Direct Optimal Control of Duffing Dynamics

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Summary

The "direct control method" is a novel concept that is an attractive alternative and competitor to the differential-equation-based methods. The direct method is equally well applicable to nonlinear, linear, time-varying, and time-invariant systems. For all such systems, the method yields explicit closed-form control laws based on minimization of a quadratic control performance measure. We present an application of the direct method to the dynamics and optimal control of the Duffing system where the control performance measure is not restricted to a quadratic form and hence may include a quartic energy term. The results we present in this report also constitute further generalizations of our earlier work in "direct optimal control methodology." The approach is demonstrated for the optimal control of the Duffing equation with a softening nonlinear stiffness.

Introduction

The study of dynamical systems via differential equations of motion is referred to as an "indirect method." On the other hand, the study of dynamic systems without any resort to or knowledge of differential equations of motion is referred to as the "direct method." In the direct method, algebraic equations of motion (AEM) take the place of the traditional differential equations of motion. The AEM are obtained by using Hamilton's Law of Varying Action (HLVA) in conjunction with the assumed-time-mode expansions of the generalized coordinates (ref. 1). The constant unknown coefficients of the assumed basis functions in time of these expansions become the generalized (algebraic) states of the dynamic system. If there are control inputs on the dynamic system, they too can be expanded in terms of assumed basis functions in time multiplied by constant unknown coefficients of expansion playing the role of generalized (algebraic) control inputs.

By virtue of the assumed-time-mode (ATM) expansions of the generalized coordinates and the controls, the variational work energy quantities in HLVA can be integrated a priori in time over any time interval. This provides a set of purely algebraic equations describing the motion in terms of the constant unknown algebraic states and the algebraic control inputs for the time interval considered.

Presently, nonlinear optimal control problems are formulated in terms of differential equations. In most cases, however, it is not possible to formulate an explicit, nonlinear optimal control law in this setting. At best, the control law must be generated numerically at the expense of much computational effort (ref. 2).

Instead of differential equations, the method applied in this report produces an explicit nonlinear algebraic optimal control law by using algebraic equations of motion and an algebraic control performance measure. The algebraic performance measure is again obtained by representing the generalized coordinates and control inputs of a dynamic system in terms of assumed-time-modes

expansions and introducing them into a time-integral control performance measure. The complexities associated with nonlinear systems are no longer a serious issue with this approach, and the solutions can be obtained directly in explicit form with relative ease.

By contrast, in a differential setting, various strategies are attempted to avoid the complexities and difficulties caused by the nonlinearities. Three such strategies are (1) linearizing the system, (2) approximating the system with a large number of linear models, and (3) converting the system to a linear equivalent using a nonlinear coordinate transformation. All of them, however, involve simplification and approximation of the original system. Such issues are eliminated at the outset when the algebraic equations of motion are used.

In the AEM setting, the generalized coordinates of a system are expanded in time over the interval (t_0, t_1) in terms of admissible time basis functions. For n generalized coordinates $q_r(t)$, these expansions can be written as

$$q_r(t) = \sum_{k=0}^{N_r} A_{rk}(t)\alpha_{rk} \quad r = 1, 2, \ldots, n \tag{1}$$

where $A_{rk}(t)$ are admissible assumed time basis functions that are independent and continuous with continuous derivatives over (t_0, t_1), and α_{rk} are unknown constant coefficients.

Hamilton's Law of Varying Action can be expressed as

$$\int_{t_0}^{t_1} (\delta T + \delta W)\,\mathrm{d}t - \delta q^T \frac{\partial T}{\partial \dot{q}}\bigg|_{t_0}^{t_1} = 0 \tag{2}$$

where T is the kinetic energy, δW is the variational work, and q is the n-dimensional generalized coordinates vector. Once the assumed-time-modes expansions are invoked, each term becomes a known function of time. This allows HLVA to be explicitly evaluated, thereby eliminating time and yielding a set of n algebraic equations of motion in which the coefficients α_{rk} become the unknown algebraic coordinates or states.

