation rate is not usually computed. However, in LES based on the dynamic procedure, the dimensional product $C_\epsilon = C\bar{\epsilon}^{1/3}$ can be predicted directly. This has motivated (Wong & Lilly 1994; Carati et al. 1995b) an alternative model based on the inertial range scaling such as

$$\text{Model A}: \qquad \tau^r_{ij} \approx -2C_\epsilon^{-4/3}\,\overline{S}^r_{ij}\,. \tag{5.2}$$

Finally, we have considered a third model for which the tensor τ^r_{ij} is given by the expression (2.8)

$$\text{Model B}: \qquad \tau^r_{ij} \approx -2C_\epsilon^{-4/3}\left(\overline{S}^r_{ij} - \langle\overline{S}^r_{ij}\rangle\right), \tag{5.3}$$

where the brackets indicate ensemble-averaging over all realisations. The possible advantages of this last model have been discussed in section 2.3.

In all three models, the sign of C (or of C_ϵ) will also determine the sign of the subgrid-scale dissipation, since a negative C corresponds to a negative eddy viscosity. In order to avoid numerical instabilities, the model parameter must then be set equal to a minimal positive value (clipping procedure, see Ghosal et al. (1995)) at points where the total viscosity (eddy plus molecular) is negative. For the Smagorinsky model, the stability condition

$$C^{-2}\left(2\overline{S}^r_{kl}\overline{S}^r_{kl}\right)^{1/2} + \nu_0 > 0 \tag{5.4}$$

depends on the realisation. This is an undesirable property since C is supposed to be a universal flow characteristic for all members of the ensemble. An alternative formulation in which C is indeed the same for all realisations results from the following stability condition

$$C^{-2}\max_r\left\{\left(2\overline{S}^r_{kl}\overline{S}^r_{kl}\right)^{1/2}\right\} + \nu_0 > 0\,. \tag{5.5}$$

In the limit of an infinite number of realisations, the maximum of the resolved strain-rate tensor amplitude would be almost unbounded. Hence, for the Smagorinsky model, it is reasonable to simply impose $C > 0$. For model A, however, the situation is different. The stability condition is naturally the same in each realisation

$$C_\epsilon^{-4/3} + \nu_0 > 0\,, \tag{5.6}$$

For simplicity, the same condition has been used for model B.

5.2. The initial conditions

In practice, initial conditions for LES can be built either by filtering a DNS (when it is available) or by generating a random velocity field satisfying some constraints (as discussed in Section 2.1). For the EADP, we have to generate R equivalent but independent fields. In the case of isotropic turbulence, the only constraint that had to be satisfied by the initial field was the energy spectrum. We thus have used Rogallo's approach (1981) to build R initial conditions with the same spectrum and independent phases. For the time-evolving plane wake, random initial conditions could be generated following the same approach as the one used for initialising the DNS. However, for the plane wake, a large number of quantities are measured and any number of them might be considered as constraints that need to be maintained by all realisations (e.g. profiles of mean velocity, turbulent kinetic energy, enstrophy, etc.). The main purpose of the present study is to test the EADP rather than evaluating an initialisation procedure. For that reason we have used a simple trick to generate R statistically identical initial fields. Our procedure is based on the fact that the observed quantities are computed through plane averages and are thus invariant under the change

$$\overline{u}_i(x, y, z, t_0) \longrightarrow \overline{u}_i(x + \delta_x, y, z + \delta_z, t_0). \tag{5.7}$$

