On the Formulation of Weakly Singular Displacement/Traction Integral Equations; and Their Solution by the MLPG Method

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Abstract: In this paper, a very simple method is used to derive the weakly singular traction boundary integral equation based on the integral relationships for displacement gradients [Okada, Rajiyah and Atluri, 1989]. The concept of the MLPG method is employed to solve the integral equations, especially those arising in solid mechanics. A moving Least Squares (MLS) interpolation is selected to approximate the trial functions in this paper. Five boundary integral solution methods are introduced: direct solution method; displacement boundary-value problem; traction boundary-value problem; mixed boundary-value problem; and boundary variational principle. Based on the local weak form of the BIE, four different nodal-based local test functions are selected, leading to four different MLPG methods for each BIE solution method. These methods combine the advantages of the MLPG method and the boundary element method.

1. Introduction

Numerical methods based on integral equation formulations of continuum mechanics, have become increasingly popular, for the solution of practical engineering problems. The integral equation methods can be considered to be derivable from the global unsymmetrical weak forms and Petrov-Galerkin approximation schemes (Zhang and Atluri, 1986, 1988). In these methods, the trial and test function-spaces are quite different from each other. The test functions correspond to fundamental solutions, in infinite space, of the differential operator of the problem, and hence are usually infinitely differentiable, except possibly at the singular points. If the fundamental solution can be derived for the entire differential operator of the problem, the integral representations involve only boundary-integrals, whose descretizations lead to the "boundary element method".

In practice, information about the tractions on an oriented surface in the continuum is often required, so it is necessary to introduce an integral relation for the tractions. Such a traction integral equation is most often derived directly from the displacement integral equation, by a direct differentiation of the displacement integral equation. The resulting relation, however, contains a *hypersingular kernel*. The numerical solution of the hypersingular traction integral equation is challenging; and various strategies have been developed to cope with these difficulties. On the other hand, a distinctly different approach was presented by Okada, Rajiyah and Atluri (1989), [wherein they use the gradients of the fundamental solution as the test functions, instead of the fundamental solution itself], in order to obtain a *singularity-reduced traction integral equation*. Li and Mear (1998), Li, Mear and Xiao (1998) also developed a weakly singular global weak-form traction integral equation. They follow a systematic approach of regularization, in which stress functions associated with certain fundamental solutions are utilized to affect a global "integration-by-parts". Based on the weakly singular global weak-form integral equations, the symmetric Galerkin boundary element method (SGBEM) [Bonnet, Maier and Polizzotto (1998)] can be developed. However, in this paper, based on the *singularity-reduced traction integral formulation* of Okada, Rajiyah and Atluri (1989), we will develop a weakly singular *local weak-form traction integral equation*, without the introduction of stress functions. Our derivation is much simpler and direct than that of Li and Mear (1998).

Meshless methods, as alternative numerical approaches to eliminate the well-known drawbacks in the finite element and boundary element methods, have attracted much attention in recent decades, due to their flexibility, and due to their potential in negating the need for the human-labor intensive process of constructing geometric meshes in a domain. Such meshless methods are especially useful in problems with discontinuities or moving boundaries. The main objective of the meshless methods is to get rid of, or at least alleviate the difficulty of, meshing and remeshing the entire structure; by only adding or deleting nodes in the entire structure, instead. Meshless methods may also alleviate some other problems associated with the finite element method, such as locking, element distortion, and others.

Recently, the meshless local Petrov-Galerkin (MLPG) method, has been developed, in two of its alternate forms, in Zhu, Zhang and Atluri (1998a, b), Atluri & Zhu (1998a, b) and Atluri, Kim and Cho (1999), for solving linear and non-linear boundary problems. This method is truly meshless, as no finite element/or boundary element meshes are required, either for the purposes of interpolation of the trial and test functions for the solution variables, or for the purpose of integration of the 'energy'. All pertinent integrals can be easily evaluated over over-lapping, regularly shaped, domains (in general, spheres in three-dimensional problems) and their boundaries. Remarkable successes of the MLPG method have been reported in solving the convection-diffusion problems [Lin & Atluri (2000)]; fracture mechanics problems [Kim & Atluri (2000), Ching & Batra (2001)]; Navier-Stokes flows [Lin & Atluri (2001)]; and plate bending problems [Gu & Liu (2001); Long & Atluri (2002)]. A comparison study of the efficiency and accuracy of a variety of meshless trial and test functions is presented in Atluri and Shen (2002a, b), based on the general concept of the meshless local Petrov-Galerkin (MLPG) method.

In summary, the MLPG method is a truly meshless method, which involves not only a meshless interpolation for the trial functions (such as MLS, PU, Shepard function or RBF), but also a meshless integration of the weak-form (i.e. all integrations are always performed over regularly shaped sub-domains such as spheres, parallelopipeds, and ellipsoids in 3-D). In the conventional Galerkin method, the trial and test functions are chosen from the same function-space. In MLPG, the nodal trial and test functions can be different: the nodal trial function may correspond to any one of MLS, PU, Shepard function, or RBF types of interpolations; and the test function may be totally different, and may correspond to any one of MLS, PU, Shepard function, RBF, a Heaviside step function, a Dirac delta function, the Gaussian weight function of MLS, or any other convenient function. Furthermore, the physical sizes of the supports of the nodal trial and test functions may be different. These features make the MLPG method very flexible. The MLPG method, based on a *local formulation*, can include all the other meshless methods based on global formulation, as special cases.

In this paper, we use the concept of the MLPG method to formulate computational approaches to solve the integral equations, especially those arising in solid mechanics. A Moving Least Squares (MLS) interpolation is used to approximat the trial functions in this paper, although this is arbitrary. Four different nodal-based local test functions are also selected, leading to four different MLPG methods for BIE. Based on the MLPG concept [Atluri and Shen (2002a)], we label these variants of the MLPG method for solving the boundary integral equations as MLPG/BIE1, MLPG/BIE 2, MLPG/BIE 5, and MLPG/BIE 6, respectively [in conformity with the labeling introduced in Atluri & Shen (2002a, b)].

The paper is organized as follows. In Section 2, we introduce the MLS approximation over the surface of a 3-D solid, using arbitrary *curvilinear co-ordinates*. In Section 3 are derived the weakly singular boundary integral equations, based on the result from Okada, Rajiyah and Atluri (1989), and the results are shown to be identical to those in Li and Mear (1998). In Section 4, five boundary integral solution methods are introduced: the direct solution method; displacement boundary-value problem; traction boundary-value problem; mixed boundary-value problem; and the boundary variational principle. Local weak forms for the boundary integral equations are introduced in each of these boundary integral equation (BIE) solution methods; *four different* nodal-based local test functions are selected; and thus *four different* MLPG methods are developed in each of those BIE solution methods. The paper concludes in Section 5.

2. Moving least–squares method (MLS) in Curvilinear Coordinates, on the Boundary of a 3-D Body

Since the nodes lie only on the boundary $\partial\Omega$ (a curved surface in 3-D space) of a 3-D body Ω, curvilinear co-ordinates are necessary to define the MLS interpolates on the surface. The moving least-square method is generally considered to be one of the best schemes to interpolate data with a reasonable accuracy. The MLS interpolation does not pass through the nodal data. Here we give a brief summary of the MLS approximation for *curvilinear co-ordinates*. For details of the MLS approximation, see Belytschko et al. (1996), Atluri, Cho and Kim (1999), and Atluri and Shen (2002a).

Consider a surface $\Gamma_{\mathbf{x}}$, which is defined as the neighborhood of a point $\mathbf{s}$ and denoted as the domain of definition of the MLS approximation for the trial function at $\mathbf{s}$, and which is located in the problem surface $\partial\Omega$. To approximate the distribution of the function u in $\Gamma_{\mathbf{x}}$, over a number of randomly located nodes $\{\mathbf{s}_I\}$, I=1, 2,..., N, the moving least squares approximant $u^h(\mathbf{s})$ of u, $\forall \mathbf{s} \in \Gamma_{\mathbf{x}}$, can be defined by

$$u^h(\mathbf{s}) = \mathbf{p}^{\mathrm{T}}(\mathbf{s})\mathbf{a}(\mathbf{s}) \quad \forall \mathbf{s} \in \Gamma_{\mathbf{x}} \tag{1}$$

where $\mathbf{p}^{\mathrm{T}}(\mathbf{s})=[p_1(\mathbf{s}), p_2(\mathbf{s}), \ldots, p_m(\mathbf{s})]$ is a complete monomial basis, and $\mathbf{a}(\mathbf{s})$ is a vector containing coefficients $a_j(\mathbf{s})$, j=1, 2, …,m which are functions of the *spatial curvilinear coordinates* $\mathbf{s}=[s^1, s^2]^{\mathrm{T}}$. The commonly used bases in 2-D are the linear basis, due to their simplicity. The linear basis assures that the MLS approximation has the linear completeness. Thus, it can reproduce any smooth function *and its first derivative* with arbitrary accuracy, as the approximation is refined. It is also possible to use other functions in a basis. For example, in problems with singular solutions, singular functions can be included in the basis.

The coefficient vector $\mathbf{a}(\mathbf{s})$ is determined by minimizing a weighted discrete L_2 norm, which can be defined as

$$J(\mathbf{s}) = \sum_{I=1}^{N} w_I(\mathbf{s})\left[\mathbf{p}^{\mathrm{T}}(\mathbf{s}_I)\mathbf{a}(\mathbf{s}) - \hat{u}^I\right]^2 \tag{2}$$

where $w_I(\mathbf{s})$ is a weight function associated with the node I, with $w_I(\mathbf{s})>0$ for all $\mathbf{s}$ in the support of $w_I(\mathbf{s})$, $\mathbf{s}_I$ denotes the value of s at node I, N is the number of nodes in $\Gamma_{\mathbf{x}}$ for which the weight functions $w_I(\mathbf{s})>0$.

Here it should be noted that $\hat{u}^I$, I=1, 2,…, N, in equation (2), are the fictitious nodal values, and not the actual nodal values of the unknown trial function $u^h(\mathbf{s})$.