Similar to the generalized coordinates, the inputs $f_r(t)$ on a system can also be expressed in terms of basis functions in time as

$$f_r(t) = \sum_{k=0}^{M_r} B_{rk}(t)\beta_{rk} \quad r = 1, 2, \ldots, m \tag{3}$$

where $B_{rk}(t)$ are admissible assumed-time-modes for the inputs, and the constants β_{rk} become the unknown algebraic input coordinates or input states (refs. 1 and 3 to 6). When substituted into HLVA, equation (2), for the variational work of the inputs, expansion equation (3), also allows a priori explicit integration in time. Then the resulting algebraic equations of motion are in terms of the unknown coefficients α_{rk} (algebraic states) and β_{rk} (algebraic inputs).

Analysis and control of dynamic systems via HLVA without resorting to differential equations of motion is called the direct method. Response studies by the direct method were first demonstrated by Bailey (refs. 7 to 9), and a direct control method (DCM) was first demonstrated by Öz and Raffie (ref. 10) as an open-loop control. Öz and Adigüzel (refs. 3 and 5) and Adigüzel and Öz (refs. 4 and 6) extended the DCM to include optimal feedback control of the algebraic states. Fuerst and Öz (ref. 11) demonstrated the DCM for optimal nonlinear control of an aerodynamic body at a high angle of attack. The DCM applies to time-variant, time-invariant, linear, and nonlinear systems with virtually the same simplicity and generality. Since only algebraic equations are dealt with, the DCM promises to be an attractive competitor and alternative to differential-equation-based methods.

AEM for Duffing Dynamics

It proves convenient to express the HLVA in nondimensional time. For an arbitrary time interval of motion (t_1, t_0), the nondimensional time τ is defined as

$$\theta\tau = t - t_0, \quad \theta = t_1 - t_0, \quad 0 \le \tau \le 1 \tag{4}$$

In terms of nondimensional time τ, the form of HLVA is

$$\int_0^1 \left(\delta T + \delta W_C + \delta W_{NC} + \delta W_D \right) d\tau - \delta q^T \left. \frac{\partial T}{\partial q'} \right|_0^1 = 0 \tag{5}$$

where subscripts C, NC, and D denote the work of conservative, nonconservative, and damping forces, respectively. A prime (′) denotes the derivatives with respect to nondimensional time τ. Any control inputs are to be included under the nonconservative work expression. By assuming the functional forms of the energy-work expressions in terms of generalized coordinates, the general form of the AEM has been derived in reference 1. Although tedious, their application to the Duffing dynamics is straight forward to obtain the AEM for the Duffing system. The Duffing system that we consider is the classical single-degree-of-freedom spring, mass, damper system with a harmonic external forcing function. The nonlinearity is attributed to a quartic displacement term in the elastic potential energy of the system in addition to the quadratic displacement term that produces the linear behavior. The damping is the result of the usual linear viscous effect.

The kinetic energy and conservative work expressions in dimensional time that are required by the HLVA for the single-degree-of-freedom Duffing system in terms of the generalized coordinate q are

$$T = \frac{1}{2}m\dot{q}^2 \quad \frac{\partial T}{\partial \dot{q}} = m\dot{q} \quad V = -W_C = {}^2V + {}^4V = \frac{1}{2}k_2 q^2 + \frac{1}{2}k_4 q^4 \tag{6}$$

where ${}^2V$ and ${}^4V$ are the potential energies corresponding to quadratic and quartic terms in the generalized coordinate q. m, k_2, and k_4 are the mass and stiffness coefficients, respectively. Note that the quartic term will produce all the nonlinearities in the sequel. The other variational work expressions are the result of the viscous damping force and the external forcing function:

$$\delta W_D = -c\dot{q}\delta q, \quad \delta W_{NC} = \delta W_f = f\,\delta q \tag{7}$$

where c is the damping coefficient and f is the external force along q.

Next we introduce an assumed-time-modes expansion for (the single degree of freedom) q in terms of the nondimensional time in the form

$$q(\tau) = A_0(\tau)x_0 + A(\tau)\alpha \quad A_0 = \begin{bmatrix} A_{10}(\tau) & A_{11}(\tau) \end{bmatrix}, \quad A = \begin{bmatrix} A_{12}(\tau) & A_{13}(\tau) \dots A_{1N}(\tau) \end{bmatrix}$$

$$x_0 = \begin{bmatrix} \alpha_0 & \alpha_1 \end{bmatrix}^T = \begin{bmatrix} q(0) & q'(0) \end{bmatrix}^T, \quad \alpha = \begin{bmatrix} \alpha_2 & \alpha_3 \dots \alpha_N \end{bmatrix}^T \tag{8}$$

