Stochastic Modeling and Generation of Partially Polarized or Partially Coherent Electromagnetic Waves

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Abstract

Many new Earth remote-sensing instruments are embracing both the advantages and added complexity that result from interferometric or fully polarimetric operation. To increase instrument understanding and functionality, a model of the signals these instruments measure is presented. A stochastic model is used as it recognizes the non-deterministic nature of any real-world measurements, while also providing a tractable mathematical framework. A stationary, Gaussian-distributed model structure is proposed. Temporal and spectral correlation measures provide a statistical description of the physical properties of coherence and polarization-state. From this relationship, the model is mathematically defined. The model is shown to be unique for any set of physical parameters. A method of realizing the model (necessary for applications such as synthetic calibration-signal generation) is given, and computer simulation results are presented. The signals are constructed using the output of a multi-input, multi-output linear filter system, driven with white noise.

1 Introduction

Radio-interferometers and polarimeters are being used more widely in remote sensing for probing the Earth's lands and oceans (e.g., [Kerr, 1998, Piepmeier and Gasiewski, 2001].) While these techniques have a rich history in the space sciences, Earth-viewing systems are relatively new. There are distinct differences in the operation and calibration techniques of space-viewing telescopes versus orbiting Earth-viewing instruments. Operationally, Earth-viewing imagers have a fraction of the integration time available compared to radio-telescopes: seconds vs. hours. Additionally, radio-telescope elements have extremely narrow beams ($\sim 0.5°$) compared to low-Earth orbiting imagers, whose wide beams ($\sim 60 - 90°$) are required to cover a reasonable swath. Perhaps more important are the calibration differences. Interferometric radio-telescopes are typically calibrated using extra-solar point sources, whereby the system can be characterized to within a common gain coefficient [Thompson et al., 1991]. Furthermore, if the flux of the point source is known, the interferometer can be absolutely calibrated. There are no obvious point sources when looking downward at the Earth, at least ones that emit energy within the protected spectrum for passive observing. Furthermore, because of their large beamwidths, Earth-viewing interferometers cannot selectively view a single extra-solar point source. (Besides, the time and operations required for regularly rotating a spacecraft for calibration would cause data loss and increase mission costs.) Therefore, a different calibration technique must be devised for orbiting Earth-imaging systems. This paper lays the groundwork for such a technique by rigorously examining the signals that these radiometers measure.

While polarimetry and interferometry are usually investigated independently, they are based upon the same idea — measuring the interdependence of two signals. For polarimetry these signals are the vertical and horizontal field amplitudes (or some equivalent pair.) Two-beam interferometry involves measuring the coherency of two signals separated in space and/or time. In order to fully understand these instruments, it is desirable to have an accurate model of the types of signal pairs they measure. This type of model can be employed in mathematical analysis, computer simulations and for the generation of synthetic calibration signals (one possible on-orbit calibration tool.)

Electromagnetic waves generally have some degree of randomness — an unpolarized component in polarimetry or an incoherent component in interferometry. For this reason it is logical to employ a stochastic method when modeling the pair of signals. The relationship between the statistics of the model and the standard terminology in polarimetry and interferometry should be well understood.

The model structure, statistical parameters and constraints are described in Section 2. Sections 3 and 4 describe how this model's temporal and spectral correlation functions can define polarimetric and interferometric properties. Using this framework, Section 5 shows how a comprehensive model can be uniquely determined from the physical properties of the modeled wave. Section 6 gives a computational method for realizing the model. A Summary is then presented followed by an Appendix containing mathematical derivations. A polarimetric example is carried through Sections 3, 5 and 6 in order to demonstrate the ideas presented.

2 Model Structure and Basic Properties

2.1 Mathematical Signal Representation

The standard literature in both polarimetry and interferometry makes extensive use of a 'narrowband' signal representation (e.g. [Hecht, 1987, Brosseau, 1998].) This representation alludes to the fact that the measured signals exist within a finite frequency band (centered on some frequency ω_0.) The general form of this representation is shown below.

$$\begin{aligned} f(t) &= a(t)\cos\omega_0 t - b(t)\sin\omega_0 t \\ &= \mathrm{Re}\{[a(t) + i\,b(t)]e^{i\omega_0 t}\} \end{aligned} \tag{1}$$

It should be noted that a narrowband representation does not imply a small fractional bandwidth.

As mentioned in the previous section, the model must describe a pair of signals. These signals will be labeled $X(t)$ and $Y(t)$ and have a narrowband definition as given below.

$$\begin{aligned} X(t) &= A(t)\cos\omega_0 t - B(t)\sin\omega_0 t \\ &= \mathrm{Re}\{P(t)e^{i\omega_0 t}\} \\ Y(t) &= C(t)\cos\omega_0 t - D(t)\sin\omega_0 t \\ &= \mathrm{Re}\{Q(t)e^{i\omega_0 t}\} \end{aligned} \tag{2}$$

The functions are capitalized to indicate that they are random processes. It will also be assumed that the model is stationary and Gaussian distributed. Stationarity is a common assumption in optics [Hecht, 1987] and a Gaussian amplitude distribution can be justified by appealing to the Central Limit Theorem. The consequences of not assuming a Gaussian distribution are discussed at the end

of Chapter 5.

In order to ensure stationarity in the above signal model (Equation 2), $X(t)$ and $Y(t)$ must be jointly-stationary. For simplicity it will be assumed that $A(t)$, $B(t)$, $C(t)$ and $D(t)$ (and hence $P(t)$ and $Q(t)$) must also be jointly-stationary. To produce a Gaussian-distributed model for $X(t)$ and $Y(t)$ it will be assumed that $A(t)$, $B(t)$, $C(t)$ and $D(t)$ are Gaussian processes, which leads to bivariate Gaussian distributions in the complex plane for $P(t)$ and $Q(t)$.

The random processes $A(t)$, $B(t)$, $C(t)$ and $D(t)$ are real, jointly-stationary and Gaussian. This means their multivariate probability density function is completely defined by their means and second

order moments (shown below.)

$$
\begin{aligned}
E[A(t)] = \mu_A \qquad & E[B(t)] = \mu_B \\
E[C(t)] = \mu_C \qquad & E[D(t)] = \mu_D \\
E[A(t)A(t-\tau)] &= R_A(\tau) \\
E[B(t)B(t-\tau)] &= R_B(\tau) \\
E[C(t)C(t-\tau)] &= R_C(\tau) \\
E[D(t)D(t-\tau)] &= R_D(\tau) \\
E[A(t)B(t-\tau)] &= R_{AB}(\tau) \\
E[A(t)C(t-\tau)] &= R_{AC}(\tau) \\
E[A(t)D(t-\tau)] &= R_{AD}(\tau) \\
E[B(t)C(t-\tau)] &= R_{BC}(\tau) \\
E[B(t)D(t-\tau)] &= R_{BD}(\tau) \\
E[C(t)D(t-\tau)] &= R_{CD}(\tau)
\end{aligned}
\tag{3}
$$

The functions above can be used to create a covariance matrix and mean vector for any set of points within the signal. As these two properties define a multivariate, Gaussian distribution, the above functions give a complete description of the model.