Substituting $\mathbf{a}(\mathbf{s})$, which is solved by minimizing J in equation (2), into equation (1), give a relation which may be written in the form of an interpolation function similar to that used in the FEM, as

$$u^h(\mathbf{s}) = \sum_{I=1}^{N} \phi^I(\mathbf{s})\hat{u}^I; \qquad u^h(\mathbf{s}_I) \equiv u^I \neq \hat{u}^I, \quad \mathbf{s} \in \Gamma_{\mathbf{x}} \tag{3}$$

where

$$\phi^I(\mathbf{s}) = \sum_{j=1}^{m} p_j(\mathbf{s})\left[\mathbf{A}^{-1}(\mathbf{s})\mathbf{B}(\mathbf{s})\right]_{jI} \tag{4}$$

where

$$\mathbf{A}(\mathbf{s}) = \sum_{I=1}^{N} w_I(\mathbf{s})\mathbf{p}(\mathbf{s}_I)\mathbf{p}^{\mathrm{T}}(\mathbf{s}_I) \tag{5}$$

$$\mathbf{B}(\mathbf{s}) = \left[w_1(\mathbf{s})\mathbf{p}(\mathbf{s}_1), w_2(\mathbf{s})\mathbf{p}(\mathbf{s}_2), \cdots, w_N(\mathbf{s})\mathbf{p}(\mathbf{s}_N)\right] \tag{6}$$

The partial derivatives of $\phi^I(\mathbf{s})$ are obtained as

$$\phi^I_{,k} = \sum_{j=1}^{m}\left[p_{j,k}\left(\mathbf{A}^{-1}\mathbf{B}\right)_{jI} + p_j\left(\mathbf{A}^{-1}\mathbf{B}_{,k} + \mathbf{A}^{-1}_{,k}\mathbf{B}\right)_{jI}\right] \tag{7}$$

where $\mathbf{A}^{-1}_{,k} = \left(\mathbf{A}^{-1}\right)_{,k}$ denotes the derivative of the inverse of A with respect to x^k, which is given by

$$\mathbf{A}^{-1}_{,k} = -\mathbf{A}^{-1}\mathbf{A}_{,k}\mathbf{A}^{-1} \tag{8}$$

The MLS approximation is well defined, only when the matrix in eqn (5) is non-singular. $\phi^I(\mathbf{s})$ is usually called the shape function of the MLS approximation, corresponding to the nodal point $\mathbf{x}_I$. From eqns (4) and (6), it may be seen that $\phi^I(\mathbf{s})=0$ when $w_I(\mathbf{s})=0$, that preserves the local character of the moving least squares approximation. The nodal shape function is complete up to the order of the basis. The smoothness of the nodal shape function is determined by that of the basis, and of the weight function.

The choice of the weight function is more or less arbitrary, as long as the weight function is positive and continuous. Both Gaussian and spline weight functions with compact supports can be considered in the present work. The Gaussian weight function corresponding to node I may be written as

$$w_I(\mathbf{s}) = \begin{cases} \dfrac{\exp\left[-(d_I/r_I)^{2k}\right] - \exp\left[-(r_I/c_I)^{2k}\right]}{1 - \exp\left[-(r_I/c_I)^{2k}\right]}, & 0 \le d_I \le r_I \\ 0, & d_I \ge r_I \end{cases} \tag{9}$$

where $d_I = |\mathbf{s} - \mathbf{s}_I|$ is the distance from node $\mathbf{s}_I$ to point $\mathbf{x}$, c_I is a constant controlling the shape of the weight function w_I (and therefore the relative weights), and r_I is the size of the support for the weight function w_I (and thus determines the support of node $\mathbf{s}_I$).

A spline weight function is defined as

$$w_I(\mathbf{x}) = \begin{cases} 1 - 6\left(\dfrac{d_I}{r_I}\right)^2 + 8\left(\dfrac{d_I}{r_I}\right)^3 - 3\left(\dfrac{d_I}{r_I}\right)^4, & 0 \le d_I \le r_I \\ 0, & d_I \ge r_I \end{cases} \tag{10}$$

The size of support, r_I of the weight function w_I associated with node I should be chosen such that r_I should be large enough to have a sufficient number of nodes covered in the domain of definition of every sample point ($n \ge m$), in order to ensure the regularity of $\mathbf{A}$. It can be easily seen that the spline weight function (10) possesses C^1 continuity. So, the MLS shape functions, and the corresponding trial functions are C^1 continuous over the entire domain.

3. Boundary Integral Equations in Linear Solid Mechanics

Let σ_{ij} be the Cartesian components of the Cauchy stress tensor, and let f_i be the body force per unit volume. The equations of linear and angular momentum balance are:

$$\sigma_{ji,j} + f_i = 0; \qquad \sigma_{ji} = \sigma_{ij} \tag{11}$$

where $(\ )_{,i}$ denotes differentiation with respect to material coordinates x_i. For a linear elastic isotropic solid, the stress-strain relations are:

$$\sigma_{ij} = E_{ijkl}\varepsilon_{kl} \tag{12}$$

$$E_{ijmn} = \lambda\delta_{ij}\delta_{mn} + \mu\left(\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm}\right) \tag{13}$$

where, λ and μ are Lame constants and δ_{ij} is the Kronecker's delta. The strain-displacement relations are

$$\varepsilon_{kl} = \frac{1}{2}\left(u_{k,l} + u_{l,k}\right) \tag{14}$$

The boundary tractions are given by:

$$t_j = \sigma_{ij}n_i \tag{15}$$

where n_i are components of a unit outward normal to the closed boundary $\partial\Omega \equiv \Gamma$.

Let u_i be the trial functions for displacement, and let $\tilde{u}_i$ be the corresponding test functions. The global weak-form of the equilibrium equation (11), can be written as:

$$\int_{\Omega} (\sigma_{ij,j} + f_i)\tilde{u}_i d\Omega = 0 \tag{16}$$

where σ_{ij} are assumed to be written as functions of u_i through eqns (12) and (13).

We assume that the test functions $\tilde{u}_i$ are the fundamental solutions, in infinite space, to the Navier equations of elasticity, i.e.

$$\left[E_{ijkl}\tilde{u}_{k,l}\right]_{,i} + \delta(\mathbf{x} - \xi)\delta_{jp}e_p = 0 \tag{17}$$

where e_p denotes the direction of the unit load at $x_m = \xi_m$. We assume that the solid is isotropic, in which case the solution u_i for (17) is readily available. Thus,

$$\tilde{u}_j = u^*_{jp} e_p \quad \text{(no sum for } p\text{)} \tag{18}$$

and

$$\tilde{t}_j = t^*_{jp} e_p \equiv n_i E_{ijkl}\left(u^*_{kp,l}\right) \tag{19}$$

Here, u^*_{jp} is the jth component of displacement at location x_m due to a unit load along the pth direction at location ξ_m. Likewise, t^*_{jp} is the jth component of traction on an oriented surface at x_m due to a unit load along the pth direction at ξ_m. By using the divergence theorem twice on eqn (16), and substituting eqn (17) in the resulting equation, one obtains the integral equation:

$$u_p(\xi) = \int_{\Gamma} \left[t_j(\mathbf{x})u^*_{jp}(\mathbf{x}, \xi) - u_j(\mathbf{x})t^*_{jp}(\mathbf{x}, \xi)\right]d\Gamma + \int_{\Omega} f_j(\mathbf{x})u^*_{jp}(\mathbf{x}, \xi)d\Omega \tag{20}$$

where Γ is the *global* boundary; and Ω is the *global* domain, enclosed by Γ. By taking the point ξ, in the limit, to the boundary, one may obtain the well-known boundary integral equation for u_p. The kernel t^*_{jp} is singular at the source point ξ, with the order of singularity being $O(1/r^2)$ in 3D or $O(1/r)$ in 2D; while the kernel u^*_{jp} is weakly singular, with the order $O(1/r)$ in 3D or $O(\ln(1/r))$ in 2D. Hence, the order of the singularity of eqn (20) is $O(1/r^2)$ in 3D or $O(1/r)$ in 2D.

It is also well known that the singular kernels u^*_{jp} and t^*_{jp} remain integrable in the limit when ξ_m tends to the boundary. By a direct differentiation of eqn (20) with respect to ξ_m, one may obtain:

$$\frac{\partial u_p}{\partial \xi_k} \equiv u_{p,k}(\xi) = \int_\Gamma \left[t_j(\mathbf{x}) \frac{\partial u^*_{jp}(\mathbf{x},\xi)}{\partial \xi_k} - u_j(\mathbf{x}) \frac{\partial t^*_{jp}(\mathbf{x},\xi)}{\partial \xi_k} \right] d\Gamma + \int_\Omega f_j(\mathbf{x}) \frac{\partial u^*_{jp}(\mathbf{x},\xi)}{\partial \xi_k} d\Omega \qquad (21)$$

In the limit as $\xi \to \Gamma$, the kernel $\partial u^*_{jp}/\partial \xi_k$ is singular with the order of singularity being $O(1/r^2)$ in 3D; the kernel $\partial t^*_{jp}/\partial \xi_k$ becomes hyper-singular, with the order of singularity being $O(1/r^3)$ in 3D, and thus becomes numerically intractable. Hence, eqn (21) becomes hyper-singular with the order of singularity being $O(1/r^3)$ in 3D.

To circumvent these difficulties, Okada, Rajiyah and Atluri (1989, 1994) developed alternate types of integral representation for displacement gradients, which will be introduced in the following, while they originally presented the non-hypersingular integral equations for the Field-Boundary Element Method in nonlinear solid mechanics.

