"Comparing Satellite Rainfall Estimates with Rain-Gauge Data: Optimal Strategies Suggested by a Spectral Model"

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Popular Summary

There are a number of satellites in orbit now that are used to produce global maps of rainfall. The satellites that orbit closest to the earth produce the most accurate estimates of rain amounts, but they generally fly over any given spot on the earth only about once per day. (The Tropical Rainfall Measuring Mission satellite (TRMM) is one such satellite). The data these "low-flying" satellites provide can therefore only be used to calibrate the estimates provided by the more distant satellites, which are less accurate but view the earth almost continuously; or the satellite data can be averaged over many visits to give some information about the average rainfall in each area.

The methods used by the satellites to make their rain estimates are quite indirect. It is important that they be checked against direct observation of rainfall from instruments on the ground. Ground-based radar systems provide useful information about rainfall over large areas, but the methods used by radar to make quantitative estimates of rainfall suffer from the fact that radars, too, observe rain from a distance. Rain gauges, on the other hand, measure rainfall directly, but only over an area the size of a dinner plate. The errors rain gauges make are fairly well understood, and so, except for their limited coverage, they are ideal for checking the satellite estimates. Comparison of satellite estimates with rain-gauge measurements is often frustrating, however, because satellites can at best attempt to measure rain amounts over areas many kilometers in diameter around a gauge, whereas rain gauges can only record what falls in an area some tens of centimeters in diameter.

This paper investigates the "noisiness" in the comparisons of satellite and gauge rain estimates due to the very different observational characteristics of the two: the satellite catches glimpses of large areas at infrequent intervals, whereas rain gauges record what happens in small areas continuously. A statistical model used in some earlier studies of TRMM is shown to be particularly relevant to addressing this question. It captures both the high variability of rain from place to place and moment to moment, and also how the level of variability diminishes when rain is averaged over area or over time. The model verifies that comparison of what a satellite sees at the moment it flies over a gauge with what the gauge records during an interval of time around the satellite's visit is indeed very noisy, but that there is a best choice for the time interval of gauge observations that should be used in the comparison. It shows, not surprisingly, that the comparisons get better when there is more than one gauge in the vicinity, but that there is still an optimal time interval of gauge data around the satellite overflight time that should be used. The optimal interval shrinks as the number of gauges increases.

The paper also shows that when only a few gauges are available, many months of averaging of both the satellite data and the gauge data are needed to reduce the noise levels in the comparisons of the two averages to tolerable levels, and, somewhat surprisingly, that on these time scales the satellite data over an area hundreds of kilometers around the gauge(s) should be averaged to reduce the noisiness of the comparisons to the lowest possible level-- rather than using only satellite data taken as close to the gauge(s) as possible. It also shows that there are more complicated ways of averaging the gauge data that will improve the quality of the satellite/gauge comparison still further. These results should be helpful in choosing the most informative way to check satellite data with data from rain gauges.

Comparing Satellite Rainfall Estimates with Rain-Gauge Data: Optimal Strategies Suggested by a Spectral Model

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Abstract. Validation of satellite remote-sensing methods for estimating rainfall against rain-gauge data is attractive because of the direct nature of the rain-gauge measurements. Comparisons of satellite estimates to rain-gauge data are difficult, however, because of the extreme variability of rain and the fact that satellites view large areas over a short time while rain gauges monitor small areas continuously. In this paper, a statistical model of rainfall variability developed for studies of sampling error in averages of satellite data is used to examine the impact of spatial and temporal averaging of satellite and gauge data on intercomparison results. The model parameters were derived from radar observations of rain, but the model appears to capture many of the characteristics of rain-gauge data as well. The model predicts that many months of data from areas containing a few gauges are required to validate satellite estimates over the areas, and that the areas should be of the order of several hundred km in diameter. Over gauge arrays of sufficiently high density, the optimal areas and averaging times are reduced. The possibility of using time-weighted averages of gauge data is explored.

1. Introduction

Satellites are the only practicable means of monitoring rainfall on a global scale, but remote-sensing methods used to estimate rainfall from space-borne instruments are inexact. Quantitative use of the satellite products requires that they be accompanied by estimates of their accuracy, and along with the decades-long effort to improve satellite rain estimates there has been a parallel effort to compare the estimates from space with more direct observations taken from the ground in order to determine the error characteristics of the satellite estimates whenever possible. An especially extensive set of such studies of satellite algorithms is reviewed by *Ebert et al.* [1996].

Since satellite-instrument estimates are inherently limited by the resolution of the instrument, loosely referred to here as the field of view (FOV), the satellite estimates represent rain rates averaged over areas of the order of the instrument resolution in size [see, for example, *Olson*, 1989], with additional blurring because the satellite estimate actually depends on the state of the column of atmosphere above the FOV-sized area rather than on the rain rate at the surface. Verifying such estimates with ground observations requires that accurate estimates of rain averaged over FOV-sized areas be provided. Hydrologists have been grappling with this type of problem since well before the needs for satellite verification arose, and it is a notoriously difficult one.

There are many ground-based approaches to estimating area-averaged rain rate. Many involve remote-sensing methods such as radar. We will principally

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concern ourselves here with estimates made with rain gauges. Rain gauges have the advantage that they measure rain in a fairly direct manner, and the errors they make are generally easily understood and to a considerable extent rigorously quantifiable. They are relatively inexpensive to deploy and are located at many sites around the world. Aside from all the mishaps to which any mechanical or electrical apparatus left outdoors is prone, they have the major disadvantage that they measure only what is falling within an area a few tens of centimeters in diameter. Inferences about what might have fallen in the tens of square kilometers around the gauge can only be made to the extent that what the gauge encounters is representative of what happened in the surrounding area.

Rain rates vary rapidly in time and space on the scale of human perceptions, as anyone who has watched rain falling over a large flat open area can attest, and the same can be said for both the larger scales accessible to radars and satellites and the smaller scales explored with acoustical and optical instruments. What a rain gauge measures can therefore represent what has occurred in its neighborhood only in an average sense at best. This question has been investigated by setting out arrays of rain gauges and comparing the rain totals obtained by each gauge. Gradients in the rain totals can sometimes persist and be explained by local topographic and meteorological influences. See, for just a few of many examples, *Court* [1960] and *Wood et al.* [2000]. The representativeness of each rain gauge must therefore be examined carefully for such influences.

The problem of how well a gauge average agrees with the average rainfall in its vicinity has been extensively studied theoretically as well as empirically. Examples of such studies include *Rodríguez-Iturbe and Mejía* [1974a, b], *Silverman et al.* [1981], and *Morrissey et al.* [1995]. A particularly interesting and extensive empirical study was carried out by *Rudolf et al.* [1994]. They showed that the rms difference of the gauge average from the true areal average appeared to depend in a simple way on the number of gauges in the area. A theoretical argument for a dependence similar to what they found is given by *McCollum and Krajewski* [1998], who also investigated the error levels in averages of rain-gauge data used to estimate areal monthly averages as a function of spatial correlation of the rain-gauge data. *Krajewski et al.* [2000] used rain-gauge correlations found in U.S. data to make quantitative estimates of error levels for areal averages of such data.

In comparing the average rain rate seen by gauges to the average of satellite estimates made in the vicinity of the gauges, a number of questions arise:

- How much disagreement between the two averages is attributable to the fact that a gauge sees a very small area continuously, whereas the satellite sees a very large area around the gauge only intermittently?

- What is the best time interval over which to compare the two averages?

- Over what area around the gauge(s) should the satellite averages extend?

- If the gauge data are available as a function of time (e.g., minute by minute, hour by hour), would it be better to use time-weighted averages of the gauge data with weightings determined by the overflight times of the satellite?

The answers to these questions are not usually obvious. An important aspect of these problems is that rain statistics change depending on how the data are averaged. Daily rain-gauge data are correlated over shorter distances than monthly rain data, as can be inferred, for example, from the correlation lengths of order 10^1–10^2 km seen for daily rainfall by *Abtew et al.* [1995] and *Ciach et al.* [1997], as opposed to the correlation lengths of many hundreds of kms seen for monthly rainfall by *Mooley and Ismail* [1982] and *Morrissey* [1991]. Small-area averages of radar-derived rain rates are correlated over shorter times than large-area averages, as was shown, for example, by *Laughlin* [1981]. Determining the answers directly from data without the aid of a statistical model can be frustrating because of the inherent noisiness of rain statistics, so that extremely long averaging times are required in order to get stable results.

One important issue in the comparison of satellite averages to averages of rain-gauge data that is not addressed by the above questions is the error inherent in the satellite estimates themselves, due to all the problems associated with remote sensing. This is discussed, for example, by *Berg and Avery* [1995].