where x_0 is the two-component initial state vector of initial displacement and initial velocity and α is the $(N-2)$-component vector of generalized unknown constant algebraic states. Furthermore, the assumed-time-mode functions have continuous derivatives with respect to time and satisfy certain boundary conditions in time:

$$A_{1k}^j(0) = \delta_{jk} \quad j,k = 0,\dots,N \tag{9}$$

where the superscript j denotes the jth order derivative with respect to nondimensional time and δ_{jk} is the Kronecker delta function. Theoretically, the choices for the assumed time modes are not unique and we are not restricted to the form given in equation (9); however, we have been using it with excellent facility in our work to date. Furthermore, the form we choose to use is simple enough and the time boundary conditions can be satisfied easily, which leads to physically meaningful interpretations of the algebraic states. Specifically, with the boundary conditions of equation (9), each generalized algebraic state satisfies

$$\alpha_k = q^k(\tau = 0), \quad \alpha_0 = q(0), \quad \alpha_1 = q'(0) \quad k = 0, \dots, N \tag{10}$$

where the superscript k denotes the kth order nondimensional time derivative. Note that α_0 and α_1 are the displacement and velocity initial conditions that are typically identified as time-dependent state variables in a differential equation setting. On the other hand, for α_k, $k \geq 2$, acceleration and higher order derivatives of generalized coordinates qualify as generalized (algebraic) states in the direct methodology.

By introducing the assumed-time-mode expansion equation (8) for q into the kinetic energy, potential energy, and other work expressions, we transform to the algebraic states α as the unknowns. Then, utilizing the form of the HLVA given by equation (5) in nondimensional time τ, we perform the required variations on α_k ($k \geq 2$) as the unknowns to obtain in the following form the AEM for the Duffing system:

$$\left[P_L + P_N(x_0)\right]\alpha + \left[R_L + R_N(x_0)\right]x_0 + N = 0 \tag{11}$$

where N and the subscript N denote terms due to nonlinearities (quartic term) in the potential energy; the term Q_f due to the external forcing function has been included in the function N for notational convenience without the loss of generality; and the subscript L denotes terms due to linearities in the potential energy (quadratic term), kinetic energy, and damping force. For control purposes, it is convenient to write the AEM equation (11) by separating into alternate forms the linear and nonlinear dynamics terms:

$$P\alpha + Rx_0 + N = 0, \quad P = P_L + P_N, \quad R = R_L + R_N$$
$$P_L\alpha + R_L x_0 + \mathbb{N} = 0, \quad \mathbb{N} = P_N\alpha + R_N x_0 + N \tag{12}$$

The explicit expressions for the various matrices in the AEM equation (12) for the Duffing dynamics for simple power series in time expansions as ATM's are given in the appendix. The form of the AEM for the Duffing dynamics equations (12) conforms to the general form of the AEM stated for general dynamic systems in references 1 and 3 to 6.

The solution procedure for the response problem via the AEM consists of solving the algebraic equations (11) and (12) for the unknown constants α and using these constants to evaluate the assumed-time-mode expansions of the generalized coordinates, equation (8) for $0 \leq \tau \leq 1$ corresponding to any time t in the interval (t_0, t_1). It must be emphasized that the above AEM pertain to any time interval (t_0, t_1), which we refer to as the transition interval. However, one may also consider the transition interval as a small time step, and equations (11) and (12) can be marched in time from one interval to the next. This procedure requires the use of continuity equations of the form

$$x_0^{(j+1)} = S_0 x_0^{(j)} + S\alpha^{(j)} \tag{13}$$

where (j) and $(j+1)$ represent two subsequent time intervals. The continuity matrices S_0 and S are deduced for any specific choice of the ATM expansions by keeping in mind that the generalized velocity and displacement at the end of time interval (j) at $\tau = 1$ are the initial velocity and displacement for the next time interval $(j+1)$ at $\tau = 0$.