Thus by using R values of (δ_x^r, δ_z^r), R initial velocity fields are produced that clearly satisfy the requirement that the LES realisations be statistically equivalent. However, this procedure does not produce statistically independent initial conditions, even with random choices for (δ_x^r, δ_z^r), because the two fields are identical and simply shifted in space. Without the subgrid model terms, all the statistics would remain identical for all times. However, the model terms will have the desirable effect of de-correlating the different members of the ensemble. This results because the universal model terms act at the same $(x - z)$ location in all the realisations, not at the same relative position in the shifted flows. Examples of this de-correlation are given in figure 5, in which the reduced maximal difference

$$Q = \frac{\max_{\text{ensemble}}(Q) - \min_{\text{ensemble}}(Q)}{\langle Q \rangle} \tag{5.8}$$

is given for various quantities Q like the wake width, the resolved turbulent kinetic energy density integrated in y, and the resolved turbulent kinetic energy dissipation rate integrated in y. For these global quantities, Q is originally 0 and rapidly reaches values of the order of 5%. We investigated the behaviour of Q for a local quantity, the maximum grid-point value of the streamwise vorticity component $\omega_{x\max}^r$. Here, Q is

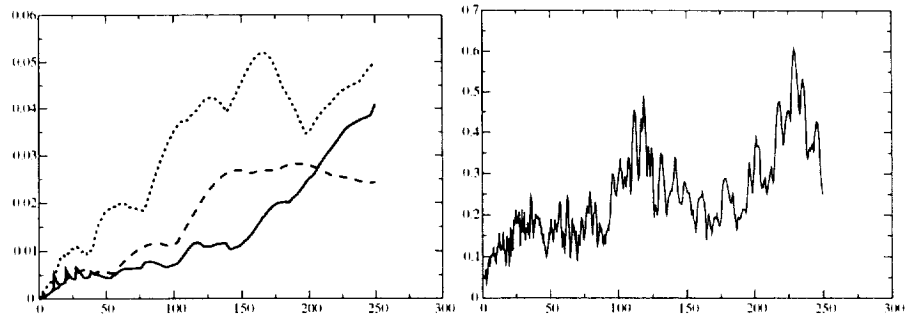

FIGURE 5. Test of de-correlation of the LES fields. The reduced maximal difference ΔQ is plotted versus time for the wake width (left, solid line), the resolved turbulent kinetic energy density integrated in y (left, dashed line), the resolved turbulent kinetic energy dissipation integrated in y (left, dotted line) and the maximum grid-point value of the x-component of the resolved vorticity, $\omega_{x\,\max}^{r}$ (right)

originally 5% and rapidly reaches values of the order of 20 to 40%. These measurements indicate a fairly rapid de-correlation of the different members of the ensemble.

Again, we stress here that this methodology used for building the initial conditions has the major advantage of guaranteeing that the statistical properties are initially identical for each realization, while results presented in figure 5 indicate reasonable de-correlations. However, when no DNS is available, it will be unavoidable to develop a suitable approach for building initial conditions that will satisfy the criteria of independence and equivalence as described in Section 2.1. In that case, the constraints to be satisfied by the initial conditions will most probably come from experimental data.

5.3. Tests of convergence

In order to test the convergence of the EADP results for increasing values of R, two types of tests were performed. First, the ensemble-averaged values of several relevant quantities in the time-evolving wake flow have been compared for various ensemble sizes. In particular, the results for (i) the wake width, (ii) the turbulent kinetic energy density integrated in y and (iii) the turbulent kinetic energy dissipation rate integrated in y are compared for $R = 4$, 8, 16, and 32. As can be seen from figure 6, only the turbulent kinetic energy integrated in y is affected by the number of realisations. However, the values obtained with 16 and 32 realisations are almost indistinguishable for all three quantities.

Second, the influence of the ensemble size on the computed eddy viscosity was examined. The profile of the mean eddy viscosity and the fraction of grid points for which the eddy viscosity has been clipped according to the criterion (5.6) are compared for the same values of R in figures 7 and 8. As seen in figure 7, the eddy viscosity profile