A number of theoretical studies of the problem of comparing averages of satellite rain estimates with averages of data from one or more gauges have been carried out. The studies by *North and Nakamoto* [1989], *North et al.* [1994], and *Yoo et al.* [1996], for example, use stochastic models in which time and space scales are interrelated. In earlier studies, error levels in satellite/gauge comparisons were estimated for specified averaging areas and time intervals. In this paper we shall

explore how they change with area and time. The questions posed above will be examined with the aid of a model of rainfall statistics that was primarily developed for studies of sampling errors in monthly averaged satellite estimates of rainfall [*Bell and Kundu*, 1996, hereinafter BK96]. It is able to describe the changes in rainfall statistics with averaging times and averaging areas mentioned above to an impressive degree, and is therefore in that respect well suited to the study of these problems. Only second-order moment statistics of rain (variances, correlations) are described by the model, however, and the model cannot address problems having to do with the higher-moment statistics of rain without additional assumptions.

Section 2 describes a framework for investigating how much satellite and gauge averages tend to differ due to their different observational characteristics. Section 3 describes the statistical model used in the study and its ability to handle different time and space scales. Section 4 explores a number of different averaging schemes and shows that there is often an optimal scale for comparison. Section 5 raises the possibility of using time-varying weighted averages of gauge data to help reduce sampling error in satellite–gauge comparisons. The results are discussed in section 6. Mathematical details of some of the calculations are given in two appendices.

2. Comparing Satellite and Gauge Averages

Comparisons of single, "instantaneous," coincident observations by satellites and rain gauges tend not to be very informative since, as we shall see, their very different sampling characteristics introduce too much uncertainty into the comparison. In addition, the instantaneous satellite estimates for single FOVs are themselves commonly believed to have errors of order 50% or more [see, for example, *Wilheit*, 1988; *Olson et al.*, 1996]. (Errors in satellite estimates for the average rain rate in a FOV will be referred to here as *retrieval errors*.) The goal in comparing averages of satellite rain-rate estimates to averages of rain-gauge data is finding evidence (or lack of it) for bias in the satellite estimates.

Biases in satellite estimates cannot be represented by a single number. They almost certainly vary with the kinds of rain being observed, the amounts, and a host of meteorological and climatological factors that will take long and patient research to unravel. The most informative comparisons of satellite and gauge averages will therefore be ones where just enough averaging is done to reduce the variability in the differences due to random sampling and retrieval error to a level where residual bias is detectable at a certain desired level. Large datasets consisting of long sequences of satellite observations in the neighborhood of rain gauges and the accompanying gauge data will need to be broken down into comparisons of subsets of observations, and indications of apparent bias examined for patterns that might indicate problems with the satellite retrievals in certain situations. For example, if the bias seemed to vary with the amount of stratiform precipitation, such a dependence might be revealed by comparing the biases seen in low-stratiform-amount and high-stratiform-amount cases. For this to be an effective approach the averaging must be sufficient to bring out the bias, but not so great that it reduces the dataset to too few cases.

As mentioned above, we expect averages of satellite data and of rain-gauge data to differ because they each include data that the other does not, referred to here as *sampling error*. There are also a number of other reasons the two averages might differ: the rain may not fall with equal probabilities at different times of the day, for example, or the mean rainfall may differ at different points (e.g., in hilly areas or near coasts). To the extent possible, the averages of the data must be adjusted to reduce the error due to these inhomogeneities to an acceptable level.

Let us first consider a simple example of the kinds of satellite/gauge comparisons one might wish to investigate, the difference between the average of satellite FOV estimates made in the neighborhood of a gauge during a single overflight of the gauge by the satellite and the average rain rate recorded by the gauge in a time interval bracketing the overflight time. Assume that the satellite overflight time occurs at $t = 0$ and that the rain gauge is located at position $\mathbf{x} = 0$, with $\mathbf{x} = \{x, y\}$. The average rain rate seen by the gauge over a time interval T is

$$R_{\mathbf{g}} = R_T(\mathbf{x} = 0) \tag{1}$$

with

$$R_T(\mathbf{x}) \equiv \frac{1}{T} \int_{-T/2}^{T/2} dt \, R(\mathbf{x}, t) \,, \tag{2}$$

where $R(\mathbf{x}, t)$ is the rain rate at point $\mathbf{x}$ at time t, and the rain rate over the gauge orifice has been approximated by the rain rate at the point $\mathbf{x} = 0$.

The satellite, on the other hand, is treated as providing an estimate of the average rain rate over an area A at time $t = 0$,

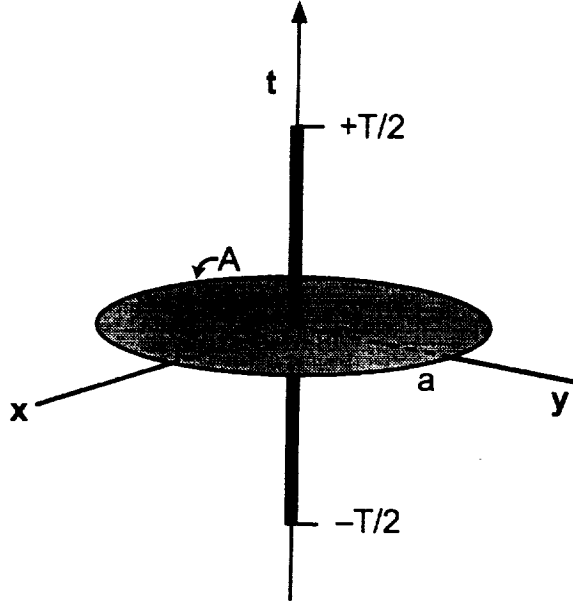$$R_{\mathbf{s}} = R_A(t = 0) \tag{3}$$

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Figure 1. Comparison of average rain rate over a period T observed by a gauge located at the center of an area A of radius a, when the area A is observed at $t = 0$ by a satellite.

with

$$R_A(t) \equiv \frac{1}{A} \int_A d^2\mathbf{x}\, R(\mathbf{x}, t) , \qquad (4)$$

where A is a circle of radius a with area $A = \pi a^2$. See the illustration in Figure 1. It would of course be possible to include several gauges and/or many satellite overflight times in the above averages, or to average over a rectangular area instead of a circular area. We will return to these questions later.

The satellite can only provide an estimate of R_s in (3) as an average of the FOV estimates that fall within the circular area A. When the area A is comparable in size to an FOV, the precise location of the gauge within the FOV matters. As we shall see later, however, our analysis suggests that averages over relatively large areas around the gauge are preferable, and, anticipating that, we will ignore the complication of specifying where the gauge is on the scale of satellite FOVs.

A bias in the satellite estimates is identified when the difference

$$\Delta R = R_s - R_g \qquad (5)$$

is larger than can be accounted for by the sampling differences of the two systems or by random retrieval and measurement errors. If there is no bias in the estimates,

ΔR should, on average, be zero. In order to test for the presence of a bias, we require an estimate of the random error components in ΔR. The mean squared difference of the averages is a useful measure of the error levels in the difference, defined as

$$
\begin{aligned}
\sigma^2 &\equiv \langle (\hat{R}'_s - \hat{R}'_g)^2 \rangle \\
&= \sigma^2_{\text{samp}} + \sigma^2_{\text{err,s}} + \sigma^2_{\text{err,g}} ,
\end{aligned} \qquad (6)
$$

where $\hat{R}_s$ and $\hat{R}_g$ are the satellite and gauge estimates of R_s and R_g in (3) and (1), respectively, with (unknown) estimation errors included; where the error variances in (6) are defined as

$$\sigma^2_{\text{samp}} = \langle (R'_s - R'_g)^2 \rangle , \qquad (7)$$

$$\sigma^2_{\text{err,s}} = \langle (\hat{R}'_s - R'_s)^2 \rangle , \qquad (8)$$

$$\sigma^2_{\text{err,g}} = \langle (\hat{R}'_g - R'_g)^2 \rangle ; \qquad (9)$$

and where the angular brackets indicate an average over an ensemble of meteorological situations similar to the one for which we are trying to estimate σ^2, and the primes indicate deviations from the ensemble mean; i.e., $R' \equiv R - \langle R \rangle$. In writing equation (6) it is implicitly assumed that sampling errors due to non-overlapping coverage, satellite retrieval errors, and gauge measurement errors are uncorrelated with each other. It is difficult to think of a plausible physical mechanism that would produce such correlations, but if reason were found to expect it, corrections for the cross-correlation effects would have to be added to the right-hand side of (6).