Assumed-Time-Mode Expansions of (Control) Inputs

The control inputs on a dynamic system are external loads designed to satisfy certain performance criteria. Similar to the generalized coordinates, the control input functions can also be represented as expansions in assumed time modes. Denoting the control forces as $f^c(t)$ for m-inputs, the control input expansions can be written as

$$f_r^c(t) = \sum_{k=0}^{M_r} B_{rk}(t)\beta_{rk} = B_r(t)\beta_r \qquad\qquad r = 1, 2, \ldots, m \qquad (14)$$

where $B_{rk}(t)$ are admissible assumed time modes for the control inputs and β_{rk} are constant unknown algebraic control input coordinates that will be determined to meet the control objectives. In equation (14), $B_r(t)$ and β_r are M_r-component vectors of ATM and algebraic input coordinates, respectively, with obvious definitions. For all m-inputs in matrix form, one has

$$f^c(t) = B(t)\beta, \quad f^c(t) = \left[f_1(t)\ldots f_m(t)\right]^T$$

$$B(t) = \text{Block Diag } B_r(t), \quad \beta = \left[\beta_1^T \beta_2^T \ldots \beta_m^T\right]^T \qquad\qquad r = 1, \ldots, m \qquad (15)$$

The control inputs can be added to the AEM simply by writing their variational work and transforming to the domain of algebraic states α:

$$\delta W_{\text{cont.}} = f^{cT}\delta q = f^{cT}A\delta\alpha = \beta^T B^T(\tau)A(\tau)\delta\alpha$$

in which transformation to nondimensional time is also assumed to be done implicitly. Adding this to the HLVA equation (5) and taking the time integrals yields the additional control term for the AEM equation (11):

$$P\alpha + Rx_0 + N + Q\beta = 0 \qquad (16)$$

where

$$Q = \int_0^1 A^T B \, d\tau \qquad (17)$$

In equation (16), α constitutes the algebraic state vector to be controlled by the algebraic control vector β.

In general, the assumed time modes B_{rk} for the inputs can be taken to be the same as the assumed time modes A_{ij} for the generalized coordinate(s). However, again, if small time-step transition intervals are to be used in a time-marching scheme, a simple choice is one in which $M_r = 0$, $k = 0$, and $B_{r0}(t) = 1$ ($r = 1, 2, \ldots, m$). That is, each input is a single-term expansion with a constant assumed-time-mode shape over each interval $0 \le \tau \le 1$. This is indeed very well suitable for (digital) implementation of the control inputs when each control coordinate β_{rk} corresponds to the value of the physical control input. This particular ATM for controls literally corresponds to a zero-order hold in sampled data systems. Indeed, the situation does not need to be any more complicated. Simplicity is sufficient and leads to an elegant implementation form for control inputs with the direct control methodology.

What remains is obtaining the algebraic control inputs β to satisfy the desired controller performance criteria. Possibly one would be interested in finding the control inputs β as functions of the algebraic states α that would by definition represent state feedback control. An explicit nonlinear feedback control solution for the nonlinear dynamics represented by the AEM (eq. (16)) is given in reference 4 by using the quadratic regulator performance measure typically used in optimal control theory. Herein, we present a more general formalism for deriving an optimal nonlinear control law that allows a general form for the control design performance index (CDPI). In particular, the general approach can accommodate quartic terms in the CDPI in addition to traditional quadratic terms in system states. This is particularly applicable to the Duffing system that has a quartic energy term which leads to nonlinear behavior. The direct optimal control solution given in the next section is applicable to a multiple-degrees-of-freedom, multiple-input system in which q represents an n-component vector of generalized coordinates.

Direct Optimal Nonlinear Control

In the interest of being brief and because of limited space, we shall present only the final form of the optimal nonlinear control law as the reader can easily verify the omitted details. The results presented in this section represent a more general form of the specific results presented in reference 4.

Consider a positive definite CDPI of the general functional form that is separated in state and control variables:

$$J_t = \frac{1}{2}\int_{t_0}^{t_1}\left[I_s(q,\dot{q})\right]+\left[I_c\left(f^c\right)\right]\mathrm{d}t = J_s + J_c \tag{18}$$

where subscripts s and c indicate the positive semidefinite state and positive definite control dependent terms of the CDPI. The CDPI can be transformed to the domain of the AEM by introducing the ATM expansions shown in equations (8) and (15) for the generalized coordinates and the input functions. However, in equations (8) for n-coordinates, the row index $A_{rk}(\tau)$ on the ATM's would run as $r = 1, \ldots, n$. In this transformation, since the time dependence of the terms is known by virtue of the ATM, the time integrals in the CDPI can be evaluated a priori to yield the equivalent algebraic performance criterion

$$J_t\left(q,\dot{q},f^c\right) = J(\alpha,\beta) = J_s(\alpha,x_0) + J_c(\beta) \tag{19}$$