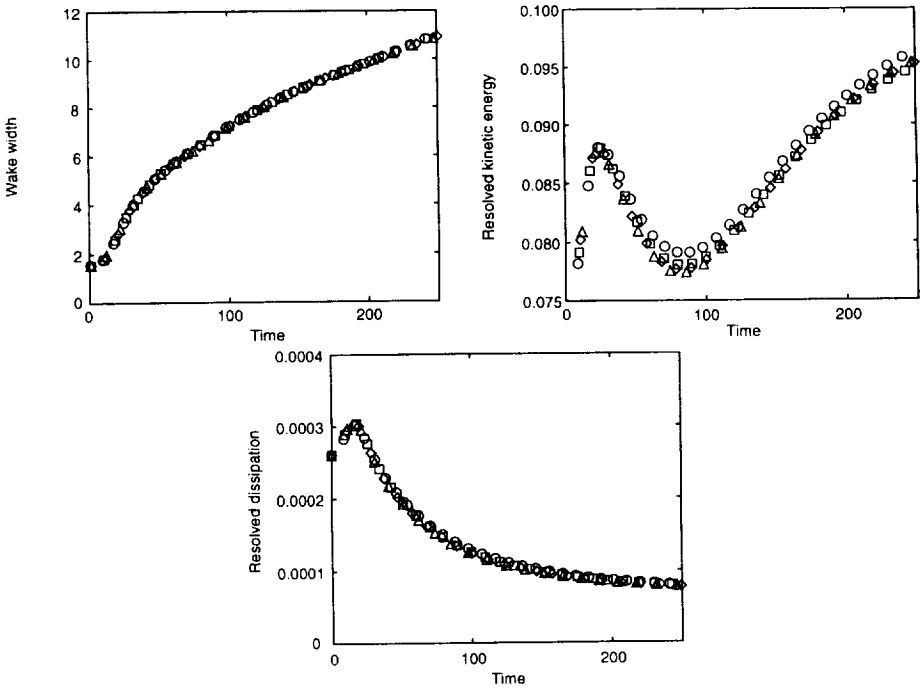

FIGURE 6. Convergence of the ensemble-averaged evolutions of the wake width (top left), the resolved turbulent kinetic energy density integrated in y (top right) and the resolved turbulent kinetic energy dissipation integrated in y (bottom). Various ensemble sizes are compared: $R = 4$ $\circ$, $R = 8$ $\square$, $R = 16$ $\diamond$, and $R = 32$ $\triangle$.

depends only weakly on the number of realisations for values of R between 4 and 32, and the profiles are nearly identical for $R = 16$ and $R = 32$. As expected, the fraction of grid points requiring clipping of the model coefficient C rapidly decreases with R (figure 8). The total fraction of clipped points integrated in y is less than 1% for $R = 16$ during the entire simulation. This, combined with the very small change in most of the ensemble-averaged quantities as R is increased from 16 to 32, supports the adoption of $R = 16$ as a reasonable ensemble size for both model testing and production LES. Because this value of R is the same as that required for the simulation of homogeneous turbulence, it seems reasonable to hope that $R = 16$ provides an adequate ensemble size for the EADP in even more complicated geometries.

The comparison between various ensemble sizes is presented here only for model A (2.4). The same conclusions concerning the convergence of the results and the appropriate value of R are obtained when either the Smagorinsky model or model B (2.5) is used.

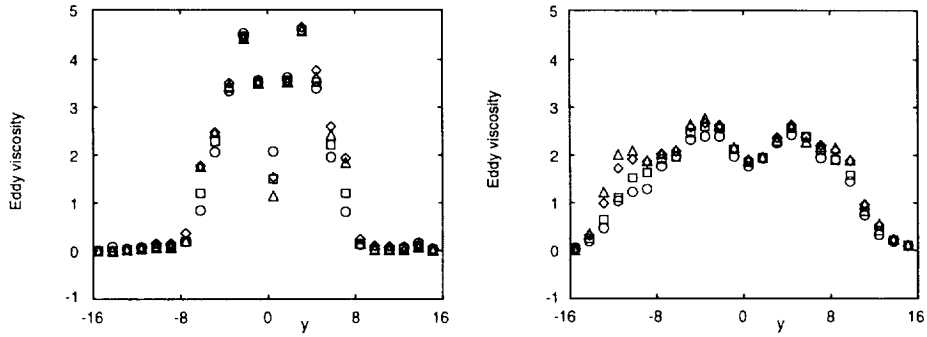

FIGURE 7. Convergence of the eddy viscosity profile for $t = 0$ (left) and for $t = 250$ (right). Various ensemble sizes are compared: $R = 4$ ○, $R = 8$ □, $R = 16$ ◇, and $R = 32$ △. The eddy viscosity is normalized by the molecular viscosity.