A typical comparison of satellite estimates with gauge averages results in a scatter plot of the satellite averages versus the gauge averages for the same areas and time periods. The quantity σ in equation (6) is a measure of the amount of scatter about the "ideal" $45°$ line on which the points would lie if the satellite estimates were perfect and the gauge averages gave the true rain rate over the area sampled by the satellite. In this paper we will concentrate on estimating the sampling error term in (6) based on models of the statistics of the "true" surface rain rates being estimated by the satellite and the gauge. Because a considerable amount of averaging (large A and T) is needed to reduce the error variance σ^2 to acceptable levels, the contributions to σ^2 by random retrieval and measurement errors represented by $\sigma^2_{\text{err,s}}$ and $\sigma^2_{\text{err,g}}$ tend to be considerably reduced, so that σ^2 is dominated by σ^2_{samp}, but this needs to be checked for each validation study.

If a good estimate of σ can be obtained, one can conclude that if $|\Delta R| > 2\sigma$, there is a strong probability ($\approx 95\%$) that a bias exists—always assuming that in-

homogeneities in the rain statistics have been compensated for and that the gauge data are accurate. The estimated bias in the satellite estimate would be

$$\text{Satellite bias} = \Delta R \pm \sigma , \qquad (10)$$

with "one-sigma" confidence limits. In the analyses that follow, determination of bias at the 10% level is used as a reasonable goal, i.e., $\sigma_{\text{samp}} < 0.1\,\bar{R}$, where $\bar{R}$ is the mean rain rate in the locality.

The sampling error variance defined in (7) can be written in terms of the space–time covariance of rain rate,

$$c(\mathbf{r}, \tau) = \langle R'(\mathbf{x} + \mathbf{r}, t + \tau) R'(\mathbf{x}, t) \rangle , \qquad (11)$$

which is assumed to depend only on the separation of the points $\{\mathbf{r}, \tau\}$. If (1) and (3) are substituted into the definition for σ_{samp}^2 in (7), we can use (11) to write (7) as

$$\sigma_{\text{samp}}^2 = c_{\text{ss}} + c_{\text{gg}} - 2c_{\text{sg}} \qquad (12)$$

with

$$
\begin{aligned}
c_{\text{ss}} &\equiv \langle (R_\text{s}')^2 \rangle \\
&= \frac{1}{A^2} \int_A d^2\mathbf{x}_1 \int_A d^2\mathbf{x}_2 \, c(\mathbf{x}_1 - \mathbf{x}_2, 0) , \quad (13)
\end{aligned}
$$

$$
\begin{aligned}
c_{\text{gg}} &\equiv \langle (R_\text{g}')^2 \rangle \\
&= \frac{1}{T^2} \int_{-T/2}^{T/2} dt_1 \int_{-T/2}^{T/2} dt_2 \, c(0, t_1 - t_2) , \quad (14)
\end{aligned}
$$

and

$$
\begin{aligned}
c_{\text{sg}} &\equiv \langle R_\text{s}' R_\text{g}' \rangle \\
&= \frac{1}{AT} \int_{-T/2}^{T/2} dt \int_A d^2\mathbf{x}\, c(\mathbf{x}, t) . \qquad (15)
\end{aligned}
$$

In the next section we describe a model for the rain-rate covariance (11) that was developed to estimate the sampling error in monthly averages of satellite data for a given area relative to the true monthly average that would have been obtained if the satellite were capable of continuous observation of the area. The model parameters were adjusted to fit the statistics of radar data over oceanic sites during two field campaigns, the Global Atmospheric Research Program (GARP) Atlantic Tropical Experiment (GATE) and the Tropical Ocean Global Atmosphere (TOGA) Coupled Ocean–Atmosphere Response Experiment (COARE). Given such a model, calculations of sampling error of satellite/gauge comparisons like the one described above can be carried out. These will be discussed in the following sections.

3. Spectral model of rain-rate covariance

The model is described in BK96. Its four parameters characterize the space–time covariance of rain rate (11) needed for calculations like the one described in the previous section. It captures an aspect of rain behavior that is not always represented in statistical models of rain: time scales of variations in area-averaged rain rate become longer as the area is increased, and spatial correlations of time-averaged rain become longer as the time interval is lengthened. This phenomenon is partially captured by statistical models describing rain as randomly created cells, within which daughter cells are grown randomly, which in turn themselves grow daughter cells, etc., with faster and faster time scales and smaller and smaller spatial scales [e.g., *Rodríguez-Iturbe et al.*, 1987; *Smith and Krajewski*, 1987]. The present model was originally motivated by a model developed by *Bell* [1987] that used spectral methods to establish the space–time statistics of the rain being modeled. It is in some respects a generalization of the diffusive model of *North and Nakamoto* [1989].

Although the model is described in detail in BK96, we review it briefly here in order to introduce the parameters needed for the computations. The space–time covariance of point rain rates (11) is expressed in terms of the Fourier transform in space and time of a spectral power function,

$$
\begin{aligned}
c(\mathbf{x}, \tau) &= (2\pi)^{-3/2} \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} dk_x \int_{-\infty}^{\infty} dk_y \\
&\quad \times e^{i(\mathbf{k}\cdot\mathbf{x} - \omega\tau)} \tilde{c}(\mathbf{k}, \omega) , \qquad (16)
\end{aligned}
$$

where $\mathbf{k}$ is a 2-dimensional wavevector $\{k_x, k_y\}$. The spectral power is given in this model by

$$\tilde{c}(\mathbf{k}, \omega) = \frac{F_0}{\omega^2 + 1/\tau_k^2} , \qquad (17)$$

$$\tau_k = \frac{\tau_0}{(1 + k^2 L_0^2)^{1+\nu}} , \qquad (18)$$

with $k = |\mathbf{k}|$. The timescale for fluctuations with spatial wavelength $2\pi/k$ is τ_k. It gets longer as the wavelength gets larger, approaching τ_0 for wavelengths longer than L_0. Spatial fluctuations in the model tend not to be correlated beyond L_0 and temporal variations tend not to be correlated longer than τ_0. Spatial variability at small scales increase as the exponent ν becomes more negative. See BK96 for details about the motivation and interpretation of the model.

The model requires 4 parameters in order to specify it completely: γ_0, ν, L_0, and τ_0, with γ_0 defined for

convenience by

$$F_0 = \sqrt{2/\pi}\,\Gamma(1+\nu)(L_0^2/\tau_0)\gamma_0 , \qquad (19)$$

where $\Gamma(n)$ is the Euler gamma function. Model parameters were obtained in BK96 that best fit the statistics of radar-derived rain-rate maps over the eastern Atlantic produced in GATE.

Kundu and Bell [2003] recently obtained model fits to radar-derived rain data collected over an area in the western tropical Pacific during TOGA COARE. Radar observations were used from two ships designated "MIT" and "TOGA" during three cruises, each providing about three weeks of data. Data used from the cruises spanned the periods

Cruise 1:	11 Nov – 10 Dec 1992,
Cruise 2:	15 Dec 1992 – 18 Jan 1993, (20)
Cruise 3:	23 Jan – 23 Feb 1993.

Separate sets of parameters were obtained for each ship for each cruise. The parameter values for which the model best fits the data statistics are given in Table 1.

Since the model parameters in Table 1 were obtained from gridded radar data with grid spacings of 4 km for GATE and 2 km for TOGA COARE and covering a time period of 2–4 weeks, it is not obvious that the model can successfully describe the smaller scale statistics of rain-gauge data nor the statistics of long time averages of gauge data, although it does quite well at fitting the data over the range of scales available in the radar data. To investigate the model's behavior on different scales, its predictions for the spatial correlation of time-averaged gauge data can be obtained by using an equation like (14) with "0" replaced by the separation of the two gauges. A formula for the spatial correlations is given in Appendix A in equation (A23).

Figure 2 shows the spatial correlations for 15-min-averaged gauge data predicted by the model using parameter values in Table 1. As can be seen from Figure 2, the model with TOGA COARE parameters predicts correlation lengths ranging from about 6 km to 13 km, whereas the GATE parameters predict a correlation length of about 33 km. There were unfortunately not enough gauges deployed during TOGA COARE and GATE to test these predictions against actual gauge data. The TOGA COARE correlation lengths are comparable to what were found for a gauge array near Melbourne, Florida, by *Habib et al.* [2001], who found correlation lengths of about 4–5 km for 15-min-averaged gauge data for August–September 1998.
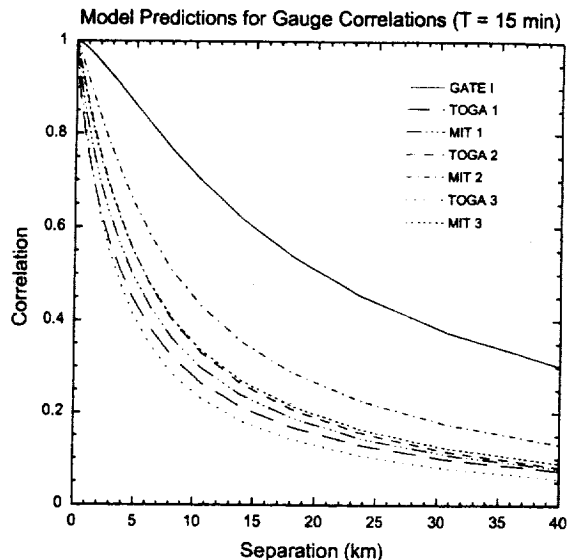

Figure 2. Predictions of spatial correlations of 15-min-averaged gauge data by spectral model using parameter values in Table 1.