The nonlinear optimal control problem is

1. Minimize $J(\alpha,\beta)$
2. Subject to the AEM

$$C(\alpha,\beta) = P\alpha + Rx_0 + N + Q\beta = 0$$

or equivalently to

$$C(\alpha,\beta) = P_L\alpha + R_L x_0 + \mathbb{N} + Q\beta = 0$$

where the unknowns are α and β. The optimality problem can be solved via the standard Lagrange multiplier method. Introducing the augmented algebraic CDPI

$$J_a(\alpha,\beta) = J(\alpha,\beta) + v^T C(\alpha,\beta) \tag{20}$$

where ν is the vector of Lagrange multipliers, the necessary conditions for optimality are

$$\frac{\partial J_a}{\partial \alpha} = 0, \quad \frac{\partial J_a}{\partial \beta} = 0 \quad \frac{\partial J_a}{\partial \nu} = 0 \tag{21}$$

which yield the solution for the optimal control:

$$\frac{\partial J_c}{\partial \beta} = \frac{\partial C^T}{\partial \beta}\left[\frac{\partial C^T}{\partial \alpha}\right]^{-1}\frac{\partial J_s}{\partial \alpha} \tag{22}$$

It should be noted that the optimal solution is valid for any arbitrary transition interval (t_0, t_1) of the motion, provided that admissible ATM expansions exist for that transition interval. Hence, with this caveat, so far nothing is implied by equation (22) regarding the smallness of the transition interval (t_0, t_1).

On the other hand, as noted previously, in practice one can also consider a time-marching approach in which each small time step can be taken to be the transition interval (t_0, t_1). In this case, the CDPI becomes a local performance measure optimized for the current time step. One can take $t_0 = t_{k-1}$ and $t_1 = t_k$ for a small k^{th} time step along the global time axis and invoke continuity conditions on the system states from one step to the next time step to study the system for arbitrarily long durations. Typical simple power series in time can be utilized as ATM's with such an approach and with this perspective, control inputs can be taken as zeroth-order expansions in time. These expansions lead to zero-order hold inputs, thus making the process attractive for digital implementation. Indeed, we have used this perspective in our work to date on this subject. The explicit form of nonlinear control laws with quadratic CDPI's employed as local performance measures for each time step are given in references 4 and 6 with illustrative examples.

On the other hand, one can also preserve the global perspective on the CDPI for an arbitrarily long transition interval (t_0, t_1) (infinite horizon control) while the system and control dynamics, and hence ATM expansions, are considered over small (local) time-finite elements i along the global time axis (t_0, t_1). In this case, it can be shown that the optimal control β at any current k^{th} small time step within an arbitrarily long transition interval (t_0, t_1) can be obtained from equation (22) in the form

$$\sum_{i=1}^{k}\left(\frac{\partial J_c}{\partial \beta}\right)_i = \left[\sum_{i=1}^{k}\left(\frac{\partial C^T}{\partial \beta}\right)_i\right]\left[\sum_{i=1}^{k}\left(\frac{\partial C^T}{\partial \alpha}\right)_i\right]^{-1}\left[\sum_{i=1}^{k}\left(\frac{\partial J_s}{\partial \alpha}\right)_i\right] \tag{23}$$

where i denotes all the previous time steps starting from the initial time t_0. In equation (23), the control input β for the current k^{th} time step utilizes the history of the control inputs, system dynamics, and performance measure up to the current time. Thus, a simple power series form of ATM's for generalized coordinates and zeroth-order ATM expansions for controls can still be utilized locally while a global CDPI is maintained. Full implications of this solution remain to be studied yet.