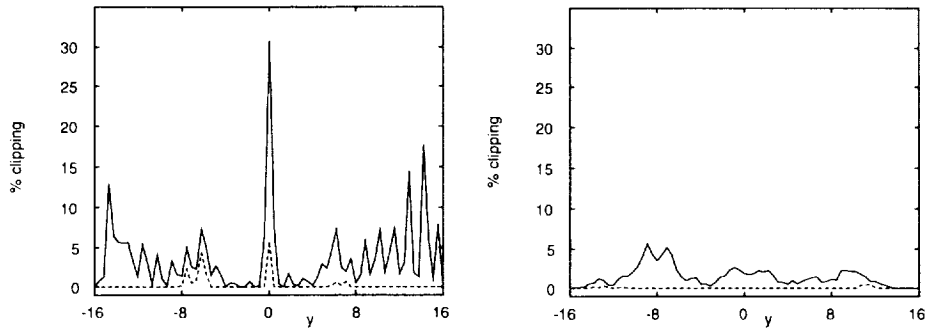

FIGURE 8. Profile of the fraction of grid points requiring clipping of the coefficient C at $t = 0$ (left) and $t = 250$ (right). Two ensemble sizes are compared: $R = 4$ solid line and $R = 32$ dashed line.

5.4. Comparison of models

As already mentioned, an important motivation for developping the EADP is the possibility of investigating new concepts in subgrid-scale modelling. Here, the filtered DNS of Moser, Rogers, & Ewing (1997) is compared with the LES predictions of Ghosal & Rogers (1997) and the predictions of the models presented in section 5.1. We have also added the results of a LES without a subgrid-scale model. In all cases, and in agreement with the conclusion of the preceding section, the simulations for the EADP have been performed with $R = 16$.

The first important conclusion is that the plane-averaged and ensemble-averaged dynamic procedures lead to indistinguishable results when they are applied with the same model. For instance, in the LES of Ghosal & Rogers, the plane-averaged dynamic procedure was implemented with the standard Smagorinsky model. Their results are identical to those obtained when the Smagorinsky model is used with the EADP. In the following