[It is interesting to note that the model also predicts a fictitious "nugget" effect when TOGA COARE parameters are used, in the sense that an exponential fit to the model spatial correlations is improved if the exponential fit has the mathematical form $r_0 \exp(-s/d_0)$ with $r_0 < 1$. The model correlations with TOGA COARE parameters suggest values of $0.90 \lesssim r_0 \leq 0.99$.]

On the other hand, correlations of monthly averaged gauge data (Figure 3) are predicted by the model to have spatial correlation lengths of about 50–80 km for TOGA COARE and about 150 km for GATE. Again, gauge statistics for these cases are unavailable, but *Krajewski et al.* [2000] found correlation lengths of order 200–460 km for summertime monthly averages from 14 U.S. gauge arrays, and *Morrissey* [1991] found correlation lengths of order 500 km for Pacific atoll rain gauges, perhaps a factor of 2 larger than what the model with the parameter values in Table 1 predicts. This suggests that the effects of large-scale variations in rainfall occurring over periods of a month or more are, not surprisingly, underestimated in the statistics for 3–4 weeks of radar data such as were used in obtaining the parameters in Table 1. Predictions using the model with the parameters in Table 1 must therefore be tempered by these considerations. We will return to this issue in the discussion at the end.

Table 1. Parameter Values for Rain-Rate Covariance Model.

Dataset	γ_0 (mm^2 h^{-2})	ν	L_0 (km)	τ_0 (h)	R (mm h^{-1})	σ_A^2 (mm^2 h^{-2})
GATE Phase I	1.0	−0.11	104.	17.6	0.50	0.461
TOGA Cruise 1	0.067	−0.335	94.06	8.30	0.139	0.039
MIT Cruise 1	0.086	−0.297	73.89	8.64	0.134	0.032
TOGA Cruise 2	0.616	−0.239	53.81	9.56	0.351	0.127
MIT Cruise 2	0.206	−0.205	70.40	13.43	0.229	0.062
TOGA Cruise 3	0.127	−0.290	61.04	12.20	0.155	0.035
MIT Cruise 3	0.180	−0.259	64.94	9.71	0.200	0.052

Parameters for the model spectrum, defined in (16), (17), and (18), from fits to radar data from Phase I of GATE and from the ships MIT and TOGA during the three TOGA COARE cruises listed in (20). Average rain rate for each dataset and model-predicted variance of area-averaged rain rate for a 314-km-diameter circle with the same area as a 2.5° × 2.5° square are given in the last two columns.
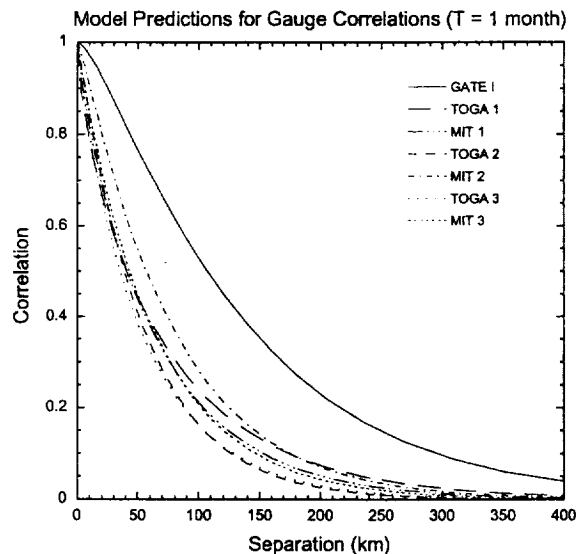

Figure 3. Predictions of spatial correlations of monthly averaged gauge data. Note change of scale of abscissa from Figure 2.

4. Sampling Errors for Validation

In this section we will explore the behavior of sampling error in the differences between satellite observations over an area surrounding one or more gauges for various averaging times T and averaging areas A. All of the calculations are done using circular areas rather than using square, grid-box shaped areas. The difference in the results should be very small if the areas are equal in magnitude. For example, the last column of Table 1 shows the variance of area-averaged rain rate calculated from the model for each set of parameters listed in the table for a 314-km-diameter circle, using (A9). At the equator, a 2.5° × 2.5° square box has the same area as the circle. When variances of area-averaged rain rate are calculated for the square area they are found to be smaller than the values for the circle by only about 1.5%.

4.1. Single Satellite Overflight, Single Gauge

As a first example, consider the problem of comparing an average of satellite estimates from a single overflight of an area A with an average rain rate seen by a gauge over an interval of time T bracketing the overflight time of the satellite, as sketched in Figure 1. The rms difference σ_{samp} between the two averages can be calculated using (12) and the model covariance (16). Results for σ_{samp}/R are plotted in Figure 4 using the GATE model parameters from Table 1. A number of conclusions are illustrated by this figure: 1) Comparisons of a single-gauge average with one satellite pass are, not surprisingly, extremely noisy, as evidenced by
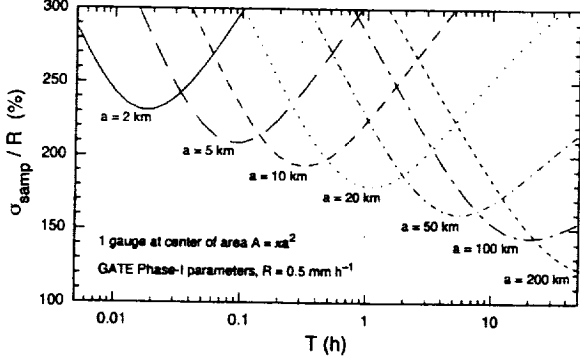
Figure 4. Model predictions of relative sampling error for comparisons of a single satellite overflight over an area A of radius a to a gauge average over an interval T bracketing the overflight time, as depicted in Figure 1.
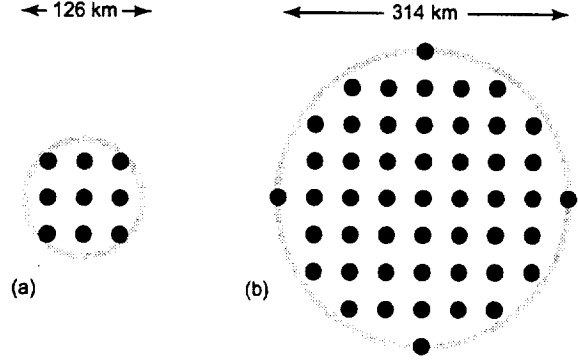

Figure 5. Idealized distribution of gauges with a density similar to that of the Oklahoma Mesonet. Inter-gauge distance is 39 km. a) Circular area with $a = 63$ km with same area as a $1° \times 1°$ grid box (9 gauges); and b) circular area with $a = 157$ km with same area as a $2.5° \times 2.5°$ grid box (49 gauges).

relative errors considerably larger than 100%, no matter what the averaging time T. 2) The comparisons become less noisy as the area averaged over increases. 3) Less obviously, for a given area A there is an optimal accumulation interval T for the gauge, and T increases as the satellite averaging area A increases.

These sampling error results depend on the rain statistics. Model predictions (not shown) for σ_{samp}/R are considerably higher when the TOGA COARE parameter values in Table 1 are used, especially for the smaller areas A. This is largely due to the fact that the TOGA COARE statistics suggest higher spatial variability at small scales than for GATE, as evidenced by the larger negative values of the exponent ν in Table 1.

4.2. Single Satellite Overflight, Gauge Array

As a second example, consider the problem of comparing a satellite average over an area A where an array of gauges is present, thus providing a better estimate of the area-averaged rain rate in A than a single gauge can. In such a case, the gauge average in equation (1) is replaced by the n-gauge average

$$R_{ng} = \frac{1}{n}\sum_{i=1}^{n} R_T(\mathbf{x}_i) \,, \qquad (21)$$

where the locations of the n gauges are specified by the positions $\mathbf{x}_i$, $i = 1,\ldots,n$. We must then calculate an expression like (7) with R_g replaced by R_{ng}. The term

c_{gg} is replaced by the double-sum expression

$$c_{ng,ng} = \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\langle R_T'(\mathbf{x}_i)R_T'(\mathbf{x}_j)\rangle \,, \qquad (22)$$

which can be calculated from the spectral model using the covariance $c_{TT}(|\mathbf{x}_i - \mathbf{x}_j|)$, an expression for which is obtained in Appendix A (equation A23). Likewise, the cross-term c_{sg} in (7) is replaced by the n-gauge expression

$$c_{s,ng} = \frac{1}{n}\sum_{i=1}^{n}\langle R_A' R_T'(\mathbf{x}_i)\rangle \,, \qquad (23)$$

which can be calculated from the model using the covariance $c_{AT}(|\mathbf{x}_i|)$ given by (A20) in Appendix A.