Instead of writing the global weak-form of the linear momentum balance relations in a scalar-form as in eqn (16), Okada and Atluri (1989, 1994) wrote the global weak-forms of the linear momentum balance relation in a three component vector form, as:

$$\int_\Omega \left(\sigma_{ij,i} + f_j \right) \tilde{u}_{j,k} d\Omega = 0 \qquad (22)$$

Assuming that the linear elastic solid is homogeneous, (22) can be rewritten as:

$$\int_\Omega \left[E_{ijmn} u_{m,ni} + f_j\right] \tilde{u}_{j,k} d\Omega = 0 \tag{23}$$

Integrating eqn (23) by parts, applying the divergence theorem, and making use of eqn (15), one obtains:

$$\int_\Gamma (t_j \tilde{u}_{j,k} + \tilde{t}_m u_{m,k} - n_k E_{ijmn} u_{m,n} \tilde{u}_{j,i}) d\Gamma + \int_\Omega \tilde{f}_m u_{m,k} d\Omega + \int_\Omega f_j \tilde{u}_{j,k} d\Omega = 0 \tag{24}$$

where

$$\tilde{f}_j = -\left(E_{ijkl} \tilde{u}_{k,l}\right)_{,i} \tag{25}$$

and

$$\tilde{t}_m = E_{ijmn} \tilde{u}_{j,i} n_m \tag{26}$$

Upon taking $\tilde{u}_i$ and $\tilde{t}_i$ to be the fundamental solutions as in eqns (18) and (19), one obtains the *global* integral relation:

$$\begin{aligned} Cu_{p,k}(\xi) &= \int_\Gamma (n_k E_{ijmn} u_{m,n} u^*_{jp,i} - t^*_{mp} u_{m,k} - t_j u^*_{jp,k}) d\Gamma - \int_\Omega f_j u^*_{jp,k} d\Omega \\ &= \int_\Gamma (n_k u_{m,n} \sigma^*_{mnp} - n_n u_{m,k} \sigma^*_{mnp} - t_j u^*_{jp,k}) d\Gamma - \int_\Omega f_j u^*_{jp,k} d\Omega \end{aligned} \tag{27}$$

In eqn (27), C=1 when $\xi_m \in \Omega$, and C=1/2 at a smooth part of the boundary. As compared to eqn (21) wherein the kernel $\partial t^*_{jp}/\partial \xi_k$ is involved in the boundary integral, in the presently derived eqn (27), only the kernels $u^*_{jp,i}$ and t^*_{mp} are involved at Γ. Note that the orders of singularity in $u^*_{jp,i}$ and t^*_{mp}, which are equal, are nevertheless smaller than that in $\partial t^*_{jp}/\partial \xi_k$. The singularities in the integrals in eqn (27) are in general tractable, and the integrals in eqn (27) can be evaluated by the method suggested by Guiggiani and Casalini

(1987) for two-dimensional case, by Guiggiani and Gigante (1990) for three-dimensional case, or by an alternate method (Okada and Atluri, 1994). Once the tractions t_j at the boundary are completely known, the displacement gradients $u_{i,j}$ can be determined from eqn (27) in the interior Ω as well as at Γ.

Now, we will further regularize eqn (27), in order to reduce its singularity to be same as the orders of singularity in u^*_{jp}. The *surface curl operator* can be defined as

$$D_m = n_i e_{ikm} \frac{\partial}{\partial x_k} \tag{28}$$

where e_{ikm} is the usual permutation symbol. The surface curl, eqn (28) is associated with the following form of the Stokes' formula:

$$\int_\Gamma D_m f(\mathbf{x}) d\Gamma(\mathbf{x}) = \oint_{\partial\Gamma} f(\mathbf{x}) dx_m \tag{29}$$

where Γ is any regular surface with the edge contour $\partial\Gamma$.

The first two terms in the extreme right side of eqn (27) can hence be written as

$$n_k u_{m,n} \sigma^*_{mnp} - n_n u_{m,k} \sigma^*_{mnp} = \sigma^*_{mnp}(n_k u_{m,n} - n_n u_{m,k}) = -\sigma^*_{nmp} e_{nkl} D_l u_m \tag{30}$$

Thus, eqn (27) can be rewritten as

$$\begin{aligned} Cu_{p,k}(\xi) &= \int_\Gamma (-\sigma^*_{nmp} e_{nkl} D_l u_m - t_j u^*_{jp,k}) d\Gamma - \int_\Omega f_j u^*_{jp,k} d\Omega \\ &= \int_\Gamma (-\sigma^*_{nmp} e_{nkl} D_l u_m - t_j u^*_{jp,k}) d\Gamma - [\int_\Gamma f_j u^*_{jp} n_k d\Gamma - \int_\Omega f_{j,k} u^*_{jp} d\Omega(\mathbf{x})] \end{aligned} \tag{31}$$

The orders of singularity in $u^*_{jp,i}$ and σ^*_{nmp}, which are equal, are $O(1/r^2)$ in 3D. So, eqn (31) is singular with the order of singular $O(1/r^2)$ in 3D. This equation (31) *is exactly the* *same as that in Li, Mear and Xiao (1998).* However, here we used a totally different and simple method to derive it, based on the traction integral equation developed by Okada, Rajiyah and Atluri (1989). In Li, Mear and Xiao (1998), a multi-valued stress function is introduced, and the derivation is based on the gradient of the displacement integral equations.

An integral relation for the traction vector then follows immediately from eqn (31) by an application of Hook's law, with the result:

$$C\sigma_{ib}(\xi) = -\int_{\Gamma}(F_{iblm}D_l u_m + t_j\sigma^*_{ibj})d\Gamma - E_{ibpk}[\int_{\Gamma} f_j u^*_{jp} n_k d\Gamma - \int_{\Omega} f_{j,k} u^*_{jp} d\Omega(\mathbf{x})] \quad (32)$$

where $F_{iblm}(\mathbf{x}-\xi) = E_{ibpk}e_{nkl}\sigma^*_{nmp}$. It is noted that the symmetry condition $u^*_{jp}(\mathbf{x}-\xi) = u^*_{pj}(\mathbf{x}-\xi)$ has been used. Eqn (32) is singular with the order of singular $O(1/r^2)$ in 3D.

3.1 Weakly singular integral equations

Using Somigliana's identity, a weakly singular displacement integral equation can be readily obtained. This can be achieved by a subtraction technique often motivated by consideration of a rigid body motion (Brebbia and Dominguez, 1992). Li, Mear and Xiao (1998) exploited the decomposition of the stress fundamental solution, in order to obtain a weakly singular displacement integral equation. Here, we will also use this method. The stress fundamental solution can be decomposed as

$$\sigma^*_{ijp}(\mathbf{x}-\xi) = e_{ikm}G^p_{mj,k}(\mathbf{x}-\xi) - H^p_{ij}(\mathbf{x}-\xi) \quad (33)$$

where

$$8\pi(1-\nu)G_{mj}^{p}(\mathbf{z}) = (1-2\nu)e_{mpj}\frac{1}{r} + e_{mtj}\frac{z_t z_p}{r^3} \tag{34}$$

and

$$4\pi H_{ij}^{p}(\mathbf{z}) = -\delta_{pj}\frac{\partial}{\partial z_i}\left(\frac{1}{r}\right) = \delta_{pj}\frac{z_i}{r^3} \tag{35}$$

where ν is the Poisson's ratio and $r(\mathbf{z}) = \|\mathbf{z}\|$.

Similar to Li, Mear and Xiao (1998), the following results will be used:

$$F_{klmj}(\mathbf{x}-\xi) = e_{lrt}\frac{\partial C_{tkmj}(\mathbf{x}-\xi)}{\partial x_r} + \frac{\partial T_{klj}(\mathbf{x}-\xi)}{\partial x_m} \tag{36}$$

where

$$8\pi(1-\nu)C_{tkmj}(\mathbf{z}) = \frac{\mu}{r}\left[-2\frac{z_k z_j}{r^2}\delta_{tm} + 2(1-\nu)\delta_{kt}\delta_{jm} + 4\nu\delta_{km}\delta_{jt} - 2\delta_{kj}\delta_{tm}\right] \tag{37}$$

and

$$8\pi(1-\nu)T_{klj}(\mathbf{z}) = \frac{\mu}{r}\left[(1-2\nu)e_{kjl} + e_{kls}\frac{z_s z_j}{r^2}\right] \tag{38}$$

where μ is the shear modulus. Thus, the traction global integral equation can be obtained from eqa. (35-38) as [same as in Li, Mear and Xiao (1998)]:

$$Ct_i(\xi) = n_b(\xi)e_{bsm}\frac{\partial}{\partial\xi_s}\int_\Gamma G^j_{mi}(\mathbf{x}-\xi)t_j(\mathbf{x})d\Gamma + n_b(\xi)e_{brt}\frac{\partial}{\partial\xi_r}\int_\Gamma C_{timj}(\mathbf{x}-\xi)D_m u_j(\mathbf{x})d\Gamma$$
$$+ n_b(\xi)\frac{1}{4\pi}\int_\Gamma \frac{(x_b-\xi_b)}{r^3}t_i(\mathbf{x})d\Gamma - n_b(\xi)E_{ibpk}[\int_\Gamma f_j u^*_{jp} n_k d\Gamma - \int_\Omega f_{j,k} u^*_{jp} d\Omega(\mathbf{x})] \tag{39}$$

Incorporating the decomposition (21) into (20), and utilizing Stokes's theorem to integrate by parts, a weakly singular integral equation is then obtained as

$$\boxed{\begin{aligned} Cu_p(\xi) = {} & \int_\Gamma t_j(\mathbf{x})u^*_{jp}(\mathbf{x},\xi)d\Gamma + \frac{1}{4\pi}\int_\Gamma n_i\frac{(\xi_i - x_i)}{r^3}u_p(\mathbf{x})d\Gamma \\ & + \int_\Gamma D_m u_j(\mathbf{x})G^p_{mj}(\mathbf{x},\xi)d\Gamma + \int_\Omega f_j(\mathbf{x})u^*_{jp}(\mathbf{x},\xi)d\Omega \end{aligned}} \tag{40}$$

Note that the kernel $n_i(\mathbf{x})(\xi_i - x_i)/r^3$ appearing in the second integral is only weakly singular since $n_i(\mathbf{x})(\xi_i - x_i)/r \to O(r)$ as $r \to 0$, hence, eqn (40) involves only $O(1/r)$ singular kernel.