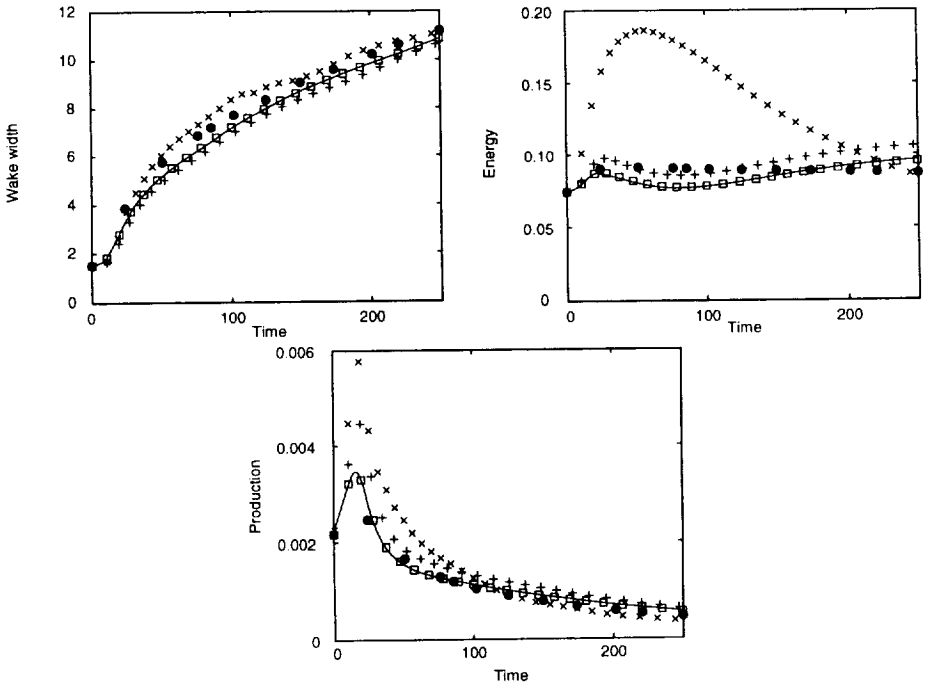

FIGURE 9. Evolution of the wake width (top left), the resolved turbulent kinetic energy density integrated in y (top right) and the resolved turbulent kinetic energy production integrated in y obtained from the filtered DNS •; the Smagorinsky model —, the model A □, the model B +, and no model ×.

comparison, the Smagorinsky case will refer to both the EADP and the plane-averaged LES of Ghosal & Rogers.

The evolutions of (*i*) the wake width, (*ii*) the resolved turbulent kinetic energy density integrated in y, and (*iii*) the resolved turbulent kinetic energy production integrated in y are presented in figure 9. The wake width is dominated by large-scale flow features and consequently is not strongly affected by the models. Actually, the prediction of the LES without a subgrid-scale model (an under-resolved DNS) provides a reasonable approximation to the value obtained by filtering the DNS data.

The turbulent kinetic energy density integrated in y is more difficult to predict using LES. Not using a subgrid-scale model results in poor prediction of resolved energy density. Model A leads to almost the same result as the Smagorinsky model. This is a general feature of the dynamic procedure that has been noted previously (Wong & Lilly 1994; Carati et al. 1995b). However, within the dynamic procedure approach, model A is computationally much cheaper to implement than the Smagorinsky model and this motivates the use of the scaling (5.2) for the sub-grid stress instead of (4.1). The model B, from

which the ensemble-averaged resolved strain-rate has been removed, leads to results that better fit the DNS data in the early stages of the simulation. At later times, however, this model is further from the filtered DNS values than model A and the Smagorinsky model. In general the predictions of all three models seem comparable.

The evolution of the turbulent kinetic energy production also shows the important role of the models. The no-model LES prediction for the resolved energy production is much too high in the early stage and too low at later times. Again, model A leads to almost the same result as the Smagorinsky model. Model B systematically over-predicts the energy production. However, it would be rather speculative to draw any definitive conclusion regarding which model (A or B) is better from the results presented here.

6. Conclusions

The tests presented here have shown that the knowledge of statistically equivalent resolved velocity fields may be useful in deriving new subgrid-scale models. We have used the additional information available from the different LES's to propose an ensemble-averaged version of the dynamic procedure. This dynamic procedure presents several advantages.

First, a local version of the ensemble-averaged dynamic model is derived in the limit of large ensemble sets. The local formulation does not rely on any homogeneity assumption and can be adapted to any geometry, unlike the classical volume- (or plane- or line-) averaged dynamic models.

Second, the practical limit of large ensemble sets is closely approached for $R \approx 16$. This is indicated by many diagnostics. For example, the PDF of the model parameter appears to be very peaked for $R = 16$ and its spatial variations decrease drastically for increasing ensemble sizes and seem to be quite mild for $R = 16$. Also, all the measured quantities, both in homogeneous turbulence and in the plane wake, are almost identical for $R = 16$ and $R = 32$. This is, of course, a major encouragement for further developping the EADP methodology. The fact that the same value $R = 16$ appears to be appropriate for both homogeneous turbulence and the plane wake suggests that this might be an adequate ensemble size for converged results even in more complex flows.

Also, the EADP reconciles the dynamic procedure with the concept of a universal parameter in turbulence modelling. This is a very desirable property since it is commonly accepted that model parameters should depend only on the external conditions of the flow and not on the particular realisation that is observed.

Considering the rapid developpment of parallel computers, the use of an ensemble of

statistically equivalent and independent LES's can be regarded as a very promising technique. This technique can be implemented with fairly small ensemble sizes. Original modelling concepts that cannot be implemented in fully inhomogeneous flows by conventional LES techniques are possible within the framework of the EADP and warrant further examination. Moreover, the additional computational cost generated by the use of R simultaneous LES's could be compensated by the fact that statistics can be accumulated much faster with the ensemble of LES's if one ensures that all the realisations are statistically independent.