Illustrative Examples

Consider the single-degree-of-freedom Duffing dynamics with a strong nonlinearity described by

$$\ddot{q} + 16\pi^2 q^2 - 200q^3 = f(t) + f^c(t)$$

The system has unstable saddle points at ± 0.8885. For the forced system, take $f(t) = 20\cos(17t)$ and f^c denotes the single control input. CDPI reflects a measure of the total potential energy of the system and hence includes a quartic penalty term:

$$J_t = \frac{1}{2}\int_{t_0}^{t_1}\left[W_q q^2 + W_{\dot{q}}\dot{q}^2 + W_{q^2} q^4 + f^{cT}(t) R_f f^{cT}(t)\right] dt$$

where W_q, $W_{\dot{q}}$, W_{q^2} are positive semidefinite state weightings and R_f is the positive definite control weighting. For simplicity, cross products between the displacements and velocities are not considered.

We use equation (23) to obtain the optimal controls and hence optimize the global CDPI for an infinite horizon control problem. For system and controller dynamics, we take local time steps of 0.004 sec. The generalized coordinate q is represented as a four-term ($N = 3$ in eq. (8)) simple power series in time ATM expansion; thus, α_2 and α_3 are the unknown algebraic states. A single-term ($M = 0$) ATM expansion (zero-order hold) is considered for the control input in equation (14); therefore, there is single algebraic control input.

We considered initial conditions in the unstable region of the phase plane beyond the saddle point; thus, the uncontrolled system is unstable. Figures 1 to 3 have initial displacements of 0.89 and initial velocities of 0.1. Figures 1 and 2 show the controlled response for the unforced and forced Duffing oscillator with quadratic displacement weighting of 1.5, velocity weighting of 0.0015, and control weighting of 9×10^{-7} in the CDPI; no weighting on the quartic displacement is considered. Figure 3 shows the controlled response of the unforced oscillator with the same CDPI weightings as those for figure 1, but a quartic displacement weighting of 1.0 is also included. Finally, figure 4 shows the controlled response of the unforced system subjected to unstable initial displacement of 2.0 and initial velocity of 1.0 with CDPI weightings of 0.9 for the quadratic displacement, 0.002 for the velocity, 18.0 for the quartic displacement, and 9×10^{-8} for the control.

Concluding Remarks

An essential feature of the direct optimal control method is the reduction of the conventional variational optimal control problem to an equivalent algebraic optimality problem from which the nonlinear optimal feedback laws are obtained in closed form and are readily applicable to simulate the closed-loop system. Nonlinearity is not a challenging issue with the direct method as it may have been with customary indirect approaches via differential equations. Furthermore, the direct method treats the time-invariant and time-varying systems alike, and the form of the technique has the potential to solve a larger class of control problems with the same simplicity than would be possible using traditional variational techniques. A case in point is the direct optimal regulator control of the Duffing dynamics illustrated herein.

Appendix—Algebraic Equations of Motion for Forced Duffing System

With the assumed time modes taken as simple power series in time $A_0 = [1 \ \tau]$, $A = [\tau^2 \tau^3 ... \tau^N]$ for the single-degree-of-freedom Duffing dynamics, one has (eqs. (11), (12), (16), (17))

$$\left[P_L + P_N(x_0)\right]\alpha + \left[R_L + R_N(x_0)\right]x_0 + N + QB = 0$$

The elements of the required matrices are

$$P_{LJI} = \frac{m}{\theta^2}\frac{i(1-i)}{i+j-1} - k_2\frac{1}{i+j+1} - \frac{c}{\theta}\frac{i}{i+j} \quad i,j = 2,3,\ldots,N$$

$$R_{LJs} = -\frac{c}{\theta}\frac{s-1}{j+1} - k_2\frac{1}{j+s} \quad s = 1,2 \,(\text{column index})$$

$$Q_{JK} = \frac{1}{j+k+1} \quad k = 0,1,\ldots,M_1$$

$$I = i-1\,(\text{column index}), \quad J = j-1\,(\text{row index}), \quad K = k+1\,(\text{column index})$$

$$R_{N_{Is}} = -2k_4\left[\frac{\alpha_0^2}{i+s} + \frac{2\alpha_0\alpha_1}{i+s+1} + \frac{\alpha_1^2}{i+s+2}\right]$$

$$i = 2,\ldots,N; \quad s = 1,2\,(\text{column index}), \quad I = i-1\,(\text{row index})$$

$$P_{N_{IJ}} = -6k_4\left[\frac{\alpha_0^2}{i+j+1} + \left(\frac{2}{i+j+2}\right)\alpha_0\alpha_1 + \frac{1}{i+j+3}\alpha_1^2\right]$$