It is also possible to find the means and second order moments for $P(t)$ and $Q(t)$ by noting that

$P(t) = A(t) + i\,B(t)$ and $Q(t) = C(t) + i\,D(t)$ (see Equations 1 and 2.)

$$\begin{aligned}
\mu_P &= E[P(t)] = E[A(t) + i\,B(t)] = \mu_A + i\,\mu_B \\
\mu_Q &= E[Q(t)] = E[C(t) + i\,D(t)] = \mu_C + i\,\mu_D \\
R_P(\tau) &= E[(A(t) + i\,B(t))(A(t-\tau) - i\,B(t-\tau))] \\
&= R_A(\tau) + R_B(\tau) - i(R_{AB}(\tau) - R_{AB}(-\tau)) \\
R_Q(\tau) &= E[(C(t) + i\,D(t))(C(t-\tau) - i\,D(t-\tau))] \\
&= R_C(\tau) + R_D(\tau) - i(R_{CD}(\tau) - R_{CD}(-\tau)) \\
R_{PQ}(\tau) &= E[(A(t) + i\,B(t))(C(t-\tau) - i\,D(t-\tau))] \\
&= R_{AC}(\tau) + R_{BD}(\tau) - i(R_{AD}(\tau) - R_{BC}(\tau))
\end{aligned} \tag{4}$$

2.2 Correlation and Spectral Functions

It is often useful to work with these second order moment functions in the Fourier domain. These frequency domain representations are known as spectral functions and will be denoted by an S i.e. $\Im\{R_{UV}(\tau)\} = S_{UV}(\omega)$, where $\Im\{\cdot\}$ denotes the Fourier transform.

Correlation functions and their corresponding spectral functions cannot take on any arbitrary form — there are theoretical restrictions on the range of legal function sets. These restrictions (given below) are most easily presented with respect to the spectral functions.

$$S_U(\omega) \geq 0 \qquad S_V(\omega) \geq 0 \qquad |S_{UV}(\omega)| \leq \sqrt{S_U(\omega) S_V(\omega)} \tag{5}$$

It can be shown that a complex, bivariate, stationary random process can be generated to have any spectral functions that satisfy Equation 5.

It was stated in the previous section that a real, stationary, multivariate, Gaussian process is uniquely described by its means and correlation functions (or equivalently spectra.) This is not the case for a complex, stationary, multivariate, Gaussian process. This can be demonstrated by examining Equation 4 and noticing that there are multiple sets of real and imaginary components ($A(t)$, $B(t)$, $C(t)$, $D(t)$) that will result in the same correlation functions for $P(t)$ and $Q(t)$ — i.e. specifying the correlation functions of $P(t)$ and $Q(t)$ does not uniquely define the model. To completely determine the model it is necessary and sufficient to define the functions given in Equation 3.

3 Polarimetry — Background and Relation to the Model

This section presents published material [Pancharatnam, 1975, Brosseau, 1998, Ishimaru, 1991] in a way that relates it to the proposed model structure.

3.1 The Stokes Parameters

Polarimetry involves studying the relationship between the horizontal and vertical fields of an electromagnetic wave. These two fields are usually written in narrowband form.

$$\begin{aligned} E_H(t) &= \mathrm{Re}\{E_1(t)e^{j\omega_0 t}\} \\ E_V(t) &= \mathrm{Re}\{E_2(t)e^{j\omega_0 t}\} \end{aligned} \tag{6}$$

The polarization state is generally measured using the 'Stokes parameters'. These parameters are defined in relation to the narrowband formulae above.

$$\begin{bmatrix} I \\ Q \\ U \\ V \end{bmatrix} = \begin{bmatrix} < |E_1(t)|^2 > + < |E_2(t)|^2 > \\ < |E_1(t)|^2 > - < |E_2(t)|^2 > \\ 2\mathrm{Re}\{< E_1(t)E_2^*(t) >\} \\ 2\mathrm{Im}\{< E_1(t)E_2^*(t) >\} \end{bmatrix} \tag{7}$$

This indicates that four parameters are needed to uniquely specify the polarization state of a wave. These four parameters are known to obey the inequality given below.

$$I \geq \sqrt{Q^2 + U^2 + V^2} \tag{8}$$

The ratio of the two terms in Equation 8 is known as the 'degree of polarization'.

Earth sensing radiometric instruments lend themselves to the use of the 'modified Stokes parameters'. These are typically used when the vertical and horizontal polarizations correspond to preferred axes tangent and normal to the Earth's surface.

$$\begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{bmatrix} = \begin{bmatrix} < |E_1(t)|^2 > \\ < |E_2(t)|^2 > \\ 2\mathrm{Re}\{< E_1(t)E_2^*(t) >\} \\ 2\mathrm{Im}\{< E_1(t)E_2^*(t) >\} \end{bmatrix} = \begin{bmatrix} \frac{I+Q}{2} \\ \frac{I-Q}{2} \\ U \\ V \end{bmatrix} \tag{9}$$

The parameters s_1 and s_2 represent the power in the horizontal and vertical channels respectively. In the past many Earth-sensing instruments have only measured these two properties. However,

recent Earth-sensing instruments are now measuring the full set of four parameters — this increases the need for a general model of polarized signals.

The inequality in Equation 8 can easily be rewritten in terms of the modified Stokes parameters. The result is given below.

$$2\sqrt{s_1 s_2} \geq \sqrt{s_3^2 + s_1^2} \tag{10}$$

It is also obvious from the definition in Equation 9 that the inequalities below are true.

$$s_1 \geq 0 \qquad s_2 \geq 0 \tag{11}$$

The polarimetric theory given above can be related to the model structure by invoking stationarity. Stationarity implies that the probability of any given polarization state is constant across the signal. In this case, stationarity also implies ergodicity which means that the expected values of the random processes are equal to the corresponding time averages. If $X(t)$ is set to model $E_H(t)$ and $Y(t)$ models $E_V(t)$ then the modified Stokes parameters can be related to the correlation functions

as shown below.

$$\begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{bmatrix} = \begin{bmatrix} < |P(t)|^2 > \\ < |Q(t)|^2 > \\ 2\mathrm{Re}\{< P(t)Q^*(t) >\} \\ 2\mathrm{Im}\{< P(t)Q^*(t) >\} \end{bmatrix} = \begin{bmatrix} E[P(t)P^*(t)] \\ E[Q(t)Q^*(t)] \\ 2\mathrm{Re}\{E[P(t)Q^*(t)]\} \\ 2\mathrm{Im}\{E[P(t)Q^*(t)]\} \end{bmatrix}$$

$$= \begin{bmatrix} R_P(0) \\ R_Q(0) \\ 2\mathrm{Re}\{R_{PQ}(0)\} \\ 2\mathrm{Im}\{R_{PQ}(0)\} \end{bmatrix} \tag{12}$$

$$= \begin{bmatrix} \int S_P(\omega)d\omega \\ \int S_Q(\omega)d\omega \\ 2\mathrm{Re}\{\int S_{PQ}(\omega)d\omega\} \\ 2\mathrm{Im}\{\int S_{PQ}(\omega)d\omega\} \end{bmatrix}$$

3.2 Frequency Distributed Stokes Parameters

The analysis above shows that the polarization state of the modeled wave is determined by the area under the spectral functions. While this is correct, a more general interpretation is given in Pancharatnam [1975]. In that paper it is shown that each frequency component of the wave can be treated as a single, independent wave with correlation properties dictated by the spectra at that frequency. The correlation functions are a measure of the interdependence between or within the

signals. As the spectral functions show how this interdependence is distributed across frequency, the polarization state at each frequency is determined by the spectral functions.

$$\begin{bmatrix} s_1(\omega) \\ s_2(\omega) \\ s_3(\omega) \\ s_4(\omega) \end{bmatrix} = \begin{bmatrix} S_P(\omega) \\ S_Q(\omega) \\ 2\mathrm{Re}\{S_{PQ}(\omega)\} \\ 2\mathrm{Im}\{S_{PQ}(\omega)\} \end{bmatrix} \tag{13}$$

It should be noted that the variable ω above represents a measure of distance from the central frequency ω_0. Also, the Stokes parameters in Equation 13 are actually densities — i.e. their units are power per unit bandwidth rather than just power.