On the other hand, we can find that

$$\begin{aligned} & n_k E_{ijmn} u_{m,n} u^*_{jp,i} - t_j u^*_{jp,k} = n_k \sigma_{ij} u^*_{jp,i} - n_i \sigma_{ij} u^*_{jp,k} \\ & = \sigma_{ij}(n_k u^*_{jp,i} - n_i u^*_{jp,k}) = \sigma_{ij} e_{imk} D_m u^*_{jp} \end{aligned} \tag{41}$$

Thus, using eqn (41), the traction integral equation (39) can also be rewritten as

$$\begin{aligned} Ct_b(\xi) = {} & n_a E_{abpk}\int_\Gamma (\sigma_{ij} e_{imk} D_m u^*_{jp} - t^*_{mp} u_{m,k})d\Gamma \\ & - n_a E_{abpk}[\int_\Gamma f_j u^*_{jp} n_k d\Gamma - \int_\Omega f_{j,k} u^*_{jp} d\Omega(\mathbf{x})] \end{aligned} \tag{42}$$

The corresponding weakly singular integral equation can then be obtained as

$$
\boxed{\begin{aligned}
Ct_b(\xi) = {} & n_a(\xi)E_{abpk}\int_{\Gamma} D_m\sigma_{ij}(\mathbf{x})e_{ikm}u^*_{jp}(\mathbf{x},\xi)d\Gamma \\
& + \frac{1}{4\pi}n_a(\xi)E_{abpk}\int_{\Gamma} n_i\frac{(\xi_i - x_i)}{r^3}u_{p,k}(\mathbf{x})d\Gamma \\
& + n_a(\xi)E_{abpk}\int_{\Gamma} D_m u_{j,k}(\mathbf{x})G^p_{mj}(\mathbf{x},\xi)d\Gamma \\
& + n_a E_{abpk}[\int_{\Gamma} f_j u^*_{jp} n_k d\Gamma - \int_{\Omega} f_{j,k}u^*_{jp}d\Omega(\mathbf{x})]
\end{aligned}} \tag{43}
$$

Note that the singular kernel appearing in the integral equation (43) is only weakly singular (Cauchy singular kernel), with the order of singular $O(1/r)$, similar to the displacement integral equation (40) and weaker than that in equation (39). In this equation, second derivatives of the shape functions are needed in constructing the global stiffiness matrix. It is important to note that even though the singularity of the integral kernels has been reduced to Cauchy type, it still requires that the derivatives of the displacements are continuous. However, the calculation of the derivatives of shape functions from the MLS approximation is quite costly. Hence, we choose to use equation (39) to develop the MLPG approach in this paper.

4. Solution Methods for BIE: the Meshless Local Petrov-Galerkin (MLPG) Approach

To solve the boundary integral equations, different methods are proposed in this section, similar to those in traditional boundary element methods (Atluri and Grannell, 1978).

To start with, we note that in a well-posed boundary-value problem in 3-D solid mechanics, at any point on the boundary, any 3 quantities out of the 6, viz, u_i (three) and t_j (3) can be prescribed and the other 3 components are the unknowns to be solved for. In a pure displacement boundary-value problem, all the 3-components of u_i are prescribed at every point on the surface of the 3-D solid; and correspondingly, all the three components of t_j at every point on $\partial\Omega$ are the unknowns. Conversely, in a pure traction boundary-value problem, all the three t_j are prescribed at every point on $\partial\Omega$, while all the 3 components of u_i at $\partial\Omega$ are unknown.

4.1 Direct solution method

This method attempts to obtain a direct numerical solution to equation (40). Upon multiplying (40) *by a continuous test function $q_p(\xi)$ over a local sub-domain* Γ_s (a part of Γ) and applying the divergence theorem, *a local "weakly singular" weak-form* of the global displacement integral equation for a system, without body force, is obtained as

$$\begin{aligned}\int_{\Gamma_s} C u_p(\xi) q_p(\xi) d\Gamma_s &= \int_{\Gamma_s} q_p(\xi) \int_{\Gamma} t_j(\mathbf{x}) u^*_{jp}(\mathbf{x},\xi) d\Gamma d\Gamma_s \\ &+ \frac{1}{4\pi}\int_{\Gamma_s} q_p(\xi) \int_{\Gamma} n_i \frac{(\xi_i - x_i)}{r^3} u_p(\mathbf{x}) d\Gamma d\Gamma_s \\ &+ \int_{\Gamma_s} q_p(\xi) \int_{\Gamma} D_m u_j(\mathbf{x}) G^p_{mj}(\mathbf{x},\xi) d\Gamma d\Gamma_s\end{aligned} \tag{44}$$

To obtain the discrete equations based on MLS interpolation, the following interpolations are used

$$u_i(\mathrm{s}) = \sum_{I=1}^{N} \phi^I(\mathrm{s}) \hat{u}_i^I \tag{45a}$$

$$t_i(\mathrm{s}) = \sum_{I=1}^{N} \phi^I(\mathrm{s}) \hat{t}_i^I \tag{45b}$$

$$q_i(\mathrm{s}) = \sum_{I=1}^{N} \psi^I(\mathrm{s}) \hat{q}_i^I \tag{45c}$$

where $\mathbf{s}$ is a point with curvilinear co-ordinates (s_1,s_2), $\phi^I(\mathbf{s})$ and $\psi^I(\mathbf{s})$ are the nodal shape functions for trial and test functions centered at node, s_1 and s_2 are functions of x_i. In general, in meshless interpolations, $\hat{u}_i^I$ are fictitious nodal values. Substitution of eqns (45) into formulations (44) leads to the following discretized system of linear equations for each node I

$$\sum_m (V_{IpJj}^m \hat{t}_j^J - U_{IpJj}^m \hat{u}_j^J) = W_{IJ} \hat{u}_p^J \tag{46}$$

where

$$V_{IpJj}^m = \int_{\Gamma_s^I} \psi^I(\xi) \int_{\Gamma^m} \phi^J(\mathbf{x}) u_{jp}^*(\mathbf{x},\xi) d\Gamma d\Gamma_s \tag{47}$$

$$U_{IpJj}^m = -\int_{\Gamma_s^I} \psi^I \int_{\Gamma^m} \{H_{nj}^p(\mathbf{x}-\xi) n_n \phi^J d\Gamma d\Gamma_s + G_{nj}^p(\mathbf{x}-\xi) D_n \phi^J\} d\Gamma d\Gamma_s \tag{48}$$

$$W_{IJ} = \int_{\Gamma_s^I} C\phi^J \psi^I d\Gamma_s \tag{49}$$

m is the number of the background mesh, and I is the nodal number. Due to the fact that the first step of the dual integration is performed on the global boundary surface Γ, a background mesh will have to be used on Γ. It is noted that the repeated indices imply summation here. The trial functions appear only in the global integration over Γ, which means that the integrand for the local integration over Γ_s will be simple. It can be found that the assembly process is not required to form a global 'stiffness' matrix.

The equations can be solved in the same way as in the conventional BEM, except that the transformations between $\hat{u}_i^I$ and $\bar{u}_i^I$, $\hat{t}_i^I$ and $\bar{t}_i^I$ must be performed, due to the fact that the MLS interpolates lack the delta function property of the usual BEM shape functions [Atluri, Kim and Cho (1999), Atluri and Shen (2002a)].

On the boundary surface on which u_i are prescribed, $\hat{u}_i^I$ can be obtained by the following transformation:

$$\hat{u}^I = \sum_{J=1}^{N} R_{IJ} \bar{u}^J \tag{50}$$

On the boundary surface on which t_i are prescribed, $\hat{t}_i^I$ can be obtained by the same transformation as the above

$$\hat{t}^I = \sum_{J=1}^{N} R_{IJ} \bar{t}^J \tag{51}$$

where $R_{IJ} = \left[\phi^J(s_I)\right]^{-1}$. The details of the transformation method for the imposition of boundary conditions can be found in [Atluri, Kim and Cho (1999), Atluri and Shen (2002a)].

Based on the local weak form, the meshless local Petrov-Galerkin (MLPG) method is used to treat the global boundary integral equations. However, due to the fact that a global weak form is used to derive the boundary integral equations, we need a background mesh to perform the global integration first. For the second step, to perform the local integration, no mesh is used.

In general, in MLPG, the nodal trial and test functions can be different, the nodal trial function may correspond to any one of: 1. MLS; 2. Partition of unity; 3. Shepard function; or 4. Radial basis function interpolations (Atluri and Shen, 2002a); and the test function may be totally different. Furthermore, the size of the sub-domains over which the nodal trial and test functions are, respectively, non-zero, may be different. Different choices for the basis functions for the trial function, and the test functions, will lead to different approximation methods. We will use the MLS interpolations introduced in Section 2, to generate the trial functions. Based on the concept of the MLPG, the test functions over each local boundary Γ_s can be chosen through a variety of ways.

As a known test function is used in the local weak form (LWF), the use of the LWF for one point (and here for one sub-surface Γ_s) will yield only one algebraic equation. It is noted that the trial functions u or t within the sub-surface Γ_s, in the interpolations without Kronecker's Delta properties, is determined by the fictitious nodal values $\hat{u}^I$ or $\hat{t}^I$, respectively, within the domain of definition for all points $\mathbf{x}$ falling within Γ_s. One can obtain as many equations as the number of nodes. Hence, we need as many local boundary surfaces Γ_s as the number of nodes in the global boundary surface, in order to obtain as many equations as the number of unknowns. In the present paper, dealing with 3D problems, the local boundary surface is chosen as a circle, centered at a node $\mathbf{x}_I$. Four test functions are chosen to formulate four different MLPG methods for BIE:

(1) the test function over Γ_s is the same as the weight function in the MLS approximation: The resultant Meshless Local Petrov-Galerkin Method for directly solving the BIE is denoted as MLPG/BIE1. In this case, just let $\psi^I = w^I$ in eqns (47), (48) and (49).

(2) the test function over Γ_s is the collocation Dirac's Delta function (collocation method): The resultant Meshless Local Petrov-Galerkin Method for directly solving the BIE is denoted as MLPG/BIE2. In this case, let $\psi^I = \delta(\xi^I)$ in eqns (47), (48) and (49), then it can be found that we get the collocation BEM. The collocation BEM can be treated simply as a special case of the MLPG approach. Chati, Mukherjee and Mukherjee (1999) developed a boundary node method by a coupling between boundary integrals equations (BIE) and Moving Least-Squares (MLS) interpolants. In fact, their method can be treated simply as a special case of the present MLPG method approach for BIE, if we let $\psi^I = \delta(\xi^I)$ in the local weak-

form. *However, here we use an entirely different regularization approach, to avoid the strong singularity.*

(3) the test function over Γ_s is the Heaviside step function (constant over each local sub-domain Γ_s): The resultant Meshless Local Petrov-Galerkin Method for solving the BIE is denoted as MLPG/BIE5. In this case, let $\psi^J=1$ in eqns (47), (48) and (49).

(4) the test function over Γ_s is identical to the trial function (Galerkin method): The resultant Meshless Local Petrov-Galerkin Method for solving the BIE is denoted as MLPG/BIE6. In the Galerkin method, the trial and test functions come from the same space. In this case, let $q_i = u_i(\mathbf{x})$, i.e. let $\psi^J=\phi^J$ in eqns (47), (48) and (49).