$$i,j = 2,3,\ldots,N; \quad I = i-1\,(\text{row index}); \quad J = j-1\,(\text{column index})$$

$$N = -\,^4V_3 - \,^4V_4\alpha + Q_f$$

$$^4V_3\left(\alpha^2, x_0\right)_K = 2k_4\alpha^T\left[\frac{\alpha_0}{i+j+k+1} + \frac{\alpha_1}{i+j+k+2}\right]\alpha$$

$$i,j,k = 2,3,\ldots,N; \quad K = k-1$$

for each k, run i,j = row, column indices, and form the scalar product of $\alpha^T[k,\alpha_0, \alpha_1]_{ij}\alpha$, where $[k,\alpha_0, \alpha_1]$ is the term in [ ] above.

$$^4V_4(\alpha)_{KL} = k_4\left[\alpha^T\left\{\frac{1}{i+j+k+l+1}\right\}\alpha\right]$$

$$i,j = 2,3,\ldots,N(\text{row,column}), \quad k,l = 2,3,\ldots,N(\text{row,column}); \quad K = k-1, L = l-1$$

for each k and l, form the scalar product $\alpha^T\{k,l\}_{ij}\alpha$ by running the indices i,j to generate the matrix in { }. Put each scalar as the kl element of the matrix [ ]$_{kl}$ where k is the row and l is the column index.

$$Q_{f_K} = \theta \int_0^1 \tau^k f(\tau) \, d\tau, \quad k = 2, 3, \ldots, N; \quad K = k - 1 \, (\text{row index})$$

$$f(\tau) = A_\Omega \cos \Omega(t_0 + \theta\tau)$$

where $f(\tau)$ is the forcing on the Duffing system with angular frequency Ω and its inclusion under the N term is merely for compactness and is of no consequence. Since $f(\tau)$ is not a function of α, the Jacobian of N that will arise in applying the optimal control solution equation (22) or equation (23) is

$$Z_N = \left[\frac{\partial N_i}{\partial \alpha_j} \right], \quad \left(Z_N^T \right)_{RK} = -4k_4 \left[\frac{\alpha_0}{r+j+k+1} + \frac{\alpha_1}{r+j+k+2} \right] \alpha$$

(for each r, k; run j as a column index of a row matrix.)

$$-3k_4 \alpha^T \left[\frac{1}{i+j+k+r+1} \right] \alpha$$

(for each r, k; run i, j as row, column indices to form the scalar product $\alpha^T [\quad] \alpha$)

$$i, j, r, k = 2, \ldots, N; \quad R = r - 1, K = k - 1$$

The continuity matrices S_0 and S of equation (13), corresponding to the simple power series in time expansion, are

$$S_0 = \text{Block Diag} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad S = \text{Block Diag} \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 2 & 3 & \cdots & N \end{bmatrix}$$

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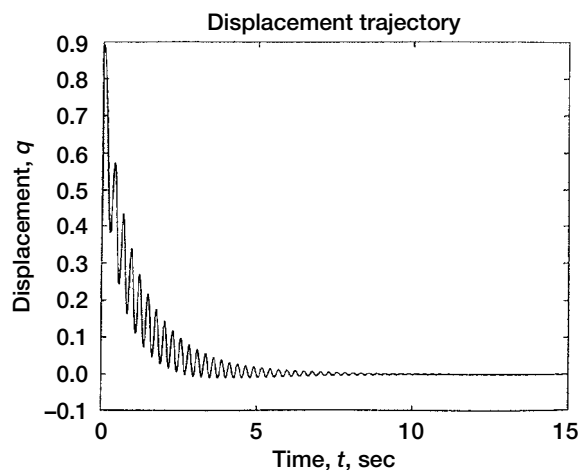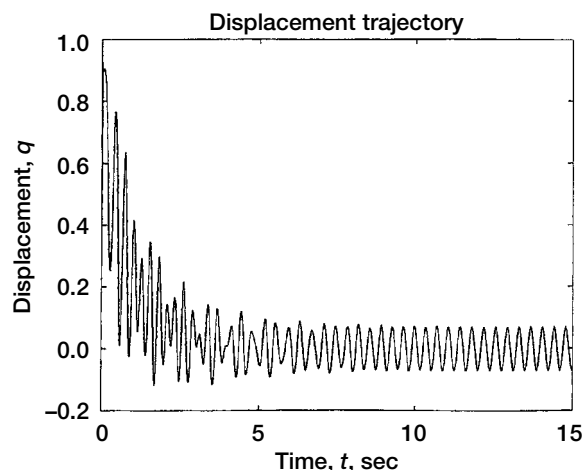
Figure 1.—Control of unforced Duffing dynamics with quadratic regulator. Initial displacement, 0.89; initial velocity, 0.1; quadratic displacement weighting, 1.5; velocity weighting, 0.0015; and control weighting, 9×10^{-7}.