To get the final wave, the components are considered to be incoherently (as they are independent) summed over frequency. This is intuitively satisfying as spectral functions sum under independent addition and Equation 13 becomes Equation 12.

If Equation 13 is substituted into the inequalities given in Equations 10 and 11 then the following inequalities are produced.

$$\begin{gathered} 2\sqrt{S_P(\omega)S_Q(\omega)} \geq 2\sqrt{\mathrm{Re}\{S_{PQ}(\omega)\}^2 + \mathrm{Im}\{S_{PQ}(\omega)\}^2} \\ \Rightarrow |S_{PQ}(\omega)| \leq \sqrt{S_P(\omega)S_Q(\omega)} \\ S_P(\omega) \geq 0 \qquad\qquad S_Q(\omega) \geq 0 \end{gathered} \tag{14}$$

It can be seen that these are the same inequalities as those stated in Equation 5. This means that the theoretical restrictions on the model given in Equation 5 are equivalent to restrictions on the modified Stokes parameters and hence do not limit the range of polarization states that can be

modeled.

The conclusion that can be drawn from this section is that the polarization state (distributed over frequency) to be modeled can be used to define the model parameters $S_P(\omega)$, $S_Q(\omega)$ and $S_{PQ}(\omega)$ (which in turn can be used to define $R_P(\tau)$, $R_Q(\tau)$ and $R_{PQ}(\tau)$.) This is demonstrated on an example signal with a contrived, frequency-dependent polarization state.

Example: The signal to be modeled has 5 distinct polarization bands within its overall bandwidth $2BW$. *The lowest fifth of its spectrum is unpolarized and has unity power density ('modified Stokes densities' are* $[0.5, 0.5, 0, 0]$*); the second fifth has a degree of polarization of 0.5, unity power density and the polarized component is left-circular (*$[0.5, 0.5, 0, -0.5]$*); the central fifth is completely linearly polarized at* 30^o *and has unity power density (*$[.75, .25, 0.866, 0]$*); the fourth section has a degree of polarization of 0.5, unity power density and the polarized component is right-circular (*$[0.5, 0.5, 0, 0.5]$*); and the highest frequency section is also unpolarized with unity power density (*$[0.5, 0.5, 0, 0]$*). By using Equation 13, the model parameters shown in Table 1 are found.* Italics used for example text

The modified Stokes parameters for the total wave can be found by performing the integrations in Equation 12. This results in the modified Stokes vector $[1.1BW, 0.9BW, 0.346BW, 0]$.

4 Interferometry — Background and Relation to the Model

4.1 Basic Coherence and Interference

Two beam interferometry measures interference and is closely related to the idea of coherence, as explained in [Hecht, 1987]. Interference is produced by the addition of two waves that have been separated in time and/or space. It is generally assumed that polarization effects can be ignored and that the space/time separation produces a time delay τ between the signals. The time delay is typically produced by a difference in propagation length and hence in time of flight. Stationarity is also usually assumed, so the two signals can be modeled as $X(t)$ and $Y(t-\tau)$ as given in Equation 2.

Interference occurs when the two beams are added. This sum is expressed below.

$$Z(t) = X(t) + Y(t-\tau) = \mathrm{Re}\{[P(t) + Q(t-\tau)e^{-i\omega_0\tau}]e^{i\omega_0 t}\} \tag{15}$$

It is the time-averaged, intensity of this sum (the square magnitude of $S(t)$) that is observed in interferometry. Constructive interference occurs when $P(t)$ and $Q(t-\tau)$ are in-phase and add to produce a large magnitude. Destructive interference occurs when they are out-of-phase and add to produce a small magnitude. This gives bright and dark interference fringes respectively.

$$\begin{aligned} I &= < Z(t)Z^*(t) > \\ &= < P(t)P^*(t) > + < Q(t-\tau)Q^*(t-\tau) > \\ &\quad +2\mathrm{Re}\{< P(t)Q^*(t-\tau)e^{i\omega_0\tau} >\} \\ &= \Gamma_{11}(0) + \Gamma_{22}(0) + 2\mathrm{Re}\{\Gamma_{12}(\tau)\} \end{aligned} \tag{16}$$

The final step makes use of ergodicity. The functions $\Gamma_{11}(\tau)$, $\Gamma_{22}(\tau)$ and $\Gamma_{12}(\tau)$ are known as the self- and mutual-coherence functions. They are commonly used to define the coherence of a pair of waves and are defined below.

$$\begin{aligned}
\Gamma_{11}(\tau) &= < P(t)P^*(t-\tau) > e^{j\omega_0\tau} = R_P(\tau)e^{j\omega_0\tau} \\
\Gamma_{22}(\tau) &= < Q(t)Q^*(t-\tau) > e^{j\omega_0\tau} = R_Q(\tau)e^{j\omega_0\tau} \\
\Gamma_{12}(\tau) &= < P(t)Q^*(t-\tau) > e^{j\omega_0\tau} = R_{PQ}(\tau)e^{j\omega_0\tau}
\end{aligned} \tag{17}$$

Equation 17 shows how the coherence properties of the signals to be modeled determine the model parameters $R_P(\tau)$, $R_Q(\tau)$ and $R_{PQ}(\tau)$ (which in turn can be used to define $S_P(\omega)$, $S_Q(\omega)$ and $S_{PQ}(\omega)$.)

4.2 Useful Functions in Coherence and Interference

There are a variety of useful measures defined by the coherence functions — a brief summary of some of them is given here.

The final term in Equation 16 is known as the interference term. Mathematically, it is this term that produces the interference fringes (the other two terms are simply constants.) So, the shape of the fringes governs the form of $\Gamma_{12}(\tau)$, which in turn determines the model parameter $R_{PQ}(\tau)$ (and $S_{PQ}(\omega)$.) The model parameters $S_P(\omega)$ and $S_Q(\omega)$ are the power spectral densities of the two signals — i.e. they determine how the power of the signals is distributed about ω_0. Also, if a signal were to interfere with itself, it would be the self-coherence functions (given by $R_P(\tau)$ or $R_Q(\tau)$) that would produce the interference term.

An important interferometric property is the fringe 'visibility function'. This function is defined by the intensity given in Equation 16.

$$V = \frac{I_{max} - I_{min}}{I_{max} + I_{min}} = \frac{2|\Gamma_{12}(\tau)|}{\Gamma_{11}(0) + \Gamma_{22}(0)} = \frac{2|R_{PQ}(\tau)|}{R_P(0) + R_Q(0)} \tag{18}$$

Here it is assumed that the oscillatory exponential term in $\Gamma_{12}(\tau)$ varies much faster than the function's magnitude. It is often the case that $\Gamma_{11}(0) = \Gamma_{22}(0)$ and the visibility function in Equation 18 reduces to the absolute value of the 'complex degree of coherence' ($V = |\gamma_{12}(\tau)|$).

$$\gamma_{12}(\tau) = \frac{\Gamma_{12}(\tau)}{\sqrt{\Gamma_{11}(0)\Gamma_{22}(0)}} = \frac{R_{PQ}(\tau)e^{i\omega_0\tau}}{\sqrt{R_P(0)R_Q(0)}} \tag{19}$$

A similar function to the complex degree of coherence is the 'complex degree of spectral coherence' (see [Goodman, 1985].) This function gives a measure of the distribution of the degree of coherency across frequency (a similar idea to the distributed polarization concept presented in Section 3.) The function is expressed in terms of the spectral representations of the coherence functions ($\mathcal{G}_{11}(\omega) = \Im\{\Gamma_{11}(\tau)\}$, $\mathcal{G}_{22}(\omega) = \Im\{\Gamma_{22}(\tau)\}$ and $\mathcal{G}_{12}(\omega) = \Im\{\Gamma_{12}(\tau)\}$.)

$$\mu_{12}(\omega) = \frac{\mathcal{G}_{12}(\omega)}{\sqrt{\mathcal{G}_{11}(\omega)\mathcal{G}_{22}(\omega)}} = \frac{S_{PQ}(\omega - \omega_0)}{\sqrt{S_P(\omega - \omega_0)S_Q(\omega - \omega_0)}} \tag{20}$$

The following properties of the spectra can be shown to be true.

$$\mathcal{G}_{11}(\omega) \geq 0 \qquad \mathcal{G}_{22}(\omega) \geq 0 \qquad 0 \leq |\mu_{12}(\omega)| \leq 1 \tag{21}$$

The above equations imply that the inequalities in Equation 5 are always satisfied. This means that the theoretical limits on the model do not limit the range of partially coherent signals that can be modeled.