The interrelationships of these developments can be illustrated as in Fig. 1. Underlying all these meshless methods for BIE is the general concept of the meshless local Petrov-Galerkin method; thus, MLPG provides a rational basis for constructing meshless methods with a greater degree of flexibility. Theoretically, as long as the union of all local domains covers the global boundary surface, the boundary integral equation will be satisfied.

In the collocation BEM, the location of collocation nodes is an important ingredient for the success of the method. However, if we employ MLPG method, this issue can be avoided.

4.2 Displacement Boundary-Value Problems in Solid Mechanics

In this case, displacements are prescribed on all the boundaries. Then, *the local weakly singular weak-form displacement integral equation* (44) for a system without body force can be rewritten as

$$\int_{\Gamma_s} q_p(\xi)\int_{\Gamma} t_j(\mathbf{x})u^*_{jp}(\mathbf{x},\xi)d\Gamma d\Gamma_s = \int_{\Gamma_s} C\bar{u}_p(\xi)q_p(\xi)d\Gamma_s$$
$$-\frac{1}{4\pi}\int_{\Gamma_s} q_p(\xi)\int_{\Gamma} n_i \frac{(\xi_i - x_i)}{r^3}\bar{u}_p(\mathbf{x})d\Gamma d\Gamma_s \tag{52}$$
$$-\int_{\Gamma_s} q_p(\xi)\int_{\Gamma} D_m\bar{u}_j(\mathbf{x})G^P_{mj}(\mathbf{x},\xi)d\Gamma d\Gamma_s$$

To obtain the discrete equations based on MLS interpolation, using interpolations (45b,c), we can obtain

$$\sum_m A^m_{IpJj}\hat{t}^J_j = F_{Ip} \tag{53}$$

where all $\hat{t}_j$ are unknowns, and

$$A^m_{IpJj} = \int_{\Gamma^I_s} \psi^I(\xi)\int_{\Gamma^m} \phi^J(\mathbf{x})u^*_{jp}(\mathbf{x},\xi)d\Gamma d\Gamma_s \tag{54}$$

$$F_{Ip} = \int_{\Gamma^I_s} C\bar{u}_p(\xi)\psi^I d\Gamma_s$$
$$-\int_{\Gamma^I_s} \psi^I \int_{\Gamma} H^P_{mj}(\mathbf{x},\xi)n_m(\xi)\bar{u}_j(\mathbf{x})d\Gamma d\Gamma_s \tag{55}$$
$$-\int_{\Gamma^I_s} \psi^I \int_{\Gamma} D_m\bar{u}_j(\mathbf{x})G^P_{mj}(\mathbf{x}-\xi)d\Gamma d\Gamma_s$$

Similar to Subsection 4.1, four test functions are chosen to formulate four different MLPG methods for the displacement BIE:

(1) MLPG/DBIE1: the test function over Γ_s is the same as the weight function in the MLS approximation. In this case, just let $\psi^I = w^I$ in eqns (53)-(55).

(2) MLPG/DBIE2: the test function over Γ_s is the collocation Dirac's Delta function (collocation method). In this case, let $\psi^I = \delta(\xi^I)$ in eqns (53-55), then it can be found that we get the collocation displacement BEM.

(3) MLPG/DBIE5: the test function over Γ_s is the Heaviside step function (constant over each local sub-domain Γ_s). In this case, let $\psi^J=1$ in eqns (53)-(55).

(4) MLPG/DBIE6: the test function over Γ_s is identical to the trial function (Galerkin method). In this case, let $q_i = t_i(\mathbf{x})$, i.e. let $\psi^J=\phi^J$ in eqns (53)-(55).

The interrelationships of these developments can also be illustrated as in Fig. 1.

4.3 Traction Boundary-Value Problem in Solid Mechanics

In this case, tractions are prescribed on all the boundaries. Upon multiplying (39) by a continuous test function $v_b(\xi)$ and applying the divergence theorem, a local weakly singular weak-form traction integral equation is obtained as

$$
\begin{aligned}
&-\int_{\Gamma_s} C\bar{t}_i(\xi) v_i(\xi) d\Gamma_s - \int_{\Gamma_s} D_m v_i(\xi) \int_{\Gamma} G^j_{mi}(\mathbf{x}-\xi) \bar{t}_j(\mathbf{x}) d\Gamma d\Gamma_s \\
&+ \oint_{\partial\Gamma_s} v_i(\xi) \int_{\Gamma} G^j_{mi}(\mathbf{x}-\xi) \bar{t}_j(\mathbf{x}) d\Gamma d\xi_m + \frac{1}{4\pi} \int_{\Gamma_s} v_i(\xi) n_b(\xi) \int_{\Gamma} \frac{(x_b-\xi_b)}{r^3} \bar{t}_i(\mathbf{x}) d\Gamma d\Gamma_s \\
&= -\oint_{\partial\Gamma_s} v_i(\xi) \int_{\Gamma} C_{timj}(\mathbf{x}-\xi) D_m u_j(\mathbf{x}) d\Gamma d\xi_t + \int_{\Gamma_s} D_t v_i(\xi) \int_{\Gamma} C_{timj}(\mathbf{x}-\xi) D_m u_j(\mathbf{x}) d\Gamma d\Gamma_s
\end{aligned}
\tag{56}
$$

where $\partial\Gamma_s$ is the edge of the local boundary surface Γ_s. It is noted that only the weakly singular kernel u^*_{jp} is involved in these equations. In the present formulations, the boundary integral equations are satisfied in all the local boundary surfaces Γ_s. Theoretically, as long as the union of all local boundary surface covers the global boundary surface, i.e., $\cup\Gamma_s \supset \Gamma$, the boundary integral equations will be satisfied in the global boundary surface.

To obtain the discrete equations based on MLS interpolation, the following interpolations are used

$$
u_i(\mathbf{s}) = \sum_{I=1}^{N} \phi^I(\mathbf{s}) \hat{u}^I_i \tag{57a}
$$

$$v_i(\mathbf{s}) = \sum_{I=1}^{N} \psi^I(\mathbf{s}) \hat{v}_i^I \tag{57b}$$

Substitution of eqns (57) into formulations (56) leads to following discretized system of linear equations for each node I

$$\sum_m H_{IpJj}^m \hat{u}_j^J = Q_{Ip} \tag{58}$$

with all $\hat{u}_j$ are unknowns, and

$$\begin{aligned} H_{IpJj}^m &= \int_{\Gamma_s^I} D_t \psi^I \int_{\Gamma^m} C_{tpnj}(\mathbf{x}-\xi) D_n \phi^J \, d\Gamma d\Gamma_s \\ &\quad - \oint_{\partial\Gamma_s^I} \psi^I \int_{\Gamma^m} C_{tpnj}(\mathbf{x}-\xi) D_n \phi^J(\mathbf{x}) \, d\Gamma d\xi_t \end{aligned} \tag{59}$$

$$\begin{aligned} Q_{Ip} &= -\int_{\Gamma_s^I} C\bar{t}_p(\xi) \psi^I \, d\Gamma_s - \int_{\Gamma_s^I} D_n \psi^I \int_{\Gamma} G_{np}^j (\mathbf{x}-\xi) \bar{t}_j(\mathbf{x}) \, d\Gamma d\Gamma_s \\ &\quad + \oint_{\partial\Gamma_s^I} \psi^I \int_{\Gamma} G_{np}^j (\mathbf{x}-\xi) \bar{t}_j(\mathbf{x}) \, d\Gamma d\xi_m \\ &\quad + \int_{\Gamma_s^I} \psi^I n_n(\xi) \int_{\Gamma} H_{nj}^p (\mathbf{x}-\xi) \bar{t}_j(\mathbf{x}) \, d\Gamma d\Gamma_s \end{aligned} \tag{60}$$

Similar to Subsection 4.1, four test functions are chosen to formulate four different MLPG methods for the traction BIE:

(1) MLPG/TBIE1: the test function over Γ_s is the same as the weight function in the MLS approximation. In this case, just let $\psi^I = w^I$ in eqns (59) and (60).

(2) MLPG/TBIE2: the test function over Γ_s is the collocation Dirac's Delta function (collocation method). In this case, let $\psi^I = \delta(\xi^I)$ in eqns (59), and (60), then it can be found that we get the collocation displacement BEM.

(3) MLPG/TBIE5: the test function over Γ_s is the Heaviside step function (constant over each local sub-domain Γ_s). In this case, let $\psi^I = 1$ in eqns (59), and (60).

(4) MLPG/TBIE6: the test function over Γ_s is identical to the trial function (Galerkin method). In this case, let $v_i = u_i(\mathbf{x})$, i.e. let $\psi^J = \phi^J$ in eqns (59), and (60).

The interrelationships of these developments can also be illustrated as in Fig. 1.

4.4 Mixed Boundary-Value Problem in Solid Mechanics

This method is developed for the special problems of fracture mechanics of linearly elastic three-dimensional solids, containing cracks. The formulation is based both on the displacement integral equation (40) and traction integral equation (39). When applied to fracture problems, the traditional boundary element methods (or the direct methods in section 4.1) become mathematically degenerate in that when the displacement integral equation is applied to points on the crack surface, information about the traction on the crack is lost (Cruse, 1988). To circumvent this difficulty an integral equation for the traction on the crack surface may be employed. Moreover, in practice, information about the tractions on an oriented surface in the continuum is often required, so it is necessary to use both the traction and displacement integral equations.