As an example of such a situation, consider the sort of comparisons that might be possible over the Oklahoma Mesonet described by *Brock et al.* [1995]. About 100 gauges are distributed over an approximate $3° \times 5°$ area, with an average inter-gauge distance of about 40 km. Suppose satellite estimates over grid-box areas of order $1° \times 1°$ or $2.5° \times 2.5°$ are compared to averages of data from gauges within these areas. In order to make the model calculations easier, we consider an idealized version of this problem in which gauges are equally spaced within circular disks with the same area as the grid boxes, i.e., with radii $a = 63$ km and $a = 157$ km respectively. Figure 5 shows a sketch of the areas with the assumed gauge positions marked. Based on this configuration, the sampling error for comparison

of a single overflight of the array by a satellite can be computed using the n-gauge analogue to equation (12),

$$\sigma^2_{ng,samp} = c_{ss} + c_{ng,ng} - 2c_{s,ng} \qquad (24)$$

using expressions (22) and (23). Relative sampling errors for comparisons over the two areal sizes are shown as smooth curves in Figure 6 as a function of the gauge averaging time T, calculated using the GATE model parameters. The optimal gauge averaging time for the larger area grid box containing 49 gauges is predicted to be about 1 h, and relative error for the comparison is 30%, much lower than for the single-gauge comparisons shown in Figure 4, as expected. The same averaging time, 1 h, is best for the $1° \times 1°$ box, though with much larger comparison errors.

When the same calculations are done with TOGA COARE model parameters (not shown), the comparison errors for regularly spaced gauge arrays are predicted to be larger than 100%, and the optimal averaging time increases to several hours.

4.3. Single Satellite Overflight, Random Gauges

It is interesting to investigate how sensitive the results are to the idealized spacing used in the previous example. It is quite easy to calculate the average of $\sigma^2_{ng,samp}$ over all possible configurations of the n gauges within the area A allowing each of the positions x_i to be arbitrarily assigned within A. Averaging over every possible configuration is equivalent to acting on expressions (22) and (23) with the averaging operation $(1/A) \int_A d^2x_i$ for each gauge i. Except for the terms involving $[R'_T(x_i)]^2$, this is equivalent to replacing $R_T(x_i)$ for each gauge by R_{AT}, the space–time average of rain rates everywhere within A over the interval T, as defined in equation (A3) in Appendix A. For $c_{ng,ng}$ one obtains

$$\langle c_{ng,ng} \rangle_x = \sigma^2_{AT} + \frac{\sigma^2_T - \sigma^2_{AT}}{n} , \qquad (25)$$

with the bracket operation on the left hand side indicating an average over gauge locations, and with σ^2_T and σ^2_{AT} defined in Appendix A by equations (A13) and (A17).

The remaining term in (24), averaged over gauge positions, is

$$\langle c_{s,ng} \rangle_x = \langle R'_A R'_{AT} \rangle , \qquad (26)$$

which is computed for the model in Appendix A, with the result given in equation (A25). Relative sampling error for the two cases studied in the previous subsection for randomly placed gauges is shown in Figure 6
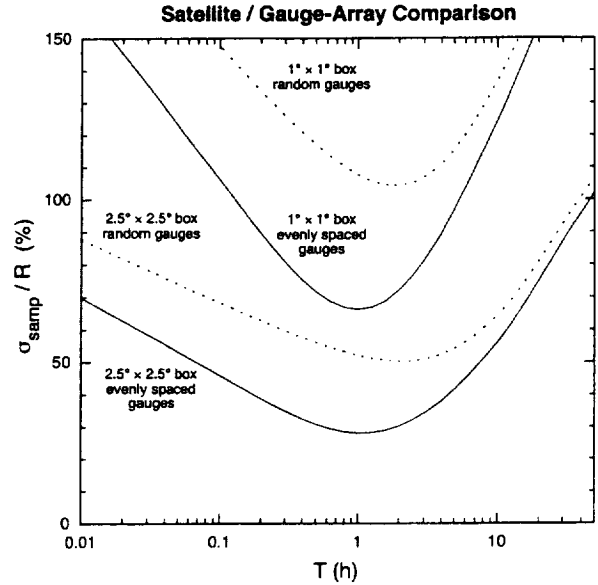

Satellite / Gauge-Array Comparison

Figure 6. Model predictions of relative sampling error for comparison of a single satellite overflight over circular areas equivalent to $1° \times 1°$ and $2.5° \times 2.5°$ boxes containing 9 and 49 gauges respectively, similar in density to that of the Oklahoma Mesonet. Smooth curves show sampling error for gauges spaced as depicted in Figure 5, plotted as a function of the time interval T over which the gauge data are averaged. Dotted curves show sampling error averaged over all possible random placements of the gauges within the areas. GATE model parameters were used in the calculations.

as dotted curves. As expected, when the gauge locations become more random, comparison error tends to increase, and, perhaps less obviously, the optimal averaging time increases as well, probably because of the tendency for there to be larger gaps between the randomly placed gauges.

4.4. [Aside:] A Classic Hydrological Problem

Rain-gauge arrays have long been used to estimate the average rain rate R_{AT} for a time period T over an area A covered by the array. (Average rainfall is given by TR_{AT}.) It is interesting to note that the mean squared error in the classic hydrological problem of determining R_{AT} with a gauge array is easily obtained for the random-gauge case just discussed, using (25) and $\langle R'_{ng} R'_{AT} \rangle_x = \sigma^2_{AT}$, as

$$\sqrt{\langle (R_{ng} - R_{AT})^2 \rangle_x} = \sqrt{\frac{\sigma^2_T - \sigma^2_{AT}}{n}}, \qquad (27)$$

which gives the approximate $n^{-1/2}$-dependence on gauge number found by *Rudolf et al.* [1994] in their studies, and indicates that the coefficient of $\sigma_T/\sqrt{n}$ they obtained is predicted to be

$$\sqrt{\langle (R_{ng} - R_{AT})^2 \rangle_x} = \left(\frac{\sigma^2_T - \sigma^2_{AT}}{\sigma^2_T} \right)^{1/2} \frac{\sigma_T}{\sqrt{n}}; \qquad (28)$$

the coefficient $(\sigma^2_T - \sigma^2_{AT})^{1/2}/\sigma_T$ depends solely on the spatial correlation of the gauge data at averaging time T. Physically plausible spatial correlations will always produce coefficients between 0 and 1. Note that equation (28) is a consequence of equation (25) alone and does not depend on the particular model we are using. When the coefficient is calculated using the model for a case analogous to the ones studied by *Rudolf et al.* [1994], assuming a disk with area equal to that of a $2.5° \times 2.5°$ grid box for monthly averaged gauge data, the model with GATE parameters predicts a coefficient value of 0.76, while with TOGA COARE parameters the model predicts coefficients ranging from 0.88 to 0.93, depending on the case.

Rudolf et al. [1994] fitted their collective results for relative mean absolute error for gauge arrays in Australia, Germany, and the USA, to an approximate form $0.865 \times \sigma_T/n^{0.555}$ (neglecting a small additive constant term). Since equation (28) is written for rms error instead of mean absolute error, the coefficient 0.865 found by *Rudolf et al.* [1994] should be multiplied by $\sqrt{\pi/2}$ to be compared with the coefficient in (28), as pointed out by *McCollum and Krajewski* [1998]. Since

$\sqrt{\pi/2} \times 0.865 = 1.1$ is greater than 1, the effects of spatial correlation of the gauges predicted by (28) do not seem to have shown up in the results obtained by *Rudolf et al.* [1994]. This may indicate the presence of greater than expected sampling error in the coefficient obtained by *Rudolf et al.* [1994], possibly due to non-normality in the distribution of the errors, or the influence of inhomogeneity effects on the statistics. *Rudolf et al.* [1994] found that sampling error appeared to decrease slightly faster with gauge number than $n^{-1/2}$. This may be due to the fact that the real gauge arrays studied by *Rudolf et al.* [1994] were not randomly distributed, since gauges in real arrays tend to be spaced a certain minimum distance apart, whereas the $n^{-1/2}$ behavior derived above depended on the randomness assumption.