This section has shown that there are a variety of related interferometric and coherency properties (e.g. the coherence functions or the interference term, the complex degree of coherency, the visibility function, etc.) that can be used to define the model. As in the polarimetric case, the functions $S_P(\omega)$, $S_Q(\omega)$ and $S_{PQ}(\omega)$ (or $R_P(\tau)$, $R_Q(\tau)$ and $R_{PQ}(\tau)$) can all be specified by the physical properties of the wave.

5 Completely Defining the Model

To completely define the model the functions in Equation 3 must be specified. In Sections 3 and 4 it was shown how the physical properties of the waves to be modeled (in either the polarimetric or interferometric case) can be used to define the correlation functions of $P(t)$ and $Q(t)$. However, in Section 2 it was shown that these correlation functions are insufficient to completely define the model (as there are many sets of the functions in Equation 3 that will result in the same correlation functions for $P(t)$ and $Q(t)$.)

It can also be shown that while extensive use has been made of the fact that $P(t)$ and $Q(t)$ are stationary (as are $A(t)$, $B(t)$, $C(t)$ and $D(t)$), this does not guarantee that $X(t)$ and $Y(t)$ are also stationary. In this section (and the corresponding Appendix) it will be shown how enforcing stationarity on $X(t)$ and $Y(t)$ allows the model to be completely defined using only the correlation functions of $P(t)$ and $Q(t)$.

5.1 Consequences of Enforcing Stationarity

The first step in enforcing stationarity is to give the constraints it places on the functions in Equation 3 (see Appendix A.)

$$\mu_A = \mu_B = \mu_C = \mu_D = 0 \tag{22}$$

$$\begin{aligned} R_A(\tau) &= R_B(\tau) \\ R_C(\tau) &= R_D(\tau) \\ R_{AC}(\tau) &= R_{BD}(\tau) \\ R_{AD}(\tau) &= -R_{BC}(\tau) \\ R_{AB}(\tau) &= -R_{AB}(-\tau) \\ R_{CD}(\tau) &= -R_{CD}(-\tau) \end{aligned} \tag{23}$$

The above equations are important because they show that rather than having four constants and ten functions to find (as in Equation 3), there are now only six functions (two of which must be odd-symmetric.)

An interesting result can be stated from Equation 23. By looking at a single signal (say $X(t)$) and its in-phase and quadrature components ($A(t)$ and $B(t)$) it can be seen that $R_A(0) = R_B(0)$ and $R_{AB}(0) = 0$. This means that at any given time the in-phase and quadrature components have the same variance and are uncorrelated. Since this is a Gaussian process, the result is that the amplitude of the signal is Rayleigh distributed and the phase is uniformly distributed. This result is stated in [Brosseau, 1998].

Equation 23 also results in $X(t)$ and $Y(t)$ being zero mean and having a well defined set of correlation functions.

$$\begin{aligned} R_X(\tau) &= R_A(\tau)\cos\omega_0\tau + R_{AB}(\tau)\sin\omega_0\tau \\ R_Y(\tau) &= R_C(\tau)\cos\omega_0\tau + R_{CD}(\tau)\sin\omega_0\tau \\ R_{XY}(\tau) &= R_{AC}(\tau)\cos\omega_0\tau + R_{AD}(\tau)\sin\omega_0\tau \end{aligned} \tag{24}$$

By substituting the relations in Equation 22 and 23 into Equation 4 it can be seen that $P(t)$ and $Q(t)$ must be zero mean and have correlation functions as given below.

$$\begin{aligned} R_P(\tau) &= 2(R_A(\tau) - i\,R_{AB}(\tau)) \\ R_Q(\tau) &= 2(R_C(\tau) - i\,R_{CD}(\tau)) \\ R_{PQ}(\tau) &= 2(R_{AC}(\tau) - i\,R_{AD}(\tau)) \end{aligned} \tag{25}$$

The equations above are significant because there is now only one real function defining each of the real and imaginary parts of the spectra of $P(t)$ and $Q(t)$ (c.f. Equation 4.)

5.2 Mathematical Model Definition

Equation 25 implies that there is now only one set of correlation functions for $A(t)$, $B(t)$, $C(t)$ and $D(t)$ that will produce a given set of correlation functions for $P(t)$ and $Q(t)$. This set of correlation functions is found by simply taking half the real and imaginary parts of $R_P(\tau)$, $R_Q(\tau)$ and $R_{PQ}(\tau)$. This results in the six functions required in Equation 23. It should be noted that Equation 23's odd-symmetry condition on $R_{AB}(\tau)$ and $R_{CD}(\tau)$ is always satisfied as $R_P(\tau)$ and $R_Q(\tau)$ are always conjugate-symmetric (this is a consequence of $S_P(\omega)$ and $S_Q(\omega)$ being real, as implied in Equation 5.)

This method of determining the correlation functions of $A(t)$, $B(t)$, $C(t)$ and $D(t)$ can also be expressed in the frequency domain.

$$\begin{aligned}
R_A(\tau) &= \tfrac{1}{2}\mathrm{Re}\{R_P(\tau)\} = \tfrac{1}{4}\{R_P(\tau) + R_P^*(\tau)\} \\
\Rightarrow S_A(\omega) &= \tfrac{1}{4}[S_P(\omega) + S_P(-\omega)] \\
R_C(\tau) &= \tfrac{1}{2}\mathrm{Re}\{R_Q(\tau)\} = \tfrac{1}{4}\{R_Q(\tau) + R_Q^*(\tau)\} \\
\Rightarrow S_C(\omega) &= \tfrac{1}{4}[S_Q(\omega) + S_Q(-\omega)] \\
R_{AB}(\tau) &= \tfrac{-i}{2}\mathrm{Im}\{R_P(\tau)\} = \tfrac{i}{4}\{R_P(\tau) - R_P^*(\tau)\} \\
\Rightarrow S_{AB}(\omega) &= \tfrac{i}{4}[S_P(\omega) - S_P(-\omega)] \\
R_{CD}(\tau) &= \tfrac{-i}{2}\mathrm{Im}\{R_Q(\tau)\} = \tfrac{i}{4}\{R_Q(\tau) - R_Q^*(\tau)\} \\
\Rightarrow S_{CD}(\omega) &= \tfrac{i}{4}[S_Q(\omega) - S_Q(-\omega)] \\
R_{AC}(\tau) &= \tfrac{1}{2}\mathrm{Re}\{R_{PQ}(\tau)\} = \tfrac{1}{4}\{R_{PQ}(\tau) + R_{PQ}^*(\tau)\} \\
\Rightarrow S_{AC}(\omega) &= \tfrac{1}{4}[S_{PQ}(\omega) + S_{PQ}^*(-\omega)] \\
R_{AD}(\tau) &= \tfrac{-i}{2}\mathrm{Im}\{R_{PQ}(\tau)\} = \tfrac{i}{4}\{R_{PQ}(\tau) - R_{PQ}^*(\tau)\} \\
\Rightarrow S_{AD}(\omega) &= \tfrac{i}{4}[S_{PQ}(\omega) - S_{PQ}^*(-\omega)]
\end{aligned} \tag{26}$$

By using the equations above and the first four equalities in Equation 23, all the functions in Equation 3 can be found. This leads to a model that is completely specified; and, that specification is unique (i.e. it is the only one that will produce the desired correlation functions for $P(t)$ and $Q(t)$ while maintaining stationarity of $X(t)$ and $Y(t)$.)