Figure 2.—Control of forced Duffing dynamics with quadratic regulator. Initial displacement,0.89; initial velocity, 0.1; quadratic displacement weighting, 1.5; velocity weighting, 0.0015; and control weighting, 9×10^{-7}.
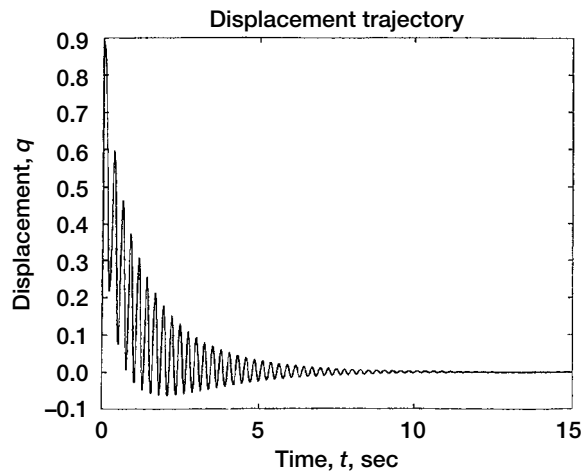

Figure 3.—Control of unforced Duffing dynamics with quartic regulator. Initial displacement, 0.89; initial velocity, 0.1,quadratic displacement weighting, 1.5; quartic displacement weighting, 1.0; velocity weighting, 0.0015; and control weighting, 9×10^{-7}.
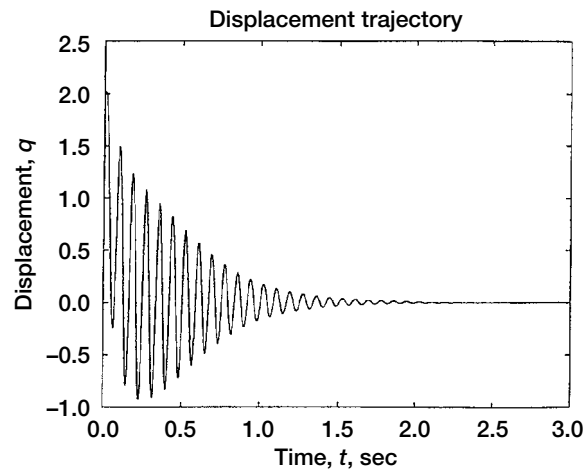

Figure 4.—Control of unforced Duffing dynamics with quartic regulator. Initial displacement, 2.0; initial velocity, 1.0; quadratic displacement weighting, 0.9; quartic displacement weighting, 18.0; velocity weighting, 0.002; and control weighting, 9×10^{-8}.

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13. ABSTRACT *(Maximum 200 words)*

The "direct control method" is a novel concept that is an attractive alternative and competitor to the differential-equation-based methods. The direct method is equally well applicable to nonlinear, linear, time-varying, and time-invariant systems. For all such systems, the method yields explicit closed-form control laws based on minimization of a quadratic control performance measure. We present an application of the direct method to the dynamics and optimal control of the Duffing system where the control performance measure is not restricted to a quadratic form and hence may include a quartic energy term. The results we present in this report also constitute further generalizations of our earlier work in "direct optimal control methodology." The approach is demonstrated for the optimal control of the Duffing equation with a softening nonlinear stiffness.

14. SUBJECT TERMS

Nonlinear systems; Energy methods; Time marching; Variational principles; Differential equations; Nonlinear equations; Softening; Duffing differential equation; Dynamic control; Optimization; Control theory; Variations; Nonlinearity; Optimal control

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