Upon multiplying (40) by a continuous test function $q_p(\xi)$ and applying the divergence theorem, a local weakly singular weak-form of the displacement integral equation for a system without body force is obtained [as in (44)] as

$$\begin{aligned}\int_{\Gamma_s} C u_p(\xi) q_p(\xi) d\Gamma_s &= \int_{\Gamma_s} q_p(\xi) \int_{\Gamma} t_j(\mathbf{x}) u^*_{jp}(\mathbf{x},\xi) d\Gamma d\Gamma_s \\ &+ \frac{1}{4\pi} \int_{\Gamma_s} q_p(\xi) \int_{\Gamma} n_i \frac{(\xi_i - x_i)}{r^3} u_p(\mathbf{x}) d\Gamma d\Gamma_s \\ &+ \int_{\Gamma_s} q_p(\xi) \int_{\Gamma} D_m u_j(\mathbf{x}) G^p_{mj}(\mathbf{x},\xi) d\Gamma d\Gamma_s\end{aligned} \tag{61}$$

Upon multiplying (39) by a continuous test function $v_b(\xi)$ and applying the divergence theorem, a local weakly singular weak-form of the traction integral equation is obtained as

$$\begin{aligned}
-\int_{\Gamma_s} Ct_i(\xi)v_i(\xi)d\Gamma_s &= \int_{\Gamma_s} D_m v_i(\xi)\int_{\Gamma} G^j_{mi}(\mathbf{x}-\xi)t_j(\mathbf{x})d\Gamma d\Gamma_s \\
&-\oint_{\partial\Gamma_s} v_i(\xi)\int_{\Gamma} G^j_{mi}(\mathbf{x}-\xi)t_j(\mathbf{x})d\Gamma d\xi_m + \int_{\Gamma_s} D_t v_i(\xi)\int_{\Gamma} C_{timj}(\mathbf{x}-\xi)D_m u_j(\mathbf{x})d\Gamma d\Gamma_s \\
&-\oint_{\partial\Gamma_s} v_i(\xi)\int_{\Gamma} C_{timj}(\mathbf{x}-\xi)D_m u_j(\mathbf{x})d\Gamma d\xi_t - \frac{1}{4\pi}\int_{\Gamma_s} v_i(\xi)n_b(\xi)\int_{\Gamma}\frac{(x_b-\xi_b)}{r^3}t_i(\mathbf{x})d\Gamma d\Gamma_s
\end{aligned} \tag{62}$$

where $\partial\Gamma_s$ is the edge of the local boundary surface Γ_s. It is noted that only the weakly singular kernel u^*_{jp} is involved in these equations.

In the absence of body forces, let the regular boundary be partitioned into a portion Γ_t on which tractions are prescribed and a portion Γ_u on displacements are prescribed such that $\Gamma=\Gamma_t+\Gamma_u$. Applying the local weak-form displacement integral equation (61) on Γ_u with $q_k=0$ on Γ_t, and, similarly, applying the local weak-form traction integral equation (62) on Γ_t with $v_k=0$ on Γ_u gives rise to the formulation

$$A(q,t)+B(q,u)=F(q) \qquad (\text{for } \Gamma_u) \tag{63}$$

$$G(v,t)+H(v,u)=Q(v) \qquad (\text{for } \Gamma_t) \tag{64}$$

where

$$A(q,t)=\int_{\Gamma_{su}} q_p(\xi)\int_{\Gamma_u} t_j(\mathbf{x})u^*_{jp}(\mathbf{x},\xi)d\Gamma d\Gamma_s \tag{65}$$

$$
\begin{aligned}
B(q,u) &= \int_{\Gamma_{su}} q_p(\xi) \int_{\Gamma_t} H^p_{mj}(\mathbf{x}-\xi) n_m(\mathbf{x}) u_j(\mathbf{x}) d\Gamma d\Gamma_s \\
&\quad + \int_{\Gamma_{su}} q_p(\xi) \int_{\Gamma_t} D_m u_j(\mathbf{x}) G^p_{mj}(\mathbf{x}-\xi) d\Gamma d\Gamma_s
\end{aligned} \tag{66}
$$

$$
\begin{aligned}
G(v,t) &= \int_{\Gamma_{st}} D_m v_i(\xi) \int_{\Gamma_u} G^j_{mi}(\mathbf{x}-\xi) t_j(\mathbf{x}) d\Gamma d\Gamma_s \\
&\quad - \oint_{\partial\Gamma_{st}} v_i(\xi) \int_{\Gamma_u} G^j_{mi}(\mathbf{x}-\xi) t_j(\mathbf{x}) d\Gamma d\xi_m \\
&\quad - \int_{\Gamma_{st}} v_i(\xi) n_m(\xi) \int_{\Gamma_u} H^i_{mj}(\mathbf{x}-\xi) t_j(\mathbf{x}) d\Gamma d\Gamma_s
\end{aligned} \tag{67}
$$

$$
\begin{aligned}
H(v,u) &= \int_{\Gamma_{st}} D_t v_i(\xi) \int_{\Gamma_t} C_{timj}(\mathbf{x}-\xi) D_m u_j(\mathbf{x}) d\Gamma d\Gamma_s \\
&\quad - \oint_{\partial\Gamma_{st}} v_i(\xi) \int_{\Gamma_t} C_{timj}(\mathbf{x}-\xi) D_m u_j(\mathbf{x}) d\Gamma d\xi_t
\end{aligned} \tag{68}
$$

and

$$
\begin{aligned}
F(q) &= \int_{\Gamma_{su}} C\bar{u}_p(\xi) q_p(\xi) d\Gamma_s - \int_{\Gamma_{su}} q_p(\xi) \int_{\Gamma_t} \bar{t}_j(\mathbf{x}) u^*_{jp}(\mathbf{x},\xi) d\Gamma d\Gamma_s \\
&\quad - \int_{\Gamma_{su}} q_p(\xi) \int_{\Gamma_u} H^p_{mj}(\mathbf{x},\xi) n_m(\xi) \bar{u}_j(\mathbf{x}) d\Gamma d\Gamma_s \\
&\quad - \int_{\Gamma_{su}} q_p(\xi) \int_{\Gamma_u} D_m \bar{u}_j(\mathbf{x}) G^p_{mj}(\mathbf{x}-\xi) d\Gamma d\Gamma_s
\end{aligned} \tag{69}
$$

$$
\begin{aligned}
Q(v) &= -\int_{\Gamma_{st}} C\bar{t}_i(\xi) v_i(\xi) d\Gamma_s - \int_{\Gamma_{st}} D_m v_i(\xi) \int_{\Gamma_t} G^j_{mi}(\mathbf{x}-\xi) \bar{t}_j(\mathbf{x}) d\Gamma d\Gamma_s \\
&\quad + \oint_{\partial\Gamma_{st}} v_i(\xi) \int_{\Gamma_t} G^j_{mi}(\mathbf{x}-\xi) \bar{t}_j(\mathbf{x}) d\Gamma d\xi_m \\
&\quad + \int_{\Gamma_{st}} v_i(\xi) n_m(\xi) \int_{\Gamma_t} H^i_{mj}(\mathbf{x}-\xi) \bar{t}_j(\mathbf{x}) d\Gamma d\Gamma_s \\
&\quad - \int_{\Gamma_{st}} D_t v_i(\xi) \int_{\Gamma_u} C_{timj}(\mathbf{x}-\xi) D_m \bar{u}_j(\mathbf{x}) d\Gamma d\Gamma_s \\
&\quad + \oint_{\partial\Gamma_{st}} v_i(\xi) \int_{\Gamma_u} C_{timj}(\mathbf{x}-\xi) D_m \bar{u}_j(\mathbf{x}) d\Gamma d\xi_t
\end{aligned} \tag{70}
$$

where $\bar{u}$ and $\bar{t}$ are the prescribed displacement and tractions, respectively, on the boundary Γ_u, and on the boundary Γ_t.

To obtain the discrete equations based on MLS interpolation, using the interpolations (45) and (57), and substituting them into formulations (63) and (64) leads to the following discretized system of linear equations for each node I

$$\sum_m (A^m_{IpJj}\hat{t}^J_j + B^m_{IpJj}\hat{u}^J_j) = F_{Ip} \tag{71}$$

$$\sum_m (G^m_{IpJj}\hat{t}^J_j + H^m_{IpJj}\hat{u}^J_j) = Q_{Ip} \tag{72}$$

where

$$A^m_{IpJj} = \int_{\Gamma^I_{su}} \psi^I(\xi) \int_{\Gamma^m_u} \phi^J(\mathbf{x}) u^*_{jp}(\mathbf{x},\xi) d\Gamma d\Gamma_s \tag{73}$$

$$B^m_{IpJj} = \int_{\Gamma^I_{su}} \psi^I \int_{\Gamma^m_t} \{ H^p_{nj}(\mathbf{x}-\xi) n_n \phi^J d\Gamma d\Gamma_s + G^p_{nj}(\mathbf{x}-\xi) D_n \phi^J \} d\Gamma d\Gamma_s \tag{74}$$

$$\begin{aligned} G^m_{IpJj} &= \int_{\Gamma^I_{st}} D_n \psi^I \int_{\Gamma^m_u} G^j_{np}(\mathbf{x}-\xi) \phi^J d\Gamma d\Gamma_s - \oint_{\partial\Gamma^I_{st}} \psi^I \int_{\Gamma^m_u} G^j_{mp}(\mathbf{x}-\xi) \phi^J d\Gamma d\xi_m \\ &- \int_{\Gamma^I_{st}} \psi^I n_n(\xi) \int_{\Gamma^m_u} H^p_{nj}(\mathbf{x}-\xi) \phi^J d\Gamma d\Gamma_s \end{aligned} \tag{75}$$

$$\begin{aligned} H^m_{IpJj} &= \int_{\Gamma^I_{st}} D_t \psi^I \int_{\Gamma^m_t} C_{tpnj}(\mathbf{x}-\xi) D_n \phi^J d\Gamma d\Gamma_s \\ &- \oint_{\partial\Gamma^I_{st}} \psi^I \int_{\Gamma^m_t} C_{tpnj}(\mathbf{x}-\xi) D_n \phi^J(\mathbf{x}) d\Gamma d\xi_t \end{aligned} \tag{76}$$

$$F_{Ip}=\int_{\Gamma^I_{su}}C\bar{u}_p(\xi)\psi^I d\Gamma_s-\int_{\Gamma^I_{su}}\psi^I\int_{\Gamma_t}\bar{t}_j(\mathbf{x})u^*_{jp}(\mathbf{x},\xi)d\Gamma d\Gamma_s$$
$$-\int_{\Gamma^I_{su}}\psi^I\int_{\Gamma_u}H^p_{mj}(\mathbf{x},\xi)n_m(\xi)\bar{u}_j(\mathbf{x})d\Gamma d\Gamma_s \tag{77}$$
$$-\int_{\Gamma^I_{su}}\psi^I\int_{\Gamma_u}D_m\bar{u}_j(\mathbf{x})G^P_{mj}(\mathbf{x}-\xi)d\Gamma d\Gamma_s$$

$$Q_{Ip}=-\int_{\Gamma^I_{st}}C\bar{t}_p(\xi)\psi^I d\Gamma_s-\int_{\Gamma^I_{st}}D_n\psi^I\int_{\Gamma_t}G^j_{np}(\mathbf{x}-\xi)\bar{t}_j(\mathbf{x})d\Gamma d\Gamma_s$$
$$+\oint_{\partial\Gamma^I_{st}}\psi^I\int_{\Gamma}G^j_{np}(\mathbf{x}-\xi)\bar{t}_j(\mathbf{x})d\Gamma d\xi_m$$
$$+\int_{\Gamma^I_{st}}\psi^I n_n(\xi)\int_{\Gamma_t}H^p_{nj}(\mathbf{x}-\xi)\bar{t}_j(\mathbf{x})d\Gamma d\Gamma_s \tag{78}$$
$$-\int_{\Gamma^I_{st}}D_t\psi^I\int_{\Gamma_u}C_{tpnj}(\mathbf{x}-\xi)D_n\bar{u}_j(\mathbf{x})d\Gamma d\Gamma_s$$
$$+\oint_{\partial\Gamma^I_{st}}\psi^I\int_{\Gamma_u}C_{tpnj}(\mathbf{x}-\xi)D_n\bar{u}_j(\mathbf{x})d\Gamma d\xi_t$$

m is the number of the background mesh, and I is the nodal number. It is noted that the repeated indices imply summation here. Note again that the trial functions appear only in the global integration, which means that the integrand for the local integration will be simple.