It can be seen that the even- and odd-symmetric portions of the spectra $S_P(\omega)$, $S_Q(\omega)$ and $S_{PQ}(\omega)$ are being taken. This results in the above spectra being conjugate-symmetric as would be

expected for the spectra of real, random processes. If the complete set of spectra are arranged to form the spectral matrix, it can be shown that the result is a Hermitian, positive semi-definite matrix (see Appendix B.) This is a general property of all spectral matrices (see [Jenkins and Watts, 1968]) and in the two-process case gives the inequalities of Equation 5.

Example: This process will be applied to the spectral functions from the example in Section 3. Using Equation 26 the results shown in Table 2 are found.

5.3 Summary of Model Design

A brief summary of the model design is presented in the form of a list.

1. Define the spectral or correlation functions of $P(t)$ and $Q(t)$ by considering the physical properties (either interferometric or polarimetric) of the waves to be modeled.

2. Apply Equation 26 to get six valid spectral or correlation functions for $A(t)$, $B(t)$, $C(t)$ and $D(t)$.

3. Apply Equation 23 to get the complete set of statistical functions (as given in Equation 3) of the random processes $A(t)$, $B(t)$, $C(t)$ and $D(t)$ and thus completely define the model.

It should be noted that there is no loss of generality in any of the steps, and that a single set of physical parameters can only produce one valid model. It can also be seen that if the Gaussian assumption is not made, then the model is not uniquely defined (as second order statistics no longer completely describe the p.d.f.). However, the previous analysis can still be used to define a stationary,

covariance-ergodic model up to second-order statistics. The Gaussian model is a special case of this where stationarity implies ergodicity and second-order statistics specify all higher orders.

6 Realizing the Model

The previous sections have developed a stochastic model for a pair of signals. This model is defined by the temporal and spectral correlation functions of $A(t)$, $B(t)$, $C(t)$ and $D(t)$. These can be found from physical signal properties and used to specify the probability density functions of the model. This is a clear mathematical model that has the potential to be useful in theoretical analysis. However, for applications such as computer simulations and synthetic signal generation it is necessary to create realizations of these random processes. It is sufficient to create realizations of the real variables $A(t)$, $B(t)$, $C(t)$ and $D(t)$, as the other variables ($P(t)$, $Q(t)$, $X(t)$ and $Y(t)$) can be created by deterministic functions of these four realizations.

As the probability density functions are known, it is possible to create a sampled realization directly. If N points of data were required, the correlation matrix for these points could be calculated and a multivariate, Gaussian random number generator applied (such as the `mvnrnd` command found in the *MATLAB* software package.) However, difficulties arise when the dimensionality of this p.d.f. is considered. There are four co-dependent outputs at every sample so a signal of length N would have a p.d.f. of dimension $4N$. For example, a signal of 125 MHz bandwidth requires a minimum sampling rate of 250×10^6 samples per second — thus a two-second signal would require 500×10^6

Emphasis used when referring to specific software

samples and result in a probability density function of dimension two billion. This is computationally impractical so a simpler method must be found.

A common technique in one dimension is to use a linear filter to shape noise into a desired spectral shape. The problem here is more complicated as the function is from one dimension (time) into four dimensions ($A(t)$, $B(t)$, $C(t)$, $D(t)$). A generalized filter structure with N inputs ($I_1(t), I_2(t), \ldots, I_N(t)$) and M outputs ($O_1(t), O_2(t), \ldots, O_M(t)$) is given below (this structure is presented in [Jenkins and Watts, 1968].)

$$\begin{aligned} O_1(t) &= \textstyle\sum_{i=1}^{N} h_{1i}(t) * I_i(t) \\ O_2(t) &= \textstyle\sum_{i=1}^{N} h_{2i}(t) * I_i(t) \\ &\vdots \\ O_M(t) &= \textstyle\sum_{i=1}^{N} h_{Mi}(t) * I_i(t) \end{aligned} \tag{27}$$

The filter responses must be chosen so that the desired correlation functions are realized. These functions are given by the expressions below [Jenkins and Watts, 1968].

$$\begin{aligned} R_{O_pO_q}(\tau) &= E[O_p(t)O_q(t-\tau)] \quad \left(= R_{O_qO_p}(-\tau)\right) \\ &= E\left[\left(\textstyle\sum_{i=1}^{N} h_{pi}(t) * I_i(t)\right)\left(\textstyle\sum_{j=1}^{N} h_{qj}(t-\tau) * I_j(t-\tau)\right)\right] \\ &= E\left[\textstyle\sum_{i=1}^{N}\sum_{j=1}^{N}\left(h_{pi}(t) * I_i(t)\right)\left(h_{qj}(t-\tau) * I_j(t-\tau)\right)\right] \\ &= \textstyle\sum_{i=1}^{N}\sum_{j=1}^{N} R_{I_iI_j}(\tau) * h_{pi}(\tau) * h_{qj}(-\tau) \end{aligned} \tag{28}$$

This set of equations has a simpler form in the Fourier domain, as the convolutions become multi-

plications.

$$\begin{aligned} S_{O_pO_q} &= \sum_{i=1}^{N}\sum_{j=1}^{N} S_{I_iI_j}(\omega)H_{pi}(\omega)H_{qj}(-\omega) \\ &= \sum_{i=1}^{N}\sum_{j=1}^{N} S_{I_iI_j}(\omega)H_{pi}(\omega)H^*_{qj}(\omega) \end{aligned} \tag{29}$$

The final step takes advantage of conjugate symmetry (which is guaranteed by the fact that $h_{mn}(t)$ is real for all mn.)

A general set of solutions for the system of equations given by Equation 29 is non-trivial as the system is non-linear. Simplifications can be made by making certain assumptions the first of which is that the input random processes are independent, white, Gaussian, unit variance and zero mean (so $S_{X_iX_j}(\omega) = \delta(i-j)$.) This ensures the outputs will be Gaussian and zero mean as required; and that Equation 29 reduces to the form given below.

$$\begin{aligned} S_{O_pO_q}(\omega) &= \sum_{i=1}^{N}\sum_{j=1}^{N} \delta(i-j)H_{pi}(\omega)H^*_{qj}(\omega) \\ &= \sum_{i=1}^{N} H_{pi}(\omega)H^*_{qi}(\omega) \end{aligned} \tag{30}$$

In this case only four output processes are required ($M = 4$.) This means we have ten independent equations one for each of the correlation functions given in Equation 3 (there are 16 equations but there is redundancy, as shown by the bracketed term in Equation 28.) A solution is presented for the case of four input processes (i.e. $N = 4$) although the method can be applied to larger dimensions provided that $M = N$ and the specified spectra satisfy the properties of a spectral matrix. The equations for this problem (as defined by Equation 30) are given below (the ω dependence has been dropped for clarity.) The outputs are denoted by A, B, C, D as before, while the inputs are numbered

1. 2. 3. 4.