4.5. Monthly Averages, Many Satellite Visits

It is clear from the above results that using rain-gauge data to validate satellite estimates at the 10% level requires averaging over more than one satellite overflight of the gauges. We turn next to comparisons of monthly averaged gauge data with averages of satellite data taken in the vicinity of the gauges during the month. In this case the time interval $T = 1$ month is specified beforehand, and we investigate how σ_{samp}/R changes with A, the area around the gauge(s) over which the satellite data are averaged.

The low earth-orbiting satellites carrying microwave instruments tend to revisit a location about once per day, at least in lower latitudes, averaging about 30 visits per month. To simplify our calculations, we assume that a satellite visits a site at regular intervals Δt, and that the visit times are given by

$$t_j = t_0 + (j - 1)\Delta t, \quad j = 1, \ldots, m. \qquad (29)$$

The satellite average to which the gauge average R_g is compared is given by

$$R_s = \frac{1}{m} \sum_{j=1}^{m} R_A(t_j), \qquad (30)$$

where $R_A(t)$ is defined in (4). Mean squared sampling error is thus given by (12), taking into account the effect of multiple satellite visits. In particular,

$$\begin{aligned} c_{ss} &= \langle (R'_s)^2 \rangle \\ &= \frac{1}{m^2} \sum_i \sum_j \langle R'_A(t_i) R'_A(t_j) \rangle. \end{aligned} \qquad (31)$$

This can be simplified using the lagged covariance of area-averaged rain rate $c_{AA}(\tau)$, defined in Appendix A

in (A31), and the identity

$$\sum_{i=1}^{m}\sum_{j=1}^{m} f(j-i) = \sum_{u=-m}^{m} (m - |u|)f(u) \qquad (32)$$

to obtain

$$c_{ss} = \frac{1}{m}\sigma_A^2 + \frac{2}{m}\sum_{u=1}^{m-1}\left(1 - \frac{u}{m}\right)c_{AA}(u\Delta t), \qquad (33)$$

with $\sigma_A^2 = c_{AA}(\tau = 0)$. Because $c_{AA}(\tau)$ falls off rapidly for $\tau \gg 1$ day, (33) is well approximated by

$$c_{ss} \approx \frac{1}{m}\left[\sigma_A^2 + 2\sum_{u=1}^{\infty} c_{AA}(u\Delta t)\right]. \qquad (34)$$

The cross term c_{sg} for this case,

$$c_{sg} = \frac{1}{m}\sum_{j=1}^{m}\langle R'_A(t_j)R'_T\rangle, \qquad (35)$$

can likewise be well approximated by

$$c_{sg} \approx \langle R'_A(0)R'_T\rangle, \qquad (36)$$

which is dealt with in equation (A19) of Appendix A. Sampling error for multiple satellite visits during $T = 1$ month can therefore be calculated from

$$\sigma_{samp}^2 \approx \sigma_T^2 + \frac{1}{m}\left[\sigma_A^2 + 2\sum_{u=1}^{\infty} c_{AA}(u\Delta t)\right] - 2c_{AT}(\mathbf{b} = 0). \qquad (37)$$

Model predictions for each of the terms in (37) are obtained in Appendix A.

Figure 7 shows the relative sampling error for a typical sampling interval of $\Delta t = 1$ day, and also for more frequent visits, down to the interval $\Delta t = 3$ h being discussed for a new satellite system called the Global Precipitation Mission (GPM). The optimal area for satellite averages being compared with a single rain gauge over one month is quite large, of the order of a 2.5° box for typical sampling intervals of once per day. As the satellite visit interval becomes shorter, the optimal area for averaging shrinks, to one smaller than a 1° box for a GPM-like case. Note that a month of averaging is still not sufficient for achieving comparisons at the 10% level. Approximately 6 months are required to reach that level for validation of a single satellite's estimates, or more gauges within the area are required. With two gauges, for instance, the required averaging time (not shown) drops to about 4 months. In a GPM-like era with the equivalent of 3-hourly satellite visits to a gauge site it may be possible to establish bias levels at the 10% level in a single month with just a few gauges in an area.
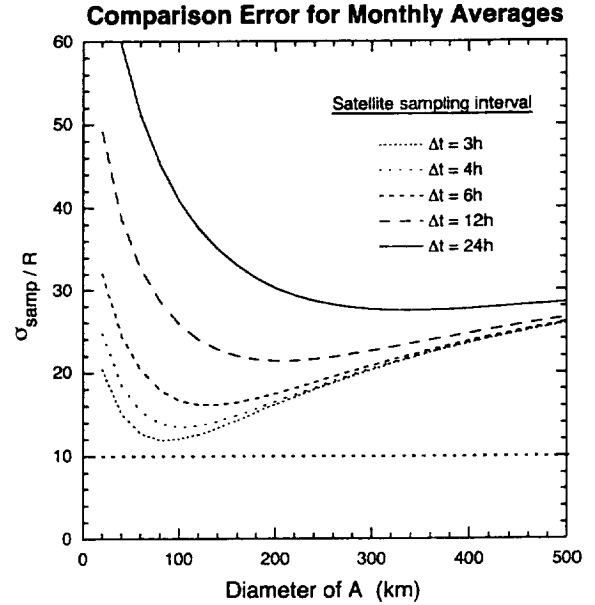

Comparison Error for Monthly Averages

Satellite sampling interval
· · · · · · · · · $\Delta t = 3h$
· · · · · · $\Delta t = 4h$
- - - - - $\Delta t = 6h$
— — — $\Delta t = 12h$
——— $\Delta t = 24h$

Figure 7. Relative sampling error predicted by model using GATE statistics for comparison of monthly averages of data from a single gauge with averages of all satellite estimates during the month for an area A around the gauge. The satellite is assumed to visit at intervals Δt. Satellites with microwave instruments typically visit at intervals $\Delta t \approx 24$ h. A dashed line indicates error at the 10% level.

5. Time-Weighted Gauge Data

The comparison of long time averages of gauge data with averages of satellite data in the previous section used straightforward averaging of the gauge data. Suppose one were to allow the weighting of the gauge data to vary in time depending on how far away in time the gauge observation is from a satellite overflight occurrence? In this section we look at a simplified version of this proposition to obtain an estimate of how much improvement in the validation procedure might be possible by using time-weighted averages of the gauge data.

As an example, assume that the gauge data are available at hourly intervals for a period H hours in length, of the order of a month or more. The gauge data are therefore provided as a sequence of hourly averages $R_T(t_j)$, $T = 1\,\mathrm{h}$, $j = 1,\ldots,H$, where $R_T(t)$ is the time-averaged gauge data over an interval T centered on time t, as defined in Appendix A in (A26). The time-weighted gauge average is

$$R_{\mathrm{g},w} = \frac{1}{H}\sum_{j=1}^{H} w_j R_T(t_j) \ . \tag{38}$$

The weighting must preserve the long-term mean rain rate, so we require the weights w_j to satisfy

$$\sum_{j=1}^{H} w_j = H \ . \tag{39}$$

We expect the weights to emphasize gauge data taken near the satellite observation times and de-emphasize data taken far from the satellite observation times.

To simplify the problem, assume that the interval between satellite visits, Δt, is an integral number of hours, and that N satellite overflights occur during the averaging period H. The satellite average rain rate is

$$R_{\mathrm{s}} = \frac{1}{N}\sum_{q=1}^{N} R_A(t_q) \ , \tag{40}$$

where the satellite visit times are given by t_q, $q = 1,\ldots,N$, at intervals $t_{q+1} - t_q = \Delta t$. The weights w_j must then be found that minimize the variance of the scatter of R_{s} about $R_{\mathrm{g},w}$,

$$\sigma^2_{\mathrm{samp},w} = \langle (R_{\mathrm{s}} - R_{\mathrm{g},w})^2 \rangle \ . \tag{41}$$

This is a standard minimization problem, requiring minimization of (41) with the constraint (39) included by adding it to $\sigma^2_{\mathrm{samp},w}$ as a Lagrange multiplier term:

$$\mathcal{L} = \langle (R_{\mathrm{s}} - R_{\mathrm{g},w})^2 \rangle - \frac{2\lambda}{H}\sum_{j=1}^{H} w_j \ . \tag{42}$$
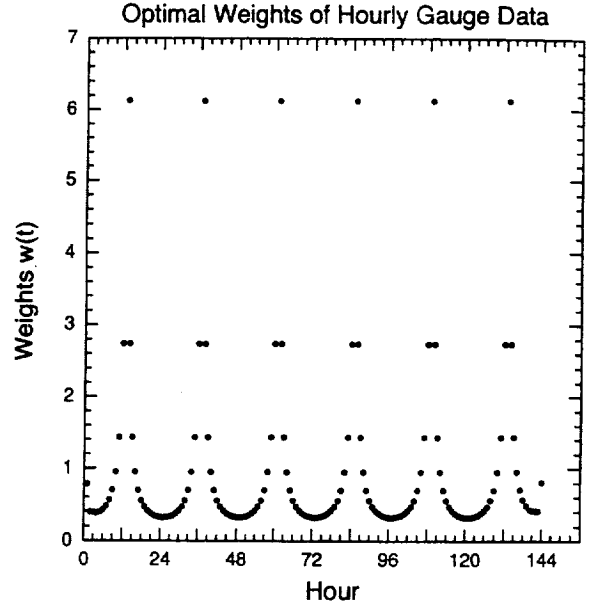

Figure 8. Optimal weights of hourly gauge data for comparison with averages of satellite data provided every 24 h over a 126-km diameter area centered on the gauge, assuming GATE statistics, for 6 days of satellite data.