Similar to Subsection 4.1, four test functions are chosen to formulate four different MLPG methods for the mixed BIE:

(1) MLPG/MBIE1: the test function over Γ_s is the same as the weight function in the MLS approximation. In this case, let $\psi^I=w^I$ in eqns (73), (74) and (77), and rewrite eqns (75), (76) and (78) as

$$G^m_{IpJj}=\int_{\Gamma^I_{st}}D_n w_I\int_{\Gamma^m_u}G^j_{np}(\mathbf{x}-\xi)\phi^J d\Gamma d\Gamma_s-\int_{\Gamma^I_{st}}w_I n_n(\xi)\int_{\Gamma^m_u}H^p_{nj}(\mathbf{x}-\xi)\phi^J d\Gamma d\Gamma_s \tag{79}$$

$$H^m_{IpJj}=\int_{\Gamma^I_{st}}D_t w_I\int_{\Gamma^m_t}C_{tpnj}(\mathbf{x}-\xi)D_n\phi^J d\Gamma d\Gamma_s \tag{80}$$

$$Q_{Ip} = -\int_{\Gamma_{su}^I} C\bar{t}_p(\xi) w_I d\Gamma_s - \int_{\Gamma_{st}^I} D_n w_I \int_{\Gamma_t} G_{np}^j(\mathbf{x}-\xi)\bar{t}_j(\mathbf{x}) d\Gamma d\Gamma_s$$
$$+ \int_{\Gamma_{st}^I} w_I n_n(\xi) \int_{\Gamma_t} H_{nj}^p(\mathbf{x}-\xi)\bar{t}_j(\mathbf{x}) d\Gamma d\Gamma_s \tag{81}$$
$$- \int_{\Gamma_{st}^I} D_t w_I \int_{\Gamma_u} C_{tpnj}(\mathbf{x}-\xi) D_n \bar{u}_j(\mathbf{x}) d\Gamma d\Gamma_s$$

Here that the test function (weight function) vanishes at $\partial\Gamma_s$ is considered.

(2) MLPG/MBIE2: the test function over Γ_s is the collocation Dirac's Delta function (collocation method). In this case, let $\psi^I=\delta(\xi^I)$ in eqns (73), (74) and (77), then it can be found that we get the collocation BEM.

(3) MLPG/MBIE5: the test function over Γ_s is the Heaviside step function (constant over each local sub-domain Γ_s). In this case, let $\psi^I=1$ in eqns (73), (74) and (77), and eqns (75), (76) and (78) can be rewritten as

$$G_{IpJj}^m = -\oint_{\partial\Gamma_{st}^I} \int_{\Gamma_u^m} G_{mp}^j(\mathbf{x}-\xi)\phi^J d\Gamma d\xi_m - \int_{\Gamma_{st}^I} n_n(\xi) \int_{\Gamma_u^m} H_{nj}^p(\mathbf{x}-\xi)\phi^J d\Gamma d\Gamma_s \tag{82}$$

$$H_{IpJj}^m = -\oint_{\partial\Gamma_{st}^I} \int_{\Gamma_t^m} C_{tpnj}(\mathbf{x}-\xi) D_n \phi^J(\mathbf{x}) d\Gamma d\xi_t \tag{83}$$

$$Q_{Ip} = -\int_{\Gamma_{su}^I} C\bar{t}_p(\xi) d\Gamma_s + \oint_{\partial\Gamma_{st}^I} \int_{\Gamma} G_{mp}^j(\mathbf{x}-\xi)\bar{t}_j(\mathbf{x}) d\Gamma d\xi_m$$
$$+ \int_{\Gamma_{st}^I} n_n(\xi) \int_{\Gamma_t} H_{nj}^p(\mathbf{x}-\xi)\bar{t}_j(\mathbf{x}) d\Gamma d\Gamma_s \tag{84}$$
$$+ \oint_{\partial\Gamma_{st}^I} \int_{\Gamma_u} C_{tpnj}(\mathbf{x}-\xi) D_n \bar{u}_j(\mathbf{x}) d\Gamma d\xi_t$$

It is well-known that the numerical integration plays an important role in the convergence of numerical solutions of meshless methods. From equation (83), it can be seen that some surface integrals over Γ_s replaced by the curve integrals over $\partial\Gamma_s$, which will improve effectiveness of this method.

(4) MLPG/MBIE6: the test function over Γ_s is identical to the trial function (Galerkin method). In the Galerkin method, the trial function and test function come from the same space. In this case, let $q_i=t_i(\mathbf{x})$ and $v_i=u_i(\mathbf{x})$, then we have

$$A^{m}_{IpJj} = \int_{\Gamma^I_{su}} \phi^I \int_{\Gamma^m_u} \phi^J u^*_{jp}(\mathbf{x},\xi)d\Gamma d\Gamma_s \tag{85}$$

$$B^{m}_{IpJj} = \int_{\Gamma^I_{su}} \phi^I \int_{\Gamma^m_t} \{H^p_{nj}(\mathbf{x}-\xi)n_n\phi^J d\Gamma d\Gamma_s + G^p_{nj}(\mathbf{x}-\xi)D_n\phi^J\}d\Gamma d\Gamma_s \tag{86}$$

$$G^{m}_{IpJj} = \int_{\Gamma^I_{st}} D_n\phi^I \int_{\Gamma^m_u} G^j_{np}(\mathbf{x}-\xi)\phi^J d\Gamma d\Gamma_s - \int_{\Gamma^I_{st}} \phi^I n_n(\xi)\int_{\Gamma^m_u} H^p_{nj}(\mathbf{x}-\xi)\phi^J d\Gamma d\Gamma_s \tag{87}$$

$$H^{m}_{IpJj} = \int_{\Gamma^I_{st}} D_t\phi^I \int_{\Gamma^m_t} C_{tpnj}(\mathbf{x}-\xi)D_n\phi^J d\Gamma d\Gamma_s \tag{88}$$

$$\begin{aligned} F_{Ip} = & \int_{\Gamma^I_{su}} C\bar{u}_p(\xi)\phi^I d\Gamma_s - \int_{\Gamma^I_{su}} \phi^I \int_{\Gamma_t} \bar{t}_j(\mathbf{x})u^*_{jp}(\mathbf{x},\xi)d\Gamma d\Gamma_s \\ & - \int_{\Gamma^I_{su}} \phi^I \int_{\Gamma_u} H^p_{mj}(\mathbf{x},\xi)n_m(\xi)\bar{u}_j(\mathbf{x})d\Gamma d\Gamma_s \\ & - \int_{\Gamma^I_{su}} \phi^I \int_{\Gamma_u} D_m\bar{u}_j(\mathbf{x})G^p_{mj}(\mathbf{x}-\xi)d\Gamma d\Gamma_s \end{aligned} \tag{89}$$

$$\begin{aligned} Q_{Ip} = & -\int_{\Gamma^I_{st}} C\bar{t}_p(\xi)\phi^I d\Gamma_s - \int_{\Gamma^I_{st}} D_n\phi^I \int_{\Gamma_t} G^j_{np}(\mathbf{x}-\xi)\bar{t}_j(\mathbf{x})d\Gamma d\Gamma_s \\ & + \int_{\Gamma^I_{st}} \phi^I n_n(\xi)\int_{\Gamma_t} H^p_{nj}(\mathbf{x}-\xi)\bar{t}_j(\mathbf{x})d\Gamma d\Gamma_s \\ & - \int_{\Gamma^I_{st}} D_t\phi^I \int_{\Gamma_u} C_{tpnj}(\mathbf{x}-\xi)D_n\bar{u}_j(\mathbf{x})d\Gamma d\Gamma_s \end{aligned} \tag{90}$$

If we replace the local boundary by the global boundary in the above equations, the formulations are same as those in SGBEM [Li, Mear and Xiao (1998)].

The interrelationships of these developments can also be illustrated as in Fig. 1. It is noted that in all the MLPG methods in this study, the usual "element assembly" process is not required, unlike the SGBEM, to form the global stiffness matrix.

Due to their flexibility, and due to their potential in negating the need for the human-labor process of constructing meshes along boundary surfaces, such MLPG methods for BIE are especially useful in those problems with discontinuities or moving boundaries. The main objective of the meshless methods is to get rid of, or at least alleviate the difficulty of, meshing and remeshing the entire boundary surface; by only adding or deleting nodes in the entire boundary surface, instead.

The MLPG methods for BIE are characterized by weakly singular kernels and meshless local weak form. SGBEM is also characterized by weakly singular kernels, but it is based on the global weak form and meshes [i.e., in SGBEM, two global integrals are involved].

In SGBEM, for a specific pair of elements, which have no common points, the ordinary Gauss rule can provide sufficient accuracy. It is not appropriate for other elements, which are coincident or have one edge or one vertex in common. To deal with weakly singular integrals, there exist methods using transformation of variables in order to weaken or cancel out the singularity by Jacobian of the transformation before applying the ordinary Gauss rule. Thus, in SGBEM, there exist 9 'weakly singular element ', which are coincident or have one edge or one vertex in common with the master element, for every master element for quadrilateral boundary elements, and 13 'weakly singular elements' for every master element for triangular boundary elements, that need the transformation of variable. This procedure will be costly. However, in the MLPG for BIE, for every (master) local boundary surface Γ_s, we can choose an appropriate background mesh and adjust the size of Γ_s, to make Γ_s only involved in one background element, that will improve the effectiveness obviously.