$$
\begin{aligned}
S_A &= |H_{A1}|^2 + |H_{A2}|^2 + |H_{A3}|^2 + |H_{A1}|^2 \\
S_{AB} &= H_{A1}H^*_{B1} + H_{A2}H^*_{B2} + H_{A3}H^*_{B3} + H_{A4}H^*_{B4} \\
S_{AC} &= H_{A1}H^*_{C1} + H_{A2}H^*_{C2} + H_{A3}H^*_{C3} + H_{A1}H^*_{C4} \\
S_{AD} &= H_{A1}H^*_{D1} + H_{A2}H^*_{D2} + H_{A3}H^*_{D3} + H_{A4}H^*_{D1} \\
S_B &= |H_{B1}|^2 + |H_{B2}|^2 + |H_{B3}|^2 + |H_{B4}|^2 \\
S_{BC} &= H_{B1}H^*_{C1} + H_{B2}H^*_{C2} + H_{B3}H^*_{C3} + H_{D1}H^*_{C4} \\
S_{BD} &= H_{B1}H^*_{D1} + H_{B2}H^*_{D2} + H_{B3}H^*_{D3} + H_{B1}H^*_{D4} \\
S_C &= |H_{C1}|^2 + |H_{C2}|^2 + |H_{C3}|^2 + |H_{C4}|^2 \\
S_{CD} &= H_{C1}H^*_{D1} + H_{C2}H^*_{D2} + H_{C3}H^*_{D3} + H_{C4}H^*_{D4} \\
S_D &= |H_{D1}|^2 + |H_{D2}|^2 + |H_{D3}|^2 + |H_{D4}|^2
\end{aligned}
\tag{31}
$$

Although these equations are non-linear they can still be solved relatively easily. The solution method developed is most succinctly expressed in matrix notation.

$$
\overline{\overline{H}} = \begin{bmatrix}
H_{A1} & H_{A2} & H_{A3} & H_{A4} \\
H_{B1} & H_{B2} & H_{B3} & H_{B4} \\
H_{C1} & H_{C2} & H_{C3} & H_{C4} \\
H_{D1} & H_{D2} & H_{D3} & H_{D4}
\end{bmatrix}
\tag{32}
$$

$$\overline{\overline{S}} = \begin{bmatrix} S_A & S_{AB} & S_{AC} & S_{AD} \\ S_{AB}^* & S_B & S_{BC} & S_{BD} \\ S_{AC}^* & S_{BC}^* & S_C & S_{CD} \\ S_{AD}^* & S_{BD}^* & S_{CD}^* & S_D \end{bmatrix} \tag{33}$$

Using this notation, Equation 31 can be rewritten as shown below.

$$\overline{\overline{S}} = \overline{\overline{H}}\,\overline{\overline{H}}^{\dagger} \tag{34}$$

For matrices the † operation represents a conjugate transpose. In order to find a suitable set of filters it is sufficient to solve Equation 34 at the frequency points of interest. It is shown in [Strang, 1976] that because $\overline{\overline{S}}$ is positive semi-definite (see Appendix B and [Jenkins and Watts, 1968]), Equation 34 can always be solved. A method for doing this is outlined below.

Because the spectral matrix $\overline{\overline{S}}$ is Hermitian, it is always diagonalizable, its eigenvalues are real and its eigenvectors are orthogonal. This allows a simple solution for Equation 34 to be found by using the diagonalized form of $\overline{\overline{S}}$. If the eigenvalues are arranged on the main diagonal in the matrix $\overline{\overline{\Lambda}}$ and the eigenvectors in the matrix $\overline{\overline{E}}$ then the following is true.

$$\begin{aligned} \overline{\overline{S}} &= \overline{\overline{E}}\,\overline{\overline{\Lambda}}\,\overline{\overline{E}}^{\dagger} \\ \overline{\overline{S}} &= \overline{\overline{E}}\sqrt{\overline{\overline{\Lambda}}}\sqrt{\overline{\overline{\Lambda}}}\,\overline{\overline{E}}^{\dagger} \\ \overline{\overline{S}} &= (\overline{\overline{E}}\sqrt{\overline{\overline{\Lambda}}})(\overline{\overline{E}}\sqrt{\overline{\overline{\Lambda}}}^{\dagger})^{\dagger} \\ \overline{\overline{S}} &= (\overline{\overline{E}}\sqrt{\overline{\overline{\Lambda}}})(\overline{\overline{E}}\sqrt{\overline{\overline{\Lambda}}})^{\dagger} \end{aligned} \tag{35}$$

$\sqrt{\overline{\overline{\Lambda}}}$ is a matrix with the square-roots of the eigenvalues on the main diagonal (so that $\sqrt{\overline{\overline{\Lambda}}}\sqrt{\overline{\overline{\Lambda}}} = \overline{\overline{\Lambda}}$.)

Positive semi-definiteness guarantees the eigenvectors are positive, so $\sqrt{\overline{\overline{\Lambda}}}$ will always be a real matrix and hence be equal to its own conjugate transpose.

By comparing Equation 35 with Equation 34 it can be seen that the filter responses can be calculated by as follows.

$$\overline{\overline{H}} = \overline{\overline{E}}\sqrt{\overline{\overline{\Lambda}}} \tag{36}$$

Equation 36 shows how Equation 30 can be solved by finding the eigenvalues and eigenvectors of the spectral matrix $\overline{\overline{S}}$ at each frequency point. It is easy to show that this method will produce filter spectra that are conjugate-symmetric, which is necessary to ensure that the filters have a real response. Reordering the eigen-values and -vectors will still result in a valid solution at a single frequency point but care should be taken not to do this when constructing functions over many frequency points. Doing so would result in a sharp discontinuity in the filter spectra produced which would increase the length of the filter impulse response.

In this section it has been shown how independent, white, Gaussian noise functions (which are easily generated) can be fed into a system of linear filters to produce realizations of the model. The filter sets are generated using Equation 36 which depends on the model parameters. This process is computationally tractable and produces results like those shown in the example below.

Example: The example functions used in Sections 3 and 5 were realized using the methodology given in this section. This process was carried out using MATLAB and plots of the resulting spectra can be seen in Figure 1. It can be seen that the frequency axes extend outside the region $[-BW, BW]$

this corresponds to oversampling. The frequency axes are normalized so that the range of unaliased frequencies falls between -0.5 *and* 0.5.

The estimates of the realized spectra were found using periodogram averaging. 100 spectra were averaged in each case and a rectangular window was used in the time domain to remove the noisy terms associated with a large $|\tau|$. *Each plot has 399 frequency points and the filters were truncated to 199 taps. Figure 1 shows that the resulting spectra agree closely with those specified. The small differences can be accounted for by the necessary truncation of the generation filters and by the fact that a finite number of spectra were used to create the periodogram average.*

7 Summary

This paper begins by showing how standard measures in polarimetry and interferometry (such as Stokes parameters and coherence functions) can be interpreted in terms of the model's temporal or spectral correlation functions. This statistical interpretation then allows a stationary, Gaussian model of the signal pair to be defined. A stationary, Gaussian model can be justified physically.

This model structure was shown not to limit the range of physical signals that can be modeled (i.e. all coherency- or polarization-states could be modeled.) Additionally, the stationarity assumption was shown to lead to a unique model for any given set of physical properties. This indicates a comprehensive, well defined stochastic model in either the polarimetric or interferometric paradigm.

To realize this signal (as would be required in such applications as synthetic calibration signal

generation), it is necessary to produce a signal pair that has the same properties as the model. This can be done by generalizing a well-known noise-shaping technique in which a white, Gaussian process is passed through a linear filter in order to color its spectrum to a desired shape. The method was corroborated by presenting results from a MATLAB implementation.