The factor $(-2/H)$ has been included with the Lagrange multiplier λ for convenience.

The optimal weights w_j are determined by the equations $\partial\mathcal{L}/\partial w_j = 0$, $j = 1,\ldots,H$, or

$$\frac{1}{H}\sum_{j'=1}^{H} c_{\mathrm{gg}}(t_{j'}-t_j)w_{j'} - \frac{1}{N}\sum_{q=1}^{N} c_{\mathrm{sg}}(t_q-t_j)-\lambda = 0 \ , \tag{43}$$

with λ determined by the constraint equation (39), where the lagged gauge covariance terms $c_{\mathrm{gg}}(t_{j'} - t_j)$ are provided in Appendix A by equation (A28), and the lagged satellite-gauge covariance terms $c_{\mathrm{sg}}(t_q - t_j)$ are provided by equation (A30). Equations (43) are linear in the weights w_j and can be solved using standard methods. Once the w_j are known, the value of $\sigma^2_{\mathrm{samp},w}$ can be computed from (41), using (33) for $\langle (R'_{\mathrm{s}})^2 \rangle$. Appendix B gives some additional information about solving (43) for the weights and obtaining the sampling error $\sigma^2_{\mathrm{samp},w}$.

Figure 8 shows the optimal weights obtained using GATE parameters for comparison of an area equivalent to a $1° \times 1°$ box with a single gauge at the center providing 1-h average rain rates, assuming that the satellite

returns every 24 h. The calculation is done for a 6-day period, and shows that once the "end effects" have subsided the weights settle into a regular repeating pattern for the interior hours. The optimal weights indicate, as expected, that it is the hour of gauge data during which the satellite visits that should be weighted most, contributing about 25% to the weighted average (6.1/24). The sampling error variance $\sigma^2_{samp,w}$, compared to what it would have been with uniform weighting, σ^2_{samp}, is reduced to about 60% of the unweighted error variance.

The amount of reduction in sampling error provided by adding time-varying weighting depends on the area A and the parameter values of the model, among other factors. For instance, if the parameters in Table 1 for MIT Cruise 1 are used, the error variance is only reduced to about 85% of the variance for uniform weighting. The reduction in variance is larger, percentagewise, for smaller areas, but the error variance also gets worse as the area shrinks, as shown in figure 4. It is not clear that generalizations about the best approach for satellite/gauge comparisons can be made, since it depends so much on the characteristics of the data in each case.

6. Discussion and Conclusions

Rain gauges provide such direct measurements of rainfall that testing remote sensing estimates of rainfall against gauge observations is extremely attractive. The high spatial and temporal variability of rain, however, makes comparisons of the two difficult. One of the choices that must be made in such comparisons is how much averaging of the gauge data and satellite data are needed in order to reduce the "noisiness" of the comparisons to a level low enough that they can be informative. A spectral model was used to examine some of these questions because it captures one of the more subtle statistical features of rain: the linking of characteristic times for changes in areal averages to the size of the averaging area. Although the spectral model parameters were adjusted to fit the statistics of radar-derived rain rates (rather than of gauge data) over two tropical oceanic regions, the model seems to capture many of the statistical characteristics of gauge data as well. The model parameters obtained from the fits to radar data may lead to underestimates of the amount of small-scale variability on time scales of a fraction of an hour and of the amount of large-scale variability on time scales of a week or more. Based on some limited experiments, it is likely that increased small-scale variability will make intercomparisons noisier, whereas increased large-scale

variability will probably make intercomparisons more informative.

The model indicates that comparisons of rain estimates from single satellite-instrument footprints in the neighborhood of a single gauge are too noisy to be of much use—a fact well documented in many examinations of such comparisons. If areal averaging of the satellite data is used to reduce sampling noise, the model indicates that there is an optimal averaging time for the gauge data for best comparisons, and that the optimal time increases with the area. With a single gauge, however, the areas and times required are too large to be practical.

The situation is improved when multiple gauges are present in the area observed by the satellite during its overflight. Even gauge densities as high as 1 per 1000 km^2 in a 2.5° × 2.5° box, however, are unlikely to bring comparison errors down to the 10% level. Averaging over multiple overflights of the gauges is required. For a typical passive-microwave-instrument-bearing satellite providing about 30 visits per month the optimal averaging area around a single gauge is about that of a 2.5° × 2.5° box, and time averaging over a substantial part of a year is required to bring sampling errors down to the 10% level. The optimal averaging area and time decreases when more gauges are present. Multiple satellites with similar instruments providing more than 1 visit per day can also decrease the averaging time required.

Finally, the improvements in satellite/gauge comparisons that might be possible if the gauge data are weighted depending on their relationship in time to the satellite overflight times indicates that substantial reduction in the scatter of the gauge and satellite averages is possible using this technique, though the amount of improvement varies considerably with the situation.

Appendix A: Details of Model Calculations

Many of the results presented in this paper require calculations of variances and covariances of spatial and temporal averages of the rain-rate field based on the point covariance function in equation (11) and the spectral model in equations (16–18). By carrying out the spatial averages over circular areas A ($A = \pi a^2$) instead of the more traditional square areas, calculations are made much simpler. A number of results useful in carrying out the calculations in a numerically efficient manner are collected in this appendix. Both the algebraic and numerical results presented in this paper were

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obtained with the help of *Mathematica* software [v. 4; see *Wolfram*, 1999] on a Macintosh computer.

To simplify notation, define the instantaneous area-averaged rain rate (identical to equation 4) at time t as

$$R_A(t) = \frac{1}{A} \int_A d^2\mathbf{x}\, R(\mathbf{x}, t) \,, \tag{A1}$$

the time-averaged rain rate for a gauge located at a point $\mathbf{b}$ relative to the center of the area A as

$$R_T(\mathbf{b}) = \frac{1}{T} \int_{-T/2}^{T/2} dt\, R(\mathbf{b}, t) \,, \tag{A2}$$

and the area-time averaged rain rate as

$$R_{AT} = \frac{1}{AT} \int_A d^2\mathbf{x} \int_{-T/2}^{T/2} dt\, R(\mathbf{x}, t) \,. \tag{A3}$$

We give as an example some of the steps needed in obtaining a simple integral expression for the variance of R_A,

$$\sigma_A^2 \equiv \langle (R_A')^2 \rangle \tag{A4}$$

$$= \langle \frac{1}{A^2} \int_A d^2\mathbf{x} \int_A d^2\mathbf{y}\, R'(\mathbf{x}, 0) R'(\mathbf{y}, 0) \rangle$$

$$= \frac{1}{A^2} \int_A d^2\mathbf{x} \int_A d^2\mathbf{y}\, c(\mathbf{x} - \mathbf{y}, 0) \tag{A5}$$

using equation (11). Substituting equation (16) we obtain

$$\sigma_A^2 = \frac{1}{A^2} \int_A d^2\mathbf{x} \int_A d^2\mathbf{y}\, (2\pi)^{-3/2}$$

$$\times \int d^2\mathbf{k} \int d\omega\, e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} \tilde{c}(\mathbf{k}, \omega) \,. \tag{A6}$$

Since $\tilde{c}(\mathbf{k}, \omega)$ does not depend on the direction of $\mathbf{k}$, the areal integrals can be done using

$$\int_A d^2\mathbf{x}\, e^{i\mathbf{k}\cdot\mathbf{x}} = \int_0^a r\, dr \int_0^{2\pi} d\phi\, e^{ikr\cos\phi}$$

$$= 2\pi a^2 J_1(ka)/(ka) \,, \tag{A7}$$

where $J_m(x)$ is the Bessel function of the first kind [see, for example, *Dwight*, 1961], and the integral over ω in (A6) with $\tilde{c}(\mathbf{k}, \omega)$ in (17) gives

$$\int_{-\infty}^{\infty} d\omega\, \tilde{c}(k, \omega) = \pi F_0 \tau_k \,. \tag{A8}$$