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REFERENCES

CARATI, D., GHOSAL, S. & MOIN, P. 1995*a* On the representation of backscatter in dynamic localization models. *Phys. Fluids* **7**, 606–616.

CARATI, D., JANSEN, K. & LUND, T. 1995*b* A family of dynamic models for large-eddy simulation. In *Annual Research Briefs - 1995* (ed. P. Moin & W. Reynolds), pp. 35–40. Stanford University and NASA Ames Research Center: Center for Turbulence Research.

CARATI, D. & ROGERS, M. 1998 Ensemble-averaged LES of a time-evolving plane wake. In *Proceedings of the 1998 Summer Program* (ed. P. Moin & W. Reynolds), pp. 325–336. Stanford University and NASA Ames Research Center: Center for Turbulence Research.

CARATI, D., WRAY, A. & CABOT, W. 1996 Ensemble-averaged dynamic modeling. In *Proceedings of the 1996 Summer Program* (ed. P. Moin & W. Reynolds), pp. 237–248. Stanford University and NASA Ames Research Center: Center for Turbulence Research.

CORRAL, R. & JIMENEZ, J. 1995 Fourier/Chebyshev methods for the incompressible navier-stokes equations in inifinite domains. *J. Comput. Phys.* **121**, 261-270.

GERMANO, M. 1992 Turbulence: the filtering approach. *J. Fluid Mech.* **238**, 325-336.

GERMANO, M., PIOMELLI, U., MOIN, P. & CABOT, W. 1991 A dynamic subgrid-scale eddy-viscosity model. *Phys. Fluids A* **3**, 1760–1765.

GHOSAL, S., LUND, T., MOIN, P. & AKSELVOLL, K. 1995 A dynamic localization model for large-eddy simulation of turbulent flows. *J. Fluid Mech.* **286**, 229–255.

GHOSAL, S. & ROGERS, M. 1997 A numerical study of self-similarity in a turbulent plane wake using large-eddy simulation. *Phys. Fluids* **9**, 1729–1739.

HENNINGSON, D., SPALART, P. & KIM, J. 1987 Numerical simulations of turbulent spots in plane Poiseille and boundary-layer flow. *Phys. Fluids* **30**, 2914–2917.

LEITH, C. 1990 Stochastic backscatter in a subgrid-scale model: Plane shear mixing layer. *Phys. Fluids A* **2**, 297–299.

LILLY, D. 1992 A proposed modification of the germano subgrid-scale closure method. *Phys. Fluids* **4**, 633–635.

MASON, P. & THOMSON, D. 1992 Stochastic backscatter in large-eddy simulations of boundary layers. *J. Fluid Mech.* **242**, 51–78.

MENEVEAU, C., LUND, T. & CABOT, W. 1996 A lagrangian dynamic subgrid-scale model of turbulence. *J. Fluid Mech.* **319**, 353–386.

MOSER, R. & ROGERS, M. 1994 Direct simulation of a self-similar plane wake. NASA technical Memorandum 108815. NASA, Ames Research Center.

MOSER, R., ROGERS, M. & EWING, D. 1997 Self-similarity of time evolving plane wakes. *J. Fluid Mech.* **367**, 255–289.

PIOMELLI, U. & LIU, J.H. 1995 Large-eddy simulation of rotating channel flow using a localized dynamic model. *Phys. Fluids* **7**, 839–848.

ROGALLO, R. 1981 Numerical experiments in homogeneous turbulence. NASA technical Memorandum 81315. NASA, Ames Research Center.

SCHUMANN, U. 1975 Subgrid scale model for finite difference simulation of turbulent flows in plane channels and annuli. *J. Comp. Phys.* **18**, 376–404.

WONG, V. & LILLY, D. 1994 A comparison of two subgrid closure methods for turbulent thermal convection. *Phys. Fluids* **6**, 1016–1023.