Appendix A Derivation of Equation 22, 23 and 24

Because $X(t)$ and $Y(t)$ are Gaussian, they will be stationary if and only if their means and second order moments are independent of time.

$$\begin{aligned} \mu_X &= E[X(t)] \\ &= E[A(t)\cos\omega_0 t - B(t)\sin\omega_0 t] \\ &= \mu_A \cos\omega_0 t - \mu_B \sin\omega_0 t \end{aligned} \tag{37}$$

For this to be independent of t, $\mu_A = \mu_B = 0 \Rightarrow \mu_X = 0$. Similarly $\mu_C = \mu_D = 0 \Rightarrow \mu_Y = 0$. This shows that zero mean processes are required by stationarity.

The second order moments must also be independent of time.

$$
\begin{aligned}
& R_X(\tau) \\
&= E[X(t)X(t-\tau)] \\
&= E[(A(t)\cos\omega_0 t - B(t)\sin\omega_0 t) \\
&\qquad (A(t-\tau)\cos\omega_0(t-\tau) - B(t-\tau)\sin\omega_0(t-\tau))] \\
&= R_A(\tau)\cos\omega_0 t\cos\omega_0(t-\tau) + R_B(\tau)\sin\omega_0 t\sin\omega_0(t-\tau) \\
&\qquad - R_{AB}(\tau)\cos\omega_0 t\sin\omega_0(t-\tau) - R_{AB}(-\tau)\sin\omega_0 t\cos\omega_0(t-\tau) \\
&= \tfrac{1}{2}R_A(\tau)[\cos\omega_0\tau + \cos\omega_0(2t-\tau)] \\
&\qquad + \tfrac{1}{2}R_B(\tau)[\cos\omega_0\tau - \cos\omega_0(2t-\tau)] \\
&\qquad - \tfrac{1}{2}R_{AB}(\tau)[\sin\omega_0\tau + \sin\omega_0(2t-\tau)] \\
&\qquad - \tfrac{1}{2}R_{AB}(-\tau)[-\sin\omega_0\tau + \sin\omega_0(2t-\tau)] \\
&= \tfrac{R_A(\tau)+R_B(\tau)}{2}\cos\omega_0\tau + \tfrac{R_{AB}(\tau)-R_{AB}(-\tau)}{2}\sin\omega_0\tau \\
&\qquad - \tfrac{R_A(\tau)-R_B(\tau)}{2}\cos\omega_0(2t-\tau) - \tfrac{R_{AB}(\tau)+R_{AB}(-\tau)}{2}\sin\omega_0(2t-\tau)
\end{aligned}
\tag{38}
$$

To remove the t dependence requires that

$$
\begin{aligned}
& R_A(\tau) = R_B(\tau) \qquad \text{and} \qquad R_{AB}(\tau) = -R_{AB}(-\tau) \\
& \Rightarrow R_X(\tau) = R_A(\tau)\cos\omega_0 t + R_{AB}(\tau)\sin\omega_0 t
\end{aligned}
\tag{39}
$$

A similar argument can be applied to $Y(t)$ to give

$$
\begin{aligned}
& R_C(\tau) = R_D(\tau) \qquad \text{and} \qquad R_{CD}(\tau) = -R_{CD}(-\tau) \\
& \Rightarrow R_Y(\tau) = R_C(\tau)\cos\omega_0 t + R_{CD}(\tau)\sin\omega_0 t
\end{aligned}
\tag{40}
$$

The cross-term between the components must also be independent of t.

$$
\begin{aligned}
& R_{XY}(\tau) \\
&= E[X(t)Y(t-\tau)] \\
&= E[(A(t)\cos\omega_0 t - B(t)\sin\omega_0 t) \\
&\qquad (C(t-\tau)\cos\omega_0(t-\tau) - D(t-\tau)\sin\omega_0(t-\tau))] \\
&= R_{AC}(\tau)\cos\omega_0 t\cos\omega_0(t-\tau) + R_{BD}(\tau)\sin\omega_0 t\sin\omega_0(t-\tau) \\
&\qquad - R_{AD}(\tau)\cos\omega_0 t\sin\omega_0(t-\tau) - R_{BC}(\tau)\sin\omega_0 t\cos\omega_0(t-\tau) \\
&= \tfrac{1}{2}R_{AC}(\tau)[\cos\omega_0\tau + \cos\omega_0(2t-\tau)] \\
&\qquad + \tfrac{1}{2}R_{BD}(\tau)[\cos\omega_0\tau - \cos\omega_0(2t-\tau)] \\
&\qquad - \tfrac{1}{2}R_{AD}(\tau)[\sin\omega_0\tau + \sin\omega_0(2t-\tau)] \\
&\qquad - \tfrac{1}{2}R_{BC}(\tau)[-\sin\omega_0\tau + \sin\omega_0(2t-\tau)] \\
&= \tfrac{R_{AC}(\tau)+R_{BD}(\tau)}{2}\cos\omega_0\tau + \tfrac{R_{AD}(\tau)-R_{BC}(\tau)}{2}\sin\omega_0\tau \\
&\qquad - \tfrac{R_{AC}(\tau)-R_{BD}(\tau)}{2}\cos\omega_0(2t-\tau) - \tfrac{R_{AD}(\tau)+R_{BC}(\tau)}{2}\sin\omega_0(2t-\tau)
\end{aligned}
\tag{41}
$$

To remove the t dependence requires that

$$
\begin{aligned}
& R_{AC}(\tau) = R_{BD}(\tau) \quad \text{and} \quad R_{AD}(\tau) = -R_{BC}(\tau) \\
& \Rightarrow R_{XY}(\tau) = R_{AC}(\tau)\cos\omega_0 t + R_{AD}(\tau)\sin\omega_0 t
\end{aligned}
\tag{42}
$$

Appendix B Proof that the Spectral Matrix $\overline{\overline{S}}$ is positive semi-definite

In order to show that the Hermitian matrix $\overline{\overline{S}}$ is positive semi-definite it is necessary to prove the following inequality.

$$< \overline{\overline{S}}x, x >= x^{*}\overline{\overline{S}}x \geq 0 \tag{43}$$

Let $x = [a, b, c, d]$.

$$x^{*}\overline{\overline{S}}x = [a^{*}\ b^{*}\ c^{*}\ d^{*}] \begin{bmatrix} S_A & S_{AB} & S_{AC} & S_{AD} \\ S_{AB}^{*} & S_B & S_{BC} & S_{BD} \\ S_{AC}^{*} & S_{BC}^{*} & S_C & S_{CD} \\ S_{AD}^{*} & S_{BD}^{*} & S_{CD}^{*} & S_D \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \tag{44}$$

Writing the $\overline{\overline{S}}$ matrix in terms of $S_P(\omega)$, $S_Q(\omega)$ and $S_{PQ}(\omega)$ (using Equation 23 and 26).

$$\begin{aligned} x^*\overline{\overline{S}}x \quad = \quad & \tfrac{1}{1}\{S_P(\omega)[aa^* + bb^* + i(a^*b - ab^*)] \\ & + S_P(-\omega)[aa^* + bb^* + i(ab^* - a^*b)] \\ & + S_Q(\omega)[cc^* + dd^* + i(c^*d - cd^*)] \\ & + S_Q(-\omega)[cc^* + dd^* + i(cd^* - c^*d)] \\ & + S_{PQ}(\omega)[a^*c + b^*d + i(a^*d - b^*c)] \\ & + S^*_{PQ}(\omega)[ac^* + bd^* + i(bc^* - ad^*)] \\ & + S_{PQ}(-\omega)[ac^* + bd^* + i(ad^* - bc^*)] \\ & + S^*_{PQ}(-\omega)[a^*c + b^*d + i(b^*c - a^*d)]\} \end{aligned} \tag{45}$$