After some algebra, one obtains from (A6)

$$\sigma_A^2 = \frac{4\gamma_0'}{\alpha^2} \int_0^{\infty} \frac{dk}{k} \frac{J_1^2(k)}{v(k/\alpha)} \tag{A9}$$

with the definitions

$$\gamma_0' = \Gamma(1 + \nu)\gamma_0 \,, \tag{A10}$$

$$\alpha = a/L_0 \,, \tag{A11}$$

and

$$v(z) = (1 + z^2)^{1+\nu} \,. \tag{A12}$$

The variance of $R_T(\mathbf{b})$ defined in (A2), which does not depend on position $\mathbf{b}$, can be calculated with an approach similar to the one above, yielding

$$\sigma_T^2 \equiv \langle R_T'^2 \rangle \tag{A13}$$

$$= \frac{2\gamma_0'}{u} \int_0^{\infty} dk\, \frac{k}{v^2(k)} h(k; u) \,, \tag{A14}$$

with

$$u \equiv T/\tau_0 \tag{A15}$$

and

$$h(k; u) \equiv 1 - \frac{1}{uv(k)}[1 - e^{-uv(k)}] \,. \tag{A16}$$

Likewise, the variance of R_{AT} defined in (A3) can be calculated to be

$$\sigma_{AT}^2 \equiv \langle R_{AT}'^2 \rangle \tag{A17}$$

$$= \frac{8\gamma_0'}{u\alpha^2} \int_0^{\infty} \frac{dk}{k} \frac{J_1^2(k)}{v^2(k/\alpha)} h(k/\alpha; u) \,. \tag{A18}$$

A number of covariances are also needed, and these are calculated in a manner similar to the example given above. The covariance of $R_A(t = 0)$ with a gauge average $R_T(\mathbf{b})$ is given by

$$c_{AT}(b) \equiv \langle R_A' R_T'(\mathbf{b}) \rangle \tag{A19}$$

$$= \frac{4\gamma_0'}{u\alpha^2} \int_0^{\infty} dk\, \frac{J_1(k) J_0(\beta k/\alpha)}{v^2(k/\alpha)}$$

$$\times [1 - e^{-(u/2)v(k/\alpha)}] \tag{A20}$$

with

$$\beta \equiv b/L_0 \,. \tag{A21}$$

The covariance of time averages of gauge data for two gauges separated by a distance $\mathbf{b}$ is given by

$$c_{TT}(b) \equiv \langle R_T'(\mathbf{b}) R_T'(0) \rangle \tag{A22}$$

$$= \frac{2\gamma_0'}{u} \int_0^{\infty} dk\, \frac{k J_0(\beta k)}{v^2(k)} h(k; u) \,. \tag{A23}$$

The covariance of spatial averages with space-time averages needed for equation (26) is given by

$$c_{A,AT} \equiv \langle R_A' R_{AT}' \rangle \tag{A24}$$

$$= \frac{8\gamma_0'}{u\alpha^2} \int_0^{\infty} \frac{dk}{k} \frac{J_1^2(k)}{v^2(k/\alpha)}$$

$$\times [1 - e^{-(u/2)v(k/\alpha)}] \,. \tag{A25}$$

The calculation of optimal time-dependent weighting of gauge data for comparison with satellite estimates requires formulas for the lagged covariance of gauge averages and satellite areal averages. First, define the gauge average centered around time t_0 as

$$R_T(t_0) \equiv \frac{1}{T} \int_{-T/2}^{T/2} dt \, R(0, t_0 + t) \, . \tag{A26}$$

The covariance of two gauge averages with averaging interval T lagged by time τ is given by

$$
\begin{aligned}
c_{TT}(\tau; T) &\equiv \langle R'_T(\tau) R'_T(0) \rangle \tag{A27} \\
&= \frac{2\gamma'_0}{u^2} \int_0^\infty dk \, k \frac{e^{-|\tau/\tau_0|v(k)}}{v^3(k)} \\
&\quad \times \sinh^2[uv(k)/2] \, , \tag{A28}
\end{aligned}
$$

valid for $|\tau| \geq T$. The lagged covariance of an instantaneous average over a circular area A at $t = 0$ with a gauge average lagged by τ is given by

$$
\begin{aligned}
c_{AT}(\tau; T) &\equiv \langle R'_A(0) R'_T(\tau) \rangle \tag{A29} \\
&= \frac{4\gamma'_0}{u\alpha^2} \int_0^\infty dk \, \frac{J_1(k) e^{-|\tau/\tau_0|v(k/\alpha)}}{v^2(k/\alpha)} \\
&\quad \times \sinh[uv(k/\alpha)/2] \, , \tag{A30}
\end{aligned}
$$

valid for $|\tau| \geq T/2$. Finally, the covariance of instantaneous area-averaged rain rate separated by a time interval τ is given by

$$
\begin{aligned}
c_{AA}(\tau) &\equiv \langle R'_A(\tau) R'_A(0) \rangle \tag{A31} \\
&= \frac{4\gamma'_0}{\alpha^2} \int_0^\infty \frac{dk}{k} \frac{J_1^2(k)}{v(k/\alpha)} \\
&\quad \times e^{-|\tau/\tau_0|v(k/\alpha)} \, . \tag{A32}
\end{aligned}
$$

Appendix B: Optimal Weight Solution

In section 5 of the paper equations (43) for the optimal weights of gauge data are obtained. They are fairly simple to solve for the weights using standard linear algebra methods. We present here a brief description of the approach we have used.

Rewrite equations (43) in a form amenable to methods for solving sets of simultaneous linear equations by defining the weight vector $\mathbf{w}$,

$$(\mathbf{w})_j = w_j \, , \tag{B1}$$

the symmetric covariance matrix,

$$(\mathbf{C})_{jj'} = c_{gg}(t_{j'} - t_j) \, , \tag{B2}$$

and the vector defined by the middle term of (43), summed over all satellite overflight times, as

$$(\mathbf{d})_j = \frac{1}{N} \sum_{q=1}^N c_{sg}(t_q - t_j) \, . \tag{B3}$$

It is also convenient to introduce a vector consisting entirely of 1's,

$$(\mathbf{1})_j = 1, \quad j = 1, \dots, H \, . \tag{B4}$$

Using the above definitions, equation (43) can be written in vector notation as

$$H^{-1} \mathbf{C} \mathbf{w} - \mathbf{d} - \lambda \mathbf{1} = 0 \, , \tag{B5}$$

with the constraint equation (39) for $\mathbf{w}$ written as

$$\mathbf{1}^\mathsf{T} \mathbf{w} = H \, , \tag{B6}$$

where the superscript T indicates matrix transpose (e.g., $\mathbf{w} \cdot \mathbf{d} = \mathbf{w}^\mathsf{T} \mathbf{d}$). Equation (B5) is readily solved for the weights as

$$\mathbf{w} = H \mathbf{C}^{-1}(\mathbf{d} + \lambda \mathbf{1}) \, , \tag{B7}$$

where λ is determined by the constraint equation (B6) and solution (B7) to be

$$\lambda = \frac{1 - \mathbf{1}^\mathsf{T} \mathbf{C}^{-1} \mathbf{d}}{\mathbf{1}^\mathsf{T} \mathbf{C}^{-1} \mathbf{1}} \, . \tag{B8}$$

The solution requires obtaining the inverse matrix $\mathbf{C}^{-1}$, which is a standard numerical problem.

Once the weights are obtained, the sampling error for the difference of the satellite average from the time-weighted gauge average can be obtained from (41) as

$$
\begin{aligned}
\sigma_{\text{samp},w}^2 &= \langle (R'_s)^2 \rangle + \langle (R'_{g,w})^2 \rangle - 2\langle R'_s R'_{g,w} \rangle \\
&= \langle (R'_s)^2 \rangle + H^{-2} \mathbf{w}^\mathsf{T} \mathbf{C} \mathbf{w} - 2H^{-1} \mathbf{w}^\mathsf{T} \mathbf{d} \\
&= c_{ss} + H^{-1} \mathbf{w}^\mathsf{T} (\mathbf{d} + \lambda \mathbf{1}) - 2H^{-1} \mathbf{w}^\mathsf{T} \mathbf{d} \\
&= c_{ss} + \lambda - H^{-1} \mathbf{w}^\mathsf{T} \mathbf{d} \tag{B9}
\end{aligned}
$$

using equations (38), (B2), and (B3) for the first step above, equation (B5) for the second step, and the constraint equation (B6) for the last step. Equation (33) provides an exact expression for c_{ss}.

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