In fact, this method can be derived directly from the local weak-form of the boundary conditions:

$$\int_{\Gamma_{us}} (u_i - \bar{u}_i) q_i \, d\Gamma = 0 \tag{91}$$

$$\int_{\Gamma_{st}} (t_i - \bar{t}_i) v_i \, d\Gamma = 0 \tag{92}$$

where, a priori, the equilibrium equations and constitutive relations are satisfied. Using equations (40) and (39) to represents u_i and t_i respectively, we can obtain the method developed in this subsection.

4.5 Boundary Variational Principle

This subsection deals with boundary solution methods, based on a boundary variational principle [Atluri and Grannell (1978)]. The functions that are assumed to be independent are: displacement $\mathbf{u}$ in the domain; boundary displacement $\tilde{\mathbf{u}}$; and boundary traction $\tilde{\mathbf{t}}$ [which is a Lagrange multipler to enforce $\mathbf{u}=\tilde{\mathbf{u}}$ at Γ].

The corresponding variational functional Π_p, for linear elasticity, is defined as follows:

$$\Pi_p = \int_\Omega \frac{1}{2} \sigma_{ij} \varepsilon_{ij} \, d\Omega - \int_{\Gamma_t} \bar{t}_i \tilde{u}_i \, d\Gamma - \int_\Gamma \tilde{t}_i (u_i - \tilde{u}_i) \, d\Gamma \tag{93}$$

where the boundary displacement $\tilde{\mathbf{u}}$ satisfies the essential boundary condition, i.e., $\tilde{\mathbf{u}} = \bar{\mathbf{u}}$ on Γ_u, $\bar{t}_i$ is the prescribed traction on the traction boundary Γ_t, and no body force is considered.

By carrying out the variations it can be shown that

$$\delta\Pi_p = \int_{\Gamma} (t_i - \tilde{t}_i)\delta u_i d\Gamma - \int_{\Omega} \sigma_{ij,j}\delta u_i \, d\Omega + \int_{\Gamma_t} (\tilde{t}_i - \bar{t}_i)\delta\tilde{u}_i d\Gamma - \int_{\Gamma} (u_i - \tilde{u}_i)\delta\tilde{t}_i d\Gamma \tag{94}$$

with the vanishing of $\delta\Pi_p$, one can also have the following equivalent integral equations:

$$\int_{\Gamma} (t_i - \tilde{t}_i)\delta u_i d\Gamma - \int_{\Omega} \sigma_{ij,j}\delta u_i \, d\Omega = 0 \tag{95}$$

$$\int_{\Gamma} (u_i - \tilde{u}_i)\delta\tilde{t}_i d\Gamma = 0 \tag{96}$$

$$\int_{\Gamma_t} (\tilde{t}_i - \bar{t}_i)\delta\tilde{u}_i d\Gamma = 0 \tag{97}$$

All $\tilde{t}$ and $\tilde{u}$ are unknown. The traction (natural) boundary condition $\tilde{t} = \bar{t}$ on Γ_t and the essential boundary condition $\tilde{u} = \bar{u}$ on Γ_u are satisfied a priori, hence, it can be ignored temporarily in the following development.

It can be seen that equations (95) and (96) hold in any sub-domain. According to the concept of the MLPG (Atluri and Shen, 2002a), we use the following weak forms on a sub-domain Ω_s and the corresponding boundary Γ_s to replace equations (95) and (96)

$$\int_{\Gamma_s} (t_i - \tilde{t}_i)v_i d\Gamma - \int_{\Omega_s} \sigma_{ij,j} v_i \, d\Omega = 0 \tag{98}$$

$$\int_{\Gamma_s} (u_i - \tilde{u}_i) q_i d\Gamma = 0 \tag{99}$$

where **v** and **q** are test functions, and the choice of different test function leads to different MLPG methods. If we take the test functions as the weight function in the MLS approximation, we will obtain the method developed in Zhang and Yao (2001), however, the local weak forms used here are a little different from those in Zhang and Yao (2001), where the boundary of the subdomain Ω_s is involved, that actually should not appear. If the subdomain Ω_s does not intersect with the boundary, equation (98) will not include the boundary integral. In equations (98) and (99), $\tilde{\mathbf{u}}$ and $\tilde{\mathbf{t}}$ on the boundary, and **v** and **q** are interpolated according to equations (45) and (57b). The **u** and **t** inside Ω and on Γ are defined as [Atluri and Grannell (1978), Zhang and Yao (2001)]

$$\begin{aligned} u_i &= \sum_{I=1}^{N} u_{ip}^I a_p^I \\ t_i &= \sum_{I=1}^{N} t_{ip}^I a_p^I \end{aligned} \tag{100}$$

where u_{jp}^I and t_{jp}^I are the fundamental solutions with the source point at a node I, a_p^I are unknown parameters. We only consider the nodes on the boundary.

As u is expressed by equation (100), the last integral on the left-hand side of equation (98) vanishes if one excludes node J from the subdomain Ω_s at which the singularity occurs. This singularity will be considered when evaluating the boundary integrals. Then by substituting equations (45), (57) and (100) into (98) and (99), and omitting the vanished terms, we can obtain the final system of equations. The equations can be solved in the same way as that in Subsection 4.1.

Similar to Subsection 4.1, four test functions are chosen to formulate four different MLPG methods for boundary variational principle:

(1) MLPG1: the test functions v_i and q_i, over Γ_s are the same as the weight function in the MLS approximation. Then it can be found that we obtian the method developed in Zhang and Yao (2001).

(2) MLPG2: the test functions v_i and q_i, over Γ_s is the collocation Dirac's Delta function (collocation method).

(3) MLPG5: the test functions v_i and q_i, over Γ_s is the Heaviside step function (constant over each local sub-domain Γ_s)

(4) MLPG6: the test functions v_i and q_i, over Γ_s is identical to the trial function (Galerkin method). In this case, let $v_i = \tilde{u}_i$ and $q_i = \tilde{t}_i$.

5. Conclusion

The meshless local Petrov-Galerkin method (MLPG) method is extended to treat the boundary integral equations in this paper. Five boundary integral solution methods are introduced: direct solution method; displacement boundary-value problem; traction boundary-value problem; mixed boundary-value problem; and boundary variational principle. Based on the local weak form of BIE, four different nodal-based local test functions are selected, leading to four different MLPG methods for each BIE solution method. These methods combine the advantage of the MLPG and the boundary element method.

A very simple method is used to derive the weakly singular traction boundary integral equation based on the integral relationships for displacement gradients [Okada and Atluri, 1989].

Numerical implementation and demonstrations of the advantages of the presently proposed MLPG method for BIE will be discussed by the authors in forthcoming papers.

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Captions

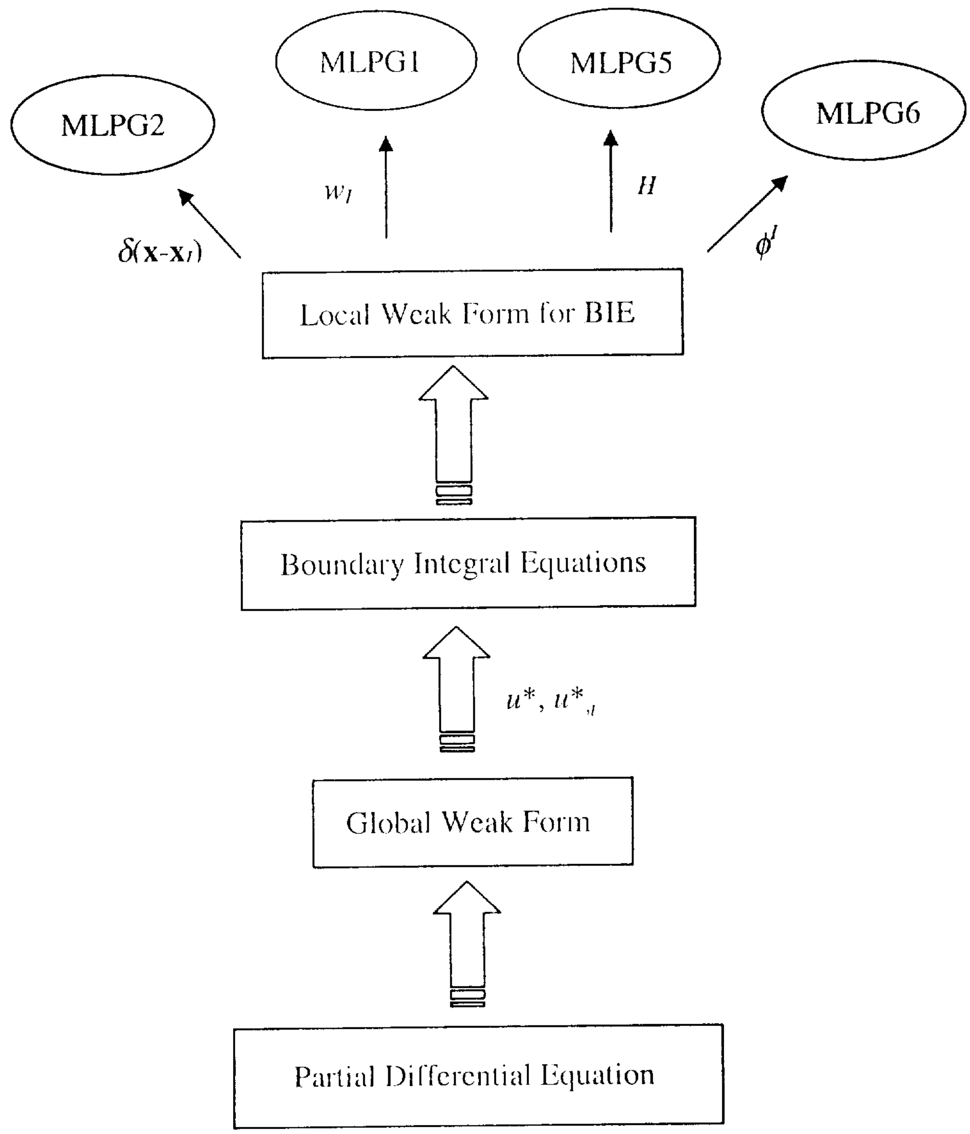

Fig.1 Interrelationship of meshless methods for BIE.