Let $t = a + ib$, $u = c + id$, $v = a - ib$, $w = c - id$.

$$
\begin{aligned}
x^* \overline{\overline{S}} x &= \tfrac{1}{4}\{S_P(\omega) tt^* + S_P(-\omega) vv^* + S_Q(\omega) uu^* + S_Q(-\omega) ww^* \\
&\quad + S_{PQ}(\omega) t^* u + S_{PQ}^*(\omega) tu^* + S_{PQ}(-\omega) cw^* + S_{PQ}^*(-\omega) v^* w\} \\
&= \tfrac{1}{4}\{S_P(\omega) tt^* + S_P(-\omega) vv^* + S_Q(\omega) uu^* + S_Q(-\omega) ww^* \\
&\quad + 2\mathrm{Re}[S_{PQ}(\omega) t^* u] + 2\mathrm{Re}[S_{PQ}(-\omega) vw^*]\} \\
&\geq \tfrac{1}{4}\{S_P(\omega)|t|^2 + S_P(-\omega)|v|^2 + S_Q(\omega)|u|^2 + S_Q(-\omega)|w|^2 \\
&\quad - 2|S_{PQ}(\omega)||t||u| - 2|S_{PQ}(-\omega)||v||w|\} \\
&\geq \tfrac{1}{4}\{S_P(\omega)|t|^2 + S_P(-\omega)|v|^2 + S_Q(\omega)|u|^2 + S_Q(-\omega)|w|^2 \\
&\quad - 2\sqrt{S_P(\omega) S_Q(\omega)}|t||u| - 2\sqrt{S_P(-\omega) S_Q(-\omega)}|v||w|\} \\
&= \tfrac{1}{4}\{(\sqrt{S_P(\omega)}|t| - \sqrt{S_Q(\omega)}|u|)^2 \\
&\quad + (\sqrt{S_P(-\omega)}|v| - \sqrt{S_Q(-\omega)}|w|)^2\} \\
&\geq 0
\end{aligned}
\tag{46}
$$

This proves Equation 43 and hence shows that $\overline{\overline{S}}$ is positive semi-definite.

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Figure 1: Example of realized spectra (plotted with desired spectra) for complex processes $P(t)$ and $Q(t)$: (a) power spectral density of $P(t)$; (b) power spectral density of $Q(t)$; (c) complex cross spectra of $P(t)$ and $Q(t)$ (real part); (d) complex cross spectra of $P(t)$ and $Q(t)$ (imaginary part).

Table 1: Example Spectral Functions found from Polarization State

Frequency Band	$S_P(\omega)$	$S_Q(\omega)$	$S_{PQ}(\omega)$
$(-\infty, -BW]$	0	0	0
$(-BW, -0.6BW]$	0.5	0.5	0
$(-0.6BW, -0.2BW]$	0.5	0.5	$-0.25i$
$(-0.2BW, 0.2BW]$	0.75	0.25	0.433
$(0.2BW, 0.6BW]$	0.5	0.5	$0.25i$
$(0.6BW, BW]$	0.5	0.5	0
(BW, ∞)	0	0	0

Table 2: Model Spectral Functions found for Example in Section 3

Frequency Band	$S_A(\omega)$ $= S_B(\omega)$	$S_C(\omega)$ $= S_D(\omega)$	$S_{AB}(\omega)$	$S_{CD}(\omega)$	$S_{AC}(\omega)$ $= S_{BD}(\omega)$	$S_{AD}(\omega)$ $= -S_{BC}(\omega)$
$(-\infty, -BW]$	0	0	0	0	0	0
$(-BW, -0.6BW]$	0.25	0.25	0	0	0	0
$(-0.6BW, -0.2BW]$	0.25	0.25	0	0	0	−0.125
$(-0.2BW, 0.2BW]$	0.375	0.125	0	0	0.217	0
$(0.2BW, 0.6BW]$	0.25	0.25	0	0	0	0.125
$(0.6BW, BW]$	0.25	0.25	0	0	0	0
(BW, ∞)	0	0	0	0	0	0

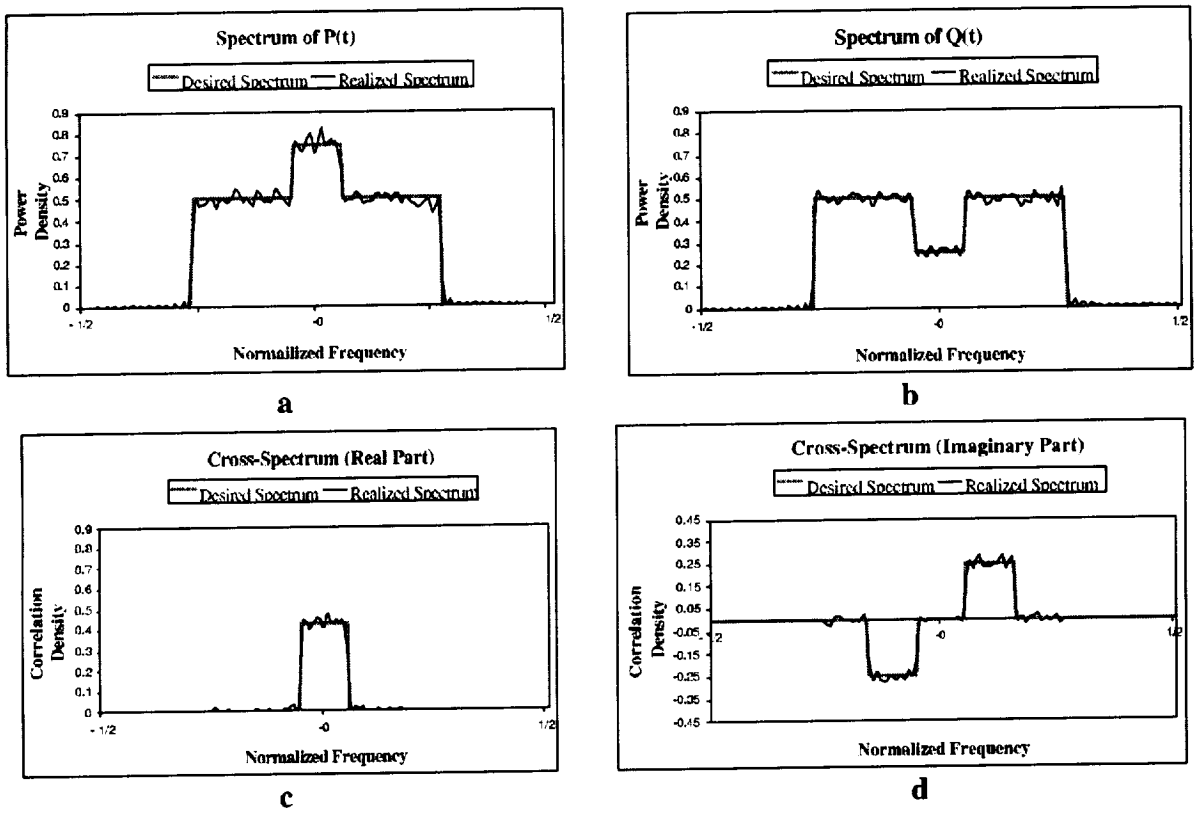

Figure 1: Example of realized spectra (plotted with desired spectra) for complex processes P(t) and Q(t): (a) power spectral density of P(t); (b) power spectral density of Q(t); (c) complex cross spectra of P(t) and Q(t) (real part); (d) complex cross spectra of P(t) and Q(t) (imaginary part)